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ABSTRACT

We study three dimensional bi-Hamiltonian systems in general and use the obtained results to classify all three dimensional Lotka-Volterra equations, which admit a bi-Hamiltonian representation.

ZUSAMMENFASSUNG

In der vorliegenden Arbeit studieren wir drei-dimensionale bi-Hamiltonsche Systeme und klassifizieren alle drei-dimensionalen Lotka-Volterra Gleichungen, welche eine bi-Hamiltonische Darstellung zulassen.

Keywords: Hamiltonian systems, Lotka-Volterra equations

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1. Introduction

In the last decade a lot of research effort has been devoted to the problem of finding constants of the motion of three dimensional dynamical systems depending on parameters. Except for some simple cases, this problem is very hard and no general methods to solve it are known up to now. Nevertheless several approaches were developed in the last years and the most important among them are: specific ansatz for a constant of the motion [1,2], Painleve analysis [3,4], the Lie symmetry method [5] and the Frobenius integrability theorem [6,7]. Knowing constants of the motion for a given dynamical system is interesting both from an analytical and numerical point of view. If a three dimensional dynamical system admits a constant of the motion, then the phase space is foliated into two dimensional flow invariant leafs and therefore certain types of irregular orbits cannot occur. For complex problems constants of the motion are a welcomed check of the used numerical scheme with respect to its accuracy and stability.

In some cases it is even possible to find for a given three dimensional dynamical system two functionally independent constants of the motion. Then the orbits are the intersections of two dimensional flow invariant surfaces and therefore non chaotic. (Indeed, chaotic behaviour is often associated with non-integrability of the dynamical system.)

The aim of the present paper is to show that almost all three dimensional dynamical systems, which admit two independent constants of the motion are so called bi-Hamiltonian systems, i.e. they can be written in Hamiltonian form in two distinct ways. The paper is organised as follows: in section 2 we explain what Hamiltonian and bi-Hamiltonian systems are. In section 3 we study three dimensional bi-Hamiltonian systems in general and in section 4 we apply the obtained results to the three dimensional Lotka-Volterra equations and classify all of them, which admit a bi-Hamiltonian representation.

2. Hamiltonian and Bi-Hamiltonian systems

Hamiltonian systems have their origin in classical mechanics. In local coordinates they are given by an ordinary differential equation of the form

$$\dot{x} = S\nabla H$$

where $S = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ (I the identity matrix) and H a smooth real valued function. It is well known that H is a constant of the motion, the so called symplectic structure S is preserved by the induced flow and the matrix of the linearisation at every fixed point can be written as the product of the skew-symmetric matrix S and a symmetric matrix, which implies a high symmetry for the fixed point eigenvalues [8,9]. The main disadvantages of the classical notion of Hamiltonian systems are the coordinate dependence and the artificial restriction to even dimensional phase spaces. These drawbacks are eliminated by the following definition.

Definition 2.1

Let $\dot{x} = f(x)$ be a smooth differential equation defined on some open subset G of R^n. It is called a Hamiltonian system, if it can be written as $\dot{x} = J\nabla H$ where

i) H is a smooth real valued function defined on G.

ii) J is a *Poisson structure matrix*, i.e. it is an x-dependent skew-symmetric matrix satisfying the Jacobi-identity $\sum_{l=1}^{n}\left(j_{il}\frac{\partial j_{mk}}{\partial x_l} + j_{kl}\frac{\partial j_{im}}{\partial x_l} + j_{ml}\frac{\partial j_{ki}}{\partial x_l}\right) = 0 \quad \forall x \in G, \quad \forall 1 \le i,m,k \le n$

iii) the matrix of the linearisation at every fixed point can be written as the product of a skew symmetric and a symmetric matrix.

This definition of a Hamiltonian system deviates from the usual one - as it can be found for instance in Ref. [10] - by the extra assumption iii). We included iii) in our definition mainly for two reasons: First of all classical Hamiltonian systems have this property and secondly without condition iii) strange examples of Hamiltonian systems can arise. For instance, let A be a $n \times n$-matrix with a zero eigenvalue, then $\dot{x} = Ax$ is Hamiltonian in the sense of Ref. [10]: Assume w.l.o.g. (without loss of generality) that the last row of A consists of zeros and take $H = x_n$ and J given by

$$j_{il} = \begin{cases} 0 & for \;\; l \neq n \\ (Ax)_i & for \;\; l = n \end{cases} \quad for \; i < l$$

$$j_{il} = -j_{li} \; for \;\; i > l$$

J is a so called Lie-Poisson structure [10].

