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Abstract

The main goal of the paper is to show the application of the projection method as a tool for the analysis of transitional dynamics of endogenous growth models, the analysis which is very often omitted in common literature on the topic. The application of the method is demonstrated on an endogenous growth model with human capital accumulation and government sector. We analyze the long-run (steady states) and the short-run effects (transitional dynamics) of different fiscal policies. The transitional dynam cs of the competitive equilibrium and the social optimum economies are compared. It is shown that when the economy starts with relatively abundant physical capital it is optimal to decrease its level very rapidly even at the cost of a big decline of consu mption for a period of time. The introduction of education subsidies can bring the economy closer to the optimum and, therefore, improve the welfare of the society.

Zusammenfassung

Im Mittelpunkt des Papiers steht eine Anwendung der "projection method" am Beispiel eines endogenen Wachstumsmodelles mit Staatssektor und Humankapitalakkumulation. Bei der "projection method" handelt es sich um ein Instrument zur Analyse der Anpassungsdynamik in endogenen Wachstumsmodellen. In der vorliegenden Arbeit werden sowohl die Eigenschaften der langfristigen Gleichgewichtsbeziehungen als auch die kurzfristige Anpassungsdynamik unter Wettbewerbsbedingungen und unter Einbeziehung externer Effekte studiert. Es zeigt sich, daß die Einführung von Ausbildungssubventionen im Falle einer an physischem Kapital reichen Volkswirtschaft die Wohlfahrt derselben unter bestimmten Bedingungen erhöht.

JEL-Classification: C6, O11, I28

Comments

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Contents

1 Introduction

Most of papers on endogenous growth theory are restricted to the steady state analysis. It is based on the assumption that balanced growth regimes can serve as good approximations of the behavior of real economies. However, there exist situations, such as wars, disasters, and the collapse of communist regimes in Central and Eastern Europe, when the relations between levels of variables do not correspond to the steady state relations. Changed government policies can be also an example of it. In these economies then may appear a transitional period during which they move back to steady states[1]. The short-run effects are getting very important and we need to study the transitional dynamics. Following the arguments for the importance of the analysis of the transition (Mulligan and Sala-i-Martin (1993)), we suggest here the projection method (Judd (1992)) as a tool which can be used to analyze it.

The application of the method is shown on the model that follows Lucas (1988) and is extended by the government sector in a similar way as in (Sorensen (1993)). Thanks to the used method the analysis is not restricted only to the study of long-run effects of fiscal policies (steady states), as it is common in recent literature (see for example Barro and Sala-i-Martin (1990), Nerlove, Razin, Sadka and Weizsäcker (1990) or Sorensen (1993)), but allows also for the analysis of short-run effects (transitional dynamics) as in the paper by Mulligan and Sala-i-Martin (1993)[2].

The rest of the paper is organized as follows. In Section 2 we shortly discuss several methods used for solving intertemporal optimizing models and mainly the principle and the implementation of the projection method. In Section 3.1 we develop the model and derive the first order conditions for decentralized economy equilibrium and social optimum as well[3]. Section 3.2 is devoted to the analysis of steady states and briefly to the selected aspects of the model's comparative statics.

Finally, we study the transitional dynamics of the model. In that section we show how to transform the model to the form which is suitable for the application of the projection

[1]When economies are imposed to structural changes then the new steady states will be different from the original ones, eg the case of the post-socialist CEE countries.

[2]In the paper they use the time elimination method for the analysis of the transitional dynamics in two sector model of endogenous growth.

[3]Because of the existence of externalities these two ones are not identical.

method and then we perform several experiments under different policy and parameter regimes.

2 Methods of Solution of Intertemporal Optimizing Models

Much of the current research in macroeconomics and mainly in neoclassical growth models is based on the intertemporal optimizing infinite-lived representative agent models. Solving these models via the determination of the first order necessary conditions by means of Pontryagin Maximum Principle or Bellman Dynamical Programming approach we face the so called two point boundary value problem which is much more difficult to solve than the Cauchy initial value problem. It is well known that analytical solutions exist only for the very special class of nonlinear problems and for the class of LQ (linear-quadratic) problems. To get a solution when the problem does not belong to the above mentioned classes, we have to apply numerical techniques. The most common methods are the following: the shooting method, the relaxation method, the perturbation method, the time elimination method, and the projection method. The first two methods, the shooting and the relaxation method (see eg Press et al (1986)), are very general and able to solve any problem. However, because they require numerical integration of the system of differential equations, at least several, if not very many, times, their application in the solved problem is too clumsy and tedious.

Fortunately, a favourable feature of the infinite optimizing problems is the fact that if both the criterion function and the dynamic constraints are autonomous, the control variables can be characterized as static (t-invariant) policy functions or feedback rules of the state variables. This property is critical for the possibility to use the last three mentioned methods. The time elimination method, suggested in the paper Mulligan and Sala-i-Martin (1991), converts the two-point boundary value problem to the initial value problem and by the application of numerical integration started at steady state gets the approximation of policy functions. Thus the approximation is given in the form of "trajectories" of the control-like variables with respect to the state-like variables. On the other hand, the perturbation and the projection method give the solution in the form of a polynomial approximation. The advantage of the solution is its quasi-analytical form. More specifically, the perturbation method (Judd (1993)) makes use of Taylor expansion series or Pade approximations of policy functions at steady state. The projection method approximates the policy functions at some predefined interval which makes them to have a good precision on the broader range.

2.1 Projection Methods

The projection methods[4] are used for solving the systems of ordinary or partial differential equations and are generally applicable to economic problems (Judd (1992)), in particular to the optimal control problems with t-invariant feedback rules. As compared to other numerical methods used for this kind of problems, the projection method[5] differs in assuming that the solution can be represented quasi-analytically in the form of polynomial approximations of policy functions.

Suppose that the model is represented by the system of differential equations. First, we must transform the problem to the form

$$\mathcal{N}(p) = 0 \tag{1}$$

where $\mathcal{N}$ is an operator and the function p is a zero of the operator $\mathcal{N}$ which means that p solves the given system of differential equations. For initial value problems a zero p of the operator $\mathcal{N}$ is a vector function of time; for boundary value problems (which is characteristic for growth models) a zero p of the operator $\mathcal{N}$ is a vector of policy functions which are functions of state variables only.

Further, we assume that the solution for the policy functions can be approximated by the formula

$$p(x) \approx \hat{p}(x; a) = \sum_{i=1}^{q} a_i \phi_i(x) \tag{2}$$

where $a = (a_1, \ldots, a_q)^T$ is the vector of q unknown coefficients and $\varphi = (\phi_1, \ldots, \phi_q)^T$ is the basis of q a priori determined analytic functions. In our case we have implemented the family of Chebyshev polynomials, that are defined over the interval $[-1, 1]$ by the formula

[4]These methods are also very often called Minimum Weighted Residuals Methods (eg Fletcher (1988)).

[5]Also the perturbation method (Judd (1993)), as we mentioned above, assumes that the solution can be approximated by some polynomials.

$T_n(x) \equiv cos(n \arccos x)$ and generated by the recursive formula $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ with $T_0(x) = 1$ and $T_1(x) = x$. The restriction to the interval $[-1, 1]$ is inessential in the sense that we can define a linear transformation which will enable us to gain the approximate solution on any interval.

The projection method gets q conditions for the q unknown coefficients in the vector a by means of q projections. In our case we have used the point projection. It means that we compute the values of the operator $\mathcal{N}$ at the q important points $\{x_i\}_{i=1}^q$ which are zeroes of the qth basis element[6]. Thus we get the system of q nonlinear equations

$$R(x_i; a) \equiv (\mathcal{N}(\hat{p}))(\mathrm{x_i}) = 0, \qquad i = 1, \ldots, q. \tag{3}$$

The vector function R is called the residual function. In this way the problem of solving the system of nonlinear differential equations was transformed into the problem of solving the system of nonlinear algebraic equations.

2.2 PROJEC: library module in GAUSS

The above briefly described kind of projection method, the orthogonal collocation method with Chebyshev polynomials (max. 9th degree of approximation), has been programmed in GAUSS 3.1. The system enabling us to solve this kind of problems is the PROJEC library unit in GAUSS.

PROJEC contains procedures for finding approximation of the vector of policy functions $\mathbf{p} = (p_1, \ldots, p_m)^T$ for a user-provided operator $\mathcal{N}$ expressing the first order conditions for the discrete/continuous optimal control problems. The approximation procedure uses as a basis the tensor product of Chebyshev polynomials that are computed on the base of the given degrees of the approximation related to the particular state variables. When we have n state variables $\mathbf{x} = (x_1, \ldots, x_n)$ and n degrees of the approximation are $q_1, \ldots, q_n$, respectively, and the one-dimensional basis of the degree q_i for the variable x_i is given by $\varphi_{q_i}(x_i) = (\phi_1(x_i), \phi_2(x_i), \ldots, \phi_{q_i}(x_i))$ $(i = 1, \ldots, n)$ then the n-dimensional polynomial basis for the model is the n-fold tensor product of the one-dimensional bases $\psi(\mathbf{x}) = \varphi_{q_1}(x_1) \otimes \varphi_{q_2}(x_2) \otimes \cdots \otimes \varphi_{q_n}(x_n) = \{\phi_{i_1}(x_1) \cdots \phi_{i_n}(x_n) | \ i_j = 1, \ldots, k_j, \ j =$

[6]This kind of the projection method is called the orthogonal collocation method.

$1, \ldots, n\}$, where $\phi_{i_j}(x_j)$ is the i_jth Chebyshev polynomial in x_j variable, and $q = \prod_{i=1}^{n} q_i$ is the number of the elements of the n-dimensional polynomial basis.

