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Abstract

The paper investigates general properties of power indices, measuring the voting power in committees. Concepts of local and global monotonicity of power indices are introduced. Shapley-Shubik, Banzhaf-Coleman, and Holler-Packel indices are analyzed and it is proved that while Shapley-Shubik index satisfies both local and global monotonicity property, Banzhaf-Coleman satisfies only local monotonicity without being globally monotonic and Holler-Packel index satisfies neither local nor global monotonicity.

Keywords
Committee, monotonicity, power index, power axioms, voting

JEL-Classifications
D71, D72, K40

Comments

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1. Introduction

Measuring of voting power in committee systems was introduced by Shapley and Shubik [1954] more than 50 years ago. Recently we can observe growing interest to different aspects of analysis of distribution and concentration of power in committees, related to development of new democratic structures in countries in transition as well as to institutional reform in European Union. Both methodological and applied problems of analysis of power are being intensively discussed during last few years (see e.g. Gambarelli [1990], [1992], Holler and Li [1995], Holubiec and Mercik [1994], Roth [1988], Turnovec [1995, 1996], Widgrén [1994, 1995]).

This paper focuses on methodological issues of power analysis. General properties of different measures of power are investigated. A model of a committee in terms of quotas and allocations of weights of committee members is used instead of traditional model of cooperative simple games. A subset of the set of all committee members is called a winning voting configuration if the total weight of all its members is at least equal to the quota. The power index is defined as a mapping from the space of all committees into the unit simplex, representing a reasonable expectation of the share of voting power given by ability to contribute to formation of winning voting configurations.

Using Allingham [1975] axiomatic characterization of power, amended by an intuitively appealing axiom of so called global monotonicity, the most widely known power indices are confronted with the five axioms of power: dummy (a member of a committee who can contribute nothing to any winning voting configuration has no power), anonymity (power does not depend on committee members' names or numbers), symmetry (members with equal contribution to any voting configuration have the same power), local monotonicity (a member with greater weight cannot have less power than a member with smaller weight), and global monotonicity (if the weight of one member is increasing while the weights of all other members are decreasing or staying the same, the power of the "growing weight" member will at least not decrease).

The most widely known and used power indices, Shapley-Shubik index [1954], Banzhaf-Coleman index [1965, 1971] and Holler-Packel index [1978, 1983], are analyzed and it is proved that while Shapley-Shubik index satisfies all five power axioms, Banzhaf-Coleman satisfies dummy, anonymity, symmetry and local monotonicity axioms without being globally monotonic and Holler-Packel index satisfies only dummy, anonymity and symmetry axioms without being locally monotonic and globally monotonic.

The paper has no ambition to answer the straightforward question: which index is right? It rather indicates that there is something missing in power analysis: a unified approach to the problems of modelling and evaluating of voting and decisional power.

2. Power in Committees

Let N = {1, ..., n} be the set of members (players, parties) and ω_i (i = 1, ..., n) be the (real, non-negative) weight of the i-th member such that

$$\sum_{i \in N} \omega_i = 1, \quad \omega_i \geq 0$$

(e.g. the share of votes of party i, or the ownership of i as a proportion of the total number of shares, etc.). Let γ be a real number such that $0 \leq \gamma \leq 1$.

The (n+1)-tuple

$$[\gamma, \boldsymbol{\omega}] = [\gamma, \omega_1, \omega_2, \ldots, \omega_n]$$

such that we shall call a committee of the size n = card N with quota γ and allocation of weights

$$\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_n)$$

(by *card* S we denote the cardinality of the finite set S, for empty set *card* $\emptyset$ = 0)

Any non-empty subset $S \subset N$ we shall call a voting configuration. Given an allocation ω and a quota γ, we shall say that $S \subset N$ is a winning voting configuration, if

$$\sum_{i \in S} \omega_i \geq \gamma$$

and a losing voting configuration, if

$$\sum_{i \in S} \omega_i < \gamma$$

(i.e. the configuration S is winning, if it has a required majority, otherwise it is losing).