In order to exclude such exotic examples of Hamiltonian systems we were forced to include condition iii) in our definition.

It has to be noted that condition iii) of definition 2.1 is automatically fulfilled, if the rank of the Poisson structure matrix J is locally constant at the fixed points; this can be proved by a generalisation of the well known Darboux theorem for classical Hamiltonian systems [11].

A $n \times n$-matrix A, which can be written as the product of a skew-symmetric and a symmetric matrix, we call *Poisson-matrix*. Then definition 2.1 implies that for a given differential equation to be Hamiltonian it is necessary that the matrix of the linearisation at every fixed point is a Poisson-matrix. A detailed analysis of Poisson matrices can be found in Ref. [12]. There the following result is proved.

Theorem 2.2

A $n \times n$-matrix A is a Poisson-matrix if and only if $A \cong -A$, i.e. there exists an invertible matrix T such that $A = -T^{-1}AT$.

An easy consequence of theorem 2.2 is the following corollary.

Corollary 2.3

If a $n \times n$-matrix A is a Poisson-matrix, then with λ also $-\lambda, \bar{\lambda}, -\bar{\lambda}$ are eigenvalues of A.

These are the familiar symmetry properties of the eigenvalues at fixed points of classical Hamiltonian systems and the above example shows that assumption iii) of definition 2.1 is crucial for this property to hold.

Sometimes it is the case that a given differential equation can be written in two distinct ways as a Hamiltonian system; if further the Poisson structure matrices are in a certain sense compatible, then it is called a bi-Hamiltonian system. Formally this situation is described by the following definition.

Definition 2.4

Let $\dot{x} = f(x)$ be a smooth differential equation defined on some open subset G of R^n. It is called a bi-Hamiltonian system, if it can be written in two distinct ways as a Hamiltonian system, i.e. $\dot{x} = J_1 \nabla H_1 = J_2 \nabla H_2$ where

i) J_1 and J_2 are not constant multiples of each other and compatible, i.e. $\alpha J_1 + \beta J_2$ is a Poisson structure matrix for all $\alpha,\ \beta \in R$.

ii) H_1 and H_2 are functionally independent for all non singular points of the differential equation.

According to a fundamental theorem of Magri [13], provided certain technical conditions are satisfied, bi-Hamiltonian systems are completely integrable systems in the sense of Liouville [8,11]. In the present work, however, we do not study bi-Hamiltonian systems in general, but we restrict our attention to the study of three dimensional bi-Hamiltonian systems, and we will see that in this special case, things become very simple, mainly for dimensional reasons.

3. Three dimensional Bi-Hamiltonian systems

Definition 2.4 implies that for a given three dimensional differential equation to be a bi-Hamiltonian system it is necessary that the linearisation at each fixed point is a Poisson matrix and that there exist two globally defined functionally independent constants of the motion. In the remaining part of this section we will show that these conditions are already sufficient for a three dimensional dynamical system for being a bi-Hamiltonian one. This is surprising because it implies that in three dimensions the two Poisson structure matrices are completely determined by the constants of the motion. How a Poisson structure matrix can be generated form a constant of the motion is answered by the following proposition.

Let $C_i \equiv \frac{\partial C}{\partial x_i}$ for the rest of this section.

Proposition 3.1

Let C, $m: G \subseteq R^3 \to R$ be arbitrary smooth functions, then

$$J = m(x)\begin{pmatrix} 0 & C_3 & -C_2 \\ -C_3 & 0 & C_1 \\ C_2 & -C_1 & 0 \end{pmatrix}$$

is a Poisson structure matrix.

Proof:

J is clearly skew-symmetric.

The Jacobi-identity reduces in the three dimensional case to the following single equation:

$$j_{12}\left(-\frac{\partial j_{13}}{\partial x_1}-\frac{\partial j_{23}}{\partial x_2}\right)+j_{13}\left(\frac{\partial j_{12}}{\partial x_1}-\frac{\partial j_{23}}{\partial x_3}\right)+j_{23}\left(\frac{\partial j_{13}}{\partial x_3}+\frac{\partial j_{12}}{\partial x_2}\right)\equiv 0$$

$$\Leftrightarrow$$

$$
\begin{aligned}
&mC_3\left(m_1C_2 + mC_{21} - m_2C_1 - mC_{12}\right) - \\
&mC_2\left(m_1C_3 + mC_{31} - m_3C_1 - mC_{13}\right) + \\
&mC_1\left(-m_3C_2 - mC_{23} + m_2C_3 + mC_{32}\right) \equiv 0
\end{aligned}
$$

q.e.d.