The first order conditions expressing the studied optimal problem are then transformed to the problem of $m \times q$ nonlinear equations for $m \times q$ unknown values of the parameters a_{ij}. For the solution of the problem the Newton method (procedure NEWTON) has been used and the user has to specify the initial values of these parameters. As a starting procedure of the solution of the problem can be used the steep descent method (procedure STEEP) which may be more efficient in the regions far from the solution. After reaching by the user specified distance from the solution, the program automatically switches to the application of the Newton method which is more efficient close to the solution.

The reader interested in the application of the module PROJEC is referred to Appendix where an example described in Section 4 is presented. The more information about it can be learned from the handbook of the projection method in (Kejak and Keuschnigg (1994)) and the description of the PROJEC library module in (Kejak (1994)).

3 Growth With Human Capital Accumulation

In the following section we setup the Lucas' human capital accumulation model with government and derive the first order necessary conditions. Then we analyze steady state and briefly review the comparative statics of a version of the model without externalities and the influence of the introduction of externalities on steady state. We also shortly mention the steady state of the social optimum economy. The core of the section is the presentation of the transformation of the system into the form convenient for the application of the projection method.

3.1 An Extended Model of Human Capital Accumulation

Suppose that the economy is populated by identical workers with the same skill level H and that they devote a fraction l of their (non-leasure) time to current production, and remaining $1 - l$ to human capital accumulation. The effective labor input in production is the $L = lH$. The economy also consists of large number of identical firms with the production function

$$F(K,H) = K^{\beta}L^{1-\beta}H_a^{\gamma} = K^{\beta}(lH)^{1-\beta}H_a^{\gamma}, \qquad 0 < \beta < 1, \qquad \gamma \geq 0 \tag{4}$$

where K is the physical capital, and the term H_a^{γ} introduces an "external effect", which is related to the average level of human capital in the economy. Because the workers are identical, average skills in equilibrium coincide with individual skills ($H_a = H$).

A linear Uzawa-Rosen type specification describes the accumulation of human capital:

$$\dot{H} = \phi(1-l)H. \tag{5}$$

where ϕ is a parameter indicating the effectiveness of education (investment in human capital).

Equation (5) implies that no human capital accumulates if no time is devoted to education ($l = 1$). If all of time endowment is devoted to education ($l = 0$), human capital H grows at its maximal rate ϕ. In between these two points, the inputs H and $1 - l$ generate constant returns.

Now we can set up the whole model of the two sector economy as a dynamic optimization problem with two decision variables - consumption C_t, and time devoted to production l_t[7]:

$$\max \quad \int_0^{\infty} e^{-\rho t}\left(\frac{C_t^{1-\theta}-1}{1-\theta}\right)dt \qquad \text{s.t.} \tag{6}$$

$$\dot{K}_t = (1-\tau_K)r_tK_t + (1-\tau_L)w_tl_tH_t + \tau w_t(1-l_t)H_t - T_t - C_t \tag{7}$$

$$\dot{H}_t = \phi(1-l_t)H_t. \tag{8}$$

This model is an extension of the original Lucas' model by the government with several fiscal instruments: labor income tax rate τ_L, capital income tax rate τ_K, education subsidy rate τ, lump sum tax T and government expenditure to output ratio κ[8] (see Sorensen (1993)).

[7]In the rest of the paper we do not use time indexes unless it is unnecessary.

[8]In the literature we can distinguish two main strands concerning government expenditure: government expenditure that does not increase production possibilities, e.g. Lucas (1990), and King and Rebelo (1990), and productive government expenditure, e.g. Barro (1990), and Barro and Sala-i-Martin (1990). In the paper we follow the former approach without productive government expenditure.

In writing the current-value Hamiltonian $\mathcal{H}$, we denote the shadow prices for physical and human capital by λ and μ, respectively:

$$\begin{aligned}\mathcal{H}(K,H;C,l;\lambda,\mu) &= \frac{C^{1-\theta}-1}{1-\theta}+\lambda\{(1-\tau_K)rK+(1-\tau_L)wlH+\tau w(1-l)H-T-C\} \\ &\quad +\mu\phi(1-l)H \qquad (9)\end{aligned}$$

To simplify the mathematical derivations, we use the following notation:

$$F_K = \frac{\partial F}{\partial K} = \beta K^{\beta-1}(lH)^{1-\beta} H_a^{\gamma} \qquad (10)$$

$$F_l = \frac{\partial F}{\partial l} = (1-\beta)K^{\beta}l^{-\beta}H^{1-\beta}H_a^{\gamma} \qquad (11)$$

$$F_H = \frac{\partial F}{\partial H} = (1-\beta)K^{\beta}l^{1-\beta}H^{-\beta}H_a^{\gamma} \qquad (12)$$

$$F_H^* = \frac{\partial F^*}{\partial H} = (1-\beta+\gamma)K^{\beta}l^{1-\beta}H^{\gamma-\beta} \qquad (13)$$

where F_K, F_l, F_H and F_H^* are the marginal productivity of physical capital, the marginal productivity of time devoted to production, the private marginal productivity of human capital, and the social marginal productivity of human capital, respectively. The two last marginal productivities differ in that the private agents take the average level of human capital H_a as given (12), while the society as whole additionaly considers the effect of the market clearing condition $H_a = H$ (13).

By using the Pontryagin Maximum Principle the first order conditions for competitive equilibrium are thus

$$\lambda = c^{-\theta} \qquad (14)$$

$$\mu\phi = \lambda w(1-\tau_L-\tau) \qquad (15)$$

$$\dot{\lambda} = \rho\lambda-\lambda[(1-\tau_K)r] \qquad (16)$$

$$\dot{\mu} = \rho\mu-\lambda w[(1-\tau_L)l+\tau(1-l)]-\mu\phi(1-l) \qquad (17)$$

In these conditions, we can eliminate the shadow prices λ and μ and get the equations for control variables. Equation (18) follows from (14) and (16) and from conditions for factor prices under perfect competition ($r = F_K - \delta$; $w = F_L = \frac{F_l}{H}$).

$$\frac{\dot{C}}{C} = \sigma[(1-\tau_K)(F_K-\delta)-\rho] \tag{18}$$

where $\sigma = \theta^{-1}$ is the intertemporal elasticity of substitution.

In this model we assume that the government budget is balanced at each time, so that

$$G = \kappa F = \tau_K r K + \tau_L w l H - \tau w(1-l)H + T. \tag{19}$$

and the equation (7) can be expressed in the form

$$\frac{\dot{K}}{K} = (1-\kappa)\frac{F}{K} - \delta - \frac{C}{K}. \tag{20}$$

Using this equation and equation (15) differentiated with respect to time as well as equation (17), we get

$$\frac{\dot{l}}{l} = (\tau_K - \kappa)\frac{F}{K} + \frac{\delta(1-\tau_K)}{\beta} - \delta - \frac{C}{K} + \frac{\phi}{\beta}\left\{(\gamma-\beta)(1-l) + l + \frac{1-\tau_L}{1-\tau_L-\tau}\right\} \tag{21}$$

The equations (18), (21) together with the laws of motion (8) and (20) and the transversality conditions determine the equilibrium trajectories of the economy.

3.2 Sustained Growth and Steady State Analysis

For steady state (or balanced growth path) we require all variables to grow at constant (possibly zero) rates. Because of limited time endowment equal to unity, the time allocation variable l cannot grow at steady state, thus $g_l = 0$. Suppose that the rate of growth of consumption is $\dot{C}/C = g_C = g_{ss}$. Equation (18) implies the constant marginal productivity of physical capital at steady state. This, in turn, (see equation (10)) requires that physical capital and human capital steady state growths are in the relation $(\beta-1)g_K + (1-\beta+\gamma)g_H = 0$. Thus the steady state growth rates satisfy the conditions:

$$g_C = g_K = g_{ss} \tag{22}$$

$$g_H = \frac{1-\beta}{1-\beta+\gamma}g_{ss} \tag{23}$$

$$g_l = 0. \tag{24}$$

From the above conditions on growth rates and the steady state versions of (8), (18), (20), and (21), we can obtain the following steady state conditions:

$$g_{ss} = \frac{\sigma}{A}\left(\phi\frac{1-\tau_L}{1-\tau_L-\tau}-\rho\right) \tag{25}$$

$$l_{ss} = 1-\frac{1-\beta}{1-\beta+\gamma}\frac{g_{ss}}{\phi} \tag{26}$$

$$(F_K)_{ss} = \delta+\phi\frac{1-\tau_L}{(1-\tau_L-\tau)(1-\tau_K)}+\frac{\gamma}{(1-\beta+\gamma)(1-\tau_K)}g_{ss} \tag{27}$$

$$\left(\frac{C}{K}\right)_{ss} = \frac{1-\kappa}{\beta}(F_K)_{ss}-\delta-g_{ss} \tag{28}$$

where $A = 1-\frac{\sigma\gamma}{1-\beta+\gamma}$.

In the following paragraphs we give the brief overview of the comparative statics of a simplified model where $\tau_K = \tau_L = \tau = \kappa = 0$.

3.2.1 Economy without externalities ($\gamma = 0$)

The steady state conditions now look as follows

$$g_{ss} = \sigma(\phi-\rho) \tag{29}$$

$$l_{ss} = 1-\frac{\sigma(\phi-\rho)}{\phi} \tag{30}$$

$$\left(\frac{C}{K}\right)_{ss} = \frac{\delta+\phi}{\beta}-\sigma(\phi-\rho)-\delta \tag{31}$$

$$(F_K)_{ss} = \delta+\phi \tag{32}$$

$$\left(\frac{K}{H}\right)_{ss} = \left(\frac{\delta+\phi}{\beta}\right)^{\frac{1}{\beta-1}} l_{ss}. \tag{33}$$

From these steady state conditions we can derive several basic properties of the model (all parameters in the model are positive, ie $\phi > 0$, $\rho > 0$, $\theta = \sigma^{-1} > 0$, and $\beta > 0$). From equation (29) we can see that

$$g_{ss} = \begin{cases} > 0, & \phi > \rho, \\ = 0, & \phi = \rho, \\ < 0, & \phi < \rho. \end{cases}$$

Property 1: Consumption, physical and human capital exhibit balanced growth g_{ss} which is positive, negative or zero if the effectiveness of investment in human capital ϕ is higher, lower, or equal to the rate of time preference ρ, respectively.