Let

$$\Gamma = \left\{(\gamma, \boldsymbol{\omega}) \in \mathbb{R}_{n+1} : \sum_{i=1}^{n} \omega_i = 1,\ \omega_i \geq 0,\ 0 \leq \gamma \leq 1\right\}$$

be the space of all committees of the size n and

$$E = \left\{ \boldsymbol{e} \in \mathbb{R}_n : \sum_{i \in N} e_i = 1,\ e_i \geq 0\ (i=1, \ldots, n)\right\}$$

be the unit simplex.

A *power index* is a vector valued function

$$\boldsymbol{\pi} : \Gamma \rightarrow E$$

that maps the space Γ of all committees into the unit simplex E. A power index represents a reasonable expectation of the share of decisional power among the various members of a committee, given by ability to contribute to formation of winning voting configurations. We shall denote by $\pi_i(\gamma, \omega)$ the share of power that the index π grants to the i-th member of a committee with weight allocation ω and quota γ. Such a share is called a *power index of the i-th member*.

3. Axiomatization of Power

A member $i \in N$ of the committee $[\gamma, \omega]$ is said to be **dummy** if he cannot benefit any voting configuration by joining it, i.e. the player i is dummy if

$$\sum_{k \in S} \omega_i \geq \gamma \Rightarrow \sum_{k \in S-\{i\}} \omega_i \geq \gamma$$

for any winning configuration $S \subseteq N$ such that $i \in S$

Example 1

Let $[\gamma; \omega] = [0.7; 0.4, 0.4, 0.2]$, then we have only one winning configuration {1, 2, 3} with the member 3:

$$\omega_1 + \omega_2 + \omega_3 = 1 > \gamma$$

and we know that

$$\omega_1 + \omega_2 = 0.8 > \gamma$$

then player 3 is dummy.

Two distinct members i and j of a committee $[\gamma, \omega]$ are called **symmetric** if their benefit to any voting configuration is the same, that is, for any S such that $i, j \notin S$

$$\sum_{k \in S \cup \{i\}} \omega_k \geq \gamma \Leftrightarrow \sum_{k \in S \cup \{j\}} \omega_k \geq \gamma$$

Obviously, if for two members i and j of a committee $[\gamma, \omega]$ holds $\omega_i = \omega_j$, then i and j are symmetric.

Example 2

Let $[\gamma; \omega] = [0.7; 0.5, 0.3, 0.2]$. There is only one configuration not containing members 2 and 3, which is {1}. We can see that

$$\omega_1 + \omega_2 = 0.8 > \gamma, \quad \omega_1 + \omega_3 = 0.7 = \gamma$$

In this case the members 2 and 3 are symmetric.

Let [γ, ω] be a committee with the set of members N and

$$\sigma : N \rightarrow N$$

be a permutation mapping. Then the committee

$$[\gamma, \sigma\boldsymbol{\omega}]$$

we shall call a **permutation** of the committee [γ, ω] and σ(i) is the new number of the member with original number i.

Example 3

Let [γ; ω] = [0.7; 0.5, 0.3, 0.2] and σ(N) = (3, 1, 2), then σ(1) = 3, σ(2) = 1, σ(3) = 2, and [γ, σω] = [0.7; 0.2, 0.3, 0.5].

Let $\pi_i(\gamma, \omega)$ be a measure of power of a member i in a committee with quota γ and allocation ω, then (assuming no additional information about the structure of the committee and specific voting rules) it is natural to expect that some minimal intuitively acceptable properties should be satisfied by a reasonable π. The following axiomatic characterization of power indices (in slightly different form) was introduced by Allingham [1975]:

Axiom D (dummy)
Let [γ; ω] is a committee and i is dummy, then

$$\pi_i(\gamma ; \boldsymbol{\omega}) = 0$$

Dummy member has no power.

Axiom A (anonymity)
Let [γ; ω] is a committee and [γ; σω] its permutation, then

$$\pi_{\sigma(i)}(\gamma, \sigma\boldsymbol{\omega}) = \pi_i(\gamma, \boldsymbol{\omega}]$$

The power is a property of committee and not of players names and numbers.

Axiom S (symmetry)

Let [γ; ω] is a committee and i and j ($i \neq j$) are symmetric, then

$$\pi_i(\gamma, \boldsymbol{\omega}) = \pi_j(\gamma, \boldsymbol{\omega})$$

The power of symmetric members is the same.

