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Abstract

In a two-stage extensive form game where followers can observe moves by leaders only with noise, pure subgame perfect Nash equilibria of the limiting game without noise may not survive arbitrarily small noise. Still, for generic games, there is always at least one subgame perfect equilibrium outcome of the game with no noise that is approximated by equilibrium outcomes of games with small noise. This, however, depends crucially on generic payoffs.

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1. Introduction

Consider a Stackelberg duopoly with firms choosing quantities (from a finite set). Under standard assumptions the firm which is the leader is better off in the subgame perfect Nash equilibrium of the Stackelberg game than it would be in the Nash equilibrium of the associated Cournot simultaneous-move game. Now suppose that the follower, instead of observing the leader's choice directly, receives a noisy, but quite accurate signal about the leader's choice. Then the only Nash equilibrium in pure strategies generates the same outcome as the Nash equilibrium of the Cournot simultaneous-move game.

This observation is due to **Bagwell** [1995]: In any (finite) generic two-player game, where one player, the "leader", moves first, and then the "follower" decides, after having observed a noisy signal about the leader's choice, the pure strategy Nash equilibrium outcomes coincide with those of the associated simultaneous-move game. This has been interpreted as showing that the first-mover advantage is eliminated when there is a slight amount of noise associated with the observation of the leader's choice.

Such an interpretation, of course, depends on the validity of noise in the signals that the follower receives. Consider, for instance, a monopolist setting a price for a (finite) number of (identical) objects which (a finite number of) consumers may buy at the quoted price or reject (and thus remain without such an object). If there is noise in the observations of the price, then some consumers may find themselves accepting a bargain that the monopolist has never offered. So, whether or not the introduction of noise into the observations of the leader's choice is a valid procedure, depends on the situation which is modeled.

But even if noisy signals make sense, subsequent analysis has shown that it is the focus on pure strategy equilibria which seems to eliminate the first-mover advantage. Call an outcome *accessible* if it is induced by a subgame perfect Nash equilibrium in the game with perfect observability and every game with sufficiently small noise in the observations has a (possibly) mixed equilibrium inducing an outcome close to the accessible one. **Hurkens and van Damme** [1994] show that for any generic two-player game the (unique) subgame perfect equilibrium outcome of the game without noise is accessible. They also show that the approximating mixed equilibria in the games with noise get selected by an appropriate equilibrium selection theory in the spirit of **Harsanyi and Selten** [1988], and are in this sense preferable to the (pure) equilibria of the "one-shot" game.

The present paper adresses the issue in the general set-up of n-player games. Any such "one-shot" game can be transformed into an extensive form game by splitting the player set into two non-empty subsets, the "leaders" and the "followers", such that leaders move first, followers observe a possibly noisy signal about the leaders' strategy combination, and finally make their choices simultaneously. Payoffs need not necessarily be generic, so extensive forms underlying each of the two interactions among leaders and among followers, respectively, are allowed for.[1]

We show that one part of **Bagwell's** [1995] observation continues to hold. In particular, equilibria of the "one-shot" game in which leaders play pure remain as equilibria in which leaders play pure after the introduction of noise. But also the result by **Hurkens and van Damme** [1994] continues to hold. If the underlying "one-shot" game has generic payoffs, then for any partition of the player set into leaders and followers there exists an accessible outcome.

By means of examples we also show that these results are binding in the following sense: For *non-generic* payoffs there are two-player games for which *not all* subgame perfect equilibrium outcomes qualify as accessible. The same class of games contains examples which have *no* accessible outcome at all. Moreover, there are games with *non-generic* payoffs and with more than two players for which even a *set-valued* generalization of accessability fails to exist. Finally, games with several followers and with *generic* payoffs may have subgame perfect equilibrium outcomes which are *not* accessible.

The paper is organized as follows: Section 2 contains definitions and notation. Section 3 gives the positive results which generalize the two results from the earlier literature. Section 4 contains the examples, all of which, except Example 1, are negative in spirit. Section 5 concludes.

2. Definitions and Notation

Let $\Gamma = ((S_i)_{i \in N}, (u_i)_{i \in N})$ be a finite *normal form game* with players $i \in N = \{1, \ldots, n\}$, (finite) pure strategy sets S_i, $S = \times_{i \in N} S_i$, and payoff functions $u_i \colon S \to \Re$. Mixed strategy sets are given by $\Delta(S_i) = \{\sigma_i \colon S_i \to \Re_+ \mid \sum_{s_i \in S_i} \sigma_i(s_i) = 1\}$, with the space of mixed strategy combinations $\Theta(S) = \times_{i \in N} \Delta(S_i)$, and payoff functions for the mixed extension $U_i \colon \Theta(S) \to \Re$ are defined as usual, $\forall\, i \in N$. For the "one-shot" game Γ let, for all $i \in N$, the *best-reply correspondence* $\beta_i^o \colon \Theta(S) \to \Delta(S_i)$ be defined by

$$\beta_i^o(\sigma) = \{\bar{\sigma}_i \in \Delta(S_i) \mid U_i(\sigma_{-i}, \bar{\sigma}_i) \geq U_i(\sigma_{-i}, \sigma_i'),\ \forall\, \sigma_i' \in \Delta(S_i)\},$$

[1] What is not allowed for is that one player moves at both stages, i.e., one of the leaders and one of the followers are agents of the same original player. Under perfect recall this would conflict with the informational assumptions on the extensive form.

and let $\beta^o = \times_{i \in N} \beta_i^o \colon \Theta(S) \to \Theta(S)$. Denote by $E(\Gamma) \subset \Theta(S)$ the set of (possibly mixed) Nash equilibria of Γ, i.e., $E(\Gamma) = \{\sigma \in \Theta(S) \mid \sigma \in \beta^o(\sigma)\}$. For convenience identify S_i (S) with the set of vertices of $\Delta(S_i)$ ($\Theta(S)$), $\forall i \in N$. The game Γ is called *generic* if $u_i(s) = u_i(s') \implies s = s'$, $\forall s, s' \in S$, $\forall i \in N$.

Partition the player set N into two non-empty sets, the set of "leaders" I and the set of "followers" $I\!I$, $I \cup I\!I = N$, $I \cap I\!I = \emptyset$, denote $S^I = \times_{i \in I} S_i$ and $S^{I\!I} = \times_{i \in I\!I} S_i$, and define the *extensive form game* G as follows: First all players $i \in I$ choose simultaneously their pure strategies $s_i \in S_i$. Then all players $i \in I\!I$ get to see $s^I \in S^I$ and choose simultaneously their pure strategies $s_i \in S_i$, $\forall i \in I\!I$. Payoffs at terminal nodes are as in Γ. The normal form game that corresponds to G is denoted by Γ_G.

Let $K > 1$ be the number of strategy combinations of leaders, $K = |S^I|$, and let T be a finite set of *types* with the same number of elements as S^I, $|T| = K$. Let Λ be the set of Markov matrices $\lambda = (\lambda_{ts})_{(t,\, s) \in T \times S^I} = (\lambda(t \mid s))_{(t,\, s) \in T \times S^I}$, i.e., λ satisfies $\lambda(t \mid s) \geq 0$, $\forall (t, s) \in T \times S^I$, and $\sum_{t \in T} \lambda(t \mid s) = 1$, $\forall s \in S^I$. The set Λ will be the parameter set for the families of games defined below.

For $i \in I\!I$ let F_i be the set of all functions $f \colon T \to S_i$ and denote by $F_i^o = \{f \in F_i \mid f(t) = f(t'),\ \forall (t, t') \in T \times T\}$ the subset of all functions which are constant in types. Product sets are accordingly denoted $F = \times_{i \in I\!I} F_i$ and $F^o = \times_{i \in I\!I} F_i^o$. Next, let $\Delta(F_i) = \{\sigma_i \colon F_i \to \Re_+ \mid \sum_{f \in F_i} \sigma_i(f) = 1\}$ be the set of mixed strategies and $\Delta(F_i^o) = \{\sigma_i \in \Delta(F_i) \mid \text{supp}(\sigma_i) \subset F_i^o\}$, for all followers $i \in I\!I$. With the product sets $\Theta(S^I) = \times_{i \in I} \Delta(S_i)$ and $\Theta(F) = \times_{i \in I\!I} \Delta(F_i)$ define $\Theta = \Theta(S^I) \times \Theta(F)$. Finally, denote $\Theta(S^{I\!I}) = \times_{i \in I\!I} \Delta(S_i)$ and $\Theta(F^o) = \times_{i \in I\!I} \Delta(F_i^o)$, the vector of strategies of followers $f = (f_i)_{i \in I\!I}$, and

$$\sigma^I(s^I) = \prod_{i \in I} \sigma_i(s_i^I),\ \forall s^I \in S^I, \quad \sigma^{I\!I}(f) = \prod_{i \in I\!I} \sigma_i(f_i),\ \forall f \in F.$$

Associated with any $\lambda \in \Lambda$ there are two games: First, a normal form game $\Gamma(\lambda) = (((S_i)_{i \in I}, (F_i)_{i \in I\!I}), (v_i^\lambda)_{i \in N})$ with (pure strategy) payoff function $v_i^\lambda \colon S^I \times F \to \Re$ defined by

$$v_i^\lambda(s^I, f) = \sum_{t \in T} \lambda(t \mid s^I)\, u_i(s^I, f(t)), \quad \forall i \in N,$$

with a mixed extension with payoff functions $V_i^\lambda \colon \Theta \to \Re$ defined by

$$V_i^\lambda(\sigma) = \sum_{(s^I,\, f) \in S^I \times F} \sigma^I(s^I)\, \sigma^{I\!I}(f)\, v_i^\lambda(s^I, f), \quad \forall i \in N.$$

Second, there is an extensive form game $G(\lambda)$ defined as follows: First leaders $i \in I$ choose simultaneously their strategies $s_i \in S_i$, $\forall i \in I$. Then, given $s^I \in S^I$, a chance move selects a type $t \in T$ according to the probability distribution $\lambda(t \mid s^I)$, $\forall t \in T$. Finally, all followers $i \in I\!I$ get to see $t \in T$ and choose simultaneously their strategies $s_i \in S_i$, $\forall i \in I\!I$, in response to the type. Payoffs at terminal nodes are as in the game $\Gamma(\lambda)$.