From (30) we can see that

$$l_{ss} = \begin{cases} < 1, & \phi > \rho, \\ = 1, & \phi = \rho, \\ > 1, & \phi < \rho. \end{cases}$$

Property 2: The steady state value of time devoted to production is lower than 1 (human capital is accumulated), higher than 1 ('disinvestment' in human capital)[9], or equal to 1 (no human capital accumulation), if the effectiveness of investment in human capital ϕ is higher, lower, or equal to the rate of time preference ρ, respectively. By comparing the conditions for g_{ss} and l_{ss} we can see that the positive growth rate implies time devoted to work lower than 1 and the negative one to the time higher than 1.

From the condition that time devoted to production cannot be negative ($l_{ss} \geq 0$), we can easily derive the condition

$$\rho \geq \phi(1 - \theta).$$

If $\theta > 1$ (ie the intertemporal elasticity of substitution $\sigma < 1$), then it is always fulfilled. The condition holds for any $\rho > 0$ when $\theta > 1$ because ϕ must be positive. These results we can summarize in the following property.

Property 3: The steady state value of time devoted to production l_{ss} cannot be negative $l_{ss} \geq 0$ for any human investment efficiency ϕ, if the degree of risk aversion is higher than

[9]In this interpretation, the total time endowment is higher than unity. $l = 1$ means that just enough of total available time (larger than 1) is allocated to education so as to keep H from falling. If even less time is allocated to education ($l > 1$) society starts to forget, human capital depreciates. Hence the interpretation of disinvestment of human capital. It is also possible to include in the model a 'depreciation' of human capital, as eg in Mulligan and Sala-i-Martin (1992), $\dot{H}/H = (1 - l)\phi - \delta_H$. However, the used form of human capital accumulation is sufficient for our purpose.

1 ($\theta > 1$). If the degree of risk aversion θ is lower than 1[10] than the above mentioned condition for ϕ must be fulfilled. Hence the intertemporal elasticity of substitution in consumption (σ) must not be too high.

As we shall see further, the sensitivity of the model to the changes in effectiveness ϕ of investment in human capital will be very important. Therefore, we derive the derivatives of steady state values with respect to ϕ.

$$\begin{aligned}
\frac{\partial g_{ss}}{\partial \phi} &= \sigma > 0 \\
\frac{\partial l_{ss}}{\partial \phi} &= -\frac{\rho\sigma}{\phi^2} < 0 \\
\frac{\partial \left(\frac{C}{K}\right)_{ss}}{\partial \phi} &= \frac{1}{\beta} - \sigma = \begin{cases} > 0, & \beta < \sigma^{-1} \\ = 0, & \beta = \sigma^{-1} \\ < 0, & \beta > \sigma^{-1}. \end{cases}
\end{aligned}$$

The derivative for the case of $\left(\frac{K}{H}\right)_{ss}$ is more complicated. Therefore, we do not write it here but just note the sign which is

$$\frac{\partial \left(\frac{K}{H}\right)_{ss}}{\partial \phi} < 0.$$

Property 4: The steady state values of the growth rate g_{ss}, the time or effort devoted to production l_{ss}, and the ratio of physical to human capital $\left(\frac{K}{H}\right)_{ss}$ depend monotonically on the changes in the effectiveness ϕ of investment in human capital; the growth rate g_{ss} positively, and the time devoted to production l_{ss} and the ratio of physical to human capital $\left(\frac{K}{H}\right)_{ss}$ negatively. The dependence of the consumption-capital ratio $\left(\frac{C}{K}\right)_{ss}$ on the effectiveness ϕ of investment in human capital is nonmonotonic; it is positive, negative, or zero for the capital share β lower, higher or equal to the degree of risk aversion θ, respectively.

[10]In the case when $\theta = 1$ the utility function $\frac{C^{1-\theta}}{1-\theta}$ is not defined, but $\lim_{\theta \to 1} \frac{C^{1-\theta}}{1-\theta} = \ln C$.

3.2.2 Economy with externalities ($\gamma \neq 0$)

In this subsection we briefly describe the influence of externalities on steady state. The version of equations (25) -(28) without fiscal instruments is

$$g_{ss} = \frac{\sigma}{A}(\phi - \rho) \quad (34)$$

$$l_{ss} = 1 - \frac{1-\beta}{1-\beta+\gamma}\frac{g_{ss}}{\phi} \quad (35)$$

$$(F_K)_{ss} = \delta + \phi + \frac{\gamma}{1-\beta+\gamma} g_{ss} \quad (36)$$

$$\left(\frac{C}{K}\right)_{ss} = \frac{(F_K)_{ss}}{\beta} - \delta - g_{ss} \quad (37)$$

where $A = 1 - \frac{\sigma\gamma}{1-\beta+\gamma}$.

The sensitivity of the long-run behavior of the model to the changes in the external factor γ can be derived by the partial derivatives of the steady state values with respect to γ

$$\frac{\partial g_{ss}}{\partial \gamma} = \frac{\sigma^2(1-\beta)(\phi-\rho)}{[1-\beta+(1-\sigma)\gamma]^2} = \begin{cases} > 0, & \phi > \rho \\ = 0, & \phi = \rho \\ < 0, & \phi < \rho \end{cases}$$

$$\frac{\partial l_{ss}}{\partial \gamma} = \frac{\sigma(1-\sigma)(1-\beta)(\phi-\rho)}{\phi[1-\beta+(1-\sigma)\gamma]^2} = \begin{cases} > 0, & (\phi > \rho)\wedge(\sigma < 1)\vee(\phi < \rho)\wedge(\sigma > 1) \\ = 0, & (\phi = \rho)\vee(\sigma = 1) \\ < 0, & (\phi < \rho)\wedge(\sigma < 1)\vee(\phi > \rho)\wedge(\sigma > 1) \end{cases}$$

$$\frac{\partial (F_K)_{ss}}{\partial \gamma} = \frac{\sigma(1-\beta)(\phi-\rho)}{[1-\beta+(1-\sigma)\gamma]^2} = \begin{cases} > 0, & \phi > \rho \\ = 0, & \phi = \rho \\ < 0, & \phi < \rho \end{cases}$$

$$\frac{\partial \left(\frac{C}{K}\right)_{ss}}{\partial \phi} = \frac{\sigma(1-\sigma)(1-\beta)(\phi-\rho)}{[1-\beta+(1-\sigma)\gamma]^2} = \begin{cases} > 0, & (\phi > \rho)\wedge(\sigma < 1)\vee(\phi < \rho)\wedge(\sigma > 1) \\ = 0, & (\phi = \rho)\vee(\sigma = 1) \\ < 0, & (\phi < \rho)\wedge(\sigma < 1)\vee(\phi > \rho)\wedge(\sigma > 1) \end{cases}$$

Property 5: The sensitivity of the steady state to the γ factor is critically dependent on the relation between the efficiency of investment in human capital ϕ and the rate of time preference ρ. From the point of view of empirical observations (eg King and Rebelo (1990)), we can consider the possibility $\phi > \rho$ and $\sigma < 1$ as highly probable. Under these conditions the influence of the externality factor on all the steady state values is positive, ie with the higher value of γ, the steady state growth g_{ss}[11], time devoted to work l_{ss}, the marginal productivity of physical capital $(F_K)_{ss}$ (gross real interest rate), and the consumption to physical capital ratio $\left(\frac{C}{K}\right)_{ss}$ are higher. The steady state values of these variables are higher because the same level of human capital is more productive in the presence of the higher external factor than of the lower one. The influence of externalities on the steady state is lower, the lower is the efficiency of human capital ϕ and the less people are patient (higher ρ).

3.2.3 Social optimum

To be able to compare the effects of different fiscal policies on the welfare of agents of the economy, it is useful to derive social optimum. In the presence of the external effect H_a^γ, optimal growth paths and competitive equilibrium paths do not coincide[12]. The first order conditions for social optimum are thus

$$\begin{aligned} \lambda &= c^{-\theta} && (38)\\ \mu\phi H &= \lambda F_l && (39)\\ \dot{\lambda} &= \rho\lambda - \lambda[F_K - \delta] && (40)\\ \dot{\mu} &= \rho\mu - \lambda F_H^* - \mu\phi(1-l). && (41) \end{aligned}$$

The first three equations (38)–(40) are equivalent to the equations (14)–(16)[13]. The last equation (41) shows that the social valuation of human capital differ from the private one (17) in the presence of the external factor $\gamma > 0$.

Using the equation for physical capital accumulation, equation (39) differentiated with respect to time, and equation (41), we get

$$\frac{\dot{l}}{l} = \frac{\delta + \frac{1-\beta+\gamma}{1-\beta}\phi}{\beta} - \frac{1-\beta+\gamma}{1-\beta}\phi(1-l) - \frac{C}{K} - \delta. \qquad (42)$$

[11]Note that g_{ss} is the growth rate only of consumption and physical capital.

[12]This is true even in the situation without distortive taxation.

[13]For the economy without distortive taxation.

Similarly as in the case of steady state conditions for the competitive economy, we can derive steady state conditions for socially optimal economy as

$$g^*_{ss} = \sigma(\phi^* - \rho) \tag{43}$$

$$l_{ss} = 1 - \frac{g^*_{ss}}{\phi^*} \tag{44}$$

$$\left(\frac{C}{K}\right)_{ss} = \frac{\delta + \phi^*}{\beta} - g^*_{ss} - \delta \tag{45}$$

$$(F_K)_{ss} = \delta + \phi^* \tag{46}$$

where $\phi^* = \frac{1-\beta+\gamma}{1-\beta}\phi$. It can be seen that the formulas are identical to those for the competitive economy when we substitute ϕ^* for ϕ.