Axiom LM (local monotonicity)

Let [γ; ω] be a committee and $\omega_i > \omega_j$, then

$$\pi_i(\gamma, \boldsymbol{\omega}) \geq \pi_j(\gamma, \boldsymbol{\omega})$$

The member with greater weight cannot have less power than the member with smaller weight.

Turnovec [1994] suggested the additional fifth axiom:

Axiom GM (global monotonicity)

Let [γ, α] and [γ, β] be two different committees of the same size such that $\alpha_k > \beta_k$ for one $k \in N$ and $\alpha_i \leq \beta_i$ for all $i \neq k$, then

$$\pi_k(\gamma, \boldsymbol{\alpha}) \geq \pi_k(\gamma, \boldsymbol{\beta})$$

If the weight of one member is increasing and the weights of all other members are decreasing or staying the same, then the power of the "growing weight" member will at least not decrease.

4. Marginality

We shall term a member i of a committee [γ, ω] to be **marginal** (essential, critical, decisive) with respect to a configuration $S \subseteq N$, $i \in S$, if

$$\sum_{k \in S} \omega_k \geq \gamma$$

$$and$$

$$\sum_{k \in S \setminus \{i\}} \omega_k < \gamma$$

A voting configuration $S \subseteq N$ such that at least one member of the committee is marginal with respect to S we shall call a critical winning configuration (CWC).

Let us denote by $C(\gamma, \omega)$ the set of all CWC in the committee with the quota γ and allocation ω. By $C_i(\gamma, \omega)$ we shall denote the set of all CWC the member $i \in N$ is marginal with respect to, and by $C_{is}(\gamma, \omega)$ the set of all CWC of the size s (by size we mean cardinality of CWC, $1 \leq s \leq n$) the member $i \in N$ is marginal with respect to. Then

$$C_i(\gamma, \boldsymbol{\omega}) = \bigcup_{s=1}^{n} C_{is}(\gamma, \boldsymbol{\omega}) = \bigcup_{s=1}^{n} P_{is}(\gamma, \boldsymbol{\omega})$$

$$C(\gamma, \boldsymbol{\omega}) = \bigcup_{i \in N} C_i(\gamma, \boldsymbol{\omega})$$

A voting configuration $S \subset N$ is said to be a minimal critical winning configuration (MCWC), if

$$\sum_{i \in S} \omega_i \geq \gamma \quad and \quad \sum_{k \in T} \omega_k < \gamma \quad for\ any\ T \subsetneq S$$

Let us denote by $M(\gamma, \omega)$ the set of all MCWC in the committee [γ, ω]. By $M_{is}(\gamma, \omega)$ we shall denote the set of all minimal critical winning configurations S of the size s such that $i \in S$, and by $M_i(\gamma, \omega)$ the set of all minimal winning configurations containing member i.

Lemma 1

A member $i \in N$ of a committee [γ, ω] is **dummy** if and only if for any s = 1, 2, ..., n

$$card C_{is}(\gamma, \boldsymbol{\omega}) = 0$$

Proof: trivial

Remark

Since for all s

$$M_{is}(\gamma, \boldsymbol{\omega}) \subseteq C_{is}(\gamma, \boldsymbol{\omega})$$

the member i ∈ N is dummy if and only if for any s = 1, 2, ..., n

$$\mathrm{card} M_{is}(\gamma, \boldsymbol{\omega}) = 0$$

Lemma 2

Let [γ; ω] is a committee and [γ; σω] its **permutation**, then for any s = 1, 2, ..., n

$$\mathrm{card}\, C_{is}(\gamma, \boldsymbol{\omega}) = \mathrm{card}\, C_{\sigma(i)s}(\gamma, \sigma\boldsymbol{\omega})$$

Proof: trivial

Remark

The same statement is apparently true for the sets $M_{is}(\gamma, \omega)$; for any s = 1, 2, ..., n

$$\mathrm{card} M_{is}(\gamma, \boldsymbol{\omega}) = \mathrm{card} M_{\sigma(i)s}(\gamma, \sigma\boldsymbol{\omega})$$

Lemma 3

If the members k, r ∈ N (k ≠ r) of a committee [γ, ω] are **symmetric**, then for any s = 1, 2, ..., n

$$\mathrm{card}\, C_{ks}(\gamma, \boldsymbol{\omega}) = \mathrm{card}\, C_{rs}(\gamma, \boldsymbol{\omega})$$