On the parameter space Λ define $d\colon \Lambda \to [0, 1]$ by

$$d(\lambda) = \frac{1}{2}[1 + \max{}_{t \in T} \min{}_{s \in S^I} \lambda(t \mid s) - \min{}_{t \in T} \max{}_{s \in S^I} \lambda(t \mid s)],$$

and denote $m_\lambda(s) = \arg\max_{t \in T} \lambda(t \mid s)$, $\forall s \in S^I$. If $d(\lambda) = 0$, then $\min_{t \in T} \max_{s \in S^I} \lambda(t \mid s) = 1$ implies from $\sum_{t \in T} \lambda(t \mid s) = 1$, $\forall s \in S^I$, that λ is a permutation of the identity matrix and thus has full rank, $\text{rank}(\lambda) = K$. If $d(\lambda) = 1$, then $\max_{t \in T} \min_{s \in S^I} \lambda(t \mid s) = 1$ implies from $\sum_{t \in T} \lambda(t \mid s) = 1$, $\forall s \in S^I$, that λ has a single row of 1's and zeros elsewhere, thus $\text{rank}(\lambda) = 1$, and there is a single type that occurs with probability 1 conditional upon any strategy combination of leaders. Define $\text{int}(\Lambda)$ as the set of all $\lambda \in \Lambda$ with $\lambda(t \mid s) > 0$, $\forall (t, s) \in T \times S^I$, i.e., all types occur with positive probability. There is *noise* in types whenever $\lambda \in \text{int}(\Lambda)$.

Clearly, $\Gamma(\lambda)$ is the normal form associated with the extensive form game $G(\lambda)$, $\forall \lambda \in \Lambda$. In particular, if $d(\lambda) = 1$, then $\Gamma(\lambda)$ is "strategically equivalent" to Γ, and if $d(\lambda) = 0$, then $\Gamma(\lambda)$ resp. $G(\lambda)$ are "strategically equivalent" to G. Despite this "strategic equivalence", the extensive form game $G(\lambda)$ is radically different to the extensive form game G, even if $d(\lambda) = 0$. The number of terminal nodes of G is $|S| = |S^I|\,|S^{I\!I}|$, while the number of terminal nodes of $G(\lambda)$ is $|S^I|^2\,|S^{I\!I}|$, for all $\lambda \in \Lambda$. Moreover, the number of nodes in an information set of a follower $i \in I\!I$ in G is at most as large as the number of strategy combinations of the other followers, while in $G(\lambda)$ the number of nodes in an information set of a follower is at least as large as the number of strategy combinations of leaders. Therefore, only $\Gamma(\lambda)$ and Γ_G may be close games, if $d(\lambda)$ is close to zero, but $G(\lambda)$ and G cannot be close to each other, at least when distance between games is distance between payoff functions.

Denoting by $\sigma_{-i} \in \Theta_{-i}$ the vector of strategies of all players except player $i \in N$ define her *best-reply correspondence* $\beta_i^\lambda\colon \Theta \to \Delta_i$ in the game $\Gamma(\lambda)$ by

$$\beta_i^\lambda(\sigma) = \{\bar{\sigma}_i \in \Delta_i \mid V_i^\lambda(\sigma_{-i}, \bar{\sigma}_i) \geq V_i^\lambda(\sigma_{-i}, \sigma_i'),\ \forall \sigma_i' \in \Delta_i\},$$

where $\Delta_i = \Delta(S_i)$, $\forall i \in I$ and $\Delta_i = \Delta(F_i)$, $\forall i \in I\!I$, $\forall \lambda \in \Lambda$, and let $\beta^\lambda = \times_{i \in N} \beta_i^\lambda\colon \Theta \to \Theta$. The set of Nash equilibria of $\Gamma(\lambda)$ is $E(\Gamma(\lambda)) = \{\sigma \in \Theta \mid \sigma \in \beta^\lambda(\sigma)\}$.

Similarily, denote by $\beta_{II}^o = \times_{i \in II} \beta_i^o \colon \Theta(S) \to \Theta(S^{II})$ the best reply correspondence of followers in the "one-shot" game Γ. A *subgame perfect* Nash equilibrium of G is some $\sigma = (\sigma^I, \sigma^{II}) \in \Theta$ such that $\sigma \in E(\Gamma(\lambda))$ for some $\lambda \in \Lambda$ with $d(\lambda) = 0$ and behavior strategy combinations $b(t) = ((b_i(s_i \mid t))_{s_i \in S_i})_{i \in II}$ induced by $\sigma^{II} \in \Theta(F)$ satisfy $b(m_\lambda(s)) \in \beta_{II}^o(s, b(m_\lambda(s)))$, $\forall s \in S^I$, where $m_\lambda \colon S^I \to T$ is one-to-one because $d(\lambda) = 0$.

Beliefs at information sets $t \in T$ of players $i \in II$ in the game $G(\lambda)$, $\lambda \in \Lambda$, are defined by

$$\mu_i(s^I \mid t) = \frac{\sigma^I(s^I)\,\lambda(t \mid s^I)}{\sum_{s \in S^I} \sigma^I(s)\,\lambda(t \mid s)}, \quad \forall s^I \in S^I,$$

whenever this quantity is defined, and arbitrary otherwise. Obviously, $\mu_i(s^I \mid t)$ is determined by the above whenever the probability that information set t is reached, $\sum_{s \in S^I} \sigma^I(s)\,\lambda(t \mid s)$, is positive. If $\lambda \in \operatorname{int}(\Lambda)$, then every information set $t \in T$ is reached with positive probability and in this case beliefs are independent of $i \in II$.

For all followers $i \in II$ denote by $F_i(t, s_i) = \{f \in F_i \mid f(t) = s_i\}$, $\forall (t, s_i) \in T \times S_i$, the set of pure strategies which prescribe s_i after t. For followers $i \in II$ every mixed strategy $\sigma_i \in \Delta(F_i)$ induces a *behavior strategy* $b_i(\,.\mid t)$ at every information set $t \in T$ by

$$b_i(s_i \mid t) = \sum_{f \in F_i(t, s_i)} \sigma_i(f), \quad \forall s_i \in S_i,\ \forall t \in T.$$

The induced probability distribution on S^{II} is given by

$$b(s^{II} \mid t) = \prod_{i \in II} b_i(s_i^{II} \mid t), \quad \forall (s^{II}, t) \in S^{II} \times T.$$

Using this notation payoffs in the mixed extension of $\Gamma(\lambda)$ can be written as

$$\begin{aligned} V_i^\lambda(\sigma) &= \sum_{s^I \in S^I} \sigma^I(s^I) \sum_{t \in T} \lambda(t \mid s^I) \sum_{s^{II} \in S^{II}} b(s^{II} \mid t)\, u_i(s^I, s^{II}) = \\ &= \sum_{(s, f) \in S^I \times F} \sigma^I(s^I)\,\sigma^{II}(f) \sum_{s^{II} \in S^{II}} [\sum_{t \in f^{-1}(s^{II})} \lambda(t \mid s^I)] u_i(s^I, s^{II}), \end{aligned}$$

for all $i \in N$, where $f^{-1}(s^{II}) = \{t \in T \mid f(t) = s^{II}\}$. Let $\Delta(S) = \{\varphi \colon S \to \Re_+ \mid \sum_{s \in S} \varphi(s) = 1\}$ be the set of probability distributions on S. The mapping $\phi \colon \Theta \times \Lambda \to \Delta(S)$ defined by

$$\phi(\sigma, \lambda) = \{\varphi \in \Delta(S) \mid \varphi(s) = \sigma^I(s^I) \sum_{t \in T} \lambda(t \mid s^I)\, b(s^{II} \mid t),$$
$$\forall s = (s^I, s^{II}) \in S\}$$

gives the *outcome* induced by $\sigma \in \Theta$ and $\lambda \in \Lambda$. Note that the outcome induced by a pure strategy combination $(s^I, f) \in S^I \times F$ need not be pure (need not be in S) unless $f \in F^o$. Also define the mapping $\Phi \colon \Delta(S) \to \Theta(S)$ which gives the marginals corresponding to an outcome by

$$\Phi(\varphi) = (\Phi_i(\varphi))_{i \in N} = \left(\left(\sum_{s_{-i} \in S_{-i}} \varphi(s_{-i}, s_i) \right)_{s_i \in S_i} \right)_{i \in N},$$

where s_{-i} denotes the (pure) strategy combination of all players except player i in Γ, and $S_{-i} = \times_{j \in N \setminus \{i\}} S_j$.