4 Transitional Dynamics

In this section we transform the system to the reduced form with transformed variables exhibiting zero steady state growth. This growth elimination process[14] is necessary for the possibility of the application of the projection method in the solution of the model. Then we introduce t-invariant policy functions into the Euler equations and show how to setup the form of residual functions which can be directly used for the implementation of the projection method in GAUSS via the PROJEC module.

The second part of the section is devoted to fiscal policy experiments with the model and their effects on the transitional dynamics. The suggested experiments and the presented analysis of their effects is far from to be complete. This is caused mainly by the primary purpose of the paper to illustrate the application of the projection method rather than fully explore the effects of different fiscal policies on the model. We plan to place a stronger emphasis on the latter aspect in further work.

4.1 Transformation of the model

When we want to use the projection method in growth models, and in principle any of the methods based on the polynomial approximation of policy functions, ie the perturbation

[14]In principle the same holds for the application of the perturbation method and the time elimination method (see Section 2).

method or the time-elimination method (see Section 2), we face the problem that many of the variables of the model exhibit balanced growth rates which exclude the possibility to find a limited region (a point or even an interval) to which we want to relate our approximation. Therefore, we have to find the transformation that enables us to express the model in transformed variables (Mulligan and Sala-i-Martin (1991) call them as control-like variables and state-like ones) which have no growth at steady state. Typically the transformation also reduces the dimensionality of the model which is favourable for the presentation of the dynamical behaviour.

Based on the relation between the steady state growth of physical and that of human capital given by equation (23), and the relation between the steady state growth of physical capital and that of consumption by (22); we can suggest the following transformations[15]

$$k_t \equiv K_t H_t^{-\left(1+\frac{\gamma}{1-\beta}\right)} \tag{47}$$

$$c_t \equiv C_t K_t^{-1}. \tag{48}$$

Therefore, we can transform the model equations (8), (18), (20), and (21) in variables K_t, H_t, C_t, and l_t to the reduced model equations in the state-like and control-like variables k_t, c_t, and l_t. We get

$$\frac{\dot{k}}{k} = (1-\kappa)\frac{F}{K} - \delta - c - \left(1 + \frac{\gamma}{1-\beta}\right)\phi(1-l) \tag{49}$$

$$\frac{\dot{c}}{c} = \left(\sigma(1-\tau_K) - \frac{1-\kappa}{\beta}\right) F_K - \sigma(\delta+\rho) + \delta(1+\sigma\tau_K) + c \tag{50}$$

$$\frac{\dot{l}}{l} = \frac{(\tau_K-\kappa)}{\beta}F_K + \frac{\delta(1-\tau_K)}{\beta} + \frac{\phi}{\beta}\frac{1-\tau_L}{1-\tau_L-\tau} - \delta - c + \phi\left(\frac{\gamma}{\beta} - 1\right)(1-l) \tag{51}$$

where the marginal product of physical capital F_K can be expressed as $F_K = \beta K^{\beta-1}(lH)^{1-\beta}H^{\gamma}$ $= \beta k^{\beta-1} l^{1-\beta}$.

From the above mentioned definitions of the variables we can see that the newly established variables have zero growth at steady state (they are constant at steady state) and

[15] In general it is not sure whether such a transformation leads to the existence of a reduced form of the model, ie that the model can be expressed only by means of the transformed variables.

we have also reduced the 4-dimensional original model to the 3-dimensional transformed model.

Now it is time to introduce policy functions and perform the time elimination proces. In our case of the two control-like variables c, l and the one state-like variable k, the aim is to specify two policy functions $c = p(k)$ and $l = q(k)$, where the functions p and q depend monotonically[16] on variable k. Using the following identities

$$\begin{aligned} p'(k_t) &= \frac{\mathrm{d}p(k_t)}{\mathrm{d}k_t} = \frac{\dot{c}_t}{\dot{k}_t} \\ q'(k_t) &= \frac{\mathrm{d}q(k_t)}{\mathrm{d}k_t} = \frac{\dot{l}_t}{\dot{k}_t} \end{aligned}$$

and equations (49)–(51) we get

$$p'(k) = \frac{\left\{\left(\sigma(1-\tau_K) - \frac{1-\kappa}{\beta}\right) F_K - \sigma(\delta+\rho) + \delta(1+\sigma\tau_K) + c\right\} c}{\left\{(1-\kappa)\frac{F}{K} - \delta - c - \left(1+\frac{\gamma}{1-\beta}\right)\phi(1-l)\right\} k} \tag{52}$$

$$q'(k) = \frac{\left\{\frac{(\tau_K-\kappa)}{\beta} F_K + \frac{\delta(1-\tau_K)}{\beta} + \frac{\phi}{\beta}\frac{1-\tau_L}{1-\tau_L-\tau} - \delta - c + \phi\left(\frac{\gamma}{\beta}-1\right)(1-l)\right\} l}{\left\{(1-\kappa)\frac{F}{K} - \delta - c - \left(1+\frac{\gamma}{1-\beta}\right)\phi(1-l)\right\} k}. \tag{53}$$

Equations (52) and (53) can be easily transformed into the form of the operator $\mathcal{N}$ suitable for the application of the projection method (see Section 2)

$$(\mathcal{N}_1(p))(k) = p'(k)\dot{k} - \dot{c} = 0 \tag{54}$$

$$(\mathcal{N}_2(q))(k) = q'(k)\dot{k} - \dot{l} = 0. \tag{55}$$

After the specification of the domain for the approximation $[k_1, k_2]$ which should include the steady state value of the state-like variable, our approximations of p and q will be parametrically given by

[16]This monotonicity follows from the properties of a stable saddle path.

$$\hat{p}(k;\mathbf{a_1}) = \sum_{i=1}^{s} a_{1i}\phi_i[\tilde{k}] \quad (56)$$

$$\hat{q}(k;\mathbf{a_2}) = \sum_{i=1}^{s} a_{2i}\phi_i[\tilde{k}] \quad (57)$$

where ϕ_i is ith Chebyshev polynom, $\tilde{k}$ is linear transformation of the interval $[k_1, k_2]$ into $[-1, 1]$, and s is the degree of approximation (the number of terms used). From equations (53)–(55) and approximations (56) and (57), the residual functions become

$$\begin{aligned} R_1(k;\mathbf{a_1},\mathbf{a_2}) &= \hat{p}'(k;\mathbf{a_1})\left\{(1-\kappa)\left(\frac{\hat{q}(k;\mathbf{a_2})}{k}\right)^{1-\beta} - \delta - \hat{p}(k;\mathbf{a_1}) - \left(1+\frac{\gamma}{1-\beta}\right)\phi(1-l)\right\}k \\ &- \left\{(\sigma\beta(1-\tau_K)+\kappa-1)\left(\frac{\hat{q}(k;\mathbf{a_2})}{k}\right)^{1-\beta} - \sigma(\delta+\rho) + \delta(1+\sigma\tau_K) + \hat{p}(k;\mathbf{a_1})\right\} \\ &\times\ \hat{p}(k;\mathbf{a_1}) = 0 \quad (58) \\ R_2(k;\mathbf{a_1},\mathbf{a_2}) &= \hat{q}'(k;\mathbf{a_2})\left\{(1-\kappa)\left(\frac{\hat{q}(k;\mathbf{a_2})}{k}\right)^{1-\beta} - \delta - \hat{p}(k;\mathbf{a_1}) - \left(1+\frac{\gamma}{1-\beta}\right)\phi(1-l)\right\}k \\ &- \left\{(\tau_K-\kappa)\left(\frac{\hat{q}(k;\mathbf{a_2})}{k}\right)^{1-\beta} + \frac{\delta(1-\tau_K)}{\beta} + \frac{\phi}{\beta}\frac{1-\tau_L}{1-\tau_L-\tau} - \delta - c\right. \\ &- \left.\hat{p}(k;\mathbf{a_1}) + \phi\left(\frac{\gamma}{\beta}-1\right)(1-l)\right\}\hat{q}(k;\mathbf{a_2}) = 0. \quad (59) \end{aligned}$$

The application of the PROJEC module via a user-defined procedure for this example can be seen in Appendix.

4.2 Policy experiments

In this section we suggest several policy experiments based on the different values of parameters for the labor income tax rate τ_L, the capital income tax rate τ_K, the education subsidy rate τ, and the government expenditure ratio κ. We also base several experiments in the presence of externalities of human capital γ.

All these experiments are solved by means of the projection method and the simulated transition paths of the model variables are given in figures. The discussion of the results is preliminary and related mainly to the benchmark case (all above mentioned p arameters are zero). The broader discussion of all other results is postponed to further papers.

Calibration of the model To study the effects of fiscal policies in the model, we have to calibrate[17] the values of the model parameters. We use the values for USA used often in the related literature. The capital income share β is specified to be between 0.25 to 1/3 and we choose the value 0.3. Following Mulligan and Sala-i-Martin (1993) the value of intertemporal elasticity of substitution σ is 0.5 (the degree of risk aversion θ is 2). The rate of depreciation δ is set to be 10 percent. Since we assume that the before tax real interest rate is 6 percent we can see[18] that the coefficient of the productivity of human capital ϕ should also be equal to 0.06 (for the case of the absence of externalities.) If we assume that the economy grows at 2 percent per year then we can compute the rate of time preference $\rho = r - g/\sigma$ as 0.02.

Experiment 1: Benchmark Case

$$(\tau = \tau_L = \tau_K = \kappa = \gamma = 0)$$

The setup of the parameters shows that this experiment is the standard version of the Lucas model with no externalities. The transitional dynamics of this case has already been described in several papers (eg Mulligan and Sala-i-Martin (1992), Kejak (1993)).