Proof:

Let k, r are symmetric and let $S \in C_{ks}(\gamma, \omega)$. We want to prove that either $S \in C_{rs}(\gamma, \omega)$, or $\{S \setminus \{k\} \cup \{r\}\} \in C_{rs}(\gamma, \omega)$, so to any configuration S from $C_{ks}(\gamma, \omega)$ there corresponds at least one configuration in $C_{rs}(\gamma, \omega)$. There are two possibilities:

a) If $r \in S$, then (from marginality of k with respect to S)

$$\sum_{S\setminus\{r\}\setminus\{k\}} \omega_i + \omega_r + \omega_k \geq \gamma, \quad \sum_{S\setminus\{r\}\setminus\{k\}} \omega_i + \omega_r < \gamma$$

and from symmetry of k and r

$$\sum_{S\setminus\{k\}\setminus\{r\}} \omega_i + \omega_k < \gamma$$

That implies marginality of r with respect to S and $S \in C_{rs}(\gamma, \omega)$.

b) If $r \notin S$, then from marginality of k with respect to S

$$\sum_{i \in S\setminus\{k\}} \omega_i + \omega_k \geq \gamma, \quad \sum_{i \in S\setminus\{k\}} \omega_i < \gamma$$

and from symmetry of k and r

$$\sum_{i \in S\setminus\{k\}} \omega_i + \omega_k \geq \gamma \quad \Rightarrow \quad \sum_{i \in S\setminus\{k\}} \omega_i + \omega_r \geq \gamma$$

and we know that

$$\sum_{i \in S\setminus\{k\}} \omega_i < \gamma$$

hence r is marginal with respect to $\{S\setminus\{k\}\cup\{r\}\}$ and $\{S\setminus\{k\}\cup\{r\}\} \in C_{rs}(\gamma, \omega)$.

Resuming cases a) and b), to any $S \in C_{ks}(\gamma, \omega)$ there corresponds one configuration in $C_{rs}(\gamma, \omega)$ and

$$card C_{ks}(\gamma, \boldsymbol{\omega}) \leq card C_{rs}(\gamma, \boldsymbol{\omega})$$

By the same way we can show that

$$card C_{rs}(\gamma, \boldsymbol{\omega}) \leq card C_{ks}(\gamma, \boldsymbol{\omega})$$

what implies

$$card C_{ks}(\gamma, \boldsymbol{\omega}) = card C_{rs}(\gamma, \boldsymbol{\omega})$$

QED.

Remark:

Using the same arguments we can show that for k, r symmetric and for any s

$$card M_{ks}(\gamma, \boldsymbol{\omega}) = card M_{rs}(\gamma, \boldsymbol{\omega})$$

Conjecture to be proved: if for all s

$$card C_{ks}(\gamma, \boldsymbol{\omega}) = card C_{rs}(\gamma, \boldsymbol{\omega})$$

then k, r are symmetric.

Lemma 4

Let $[\gamma, \omega]$ be a committee such that $\omega_k > \omega_r$, then for any s = 1, 2, ..., n

$$card C_{ks}(\gamma, \boldsymbol{\omega}) \geq card C_{rs}(\gamma, \boldsymbol{\omega})$$

Proof:

For any s = 1, 2, ..., n and any voting configuration $S \in C_{rs}(\gamma, \omega)$ we can consider two cases: a) $k \in S$, b) $k \notin S$. By definition of marginality

$$\omega_r + \sum_{i \in S \setminus \{r\}} \omega_i \geq \gamma, \qquad \sum_{i \in S \setminus \{r\}} \omega_i < \gamma$$

a) If $k \in S$, then from marginality of r with respect to S and assumption of the lemma it follows that

$$\sum_{i \in S} \omega_i = \omega_k + \omega_r + \sum_{i \in S \setminus \{r\} \setminus \{k\}} \omega_i \geq \gamma$$

and

$$\sum_{i\in S\setminus\{k\}} \omega_i = \omega_r + \sum_{i\in S\setminus\{r\}\setminus\{k\}} \omega_i < \gamma$$

what implies that $S \in C_{ks}(\gamma, \omega)$.

b) If $k \notin S$, then from marginality of r with respect to S and $\omega_k > \omega_r$ it follows that

$$\omega_k + \sum_{i\in S\setminus\{r\}} \omega_i \geq \omega_r + \sum_{i\in S\setminus\{r\}} \omega_i \geq \gamma, \qquad \sum_{i\in S\setminus\{r\}} \omega_i < \gamma$$

what implies that $\{S \setminus \{r\}\} \cup \{k\} \in C_{ks}(\gamma, \omega)$.