The issue here is whether a given subgame perfect Nash equilibrium of G induces an outcome which is close to some outcome induced by an equilibrium of $\Gamma(\lambda)$ for sufficiently small noise. Call an outcome $\varphi \in \Delta(S)$ *accessible* if
(i) there exists a subgame perfect Nash equilibrium of G which induces $\varphi \in \Delta(S)$, and
(ii) for any sequence $\{\lambda_r\}_{r=1}^{\infty}$ with $\lambda_r \in \text{int}(\Lambda)$, $\forall r$, and $d(\lambda_r) \to_{r \to \infty} 0$, there exists an associated sequence $\{\sigma^r\}_{r=1}^{\infty}$ with $\sigma^r \in E(\Gamma(\lambda_r))$, $\forall r$, such that $\phi(\sigma^r, \lambda_r) \to_{r \to \infty} \varphi \in \Delta(S)$.

3. Positive Results

The first result makes precise what is meant by "strategic equivalence" between $\Gamma(\lambda)$ and Γ for $d(\lambda) = 1$ and between $\Gamma(\lambda)$ and Γ_G for $d(\lambda) = 0$:

Proposition 1. *(a) If $d(\lambda) = 1$, then $\sigma \in E(\Gamma(\lambda))$ if and only if $\Phi(\phi(\sigma, \lambda)) \in E(\Gamma)$.*
(b) If $d(\lambda) = 0$, then $\sigma \in E(\Gamma(\lambda))$ if and only if $\sigma \in E(\Gamma_G)$.

Proof: (a) Assume $d(\lambda) = 1$, so there exists a unique type $t_o \in T$ such that $\lambda(t_o \mid s) = 1$, $\forall s \in S^I$, and, therefore, $v_i(s^I, f) = u_i(s^I, f(t_o))$, $\forall i \in N$. If $\sigma \in E(\Gamma(\lambda))$, then

$$V_i^{\lambda}(\sigma) = \sum_{(s^I, s^{II}) \in S} \sigma^I(s^I)\, b(s^{II} \mid t_o)\, u_i(s^I, s^{II}) = U_i(\sigma^I, b(t_o)), \quad \forall i \in N,$$

where $b(t_o) = ((b_i(s_i \mid t_o))_{s_i \in S_i})_{i \in II}$ denotes the behavior strategy combination induced by $\sigma^{II} \in \Theta(F)$ after the type t_o, implies from $\phi(\sigma, \lambda)(s) = \sigma^I(s^I)\, b(s^{II} \mid t_o)$, $\forall s = (s^I, s^{II}) \in S$, that $\Phi(\phi(\sigma, \lambda)) \in E(\Gamma)$. Conversely, if $\bar{\sigma} \in E(\Gamma)$ and $\varphi(s) = \bar{\sigma}^I(s^I)\, \bar{\sigma}^{II}(s^{II})$, $\forall s = (s^I, s^{II}) \in S$, then for any strategy combination $\sigma^{II} \in \Theta(F)$ for the followers which induces $b(t_o) = \bar{\sigma}^{II}$ one has $(\bar{\sigma}^I, \sigma^{II}) \in E(\Gamma(\lambda))$ and $\phi((\bar{\sigma}^I, \sigma^{II}), \lambda) = \varphi$.

(b) If $d(\lambda) = 0$, then for each $s \in S^I$ the most likely type $m_\lambda(s) \in T$ is one-to-one from strategy combinations of leaders to types and occurs with probability 1, i.e., $m_\lambda(s) = m_\lambda(s') \implies s = s', \forall s, s' \in S^I$, and $\lambda(m_\lambda(s) \mid s) = 1, \forall s \in S^I$. Thus, one can replace for all followers $f_i \in F_i$ by $f_i \circ m_\lambda$, so $\Gamma(\lambda)$ and Γ_G are the same game. ∎

If $d(\lambda) < 1$, even if $\lambda(t \mid s) = 1/K, \forall (t, s) \in T \times S^I$, where $d(\lambda) = 1/2$, followers may correlate their strategies and, therefore, such a game $\Gamma(\lambda)$ need *not* be "strategically equivalent" to Γ.

Since followers move simultaneously in $G(\lambda)$, any information set $t \in T$ is reached with (strictly) positive probability, whenever $\lambda \in \text{int}(\Lambda)$. Therefore, any Nash equilibrium of $\Gamma(\lambda)$ induces a *sequential* equilibrium [**Kreps and Wilson**, 1982] of $G(\lambda)$, for any $\lambda \in \text{int}(\Lambda)$. Sequential equilibria of $G(\lambda)$ encompass beliefs for all followers. These beliefs are governed, within the present structure, by the types which followers observe *and* by what leaders (are supposed to) do in equilibrium. Any Nash equilibrium can be thought of as a situation where every player learns the strategy combination of the opponents from an umpire (and behaves optimally). If, within the present setup, a follower receives conflicting information from the observation of the type and from the umpire's recommendation, then, if the umpire's recommendation for leaders is deterministic (a pure strategy combination) and types are (non-degenerate) random, the follower will trust the umpire rather than the observation of the type. If leaders could commit to (a sufficiently large level of) strategy trembles, they could offset this effect and guarantee (almost) subgame perfect equilibrium payoffs. This possibility absent, one obtains:

PROPOSITION 2. *For all $\lambda \in int(\Lambda)$: There exists $\sigma^{II} \in \Theta(F^o)$ such that $(s^I, \sigma^{II}) \in E(\Gamma(\lambda))$ if and only if there exists $\bar{\sigma}^{II} \in \Theta(S^{II})$ such that $(s^I, \bar{\sigma}^{II}) \in E(\Gamma)$.*

PROOF: First suppose $\exists (\bar{s}^I, \sigma^{II}) \in E(\Gamma(\lambda))$, for some $\lambda \in \text{int}(\Lambda)$, with $\sigma^{II} \in \Theta(F^o)$. Then beliefs at all information sets are $\mu(\bar{s}^I \mid t) = 1$, $\forall t \in T$, and behavior strategies $b(t) = ((b_i(s_i \mid t))_{s_i \in S_i})_{i \in II}$ induced by σ^{II} are independent of $t \in T$ by $\sigma^{II} \in \Theta(F^o)$. Consequently, $\bar{\sigma}^{II} = b(t) \in \beta^o_{II}(\bar{s}^I, \bar{\sigma}^{II})$, $\forall t \in T$, and

$$
\begin{aligned}
V_i^\lambda(\bar{s}^I, \sigma^{II}) &= \sum_{t \in T} \lambda(t \mid \bar{s}^I) \sum_{s^{II} \in S^{II}} b(s^{II} \mid t)\, u_i(\bar{s}^I, s^{II}) = U_i(\bar{s}^I, \bar{\sigma}^{II}) \geq \\
\geq V_i^\lambda(\bar{s}^I_{-i}, s_i, \sigma^{II}) &= \sum_{t \in T} \lambda(t \mid \bar{s}^I_{-i}, s_i) \sum_{s^{II} \in S^{II}} b(s^{II} \mid t)\, u_i(\bar{s}^I_{-i}, s_i, s^{II}) = \\
&= U_i(\bar{s}^I_{-i}, s_i, \bar{\sigma}^{II}), \quad \forall s_i \in S_i, \forall i \in I,
\end{aligned}
$$

imply $(\bar{s}^I, \bar{\sigma}^{I\!I}) \in E(\Gamma)$.

Conversely, assume $(\bar{s}^I, \bar{\sigma}^{I\!I}) \in E(\Gamma)$ and let $\sigma^{I\!I} \in \Theta(F^o)$ be such that it induces $b(t) = ((b_i(s_i \mid t))_{s_i \in S_i})_{i \in I\!I} = \bar{\sigma}^{I\!I}$, $\forall t \in T$. Then by construction $b(t) \in \beta^o_{I\!I}(\bar{s}^I, b(t))$, $\forall t \in T$, and the above equalities and inequalities imply that $(\bar{s}^I, \sigma^{I\!I}) \in E(\Gamma(\lambda))$, for all $\lambda \in \text{int}(\Lambda)$. ∎

The above Proposition generalizes **Bagwell's** [1995] observation in the following sense:

COROLLARY 1. *If Γ is a generic game and G has a single follower, $I\!I = \{i\}$, then the pure strategy equilibrium outcomes of Γ and $\Gamma(\lambda)$ coincide for all $\lambda \in int(\Lambda)$.*

PROOF: If Γ is generic and $I\!I$ is a singleton set, $I\!I = \{i\}$, then the follower has a unique (and, therefore, pure) best reply in any proper subgame of G. In particular, if $(s^I, \sigma^{I\!I}) \in E(\Gamma(\lambda))$, for some $\lambda \in \text{int}(\Lambda)$, then $\mu(s^I \mid t) = 1$, $\forall t \in T$, implies

$$b(\arg\max_{s_i \in S_i} u_i(s^I, s_i) \mid t) = 1, \quad \forall t \in T,$$

and, therefore, $\sigma^{I\!I} \in \Delta(F^o_i)$. The rest of the statement of the Corollary follows from Proposition 2. ∎