If we compare the results obtained from the projection method for different values of parameters, the solutions can be separated into three qualitatively different groups which are consistent with the results obtained by Mulligan and Sala-i-Martin (1992). They found that the key factor determining the slope of the policy functions is the relation between capital share β and the degree of risk aversion θ (the inverse of the intertemporal elasticity of substitution σ). We describe the transition paths in the three cases and demonstrate them with figures by the simulation exercises.

We assume that the economy is initially at the situation when physical capital is relatively scarce, therefore, the ratio of physical to human capital K/H[19] is lower than the steady state level (low level of $K/H = k$ reflects low real wages) and the economy starts to accumulate relatively more physical capital than human one. It does it in two ways.

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[17] The calibration is a methodology useful for aggregate models to be consistent with existing microeconomic and macroeconomic evidence (see eg Mehra and Prescott (1985)).

[18] $(F_K)_{ss} = \delta + \phi\frac{1-\beta+(2-\sigma)\gamma}{1-\beta+(1-\sigma)\gamma} - \frac{\gamma\rho}{1-\beta+(1-\sigma)\gamma}$ and $(F_K)_{ss} = \delta + \phi$ when $\gamma = 0$ (see Section 3.2.1)

[19] In this case, when $\gamma = 0$, state-like variable k has the meaning of the ratio of physical to human capital.

Case 1 ($\theta > \beta$) When the coefficient of risk aversion θ is relatively high with respect to the capital share β (low intertemporal elasticity of substitution σ), people are not willing to reduce consumption, which is a substitute for physical capital, and have to the increase work effort to rebuild physical capital. This effect is called a wealth or consumption smoothing effect. The corresponding transition paths for the ratio of physical to human capital k, the consumption-capital ratio c and the effort devoted to work l are shown on Fig.1a), b) and c), respectively.

Therefore, the policy functions p, q are downward sloping (with respect to variable k) and for that case they are shown on Fig.1d) and e).

Case 2 ($\theta < \beta$) [20]

In that case the building of higher physical capital accumulation k is based on the fact that people have relatively low willingness to smooth consumption (low θ) and/or relatively low wages for low k (the capital share β determines how low real wages are for a given level of k). Therefore, the accumulation of physical capital is based rather on saving than on high work effort. This effect is called a substitution or wage rate effect and the corresponding transitional paths of the model variables are shown in Figs.2a)-c).

The policy functions p, q are upward sloping (with respect to variable k) and are shown in Figs.2d) and e).

Case 3 ($\theta = \beta$) This case says that when the coefficient of risk aversion θ is equal to the capital share β people are indifferent with respect to the level of capital ratio k and policy functions are horizontal (they do not depend on the level of the ratios of the two types of capital).

Below, we limit our experiments to the Case 1 because most estimations of the parameters of the coefficient of risk aversion and the capital share described in the literature belong to this case. The following experiments assume that the economy initially starts at the situation when physical capital is relatively abundant. The transformed variable k is thus higher than the steady state level and the economy starts to accumulate relatively more human capital than physical one.

[20] This possibility is more theoretical than practical since empirical evidence suggests $\theta \geq 1$. It seems, however, that post-socialist countries such as the Czech Republic are characterized by a high willingness to substitute consumption (high savings rates) at low real wages.

The transitional paths of Experiments 1–5, when externalities are not present in the model, are shown in Fig.3. The level variables: human capital $\ln H$, physical capital $\ln K$, consumption $\ln C$, and time devoted to production l[21] are represented in Figs.3a)–3d), respectively. The growth rates of level variables are at Figs.3e)-3h).

Experiment 2: government expenditure

$$(\kappa \;=\; 0.15)$$

It can be seen from the figures that the introduction of government spending has substantive influence only on the level of consumption (Fig.3c) and Fig.3f)). Since we assume that government expenditure is not productive the only influence is crowding out the private consumption. The growth rates are almost unchanged.

Experiment 3: education subsidies

$$(\tau \;=\; 0.1)$$

Introduction of the education subsidies motivates people to devote more time to education than to work (Fig.3d)[22]. It causes decline in output and consumption (Fig.3c). The effect of more time devoted to education is an increase in the level and the growth of human capital (Figs.3a) and 3e)). The decline of physical capital is initially faster and despite the fact that in the long-run the growth rate is higher (Fig.3f)), its level remains lower for a very long time (Fig.3b)). Time devoted to production is permanently lower which indicates higher balanced growth. The growth rate of consumption is higher (Fig.3g) but its initial deterioration is so big that the level remains for a rather long time lower (Fig.3c) and the social welfare is not improved.

Experiment 4: capital income tax

$$(\tau_K \;=\; 0.1)$$

As it can be seen from the figures, the introduction of capital income tax causes the increase in the price of physical capital and, therefore, the decrease of its employment

[21]The first three variables exhibit the balanced growth rate at steady state.

[22]All the effects are described in the relation to the benchmark case.

and its substitution by other factors of production: the decline in the time devoted to production and, therefore, also in output are small (Fig.3d)). The consumption is initially slightly higher which means that the physical capital is decreased by dissaving. The small changes in time devoted to work indicate the small changes in growth rates (Figs.3e) – 3h)) and negligible changes in the level of human capital (Figs.3a)). The level of physical capital declines faster and is permanently lower (Fig.3b)) in relation to the benchmark economy.

Experiment 5: labor income tax

$$(\tau_L \;=\; 0.1)$$

It turns out that the introducing the labor income tax as an only one element of distortion in the economy has in the situation of the inelastic labor supply no influence on the behaviour of the economy (Fig.3 and 4.)

Experiment 6: benchmark economy with externalities

$$(\gamma \;=\; 0.4)$$

This experiment produces a benchmark for comparing the effects of isolated fiscal policies in the presence of externalities.

Experiment 7: externalities and government expenditure

$$(\gamma \;=\; 0.4; \;\; \kappa \;=\; 0.15)$$

Experiment 8: externalities and education subsidies

$$(\gamma \;=\; 0.4; \;\; \tau \;=\; 0.1)$$

Experiment 9: externalities and capital income tax

$$(\gamma \;=\; 0.4; \;\; \tau_K \;=\; 0.1)$$

Experiment 10: externalities and labor income tax

$$(\gamma \ = \ 0.4; \ \tau_L \ = \ 0.1)$$

The experiments 6–10 correspond to the former ones with the extension of the human capital externalities. The new trajectories are drawn in Fig.5. By comparing Fig.3 and Fig.5 we can see that the relations between variables remained very similar. It is visible that the growth rates and the levels of human capital are now lower because human capital is more productive: time devoted to production is higher. The economy exhibits higher growth rates and the levels in the other variables. Thus we can say that the existence of spilover effects in human capital accumulation process leads to the lower level of human capital in the faster growing economy.

The competitive economies with and without externalities (experiment 6 and experiment 1, respectively) are compared in Fig.4. It turns out that the levels of physical capital, consumption and working time and the growth rates of physical capital and consumption are higher in the economy with externalities. On the other hand, the opposite is true for the levels and rates of human capital.

Experiment 11: social optimum As we already mentioned above, the social optimum solution does not coincide with the competitive equilibrium in the presence of externalities. The comparison of these solutions is presented in Fig.6. The long-run growth rates of all variables, except the working time that has zero growth in the long-run, exhibit higher values. It is also visible that is optimal to decrease the level of physical capital much rapidly even at the cost of a big decline of consumption for some time. Fig.6 shows also that education subsidies can improve the behavior of the competitive economy and increase long-run growth rates.

5 Conclusions

The paper presented the projection method as a powerful tool for the study of transitional dynamics of endogenous growth models. The basic principles of the method were described and the practical application of the method implemented in GAUSS were demonstrated on an example of endogenous growth model. The used model was the Lucas' model with human capital accumulation and government sector. The paper contains the derivation of the first order necessary conditions for decentralized economy and social optimum and the brief steady state analysis. More attention was devoted to the formulation of the problem in the form suitable for the application of the projection method and the simulation of the effects of different fiscal policies. The presentation of the obtained transitional dynamics in the graphical form and the brief description of the results were also included in the paper.

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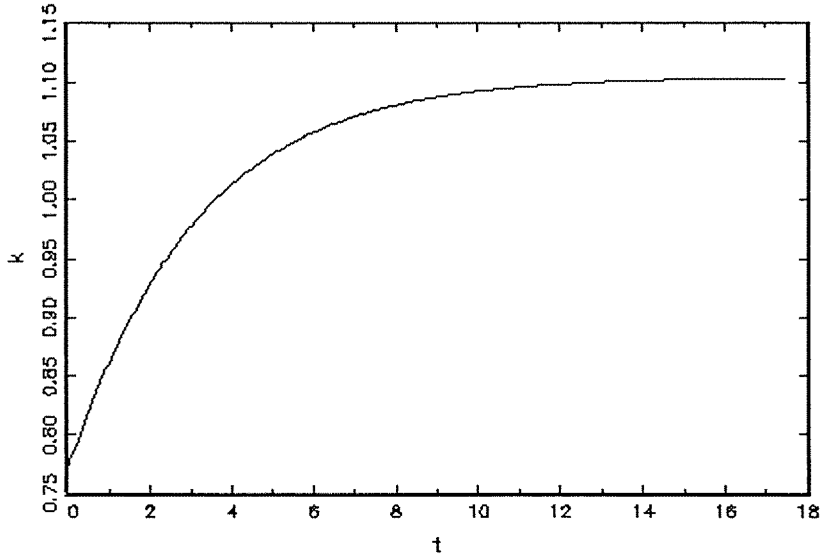
Appendix. Application of PROJEC: GAUSS code