Resuming cases a) and b) we receive

$$S \in C_{rs}(\gamma, \boldsymbol{\omega}) \Rightarrow \begin{cases} S \in C_{ks}(\gamma, \boldsymbol{\omega}) & \text{if } k \in S \\ \{S\setminus\{r\}\}\cup\{k\} \in C_{ks}(\gamma, \boldsymbol{\omega}) & \text{if } k \notin S \end{cases}$$

hence

$$card[C_{ks}(\gamma, \boldsymbol{\omega})] \geq card[C_{rs}(\gamma, \boldsymbol{\omega})]$$

QED.

Remark:

Let $[\gamma, \omega]$ be a committee such that $\omega_k > \omega_r$, then

$$\text{card}\, C_k(\gamma, \boldsymbol{\omega}) = \sum_{s=1}^{n} \text{card}\, C_{ks}(\gamma, \boldsymbol{\omega}) \geq \sum_{s=1}^{n} \text{card}\, C_{rs}(\gamma, \boldsymbol{\omega}) = \text{card}\, C_r(\gamma, \boldsymbol{\omega})$$

Lemma 5

Let $[\gamma, \alpha]$ and $[\gamma, \beta]$ be two committees of the size n such that for one $k \in N$

$$\alpha_k > \beta_k$$

and for all $i \neq k$

$$\alpha_i \leq \beta_i$$

then for any $s = 1, 2, \ldots, n$

$$\mathrm{card}\, C_{ks}(\gamma, \boldsymbol{\alpha}) \geq \mathrm{card}\, C_{ks}(\gamma, \boldsymbol{\beta})$$

Proof:

Let $S \in C_{ks}(\gamma, \beta)$, then by definition of marginality of k with respect to S

$$\beta_k + \sum_{i \in S \setminus \{k\}} \beta_i \geq \gamma, \qquad \sum_{i \in S \setminus \{k\}} \beta_i < \gamma$$

From definition of a committee

$$\alpha_k = 1 - \sum_{i \in N \setminus \{k\}} \alpha_i$$

$$\beta_k = 1 - \sum_{i \in N \setminus \{k\}} \beta_i$$

and from assumption of the lemma it follows that

$$\alpha_k - \beta_k = \sum_{i \in N \setminus \{k\}} (\beta_i - \alpha_i) \geq \sum_{i \in S \setminus \{k\}} (\beta_i - \alpha_i) \geq 0$$

Then

$$\alpha_k + \sum_{i \in S \setminus \{k\}} \alpha_i \geq \beta_k + \sum_{i \in S \setminus \{k\}} \beta_i \geq \gamma$$

$$\sum_{i \in S \setminus \{k\}} \alpha_i \leq \sum_{i \in S \setminus \{k\}} \beta_i < \gamma$$

(marginality of k with respect to S in the committee $[\gamma, \beta]$ implies marginality of k with respect to S in the committee $[\gamma, \alpha]$). Thus

$$S \in C_{ks}(\gamma, \boldsymbol{\beta}) \Rightarrow S \in C_{ks}(\gamma, \boldsymbol{\alpha})$$

and

$$C_{ks}(\gamma, \boldsymbol{\beta}) \subseteq C_{ks}(\gamma, \boldsymbol{\alpha})$$

and we have

$$\mathrm{card}\, C_{ks}(\gamma, \boldsymbol{\alpha}) \geq \mathrm{card}\, C_{ks}(\gamma, \boldsymbol{\beta})$$

QED.