For games G with more than one follower it is, of course, *not* true that the pure strategy equilibrium outcomes of Γ and $\Gamma(\lambda)$ coincide. Certainly, from the "if" part of Proposition 2, every equilibrium of Γ at which leaders play a pure strategy combination always corresponds to some equilibrium of $\Gamma(\lambda)$ at which leaders play the *same* pure strategy combination, for all $\lambda \in \text{int}(\Lambda)$. But, for $\lambda \in \text{int}(\Lambda)$, the game $\Gamma(\lambda)$ may have *more* equilibria at which leaders play a pure strategy combination, because follower may correlate their strategies conditional on types. A necessary condition for these extra equilibria of $\Gamma(\lambda)$ at which leaders play pure is given by the next result:

PROPOSITION 3. *If there exists $\varepsilon > 0$ such that for all $\lambda \in int(\Lambda)$ with $d(\lambda) < \varepsilon$ there exists $(s^I, \sigma^{I\!I}) \in E(\Gamma(\lambda) \cap (S^I \times \Theta(F))$, then there exists $\bar{b} \in \beta^o_{I\!I}(s^I, \bar{b})$ such that*

$$U_i(s^I, \bar{b}) \geq \max_{s_i \in S_i} \min_{b \in \beta^o_{I\!I}(s^I, b)} U_i(s^I_{-i}, s_i, b), \quad \forall i \in I.$$

PROOF: Assume $\exists \varepsilon > 0$ such that $\forall \lambda \in \text{int}(\Lambda)$ with $d(\lambda) < \varepsilon$ there is $\sigma^{I\!I} \in \Theta(F)$ such that $(s^I, \sigma^{I\!I}) \in E(\Gamma(\lambda))$. When leaders play pure, namely s^I, beliefs are $\mu(s^I \mid t) = 1$, $\forall t \in T$. The equilibrium property, therefore, implies $((b_i(s_i \mid t))_{s_i \in S_i})_{i \in I\!I} = b(t) \in \beta^o_{I\!I}(s^I, b(t))$, $\forall t \in T$, for followers, and

$$V_i^\lambda(s^I, \sigma^{II}) = \sum_{t\in T} \lambda(t \mid s^I)\, U_i(s^I, b(t)) \geq \max{}_{s_i\in S_i} V_i^\lambda(s_{-i}^I, s_i, \sigma^{II}) =$$

$$= \max{}_{s_i\in S_i} \sum_{t\in T} \lambda(t \mid s_{-i}^I, s_i)\, U_i(s_{-i}^I, s_i, b(t)) \geq$$

$$\geq \max{}_{s_i\in S_i} \min{}_{b\in\beta_{II}^o(s^I, b)} U_i(s_{-i}^I, s_i, b),$$

for all leaders $i \in I$. Since the payoff on the left hand side of the first of these two inequalities converges to $U_i(s^I, b(m_\lambda(s^I)))$ as $d(\lambda)$ goes to zero, $\forall\, i \in I$, this establishes the desired result. ∎

So, in the sense that followers play a correlated equilibrium against the pure strategy combination of leaders also these equilibria of $\Gamma(\lambda)$, at which leaders play pure and which are not in $E(\Gamma)$, are tied to the "one-shot" game Γ. In a case which is of particular interest for applications Proposition 3 has the following consequence:

COROLLARY 2. *If G has a unique subgame perfect equilibrium at which leaders play pure with associated outcome $(\bar{s}^I, b) \in S^I \times \Theta(S^{II})$ and $(\bar{s}^I, \sigma^{II}) \in E(\Gamma(\lambda))$ for some $\lambda \in int(\Lambda)$ and some $\sigma^{II} \in \Theta(F)$, then $(\bar{s}^I, b) \in E(\Gamma)$.*

PROOF: Because $\lambda \in \text{int}(\Lambda)$, every information set is reached with positive probability, $\lambda(t \mid \bar{s}^I) > 0$, $\forall t \in T$. If leaders choose $\bar{s}^I \in S^I$ in the game $\Gamma(\lambda)$, then $\mu(\bar{s}^I \mid t) = 1$, $\forall t \in T$, uniformly for all followers. By the assumption of a unique and pure subgame perfect equilibrium in G, there exists one but only one $b \in \beta_{II}^o(\bar{s}^I, b)$. But then the necessary condition in Proposition 3 becomes the equilibrium condition for Γ. ∎

This seems to indicate that pure subgame perfect equilibria of G do not survive the introduction of a small amount of noise in types, unless they are equilibria of the "one-shot" game Γ. But **Hurkens and van Damme** [1994] have shown that for generic 2-player games there is always a (mixed) equilibrium of $\Gamma(\lambda)$, for $\lambda \in \text{int}(\Lambda)$, the *outcome* of which converges to the pure subgame perfect equilibrium outcome of G as $d(\lambda)$ converges to zero. While their proof relies on a single follower and on inequalities describing the (single) leader's optimal choice, the generalization below uses well-known results from game theory to establish an analoguous conclusion for the general n-player case.

THEOREM 1. *If Γ is a generic normal form game, then for any partition of the player set into leaders and followers there exists an accessible outcome.*

PROOF: First observe that the number of terminal nodes of the extensive form game G equals the number of strategy combinations of the normal form game Γ, $\prod_{i\in N}|S_i|$. Hence, if Γ is a generic normal form game, then G is a generic extensive form game. Generic extensive form games have only a finite number of equilibrium outcomes [**Kreps and Wilson**, 1982, Theorem 2], so the outcome is constant across each of the finitely many components of the set of Nash equilibria of the associated normal form [**Kohlberg and Mertens**, 1986, Proposition 1]. Moreover, every game has an essential component which contains a hyperstable set [**Kohlberg and Mertens**, 1986, p. 1022] by Zorn's Lemma. Every hyperstable set contains a proper equilibrium [**Myerson**, 1978; **Kohlberg and Mertens**, 1986, Proposition 3]. Since every proper equilibrium induces a sequential equilibrium in any extensive form corresponding to the given normal form [**van Damme**, 1984; **Kohlberg and Mertens**, 1986, p. 1009], among the outcomes associated with an essential component of a normal form game there is at least one that corresponds to a sequential equilibrium of any associated extensive form game.

For any $\lambda \in \Lambda$ with $d(\lambda) = 0$ the games $\Gamma(\lambda)$ and Γ_G are identical up to a relabelling of the followers' strategies. Let $C \subset E(\Gamma_G)$ be an essential component. Because Γ is generic, so is G, and there is a single outcome corresponding to C which is induced by some subgame perfect Nash equilibrium of G, because C contains a proper equilibrium. But since C is essential, every game $\Gamma(\lambda)$ with $\lambda \in \text{int}(\Lambda)$ such that $d(\lambda)$ is sufficiently close to zero has a Nash equilibrium close to C, because d is continuous. Since the mapping ϕ is continuous, the outcomes induced by those Nash equilibria of $\Gamma(\lambda)$ close to C must be close to the outcome induced by (all equilibria in) C. ∎

4. EXAMPLES

The following example is representative for all generic games with two players (which by generic payoffs in G have a unique and pure subgame perfect Nash equilibrium). It illustrates the concepts used in this paper and the fact that approximation of subgame perfect equilibrium *in strategies* cannot be hoped for, not even in generic two-player games.

EXAMPLE 1: Let $N = \{1, 2\}$ and $S_i = \{s_i^1, s_i^2\}$, $\forall i \in N$, with payoff function $u(s_1^1, s_2^1) = (1, 2)$, $u(s_1^2, s_2^1) = (2, -1)$, $u(s_1^1, s_2^2) = (-1, 0)$, and $u(s_1^2, s_2^2) = (0, 1)$. This game Γ has a unique Nash equilibrium with payoff $(0, 1)$, where player 1 chooses her dominant strategy s_1^2. But with $I = \{1\}$ and $I\!I = \{2\}$ the extensive form game G has a unique subgame perfect Nash equilibrium with payoff $(1, 2)$, where player 1 chooses s_1^1. The latter can never be an equilibrium of $\Gamma(\lambda)$ for any $\lambda \in \text{int}(\Lambda)$.

Let $T = \{1, 2\}$, denote $\lambda(t \mid s_1^j) = \lambda_{tj}$, $\forall (t, j) \in T \times T$, and $f_j(t) = s_2^j$, $\forall t \in T$, $\forall j = 1, 2$, $f_3(t) = s_2^{3-t}$, $f_4(t) = s_2^t$, $\forall t \in T$, such that $F_2 = \{f_1, f_2, f_3, f_4\}$. Then $\Gamma(\lambda)$ is the bimatrix game in Table 1:

Tab.1.