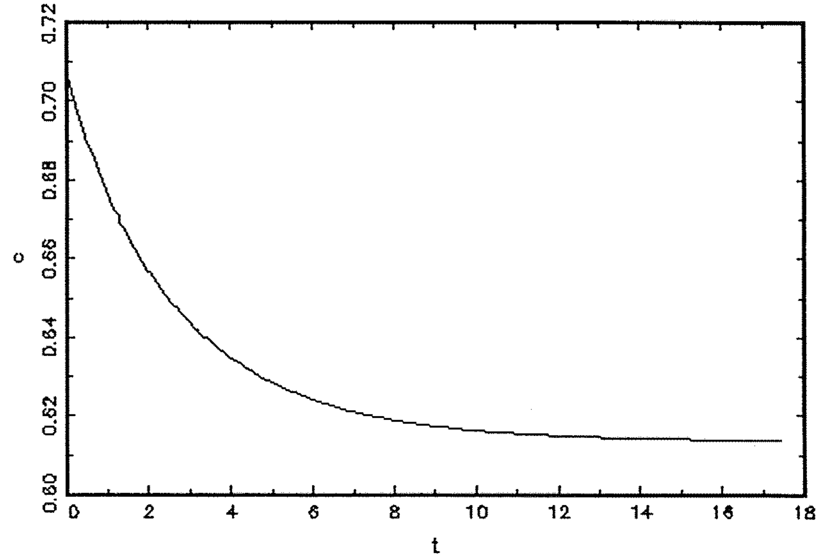
```
/* LUCAS   ------- Lucas' model */

library projec,pgraph;

/* Parameters of model */
sigc     = .5;               /* intertemporal elast. of substitution */
beta     = .3;               /* capital income share */
delta    = .08;              /* depreciation rate */
r        = .06;              /* real interest rate */
gI       = .02;              /* low development ss growth */
theta    = 1/sigc;           /* coef. of risk aversion */
phi      = r;                /* productivity of human capital */
rho      = r-gI/sigc;        /* rate of time preference */
AA       = 1;
gam      = 0;
tau      = 0;
tau_K    = 0;
tau_L    = 0;
kap      = 0;

tend = 350;
step = .05;

/* calibration on steady state */
ca       = 1-sigc*gam/(1-beta+gam);
gss      = sigc*((1-tau_L)/(1-tau_L-tau)*phi-rho)/ca;
lss      = 1-(1-beta)/(1-beta+gam)*gss/phi;
phi_bar = phi*(1-tau_L)/(1-tau_l-tau)/(1-tau_K);
gam_bar = gam/(1-beta+gam)/(1-tau_K);
mpk      = delta+phi_bar+gam_bar*gss;
kss      = lss*(mpk/beta)^(1/(beta-1));
css      = (1-kap)/beta*mpk-delta-gss;

xss      = kss|css|lss;
"Check ss values : " sumc(NONLIN(xss));

nst      = 1;    /* number of state variables */
ncon     = 2;    /* number of control variables */
nfc      = 9;    /* degree of approximation */
mcons    = nst~nfc~ncon;
x        = { .5,  1.5 };         /* interval of state variables */
x        = x*kss;
a0       = zeros(nst+1,ncon);    /* initial linear guess */

/* initial value of parameters */
A    = gradp(&NONLIN,XSS);
{ev, evi, em, emi} = eigrg2(A);
pol_c  = em[2,1]/em[1,1];    /* Policy functions for stable saddle path */
pol_l  = em[3,1]/em[1,1];
```

```
a0[1,1] = css-pol_c*kss;
a0[2,1] = pol_c;
a0[1,2] = lss-pol_l*kss;
a0[2,2] = pol_l;

PROJSET;
_prmeth = 1;
_prsave = 0;
_prinit = 0;
_prldfn = "a1_9";
{a,ret} = PROJEC(&_FRES,mcons,x,a0);




proc NONLIN(X);
local K,C,L,Kdot,Cdot,Ldot,F_K;
    K    = X[1];
    C    = X[2];
    L    = X[3];
    F_K  = (K/L)^(beta-1);
Kdot = ((1-kap)*F_K-C-delta-(1+gam/(1-beta))*phi*(1-L))*K;
Cdot = ((sigc*beta*(1-tau_K)-1+kap)*F_K-sigc*(delta+rho)+delta+C)*C;
Ldot = (tau_K-kap)*F_K+delta*(1-tau_K)/beta+phi*(1-tau_L)/(beta*(1-tau_L-tau));
Ldot = (Ldot-delta-C+(gam/beta-1)*(1-L)*phi)*L;
retp(Kdot|Cdot|Ldot);
endp;




/* procedure for residual function */
proc _FRES(k,a);
local par,R1,R2,c,l,d_c,d_l,F_K,kdot,cdot,ldot;
par = reshape(a,2,nfc)';
c   = __APROX(k,par[.,1]);                       /* c(k) */
l   = __APROX(k,par[.,2]);                       /* l(k) */
d_c = __DAPROX(k,par[.,1],1);                    /* c'(k) */
d_l = __DAPROX(k,par[.,2],1);                    /* l'(k) */
F_K = __GPOW(l/k,1-beta);                        /* F/K */
kdot= ((1-kap)*F_K-delta-c-(1+gam/(1-beta))*phi*(1-l))*k;
cdot= ((sigc*(1-tau_K)*beta+kap-1)*F_K-sigc*(delta+rho)+delta+c)*c;
ldot= ((tau_K-kap)*F_K+delta*(1-tau_K)/beta+phi*(1-tau_L)/(beta*(1-tau_L-tau))
      -c-delta+(gam/beta-1)*phi*(1-l))*l;
R1  = d_c*kdot-cdot;
R2  = d_l*kdot-ldot;
retp(R1|R2);
endp;
```

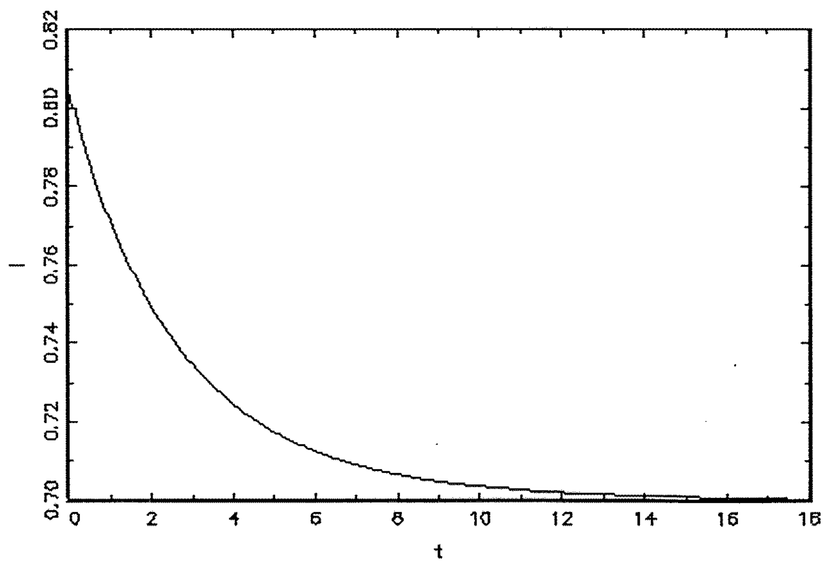
Lucas's model $(\theta > \beta)$ Fig.1a)



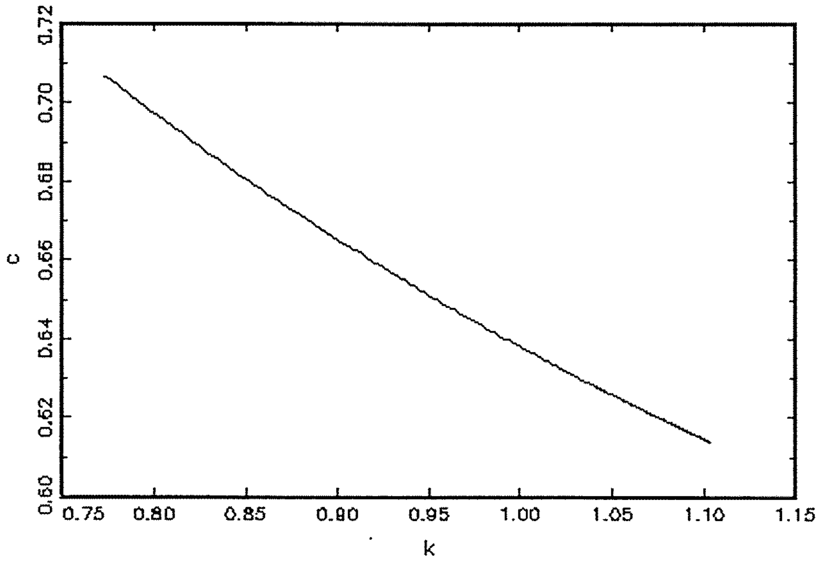
Lucas's model $(\theta > \beta)$ Fig.1b)



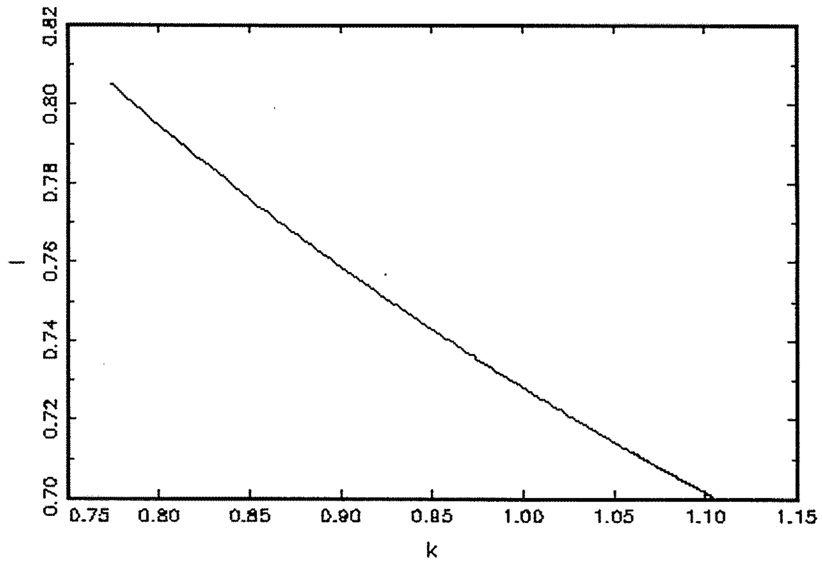
Lucas's model $(\theta > \beta)$ Fig.1c)