Now we are able to prove the main result of this section.

Theorem 3.2

A three dimensional differential equation $\dot{x} = f(x)\ \ x \in G \subseteq R^3$ is a bi-Hamiltonian system if and only if it admits two functionally independent constants of the motion and the Jacobian of each fixed point is a Poisson matrix.

Proof:

One direction of this theorem is trivial. If a three dimensional differential equation is a bi-Hamiltonian system, then by definition it has two functionally independent constants of the motion and the Jacobian of each fixed point is a Poisson matrix.

It is the converse which is of interest. If a three dimensional differential equation admits two functionally independent constants of the motion *K* and *H*, then $\nabla K, \nabla H$ are linearly independent and orthogonal to *f(x)*. Since we are in three dimensions this implies that the cross product of $\nabla K, \nabla H$ is, up to an x-dependent scalar *m(x)* equal to *f(x)*, i.e. $f(x) = m(x)\nabla K \times \nabla H$. And the rest of the proof follows from the following observation:

$$
f(x) = m(x)\nabla K \times \nabla H = m(x)\begin{pmatrix} 0 & K_3 & -K_2 \\ -K_3 & 0 & K_1 \\ K_2 & -K_1 & 0 \end{pmatrix}\begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix} = m(x)\begin{pmatrix} 0 & -H_3 & H_2 \\ H_3 & 0 & -H_1 \\ -H_2 & H_1 & 0 \end{pmatrix}\begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix}
$$

That $m(x)\begin{pmatrix} 0 & K_3 & -K_2 \\ -K_3 & 0 & K_1 \\ K_2 & -K_1 & 0 \end{pmatrix}$ and $m(x)\begin{pmatrix} 0 & -H_3 & H_2 \\ H_3 & 0 & -H_1 \\ -H_2 & H_1 & 0 \end{pmatrix}$ are Poisson structure matrices follows form proposition 3.1. They are not constant multipliers of each other, since K and H are functionally independent. That they are compatible Poisson structures can be seen by applying proposition 3.1 to the following smooth function:

$$\alpha K(x) + \beta H(x)\ , \quad \alpha,\ \beta \in R$$

Finally recall that the Jacobian of each fixed point is a Poisson matrix by assumption.

q.e.d.

Note that, if *m(x)* does not change sign, then $\frac{1}{m(x)} dx_1 \wedge dx_2 \wedge dx_3$ is as an invariant volume form of the underlying differential equation, since $div(\nabla K \times \nabla H) \equiv 0$.

In the next section we will show for a very prominent example how powerful a local analysis can be in order to detect submanifolds in the parameter space on which a given dynamical system is a Hamiltonian or even a bi-Hamiltonian one.

4. Three dimensional Bi-Hamiltonian Lotka Volterra equations: a classification

The 3-dimensional Lotka-Volterra equations are given by

$$\dot{x}_i = x_i\left(b_i + \sum_{j=1}^{3} a_{ij}x_j\right) \quad i = 1,2,3 \tag{1}$$

where b_i, a_{ij} $(i, j = 1,2,3)$ are arbitrary parameters.

This class of differential equations is of interest in different branches of science. In physics they are important for such problems as mode coupling of waves in laser [14] and plasma physics [15]. In chemistry they were used by Lotka [16] to investigate autocatalytic chemical reactions. And their role in theoretical biology and evolutionary game theory is well described for instance in Ref [17]. So the variables x_i may represent, among other things, frequencies of biological species or chemical components and, as such, belong to R_0^+. Therefore it is natural to restrict the analysis of (1) to the non negative orthant. The special structure of (1) implies that the boundary and the interior of the non negative orthant are flow invariant. It is easy to analyse under what conditions (1) is a Hamiltonian system on the boundary, since it is two dimensional there (see for instance Ref. [18]). So we can restrict ourselves to the interior of the non negative orthant, denoted by R_+^3. In the remaining part of this section we study the consequences of our assumption iii) in definition 2.1. We will see that this local property already determines the global dynamics of (1).