Remark:

Let [γ, α] and [γ, β] be two committees of the size n such that

$$\alpha_k > \beta_k$$
$$\alpha_i \leq \beta_i \quad for\ i \neq k$$

then

$$C_k(\gamma,\boldsymbol{\beta}) = \bigcup_{s=1}^{n} C_{ks}(\gamma,\boldsymbol{\beta}) \subseteq \bigcup_{s=1}^{n} C_{ks}(\gamma,\boldsymbol{\alpha}) = C_k(\gamma,\boldsymbol{\alpha})$$

hence

$$card C_k(\gamma,\boldsymbol{\alpha}) \geq card C_k(\gamma,\boldsymbol{\beta})$$

5. Properties of Power Indices

The three most widely known power indices were proposed by Shapley and Shubik (1954), Banzhaf and Coleman (1965, 1971), and Holler and Packel (1978, 1983). All of them measure the power of each member of a committee as a weighted average of the number of his marginalities in CWC or MCWC.

The Shapley-Shubik (SS) power index assigns to each member of a committee the share of power proportional to the number of permutations of members in which he is pivotal. A permutation is an ordered list of all members. A member is in a pivotal position in a permutation if in the process of forming this permutation by equiprobable additions of single members he provides the critical weight to convert the losing configuration of the preceeding members to the winning one. It is assumed that all winning configurations are possible and all permutations are equally likely. Thus SS power index assigns to the i-th member of a committee with quota γ and weight allocation ω the value of his share of power

$$\pi_i^{SS}(\gamma,\boldsymbol{\omega}) = \sum_{S\subseteq N} \frac{(\mathrm{card}S-1)!\,(\mathrm{card}N-\mathrm{card}S)!}{(\mathrm{card}N)!} \qquad (1)$$

where the sum is extended to all winning configurations S for which the i-th member is marginal.

The Banzhaf-Coleman (BC) power index assigns to each member of a committee the share of power proportional to the number of critical winning configurations for which the member is marginal. It is assumed that all critical winning configurations are possible and equally likely. Banzhaf suggested the following measure of power distribution in a committee:

$$\pi_i^{BC}(\gamma,\boldsymbol{\omega}) = \frac{\mathrm{card}\,C_i(\gamma,\boldsymbol{\omega})}{\sum_{k\in N} \mathrm{card}\,C_k(\gamma,\boldsymbol{\omega})} \tag{2}$$

(so called normalized BC-power index).

The *Holler-Packel* (HP) power index assigns to each member of a committee the share of power proportional to the number of minimal critical winning configurations he is a member of. It is assumed that all winning configurations are possible but only minimal critical winning configurations are being formed to exclude free-riding of the members that cannot influence the bargaining process. "Public good" interpretation of the power of MCWC (the power of each member is identical with the power of the MCWC as a whole, power is indivisible) is used to justify HP index. Holler [1978] suggested and Holler and Packel [1983] axiomatized the following measure of power distribution in a committee:

$$\pi_i^{HP}(\gamma,\boldsymbol{\omega}) = \frac{\mathrm{card}\,M_i(\gamma,\boldsymbol{\omega})}{\sum_{k\in N} \mathrm{card}\,M_k(\gamma,\boldsymbol{\omega})} \tag{3}$$

Example 4

Let $[\gamma, \omega] = [0.6; 0.5, 0.3, 0.2]$. The set of CWC and MCWC member 1 is marginal with respect to:

$$C_1(\gamma,\boldsymbol{\omega}) = (\{1,2\},\{1,3\},\{1,2,3\})$$
$$M_1(\gamma,\boldsymbol{\omega}) = (\{1,2\},\{1,3\})$$

The set of CWC and MCWC member 2 is marginal with respect to:

$$C_2(\gamma,\boldsymbol{\omega}) = M_2(\gamma,\boldsymbol{\omega}) = (\{1,2\})$$

The set of CWC and MWC member 3 is marginal with respect to:

$$C_3(\gamma, \boldsymbol{\omega}) = M_3(\gamma, \boldsymbol{\omega}) = (\{1, 3\})$$