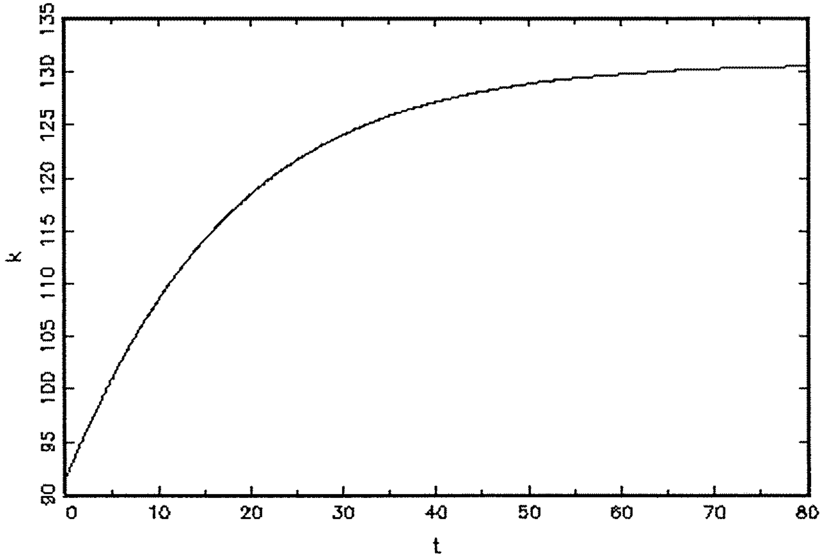
Lucas's model $(\Theta > \beta)$ Fig.1d)



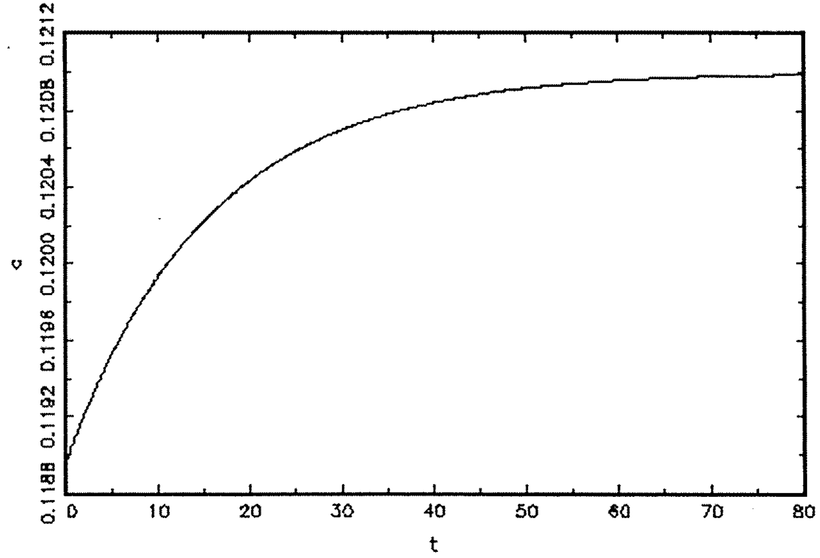
Lucas's model $(\Theta > \beta)$ Fig.1e)



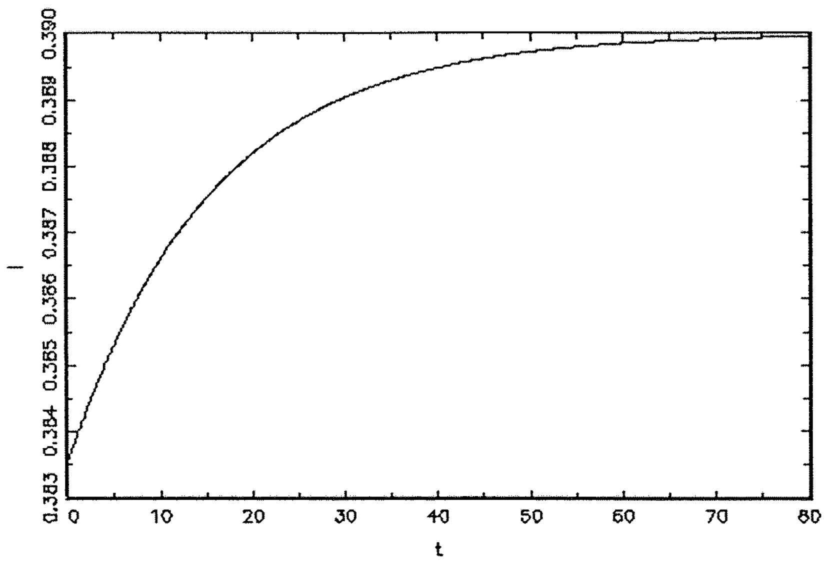
Lucas's model $(\Theta<\beta)$ Fig.2a)



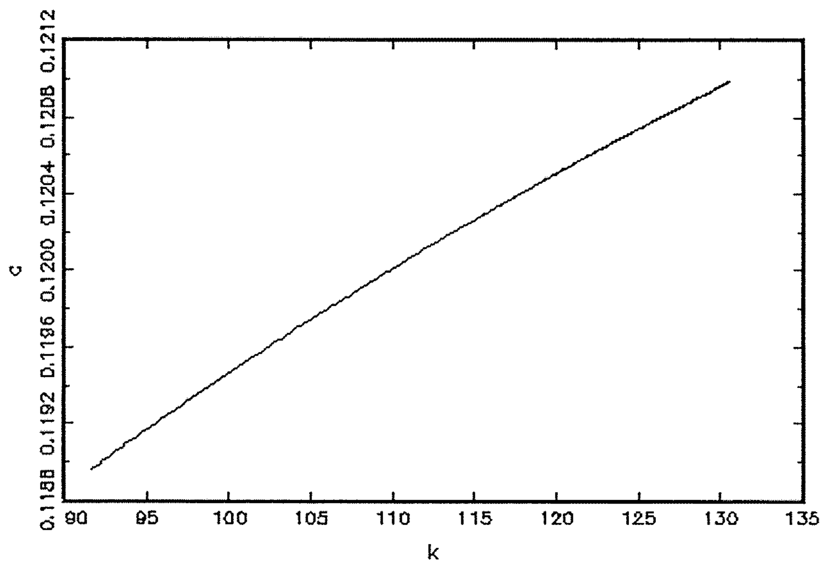
Lucas's model $(\Theta<\beta)$ Fig.2b)



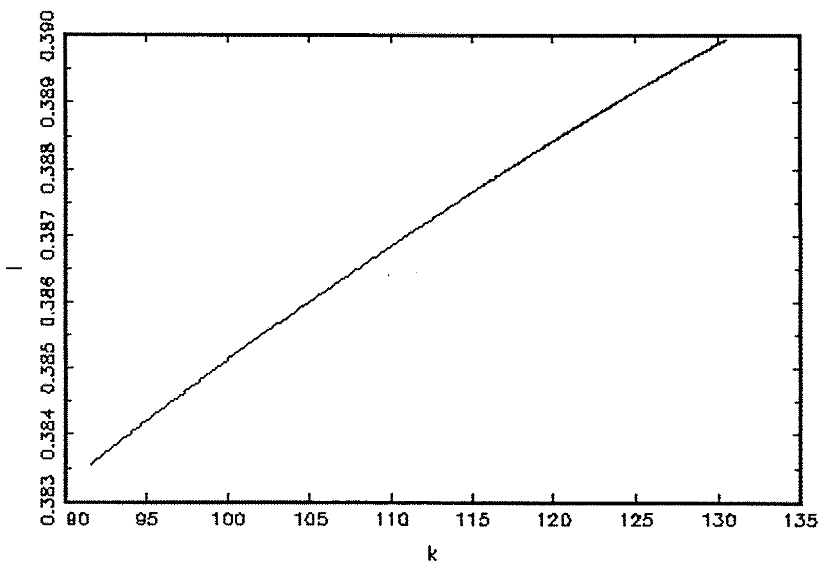
Lucas's model $(\Theta<\beta)$ Fig.2c)