So let us assume that (1) admits at least one interior fixed point p, i.e. all components of p are positive and p satisfies the following equation:

$$Ap + b = 0 \tag{2}$$

The Jacobian at such a fixed point p is given by

$$Jac(p) = \begin{pmatrix} a_{11}p_1 & a_{12}p_1 & a_{13}p_1 \\ a_{21}p_2 & a_{22}p_2 & a_{23}p_2 \\ a_{31}p_3 & a_{32}p_3 & a_{33}p_3 \end{pmatrix} \quad (3)$$

From the definition of a Poisson matrix with the help of theorem 2.2 and corollary 2.3 one can easily deduce that a three dimensional matrix is a Poisson matrix if and only if the determinant and the trace of this matrix are equal to zero. The condition $\det(Jac(p)) = 0$ implies - since p is by assumption an interior fixed point - that

$$det(A)=0 \quad (4)$$

(4) together with (2) yields that we have a fixed point line in the interior of the first orthant and hence the trace of (3) has to be equal to zero for all interior fixed points, i.e.:

$$\sum_{i=1}^{3} a_{ii}p_i = 0 \text{ for all interior fixed points p} \quad (5)$$

Condition (4) implies that the rank of the interaction matrix A is ≤ 2. Since the two cases $rank(A) = 0$ and $rank(A) = 1$ are trivial, we omit them. What remains is the $rank(A) = 2$ case. If $rank(A) = 2$, then the conditions (2) and (4) imply that (1) has a fixed point line in the interior of the first orthant. Further there exists an $\alpha \in R^3$ not the null vector, such that:

$$A^T \alpha = 0 \quad (6)$$

From (6), (4) and (2) follows that the vector $\alpha \in R^3$ determined by (6) is orthogonal to the b vector, i.e.

$$\langle \alpha | b \rangle = 0 \quad (7)$$

since $\langle \alpha | b \rangle = \langle \alpha | -Ap \rangle = \langle A^T \alpha | -p \rangle = \langle 0 | p \rangle = 0$.

The equations (6) and (7) together with corollary 4.6. of Ref [18] yield the following proposition, which is our first global result.

Proposition 4.1

If the Jacobian of all interior fixed points of the three dimensional Lotka Volterra equations is a Poisson matrix, then they admit a constant of the motion of the following form:

$$C(x) = \prod_{i=1}^{3} x_i^{\alpha_i} \tag{8}$$

where $\alpha \in R^3$ is given by (6).

Since $\alpha \in R^3$ is not the null vector, we can assume - after an appropriate labelling of the variables - w.l.o.g. that $\alpha_3 = 1$. This has two nice consequences: first the third row of the interaction matrix A can be represented as a linear combination of the first and second one and secondly we get from (8) the following coordinate transformation

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \dfrac{C}{x_1^{\alpha_1} x_2^{\alpha_2}} \end{pmatrix} \tag{9}$$

The coordinate transformation (9) can be used to bring the three dimensional Lotka Volterra equations into a form, which is appropriate for our purposes. But before doing this, let us collect some more results.

Since the third row of the interaction matrix A is a linear combination of the first and second one the fixed point equation (2) reduces to:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = -\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \tag{10}$$

$rank(A) = 2$ implies further that at least one 2×2-sub matrix of the matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \tag{11}$$

is non-singular. We demonstrate the consequences of this fact exemplary for the case

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \neq 0 \tag{12}$$

Condition (12) together with equation (10) yields

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \begin{pmatrix} -b_1 - a_{13}p_3 \\ -b_2 - a_{23}p_3 \end{pmatrix} \tag{13}$$

where p_3 is arbitrary.

Inserting (13) into (5) gives after some algebraic computations the following result:

$$-\ b_1a_{22}(a_{11} - a_{21}) + b_2a_{11}(a_{22} - a_{12}) - p_3 \det(A_1, A_2, -diag(A)) = 0, \text{ for all } p_3 > 0 \tag{14}$$

where $diag(A) = (a_{11}, a_{22}, a_{33})$ and A_i denotes the *i-th* row of A.