$(v_1^\lambda, v_2^\lambda)$	f_1	f_2	f_3	f_4
s_1^1	$(1, 2)$	$(-1, 0)$	$(\lambda_{21} - \lambda_{11}, 2\lambda_{21})$	$(\lambda_{11} - \lambda_{21}, 2\lambda_{11})$
s_1^2	$(2, -1)$	$(0, 1)$	$(2\lambda_{22}, \lambda_{12} - \lambda_{22})$	$(2\lambda_{12}, \lambda_{22} - \lambda_{12})$

For $\lambda_{tt} > 1/2$, $\forall t \in T$, and $\lambda \in \text{int}(\Lambda)$ the follower's strategy f_3 is strictly dominated and the game $\Gamma(\lambda)$ has three Nash equilibria: First (s_1^2, f_2), second

$$\sigma_1(s_1^1) = \frac{\lambda_{22}}{\lambda_{21} + \lambda_{22}}, \quad \sigma_2(f_1) = 1 - \sigma_2(f_4) = 1 - \frac{1}{2(\lambda_{11} - \lambda_{12})},$$

and third

$$\sigma_1(s_1^1) = \frac{\lambda_{12}}{\lambda_{11} + \lambda_{12}}, \quad \sigma_2(f_2) = 1 - \sigma_2(f_4) = 1 - \frac{1}{2(\lambda_{11} - \lambda_{12})}.$$

The second of those equilibria converges to $\sigma_1(s_1^1) = 1$ and $\sigma_2(f_1) = 1 - \sigma_2(f_4) = 1/2$, as $\lambda_{tt} \to 1$, $\forall t \in T$, and thus converges *in outcomes* to the *outcome* of the subgame perfect equilibrium of G. In fact, the outcome of the subgame perfect equilibrium (s_1^1, f_4) is supported by all equilibria in the connected component of Nash equilibria

$$\{\sigma \in \Theta \mid \sigma_1(s_1^1) = 1, \ \sigma_2(f_1) + \sigma_2(f_4) = 1, \ \sigma_2(f_4) \geq 1/2\} \subset E(\Gamma_G).$$

Two observations are worth noting about this example: First, the subgame perfect equilibrium of G *cannot* be supported *in strategies* for interior noise, because the follower is mislead by beliefs at the off-equilibrium information set. Second, for interior noise all three equilibria can be shown to be *regular* [cf. **Harsanyi**, 1973; **van Damme**, 1987; **Ritzberger**, 1994] and, therefore, strictly perfect [**Okada**, 1981], (trembling hand) perfect [**Selten**, 1975], and proper [**Myerson**, 1978]. In fact, for interior noise, each of the three equilibria is a singleton strategically stable set in the sense of **Kohlberg and Mertens** [1986]. This illustrates that selecting an equilibrium from $E(\Gamma(\lambda))$, $\lambda \in \text{int}(\Lambda)$, by some refinement does *not* necessarily yield an accessible outcome in the limit as the noise vanishes. As a matter of fact, it also shows that none of the refinement concepts which imply sequential outcomes is u.h.c. on the space of normal form games.

While for generic two-player games uniqueness of the subgame perfect equilibrium outcome of G yields from Theorem 1 that this outcome is accessible, multiplicity of subgame perfect equilibrium outcomes of G changes things.

EXAMPLE 2: For a non-generic game Γ not all subgame perfect equilibrium outcomes may be accessible, *even if* there are only two players.

Let Γ be given by the gamein Table 2, where $S_i = \{s_i^1, s_i^2, s_i^3\}$, $\forall i \in N = \{1, 2\}$:

Tab.2.

(u_1, u_2)	s_2^1	s_2^2	s_2^3
s_1^1	$(z, 0)$	$(-z, 0)$	$(0, -1)$
s_1^2	$(0, 0)$	$(0, 1)$	$(0, 0)$
s_1^3	$(2, -1)$	$(2, 0)$	$(-1, 1)$

where $z \in \{-1, 1\}$. The game G with $I = \{1\}$ and $I\!I = \{2\}$ has infinitely many subgame perfect equilibrium outcomes: Either player 1 cooses s_1^2 and player 2 responds with s_2^2, or player 1 chooses s_1^1 and player 2 responds by randomizing between s_2^1 and s_2^2 with probability at least 1/2 on the strategy that gives player 1 positive payoff, or player 1 randomizes between s_1^1 and s_1^2 and player 2 responds after s_1^2 with s_2^2 and randomizes after s_1^1 between s_2^1 and s_2^2 precicely with probability 1/2. Let $\lambda_{tj} = \lambda(t \mid s_1^j)$, $\forall j = 1, 2, 3$, $\forall t \in T = \{1, 2, 3\}$. Player 2's conditional payoff given type $t \in T$ from playing the behavior strategy $b_2(s_2^1 \mid t) = x_t$, $b_2(s_2^2 \mid t) = y_t$, and $b_2(s_2^3 \mid t) = 1 - x_t - y_t$ is

$$\begin{aligned} V_2^\lambda(\sigma \mid t) = {} & x_t[\sigma_1(s_1^1)(\lambda_{t1} + 2\,\lambda_{t3}) + 2\,\sigma_1(s_1^2)\,\lambda_{t3} - 2\,\lambda_{t3}] + \\ & + y_t[\sigma_1(s_1^1)(\lambda_{t1} + \lambda_{t3}) + \sigma_1(s_1^2)(\lambda_{t2} + \lambda_{t3}) - \lambda_{t3}] + \\ & + (1 - \sigma_1(s_1^1) - \sigma_1(s_1^2))\lambda_{t3} - \sigma_1(s_1^1)\,\lambda_{t1}\,. \end{aligned}$$

The key observation is the following: If it is ever optimal for player 2 to choose $x_t > 0$ at some $t \in T$, for interior noise, i.e., for $\lambda_{tj} > 0$, $\forall (t, j) \in T \times T$, then

$$0 \geq -(1 - \sigma_1(s_1^1) - \sigma_1(s_1^2))\lambda_{t3} \geq \sigma_1(s_1^2)\,\lambda_{t2} \geq 0$$

implies $\sigma_1(s_1^1) = 1$. Denoting $X_j = \sum_{t \in T} \lambda_{tj}\, x_t$ and $Y_j = \sum_{t \in T} \lambda_{tj}\, y_t$, $\forall j \in T$, player 1's payoff is given by

$$\begin{aligned} V_1^\lambda(\sigma) = {} & \sigma_1(s_1^1)[1 - 3(X_3 + Y_3) + z(X_1 - Y_1)] + \\ & + \sigma_1(s_1^2)[1 - 3(X_3 + Y_3)] + 3(X_3 + Y_3) - 1\,. \end{aligned}$$

Now consider the case $z = 1$ and suppose there is an equilibrium of $\Gamma(\lambda)$ with $\sigma_1(s_1^1) > 0$ for interior noise. If $\sigma_1(s_1^1) = 1$, then in equilibrium $x_t + y_t = 1$, $\forall t \in T$, must hold which implies $\sigma_1(s_1^1) = \sigma_1(s_1^2) = 0$, because $1 - 3(X_3 + Y_3) = -2$, contradicting the hypothesis. If $\sigma_1(s_1^1) \in (0, 1)$, then in equilibrium $x_t = 0$, $\forall t \in T$, such that $\sigma_1(s_1^1) > 0$ implies under $z = 1$ that $y_t = 0$, $\forall t \in T$. But then $\sigma_1(s_1^1) + \sigma_1(s_1^2) = 1$ holds which implies from

$$V_2^\lambda(\sigma \mid t) = x_t\, \sigma_1(s_1^1)\, \lambda_{t1} + y_t[\sigma_1(s_1^1)\lambda_{t1} + (1 - \sigma_1(s_1^1))\lambda_{t2}] - \sigma_1(s_1^1)\, \lambda_{t1}$$

that $y_t = 1$, $\forall t \in T$, a contradiction. Thus, for $z = 1$, no outcome with $\sigma_1(s_1^1) > 0$ is accessible. In particular, the leader's best subgame perfect equilibrium outcome $(s_1^1, s_2^1) \in S$ is not accessible.

Next consider the case $z = -1$. If in equilibrium $\sigma_1(s_1^2) > 0$, then $x_t = 0$, $\forall t \in T$, and under $z = -1$ also $y_t = 0$ must hold for all $t \in T$, because otherwise s_1^1 would be strictly better than s_1^2 for player 1. But then $\sigma_1(s_1^1) + \sigma_1(s_1^2) = 1$ holds which again implies $y_t = 1$, $\forall t \in T$, a contradiction. Thus, for $z = -1$, no outcome with $\sigma_1(s_1^2) > 0$ is accessible. In particular, the subgame perfect equilibrium outcome which maximizes the minimal payoff to the leader across equilibria of proper subgames is not accessible.

Still for Example 2 there always exists an accessible outcome. This, however, does not generally hold true.

EXAMPLE 3: If Γ is non-generic there may not exist an accessible outcome at all, *even if* there are only two players.

Let Γ be given by the 2×2-game in Table 3, where $S_i = \{s_i^1, s_i^2\}$, $\forall i \in N = \{1, 2\}$:

Tab.3.