SS-index (weighted sum of number of pivotal position in one member, two member and three member configurations):

$$\pi_1^{SS}(\gamma, \boldsymbol{\omega}) = 0\left(\frac{1}{3}\right) + 2\left(\frac{1}{6}\right) + 1\left(\frac{1}{3}\right) = \frac{2}{3}$$

$$\pi_2^{SS}(\gamma, \boldsymbol{\omega}) = 0\left(\frac{1}{3}\right) + 1\left(\frac{1}{6}\right) + 0\left(\frac{1}{3}\right) = \frac{1}{6}$$

$$\pi_3^{SS}(\gamma, \boldsymbol{\omega}) = 0\left(\frac{1}{3}\right) + 1\left(\frac{1}{6}\right) + 0\left(\frac{1}{3}\right) = \frac{1}{6}$$

BC-index (ratio of the number of CWC a particular member is marginal with respect to the total number of all CWC of all members):

$$\pi_1^{BC}(\gamma, \boldsymbol{\omega}) = \frac{3}{5}$$

$$\pi_2^{BC}(\gamma, \boldsymbol{\omega}) = \frac{1}{5}$$

$$\pi_3^{BC}(\gamma, \boldsymbol{\omega}) = \frac{1}{5}$$

HP-index (ratio of the number of MCWC containing a particular member to the total number of all MCWC of all members):

$$\pi_1^{HP}(\gamma, \boldsymbol{\omega}) = \frac{2}{4}$$

$$\pi_2^{HP}(\gamma, \boldsymbol{\omega}) = \frac{1}{4}$$

$$\pi_3^{HP}(\gamma, \boldsymbol{\omega}) = \frac{1}{4}$$

Which index is right? This is an issue for a rather extensive discussion that can lead to refinement of the original model. Each of the indices apparently answers a slightly different question and the problem is to formulate explicitly the relevant question as a part of the model of a committee. In this paper we are not going to follow this direction. The purpose of the paper is to confront the three indices with the five axioms of power introduced in section 2.

Theorem 1

Shapley-Shubik power index satisfies axioms D, A, S, LM and GM.

Proof

It is obvious that formula (1) for SS-index can be rewritten in the following way:

$$\pi_i^{SS}(q, \mathbf{w}) = \sum_{s=1}^{n} \frac{(s-1)!\,(n-s)!}{n!} \operatorname{card} C_{is}(\gamma, \boldsymbol{\omega})$$

Then from lemma 1 it follows that axiom D is satisfied, from lemma 2 axiom S, from lemma 3 axiom A, from lemma 4 axiom LM and from lemma 5 axiom GM.

Remark

Axioms D and A were explicitly used in axiomatic development of SS-power index (see Shapley and Shubik [1954]).

Theorem 2

Banzhaf-Coleman power index satisfies axioms D, A, S and LM.

Proof

It is obvious that formula (2) for BC power index can be rewritten in the following way:

$$\pi_i^{BC}(q, \mathbf{w}) = \frac{\sum_{s=1}^{n} \operatorname{card} C_{is}(\gamma, \boldsymbol{\omega})}{\sum_{k \in N} \sum_{s=1}^{n} \operatorname{card} C_{ks}(\gamma, \boldsymbol{\omega})}$$

Then from lemma 1 follows axiom D, from lemma 2 axiom A, from lemma 3 axiom S, and from lemma 4 axiom LM.

Remark

The following example shows that Banzhaf-Coleman power index is not globally monotonic: Let us consider two committees of the same size n = 5

$$[\gamma, \boldsymbol{\alpha}] = [9/13;6/13,4/13,1/13,1/13,1/13]$$
$$[\gamma, \boldsymbol{\beta}] = [9/13;5/13,5/13,1/13,1/13,1/13]$$

The weight of the first member in the committee [γ, α] is greater than his weight in the committee [γ, β] while the weights of all other members in [γ, α] are less or equal than in [γ, β]. It can be easily verified that

$$\boldsymbol{\pi}^{BC}(\gamma, \boldsymbol{\alpha}) = \left(\frac{9}{19}, \frac{7}{19}, \frac{1}{19}, \frac{1}{19}, \frac{1}{19}\right)$$
$$\boldsymbol{\pi}^{BC}(\gamma, \boldsymbol{\beta}) = \left(\frac{1}{2}, \frac{1}{2}, 0, 0, 0\right)$$

Theorem 3

Holler-Packel power index satisfies axioms D, A and S.