(14) is fulfilled if and only if

$$\det(A_1, A_2, -diag(A)) = 0 \tag{15}$$

and

$$-\ b_1a_{22}(a_{11} - a_{21}) + b_2a_{11}(a_{22} - a_{12}) = 0 \tag{16}$$

Equation (15) together with the fact that the third row of A is a linear combination of the first and second one implies that *-diag(A)* is a linear combination of the row vectors of the matrix A; i.e. there exists a $\beta \in R^3$, such that

$$A^T\beta = -diag(A) \tag{17}$$

This $\beta \in R^3$ is orthogonal to the b vector, i.e.

$$\langle \beta | b \rangle = 0 \tag{18}$$

since $\langle \beta | b \rangle = \langle \beta | -Ap \rangle = \langle -A^T\beta | p \rangle = \langle diag(A) | p \rangle = \sum_{i=1}^{3} a_{ii}p_i = 0$.

From (6) follows further that:

$$-diag(A) = A^T\beta = A^T\beta + \lambda A^T\alpha = A^T(\lambda\alpha + \beta)$$

where $\lambda \in R$ is arbitrary and therefore we can assume w.l.o.g. that $\beta_3 = 1$ (set $\lambda = 1 - \beta_3$ and recall that we assumed $\alpha_3 = 1$).

In proposition 4.1 of Ref. [18] we have proved that the n-dimensional Lotka Volterra equations are volume preserving with density function $\prod_{i=1}^{n} x_i^{\beta_i - 1}$ whenever the conditions $A^T\beta = -diag(A)$ and $\langle \beta | b \rangle = 0$ are fulfilled. This fact together with (17) and (18) yields our next global result.

Proposition 4.2

If the Jacobian of all interior fixed points of the three dimensional Lotka Volterra equations is a Poisson matrix, then they are volume preserving with a density function of the following form:

$$x_1^{\beta_1 - 1} x_2^{\beta_2 - 1} \tag{19}$$

Since multiplying a differential equation with a density function does not change the phase portrait, we can multiply (1) with (19). Further we can apply the coordinate transformation (9) to these differential equations. After these manipulations the three dimensional Lotka Volterra equations are given by the following system of differential equations:

$$\begin{aligned} \dot{x}_1 &= x_1^{\beta_1} x_2^{\beta_2 - 1}\left(b_1 + a_{11}x_1 + a_{12}x_2 + a_{13}\frac{c}{x_1^{\alpha_1} x_2^{\alpha_2}} \right) \\ \dot{x}_2 &= x_1^{\beta_1 - 1} x_2^{\beta_2}\left(b_2 + a_{21}x_1 + a_{22}x_2 + a_{23}\frac{c}{x_1^{\alpha_1} x_2^{\alpha_2}} \right) \\ \dot{c} &= 0 \end{aligned} \tag{20}$$

As it should be, in this new coordinates, the three dimensional Lotka Volterra equations have a trivial constant of the motion, namely:

$$K = c \tag{21}$$

The constant of the motion (21) generates a foliation of the phase space into two dimensional flow invariant leafs. The dynamics on such a leaf is given by the following two dimensional differential equation:

$$\begin{aligned} \dot{x}_1 &= x_1^{\beta_1} x_2^{\beta_2 - 1}\left(b_1 + a_{11}x_1 + a_{12}x_2 + a_{13}\frac{c}{x_1^{\alpha_1} x_2^{\alpha_2}} \right) \\ \dot{x}_2 &= x_1^{\beta_1 - 1} x_2^{\beta_2}\left(b_2 + a_{21}x_1 + a_{22}x_2 + a_{23}\frac{c}{x_1^{\alpha_1} x_2^{\alpha_2}} \right) \end{aligned} \tag{22}$$

Calculating the divergence of (22) yields:

$$x_1^{\beta_1 - 1} x_2^{\beta_2 - 1}\left[\beta_1 b_1 + \beta_2 b_2 + (\beta_1 a_{11} + \beta_2 a_{21} + a_{11})x_1 + (\beta_1 a_{12} + \beta_2 a_{22} + a_{22})x_2 + (\beta_1 a_{13} + \beta_2 a_{23} - \alpha_1 a_{13} - \alpha_2 a_{23})\frac{c}{x_1^{\alpha_1} x_2^{\alpha_2}} \right] \tag{23}$$