(u_1, u_2)	s_2^1	s_2^2
s_1^1	$(0, 1)$	$(1, -1)$
s_1^2	$(1, 0)$	$(0, 2)$

Define G by setting $I = \{1\}$ and $I\!I = \{2\}$. The game G has a whole set of subgame perfect equilibria: The leader randomizes with probability $\sigma_1(s_1^1) = \sigma_1 \in [0, 1]$ on strategy s_1^1 and the follower chooses s_2^1 in the subgame starting after s_1^1 and chooses s_2^2 in the subgame starting after s_1^2. Denoting $b_2(s_2^1 \mid t) = x_t$, $\forall t \in T = \{1, 2\}$, and $\lambda(t \mid s_1^j) = \lambda_{tj}$, $\forall (t, j) \in T \times T$, the follower's conditional payoff given type $t \in T$ can be written as

$$V_2^\lambda(\sigma \mid t) = 2\, x_t[\sigma_1(\lambda_{t1} + \lambda_{t2}) - \lambda_{t2}] + 2(1 - \sigma_1)\lambda_{t2} - \sigma_1\lambda_{t1}\,.$$

For sufficiently small noise, $\lambda_{tj} > 0, \forall (t, j) \in T \times T$, but $\lambda_{tj} \approx 0, \forall j \neq t$, $\lambda_{tt} \approx 1, \forall t \in T$, the follower's behavior in equilibrium thus satisfies

$$(x_1, x_2) = \begin{cases} (0, 0), & \text{if } \sigma_1 < \lambda_{12}/(\lambda_{11} + \lambda_{12}), \\ (1, 0), & \text{if } \lambda_{12}/(\lambda_{11} + \lambda_{12}) < \sigma_1 < \lambda_{22}/(\lambda_{21} + \lambda_{22}), \\ (1, 1), & \text{if } \lambda_{22}/(\lambda_{21} + \lambda_{22}) < \sigma_1 . \end{cases}$$

Denoting $X_j = \sum_{t \in T} \lambda_{tj} x_t, \forall j \in T$, the leader's payoff can be written as

$$V_1^{\lambda}(\sigma) = \sigma_1 [1 - X_1 - X_2] + X_2 .$$

Thus $\sigma_1 = 1$ implies $x_1 = x_2 = 1$ which in turn implies that $\sigma_1 = 0$ is the leader's unique best choice; similarily, $\sigma_1 = 0$ implies $x_1 = x_2 = 0$ which in turn implies that $\sigma_1 = 1$ is the leader's unique best choice. Therefore, in equilibrium $0 < \sigma_1 < 1$ implies $X_1 + X_2 = 1$. If now $\lambda_{11} + \lambda_{12} > 1$, then in equilibrium $\sigma_1 = \lambda_{12}/(\lambda_{11} + \lambda_{12}) \approx 0$ (which implies a payoff close to 2 for the follower). If, on the other hand, $\lambda_{11} + \lambda_{12} < 1$, then in equilibrium $\sigma_1 = \lambda_{22}/(\lambda_{21} + \lambda_{22}) \approx 1$ (which implies a payoff close to 1 for the follower). Consequently no *single* outcome is approximated by the (outcome induced by a) mixed equilibria of the game with (small) noise.

Still in the game from Example 3 there at least exists a closed (and connected) *set* of outcomes that satisfies the definition of accessability *as a set*.[2] While it seems possible that the existence of such a set-valued generalization of accessability may hold for all two-player games, the next example shows that this is false in the general case.

EXAMPLE 4: For a non-generic game Γ there may not even exist a *set* of outcomes such that every point in the set is supported by some subgame perfect equilibrium of G *and* for every sufficiently small noise there is an equilibrium of $\Gamma(\lambda)$ with outcome close to the set.

Let Γ be the 4-player game in Table 4, where $S_i = \{s_i^1, s_i^2\}, \forall i \in N = \{1, 2, 3, 4\}$, $u = (u_1, u_2, u_3, u_4)$, and $u^{kh} = u(s_1^k, s_2^h, .), \forall k, h \in \{1, 2\}$:

Tab.4.

u^{11}	s_4^1	s_4^2	u^{12}	s_4^1	s_4^2
s_3^1	(0, 0, 0, 0)	(0, 0, 0, 6)	s_3^1	(1, −5, 0, 0)	(1, −5, 0, 0)
s_3^2	(0, 0, 0, 0)	(0, 0, 6, 0)	s_3^2	(1, −5, 0, 0)	(1, −5, 0, 0)

[2]In a suitably modified definition of an *accessible set* of outcomes, of course, the outcomes induced by equilibria of $\Gamma(\lambda)$, for $\lambda \in \text{int}(\Lambda)$, need not converge. Only the (Hausdorff) distance to the accessible set would have to converge to zero.

u^{21}	s_4^1	s_4^2	u^{22}	s_4^1	s_4^2
s_3^1	(−5, 0, 1, 0)	(−5, 0, 0, 5)	s_3^1	(0, 15, 6, 0)	(0, 15, 0, 0)
s_3^2	(1, 0, 0, 1)	(1, 0, 5, 0)	s_3^2	(0, −1, 0, 6)	(0, −1, 0, 0)

Define G by letting players 1 and 2 be the leaders, $I = \{1, 2\}$, and players 3 and 4 be the followers, $I\!I = \{3, 4\}$. Then G has a single connected set of subgame perfect equilibria where player 1 randomizes between her pure strategies with probability between 3/4 and 1 on s_1^1 and player 2 chooses s_2^1 with certainty, $(\sigma_1(s_1^1), \sigma_2(s_2^1)) \in [3/4, 1] \times \{1\}$. The highest expected payoff that follower $i = 4$ can obtain in any of the subgame perfect equilibria of G is 5/24. This set is part of a larger connected component of Nash equilibria of G. Denote $b_3(s_3^1 \mid t) = x_t$ and $b_4(s_4^1 \mid t) = y_t$, $\forall t \in T = \{1, 2, 3, 4\}$, $\sigma_1(s_1^1) = \sigma_1$ and $\sigma_2(s_2^1) = \sigma_2$, and $\lambda(t \mid s_1^k, s_2^h) = \lambda_{t,\,2k+h-2}$, $\forall k, h \in \{1, 2\}$. Player 3's conditional payoff given type $t \in T$ is given by

$$V_3^\lambda(\sigma \mid t) = x_t[6y_t(\sigma_1\sigma_2\lambda_{t1} + (1-\sigma_1)(\sigma_2\lambda_{t3} + (1-\sigma_2)\lambda_{t4}))- \\ -\sigma_2(6\sigma_1\lambda_{t1} + 5(1-\sigma_1)\lambda_{t3})] + (1-y_t)\sigma_2(6\sigma_1\lambda_{t1} + 5(1-\sigma_1)\lambda_{t3}),$$

and player 4's conditional payoff given type $t \in T$ is given by

$$V_4^\lambda(\sigma \mid t) = y_t[(1-\sigma_1)(\sigma_2\,\lambda_{t3} + 6(1-\sigma_2)\lambda_{t4}) - 6\,x_t(\sigma_1\,\sigma_2\,\lambda_{t1} + \\ +(1-\sigma_1)(\sigma_2\,\lambda_{t3} + (1-\sigma_2)\lambda_{t4}))] + x_t\,\sigma_2(6\,\sigma_1\,\lambda_{t1} + 5(1-\sigma_1)\lambda_{t3}).$$

Inspection of these conditional payoffs reveals that against any strategy combination of the leaders which happens to satisfy $(\sigma_1, \sigma_2) \in (0, 1]^2$, except if $\sigma_1 = 1$, followers play a "Matching Pennies" game, governed by indifference between the two pure strategies. If $\sigma_1 = 1$ and $\sigma_2 > 0$, then $x_t = 0$ and $y_t \in [0, 1]$ holds in equilibrium. In any case $(\sigma_1, \sigma_2) \in (0, 1]^2$, therefore, implies in equilibrium

$$x_t = \frac{(1-\sigma_1)[\sigma_2\,\lambda_{t3} + 6(1-\sigma_2)\lambda_{t4}]}{6[\sigma_1\,\sigma_2\,\lambda_{t1} + (1-\sigma_1)(\sigma_2\,\lambda_{t3} + (1-\sigma_2)\lambda_{t4}]}, \quad \forall t \in T,$$

follows from the required indifference of follower 4 in equilibrium. Denoting $X_j = \sum_{t \in T} \lambda_{tj}\,x_t$, $\forall j = 1, 2, 3, 4$, the leaders' payoffs can be written

$$V_1^\lambda(\sigma) = \sigma_1[1 - 2\,\sigma_2(1 - 3\,X_3)] + \sigma_2(1 - 6\,X_3), \\ V_2^\lambda(\sigma) = \sigma_2[4\,\sigma_1(1 + 4\,X_4) + 1 - 16\,X_4] - 1 - 4\,\sigma_1 - 16(1-\sigma_1)X_4.$$

So the leaders' payoffs depend only on x_t, viz. player 3's strategy, and player 1's payoff dependes only on X_3, while player 2's payoff depends only on X_4. If in equilibrium $\sigma_1 > 0$ and $X_4 \leq 1/16$ would hold, then $\sigma_2 = 1$ is the only optimal choice for player 2 which would imply $x_t < 1/6$, $\forall t \in T$, from the above explicit formula for x_t; but then $2\,\sigma_2(1 - 3\,X_3) > 2(1 - 3/6) = 1$ implies that only $\sigma_1 = 0$ is optimal for player 1 - a contradiction. Similarily, if in equilibrium $X_3 \geq 1/3$ would hold, then $\sigma_1 = 1$ is player 1's unique best choice which implies that $\sigma_2 = 1$ is player 2's unique best choice which in turn implies $x_t = 0$, $\forall t \in T$ - again a contradiction. Thus in all equilibria with $(\sigma_1, \sigma_2) \in (0, 1]^2$ one must have $X_3 < 1/3$ and $X_4 > 1/16$. But this implies, from the leaders' payoffs, that leaders are also playing a game of the "Matching Pennies" type with a unique completely mixed equilibrium governed by indifferences.