Proof:

It is obvious that formula (3) for HP power index can be rewritten in the following way:

$$\pi_i^{HP}(q, \mathbf{w}) = \frac{\sum_{s=1}^{n} \mathrm{card} M_{is}(\gamma, \boldsymbol{\omega})}{\sum_{k\in N} \sum_{s=1}^{n} \mathrm{card} M_{ks}(\gamma, \boldsymbol{\omega})}$$

Then from remark to lemma 1 follows axiom D, from remark to lemma 2 axiom A, and from remark to lemma 3 axiom S.

Remark

a) Axioms D and A were explicitly used in axiomatic development of HP index (Holler and Packel [1983]).

b) Holler and Packel [1983] provided following example showing that HP index is not locally monotonic:

Let us consider a committee of the size n = 5

$$[\gamma, \boldsymbol{\omega}] = [0.51;0.35,0.20,0.15,0.15,0.15]$$

The weight of the second member is greater than the weight of the third, fourth and fifth member. It can be easily verified that

$$\pi^{HP}(\gamma,\boldsymbol{\omega}) = \left(\frac{4}{15},\frac{2}{15},\frac{3}{15},\frac{3}{15},\frac{3}{15}\right)$$

so having the greater weight the second member has smaller value of evaluation of power than third, fourth and fifth member.

c) Using the same example as in remark to theorem 2 we can show that HP index is not globally monotonic.

Reconsidering two committees

$$[\gamma,\boldsymbol{\alpha}] = [9/13;6/13,4/13,1/13,1/13,1/13]$$
$$[\gamma,\boldsymbol{\beta}] = [9/13;5/13,5/13,1/13,1/13,1/13]$$

with the weight of the first member in the committee $[\gamma, \alpha]$ greater than his weight in the committee $[\gamma, \beta]$ while the weights of all other members in $[\gamma, \alpha]$ are less or equal than in $[\gamma, \beta]$, it can be easily verified that

$$\pi^{HP}(\gamma,\boldsymbol{\alpha}) = \left(\frac{2}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6}\right)$$
$$\pi^{HP}(\gamma,\boldsymbol{\beta}) = \left(\frac{1}{2},\frac{1}{2},0,0,0\right)$$

Theorem 4

If a power index satisfies axioms S and GM, then it also satisfies LM.

Proof:

Let π satisfies symmetry and global monotonicity axioms. Assume at the same time that π does not satisfy local monotonicity. Then, there exists a vector of weights

$$(\sigma_1,\ldots,\sigma_r,\ldots,\sigma_s,\ldots,\sigma_n)$$

and a quota γ such that $\sigma_r > \sigma_s$ and

$$\pi_r(\gamma,\boldsymbol{\sigma}) < \pi_s(\gamma,\boldsymbol{\sigma})$$

Let us set

$$\lambda_i = \sigma_i \qquad if\ i \neq r, s$$

$$\lambda_r = \sigma_r - \frac{\sigma_r - \sigma_s}{2}$$

$$\lambda_s = \sigma_s + \frac{\sigma_r - \sigma_s}{2}$$

Then, by global monotonicity

$$\pi_s(\gamma, \boldsymbol{\lambda}) \geq \pi_s(\gamma, \boldsymbol{\sigma})$$

$$\pi_r(\gamma, \boldsymbol{\lambda}) \leq \pi_r(\gamma, \boldsymbol{\sigma})$$

and, by symmetry

$$\lambda_r = \lambda_s \quad \Rightarrow \quad \pi_r(\gamma, \boldsymbol{\lambda}) = \pi_s(\gamma, \boldsymbol{\lambda})$$

Then, from our assumption that local monotonicity is not satisfied, we obtain

$$\pi_r(\gamma, \boldsymbol{\lambda}) = \pi_s(\gamma, \boldsymbol{\lambda}) \geq \pi_s(\gamma, \boldsymbol{\sigma}) > \pi_r(\gamma, \boldsymbol{\sigma})$$

and

$$\pi_s(\gamma, \boldsymbol{\lambda}) = \pi_r(\gamma, \boldsymbol{\lambda}) \leq \pi_r(\gamma, \boldsymbol{\sigma}) < \pi_s(\gamma, \boldsymbol{\sigma})$$

what contradicts global monotonicity assumption of the lemma. Hence, global monotonicity and symmetry implies local monotonicity. QED.

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