The expression (23) implies that the divergence is identical to zero if and only if $A^T\beta = -diag(A)$ and $\langle\beta|b\rangle = 0$. But these conditions are automatically fulfilled, since they coincide with (17) and (18). Hence the differential equations (22) are a divergence free two dimensional system and therefore Hamiltonian with the canonical Poisson structure matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and the Hamilton function H given by:

$$H(x_1,x_2) = \int x_1^{\beta_1} x_2^{\beta_2-1}\left(b_1 + a_{11}x_1 + a_{12}x_2 + a_{13}\frac{c}{x_1^{\alpha_1}x_2^{\alpha_2}}\right)dx_2 + f(x_1) = -\int x_1^{\beta_1-1} x_2^{\beta_2}\left(b_2 + a_{21}x_1 + a_{22}x_2 + a_{23}\frac{c}{x_1^{\alpha_1}x_2^{\alpha_2}}\right)dx_1 + g(x_2) \quad (24)$$

for suitable chosen $f(x_1)$ and $g(x_2)$ and α_1, α_2 given by (6) and β_1, β_2 given by (17).

Integration of (24) is straightforward and hence not carried out here. (But note that an inspection of (24) immediately implies that functions of the form x^α have to be integrated and hence in the constant of the motion only terms of the form x^β or $\ln x$ can appear)

H is clearly also a constant of the motion of the full three dimensional system (20) and functionally independent of the constant of the motion (21).Hence we get a further global result, namely:

Proposition 4.3

If the Jacobian of all interior fixed points of the three dimensional Lotka Volterra equations is a Poisson matrix, then they admit a second constant of the motion of the form (24), which is functionally independent of the constant of the motion (21).

Finally let us give also a dynamical interpretation of the condition (16). It is easy to verify (see for instance Ref. [18]) that (16) is equivalent to the fact that the two dimensional subsystem on the x_1, x_2-face is - after multiplying it with an integrating factor of the form (19) - divergence free and hence Hamiltonian. But this implies the following proposition.

Proposition 4.4

If the Jacobian of all interior fixed points of the three dimensional Lotka Volterra equations is a Poisson matrix, then at least one two dimensional subsystem is a Hamiltonian one.

Putting all the results obtained in this section together we get with the help of theorem 3.2 the following classification theorem for three dimensional Lotka Volterra equations.

Main Theorem 4.5

Consider a three dimensional Lotka Volterra equation with at least one interior fixed point, then the following statements are equivalent:

i) It is a bi-Hamiltonian system.

ii) It is a Hamiltonian system.

iii) The Jacobian of all interior fixed points is a Poisson matrix.

iv) It has a fixed point line in the first orthant and the the conditions (15) and (16) are satisfied.

v) It has a fixed point line in the first orthant and the trace of the Jacobian is equal to zero at all fixed points.

vi) It has a fixed point line in the first orthant, it is volume preserving with a density function of the form (19) and at least one two dimensional subsystem is a Hamiltonian one.

Note that in general only the implications $i) \Rightarrow ii) \Rightarrow iii)$ hold, but not their converse. Condition iv) of the main theorem 4.5 implies the following corollary.

Corollary 4.6

In the class of three dimensional Lotka Volterra equations Hamiltonian and bi-Hamiltonian systems form in general a codimension 4 phenomenon.

The above results imply that the foliation of the phase space is symplectic. The dynamics on a symplectic leaf is characterised by the eigenvalues at the fixed points: are the eigenvalues at a fixed point real, then we have a saddle point with a stable and unstable manifold. On the other hand, if the eigenvalues at a fixed point are purely imaginary, then the fixed point is a center, i.e. surrounded by a continuum of periodic orbits. Whether the fixed point eigenvalues are real or purely imaginary is determined by the sign of the following algebraic expression (to see this, calculate the eigenvalues of (3)):

$$p_1p_2(a_{11}a_{22}-a_{12}a_{21})+p_1p_3(a_{11}a_{33}-a_{13}a_{31})+p_2p_3(a_{22}a_{33}-a_{23}a_{32}) \qquad (25)$$

If (25) is positive, then the eigenvalues are purely imaginary and if (25) is negative, then the eigenvalues are real.

5. Conclusions

In the present work we studied three dimensional bi-Hamiltonian systems in general and we used the obtained results to classify all three dimensional Lotka Volterra equations, which admit a bi-Hamiltonian structure. We proved that for this class of differential equations bi-Hamiltonian systems form in general a codimension 4 phenomenon. We gave a dynamic interpretation of the four conditions defining the " Hamiltonian " codimension 4 manifold in the parameter space and further we found out that there is no difference between Hamiltonian and bi-Hamiltonian Lotka Volterra equations. All these results were obtained from a detailed local analysis around the fixed points of the system.

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