Finally, if $\sigma_2 = 0$, then followers in equilibrium must play $x_t = 1$ and $y_t \in [0, 1]$, $\forall t \in T$, implying that $\sigma_1 = 1$ is the unique best choice for player 1 and, therefore, $\sigma_2 = 1$ is player 2's unique best choice, contradicting the assumption $\sigma_2 = 0$. Similarily, $\sigma_1 = 0$ in equilibrium implies $x_t \geq 1/6$, $\forall t \in T$, and, therefore, $X_j \geq 1/6$, $\forall j = 1, 2, 3, 4$, which implies from $1 - 2\,\sigma_2(1 - 3\,X_3) \geq 1 - \sigma_2$ that $\sigma_2 = 1$ if $\sigma_1 = 0$ is an optimal choice; but at $\sigma_1 = 0$ one has that $1 - 16\,X_4 \leq -5/3$ implies $\sigma_2 = 0$, a contradiction. It follows that *all* equilibria of the game $\Gamma(\lambda)$ with noise have a "Matching Pennies" structure with respect to the leaders. All equilibria must, therefore, be solutions to the following system of six equations:

$$\begin{aligned} 6\,x_t[\sigma_1\,\sigma_2\,\lambda_{t1} + (1 - \sigma_1)(\sigma_2\,\lambda_{t3} + (1 - \sigma_2)\lambda_{t4})] - & \\ -(1 - \sigma_1)[\sigma_2\,\lambda_{t3} + 6(1 - \sigma_2)\lambda_{t4}] &= 0, \quad \forall t \in T, \\ 4\,\sigma_1[1 + 4\sum_{t \in T} \lambda_{t4}\,x_t] + 1 - 16\sum_{t \in T} \lambda_{t4}\,x_t &= 0\,, \\ 1 - 2\,\sigma_2[1 - 3\sum_{t \in T} \lambda_{t3}\,x_t] &= 0\,. \end{aligned}$$

The equation for σ_1 implies that in any equilibrium $\sigma_1 \leq 3/4$. Thus if there is any equilibrium of the game $\Gamma(\lambda)$ with noise inducing an outcome close to the set of subgame perfect equilibrium outcomes, then this equilibrium must converge to $(\sigma_1, \sigma_2) = (3/4, 1)$. Now consider the noise structure

$$\lambda = (\lambda_{tj})_{(t,\,j) \in T \times T} = \begin{pmatrix} 1 - 7\,\varepsilon & \varepsilon & \varepsilon^2 & \varepsilon \\ \varepsilon & 1 - 3\,\varepsilon & \varepsilon^2 & \varepsilon \\ \varepsilon & \varepsilon & 1 - 3\,\varepsilon^2 & \varepsilon \\ 5\,\varepsilon & \varepsilon & \varepsilon^2 & 1 - 3\,\varepsilon \end{pmatrix},$$

with $\varepsilon \in (0, 1/15)$. With this λ the solutions to the above six equations can be regarded as functions of ε. For the outcome induced by a sequence $\{(\sigma_1(\varepsilon),\ \sigma_2(\varepsilon),\ (x_t(\varepsilon))_{t\in T})\}_{\varepsilon\searrow 0}$ to converge to an outcome supported by a subgame perfect equilibrium of G it is necessary that $\sigma_1(0) = 3/4$ and $\sigma_2(0) = 1$. Clearly,

$$x_3(\varepsilon) = \frac{(1-\sigma_1)[(1-3\,\varepsilon^2)\sigma_2 + 6\,\varepsilon(1-\sigma_2)]}{6[\varepsilon\,\sigma_1\sigma_2 + (1-3\,\varepsilon^2)(1-\sigma_1)\sigma_2 + \varepsilon(1-\sigma_1)(1-\sigma_2)]} \longrightarrow_{\varepsilon\searrow 0} \frac{1}{6}$$

holds, where $\sigma_i = \sigma_i(\varepsilon)$, $\forall\, i \in I$. The limit of $x_4(\varepsilon)$ as $\varepsilon \searrow 0$, however, is indeterminate. Now observe that the reduced system of (the three) equations for $(x_3,\ \sigma_1,\ \sigma_2)$ at fixed values $(x_1,\ x_2,\ x_4) = (\bar{x}_1,\ \bar{x}_2,\ \bar{x}_4) \in [0,\ 1]^3$ has a non-singular Jacobian matrix with determinant $72(1-\sigma_1(0))(1-4\bar{x}_4)(4x_3(0)-1) = -24(1-\sigma_1(0))(1-4\bar{x}_4) \neq 0$ at $\varepsilon = 0$, whenever $\bar{x}_4 \neq 1/4$, and, therefore, has a non-singular Jacobian matrix at all sufficiently small $\varepsilon > 0$ if $\bar{x}_4 \neq 1/4$. Hence, it is justified to invoke l'Hospital's rule to determine the limit of $x_4(\varepsilon)$ as $\varepsilon \searrow 0$. So again with $\sigma_i = \sigma_i(\varepsilon)$, $\forall\, i \in I$:

$$x_4(0) = \frac{1}{6}\lim_{\varepsilon\searrow 0} \frac{(1-\sigma_1)[\varepsilon^2\,\sigma_2 + 6(1-3\,\varepsilon)(1-\sigma_2)]}{5\varepsilon\sigma_1\sigma_2 + \varepsilon^2(1-\sigma_1)\sigma_2 + (1-3\,\varepsilon)(1-\sigma_1)(1-\sigma_2)} =$$

$$= \frac{1-\sigma_1(0)}{6}\,\frac{-6\,\sigma_2'(0)}{5\sigma_1(0) - (1-\sigma_1(0))\sigma_2'(0)} = \frac{(1-\sigma_1(0))\sigma_2'(0)}{(1-\sigma_1(0))\sigma_2'(0) - 5\sigma_1(0)}\,.$$

Since the derivative at $\varepsilon = 0$ of $\sigma_2(\varepsilon)$ is given by $\sigma_2'(0) = 6\,x_3'(0)$ and $x_3'(0) = -\sigma_1(0)/[6(1-\sigma_1(0))]$, this yields

$$\sigma_2'(0) = -\frac{\sigma_1(0)}{1-\sigma_1(0)} \implies x_4(0) = \frac{1}{6}\,.$$

But then from the equation for $\sigma_1(\varepsilon)$ it follows that

$$\sigma_1(0) = \frac{16\,x_4(0) - 1}{16\,x_4(0) + 4} \implies \sigma_1(0) = \frac{1}{4}\,.$$

Since this violates the necessary condition $\sigma_1(0) = 3/4$, the only equilibrium of $\Gamma(\lambda)$ with the above noise structure does not induce an outcome close to any of the outcomes supported by subgame perfect equilibria of G.

In fact, the unique equilibrium of $\Gamma(\lambda)$ with the above noise structure does not even induce *payoffs* close to some subgame perfect equilibrium payoff. As ε approaches zero, the payoff to follower $i = 4$ approaches

15/24 = 5/8 which exceeds the highest payoff to follower 4 in any of the subgame perfect equilibria of G.

Finally, we return to the question whether *all* subgame perfect equilibrium outcomes of G for a *generic* game Γ might be accessible. The final example shows that this is not the case.

EXAMPLE 5: With more than two players there may exist subgame perfect equilibrium outcomes which are *not* accessible, even for *generic* games Γ.

Consider the generic 3-player game in Table 5, where $S_i = \{s_i^1, s_i^2\}$, $\forall i \in N$, and $u = (u_1, u_2, u_3)$:

Tab.5.

$u(s_1^1, .)$	s_3^1	s_3^2	$u(s_1^2, .)$	s_3^1	s_3^2
s_2^1	$(0, 2, 2)$	$(4, -1, 1)$	s_2^1	$(\frac{1}{6}, \frac{3}{2}, -\frac{1}{4})$	$(-\frac{1}{2}, 0, 0)$
s_2^2	$(2, 1, -1)$	$(-2, -2, -2)$	s_2^2	$(1, 3, 3)$	$(-1, -\frac{1}{2}, \frac{3}{2})$

The associated game G, where player 1 is the leader, $I = \{1\}$, and players 2 and 3 are the followers, $I\!I = \{2, 3\}$, has two subgame perfect equilibrium outcomes, (s_1^1, s_2^1, s_3^1) and (s_1^2, s_2^2, s_3^1). (The payoff to the leader from the mixed equilibrium in the subgame after s_1^2 is $-5/14 < 0$ and thus cannot be supported by (s_2^1, s_3^1) after s_1^1.) Again denote $\lambda_{tj} = \lambda(t \mid s_1^j)$, $\forall j = 1, 2$ and $\forall t \in T = \{1, 2\}$, denote $x_t = b_2(s_2^1 \mid t)$ and $y_t = b_3(s_3^1 \mid t)$, $\forall t \in T$, and $\sigma = \sigma_1(s_1^1)$. The conditional payoff to follower 2 given type $t \in T$ is given by

$$V_2^\lambda(\sigma \mid t) = x_t[\sigma\,\lambda_{t1} + \frac{1}{2}(1-\sigma)\lambda_{t2} - 2\,y_t(1-\sigma)\lambda_{t2}] + \\ + y_t(3\,\sigma\,\lambda_{t1} + \frac{7}{2}(1-\sigma)\lambda_{t2}) - 2\,\sigma\,\lambda_{t1} - \frac{1}{2}(1-\sigma)\lambda_{t2}\,,$$

and the conditional payoff to follower 3 given type $t \in T$ is given by

$$V_3^\lambda(\sigma \mid t) = y_t[\sigma\,\lambda_{t1} + \frac{3}{2}(1-\sigma)\lambda_{t2} - \frac{9}{4}\,x_t(1-\sigma)\lambda_{t2}] + \\ + x_t(3\,\sigma\,\lambda_{t1} - \frac{3}{2}(1-\sigma)\lambda_{t2}) - 2\,\sigma\,\lambda_{t1} + \frac{3}{2}(1-\sigma)\lambda_{t2}\,.$$

What followers in equilibrium do given type $t \in T$ thus depends on σ in the following way: Since

$$\frac{2\,\sigma\,\lambda_{t1} + (1-\sigma)\lambda_{t2}}{4(1-\sigma)\lambda_{t2}} \leq 1 \iff \sigma \leq \frac{3\,\lambda_{t2}}{2\,\lambda_{t1} + 3\,\lambda_{t2}}, \quad \text{and}$$

$$\frac{4\,\sigma\,\lambda_{t1} + 6(1-\sigma)\lambda_{t2}}{9(1-\sigma)\lambda_{t2}} \leq 1 \iff \sigma \leq \frac{3\,\lambda_{t2}}{4\,\lambda_{t1} + 3\,\lambda_{t2}} < \frac{3\,\lambda_{t2}}{2\,\lambda_{t1} + 3\,\lambda_{t2}},$$

for any interior noise, first, $\sigma > 3\,\lambda_{t2}/(2\,\lambda_{t1} + 3\,\lambda_{t2})$ implies $(x_t,\, y_t) = 1,\, 1)$, second, $\sigma = 3\,\lambda_{t2}/(2\,\lambda_{t1} + 3\,\lambda_{t2})$ implies $x_t \in [0,\, 1]$ and $y_t = 1$, third, $3\,\lambda_{t2}/(4\,\lambda_{t1} + 3\,\lambda_{t2}) < \sigma < 3\,\lambda_{t2}/(2\,\lambda_{t1} + 3\,\lambda_{t2})$ implies $(x_t,\, y_t) = (0,\, 1)$, fourth, $\sigma = 3\,\lambda_{t2}/(4\,\lambda_{t1} + 3\,\lambda_{t2})$ implies either $(x_t,\, y_t) = (0,\, 1)$ or $x_t = 1$ and $y_t \in [0,\, 5/8]$, and, finally, $\sigma < 3\,\lambda_{t2}/(4\,\lambda_{t1} + 3\,\lambda_{t2})$ implies either $(x_t,\, y_t) = (0,\, 1)$ or $(x_t,\, y_t) = (1,\, 0)$ or

$$(x_t,\, y_t) = \left(\frac{4\,\sigma\,\lambda_{t1} + 6(1-\sigma)\lambda_{t2}}{9(1-\sigma)\lambda_{t2}},\, \frac{2\,\sigma\,\lambda_{t1} + (1-\sigma)\lambda_{t2}}{4(1-\sigma)\lambda_{t2}}\right).$$

More compactly the structure of equilibrium behavior given type $t \in T$ as a function of σ can be summarized by the implication

$$y_t < 1 \implies y_t \leq \frac{9}{8}\,x_t - \frac{1}{2}\,,$$

which holds, because for $\sigma \in [0,\, 1]$

$$(x_t,\, y_t) = \left(\frac{4\,\sigma\,\lambda_{t1} + 6(1-\sigma)\lambda_{t2}}{9(1-\sigma)\lambda_{t2}},\, \frac{2\,\sigma\,\lambda_{t1} + (1-\sigma)\lambda_{t2}}{4(1-\sigma)\lambda_{t2}}\right) \implies$$
$$\implies y_t = \frac{9}{8}\,x_t - \frac{1}{2}\,.$$

Turning to the leader's payoff, this can be written as

$$V_1^\lambda(\sigma) = \sigma[\sum_{t\in T} \lambda_{t1}(6\,x_t + 4\,y_t - 8\,x_t\,y_t) - \sum_{t\in T} \lambda_{t2}(\frac{1}{2}\,x_t + 2\,y_t -$$
$$- \frac{4}{3}\,x_t\,y_t) - 1] + \sum_{t\in T} \lambda_{t2}(\frac{1}{2}\,x_t + 2\,y_t - \frac{4}{3}\,x_t\,y_t) - 1\,.$$

If $(s_1^1,\, s_2^1,\, s_3^1)$ is accessible, then certainly $\sigma > 3\,\lambda_{12}/(2\,\lambda_{11} + 3\,\lambda_{12})$ must hold, because otherwise sufficiently small noise, $\lambda_{tt} \approx 1$, $\forall t \in T$, and $\lambda_{tj} \approx 0$, $\forall j \neq t$, means that σ converges to zero. Consequently, $(x_1,\, y_1) = (1,\, 1)$ must hold. If σ would equal 1, then also $(x_2,\, y_2)$ would equal $(1,\, 1)$ which would imply $V_1^\lambda(\sigma) = (1-\sigma)/6$ or $\sigma = 0$, a contradiction. Thus accessability of $(s_1^1,\, s_2^1,\, s_3^1)$ requires $0 < \sigma < 1$ and, therefore, with $(x_1,\, y_1) = (1,\, 1)$,

$$2\,\lambda_{11} - \frac{7}{6}\,\lambda_{12} - 1 - (\frac{1}{2}\,\lambda_{22} - 6\,\lambda_{21})x_2 = [2\,\lambda_{22} - 4\,\lambda_{21} - (\frac{4}{3}\,\lambda_{22} - 8\,\lambda_{21})x_2]y_2\,.$$

If y_2 would equal 1 this would imply $2\,\lambda_{21} - 1 + \frac{5}{6}\,\lambda_{12} + (\frac{5}{6}\,\lambda_{22} - 2\,\lambda_{21})x_2 = 0$ which is equivalent to $x_2 = (6 - 12\,\lambda_{21} - 5\,\lambda_{12})/(5\,\lambda_{22} - 12\,\lambda_{21}) > 1$

which is impossible. Hence, $y_t < 1$ implies $y_t \leq \frac{9}{8} x_t - \frac{1}{2}$ and, therefore,

$$2\lambda_{11} - \frac{7}{6}\lambda_{12} - 1 - (\frac{1}{2}\lambda_{22} - 6\lambda_{21})x_2 \leq [2\lambda_{22} - 4\lambda_{21} -$$
$$- (\frac{4}{3}\lambda_{22} - 8\lambda_{21})x_2](\frac{9}{8}x_2 - \frac{1}{2}) \quad \Longleftrightarrow$$
$$\Longleftrightarrow (\frac{3}{2}\lambda_{22} - 9\lambda_{21})(x_2)^2 - (\frac{41}{12}\lambda_{22} - \frac{29}{2}\lambda_{21})x_2 +$$
$$+ 2 - 4\lambda_{21} - \frac{13}{6}\lambda_{12} \leq 0\,.$$

The final quadratic expression is an upwards opening parabola in x_2. But, because $(\frac{41}{12}\lambda_{22} - \frac{29}{2}\lambda_{21})^2 - 4(\frac{3}{2}\lambda_{22} - 9\lambda_{21})(2 - 4\lambda_{21} - \frac{13}{6}\lambda_{12}) \approx -.326 < 0$ if $\lambda_{tt} \approx 1$, $\forall t \in T$, the quadratic expression does not have a real root for sufficiently small noise. It follows that for arbitrarily small noise there is no equilibrium with an outcome in a neighborhood of (s_1^1, s_2^1, s_3^1). In fact the *only* accessible outcome in this example is (s_1^2, s_2^2, s_3^1).

Conclusions

This paper has shown that imperfect observability of commitments need not necessarily eliminate a first-mover advantage in an extensive form game: For every generic game there is an accessible outcome, independently of how the player set is split into leaders and followers. But this conclusion depends heavily on generic payoffs. Moreover, the main theorem of this paper is strictly an existence result in the sense that not all subgame perfect equilibrium outcomes may qualify as accessible.

The challenge to pure subgame perfect Nash equilibrium by the introduction of noisy observations is brought about by misguided beliefs at information sets off the equilibrium path. The source of this misperception on the part of followers rests with the solution concept: If observations on the leaders' choices are noisy, but the umpire's recommendation is not, followers trust the umpire rather than what they see.

One way to put the present results is to say that the correspondence that maps the class of two-stage extensive form games studied here into sequential equilibria is not u.h.c. on the space of associated normal form games. But *no* refinement concepts satisfies such a continuity property. And that subgame perfect equilibria may not survive a change of the extensive form which eliminates all proper subgames cannot come as a surprise.

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