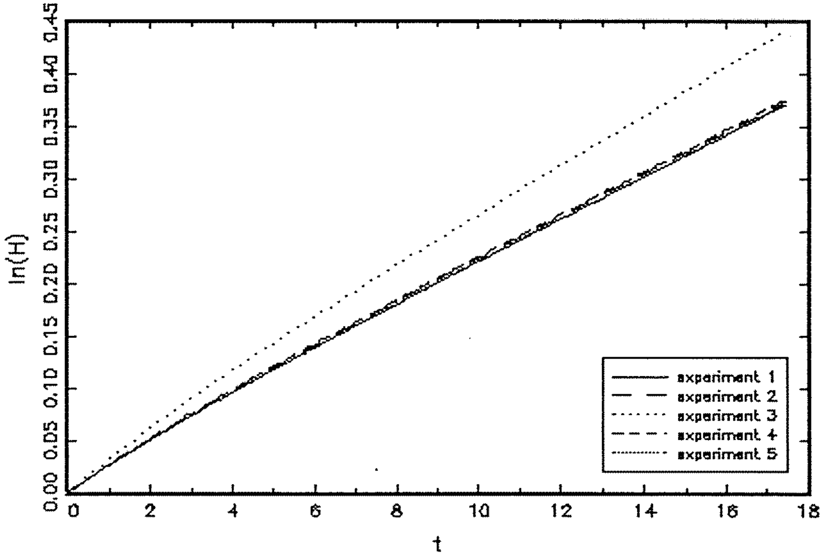
Lucas's model (θ<β) Fig.2d



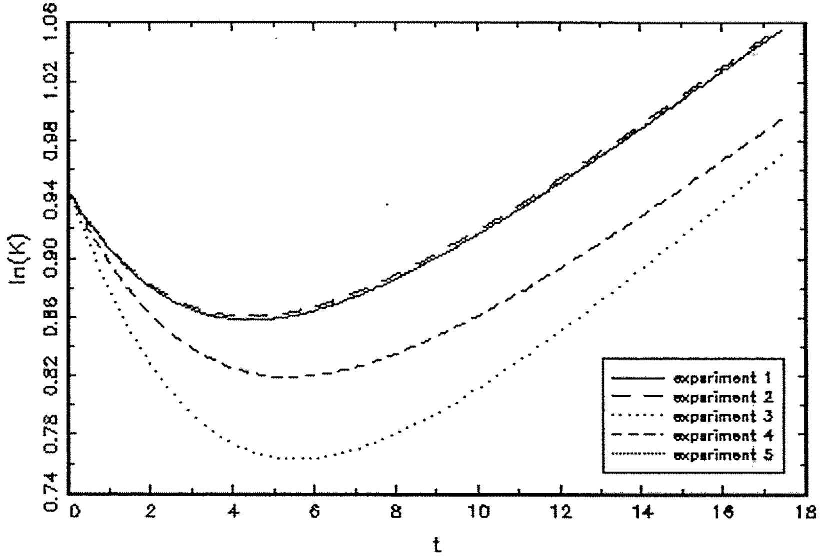
Lucas's model (θ<β) Fig.2e



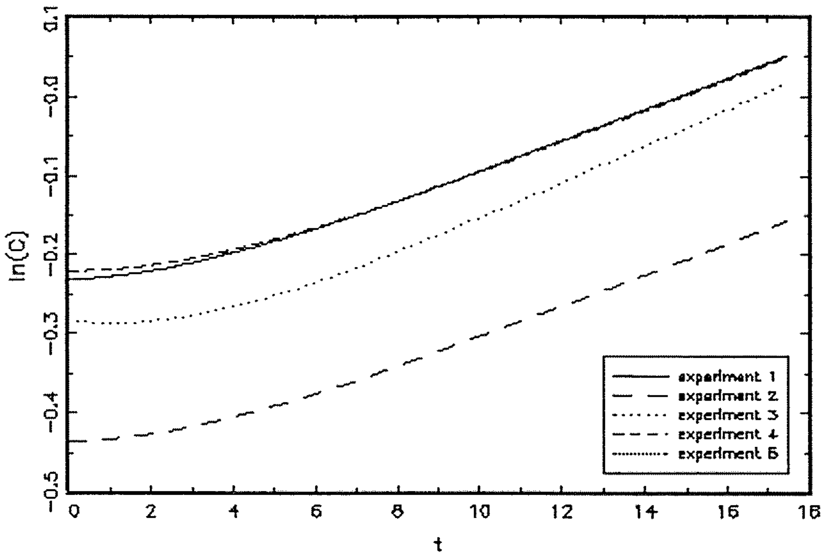
Lucas' model ($\gamma=0$) Fig.3a)



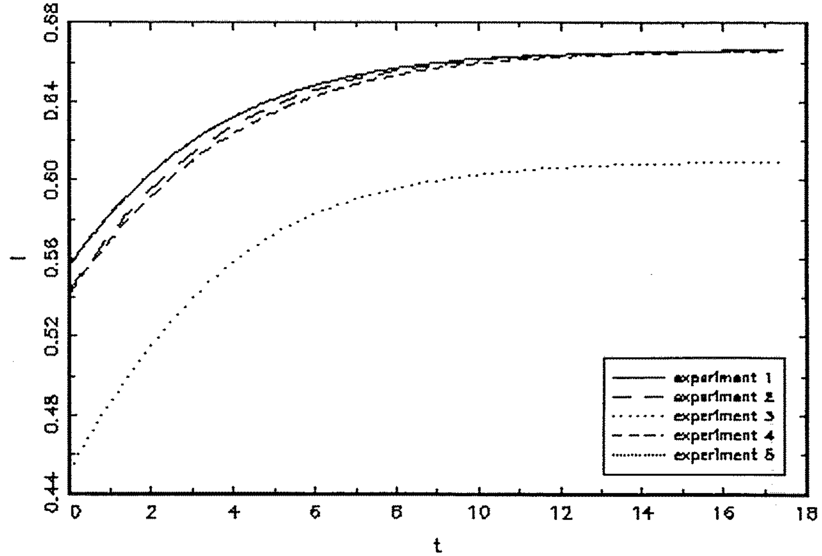
Lucas' model ($\gamma=0$) Fig.3b)



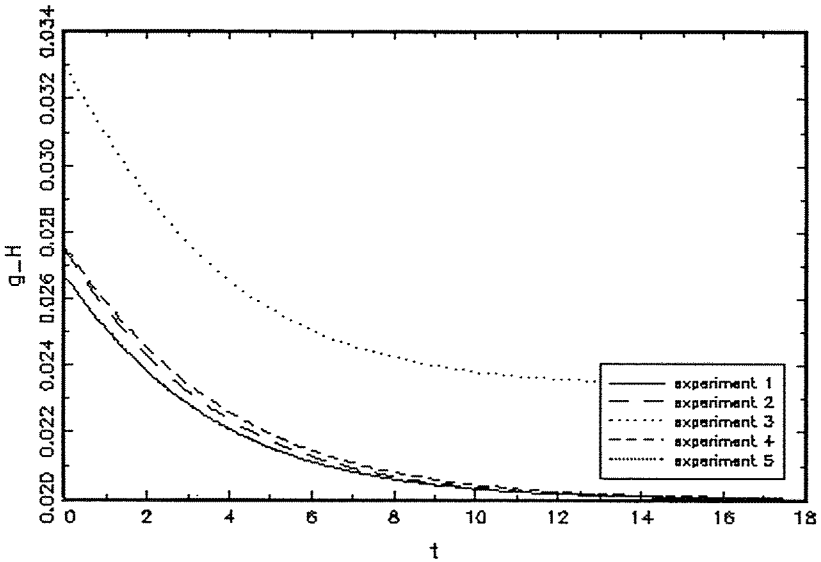
Lucas' model ($\gamma=0$) Fig.3c)



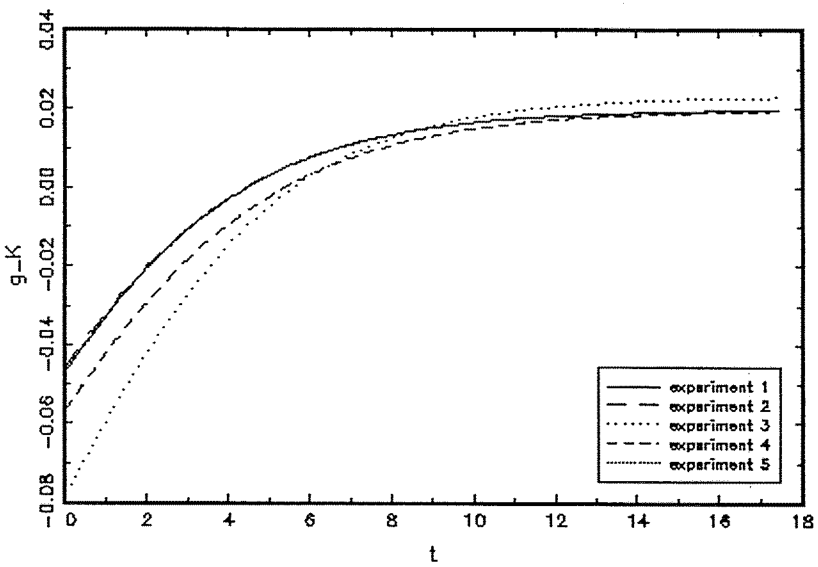
Lucas' model ($\gamma=0$) Fig.3d)



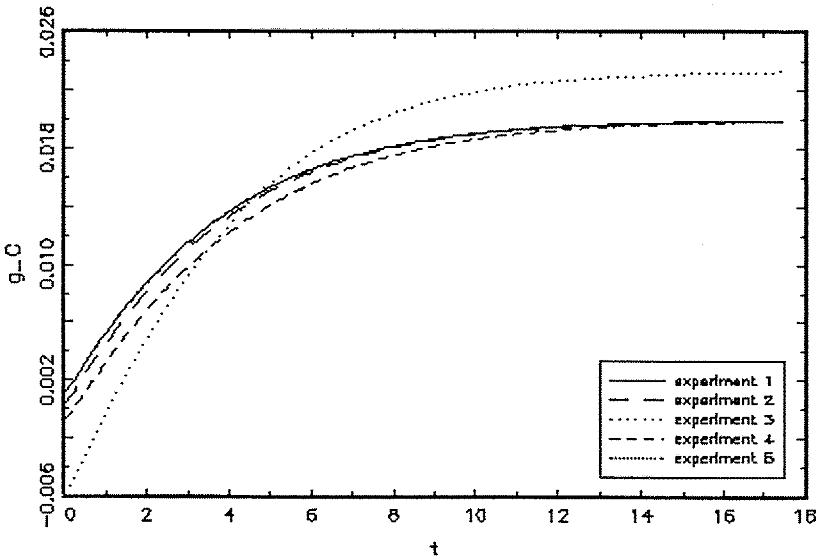
Lucas' model ($\gamma=0$) Fig.3e)



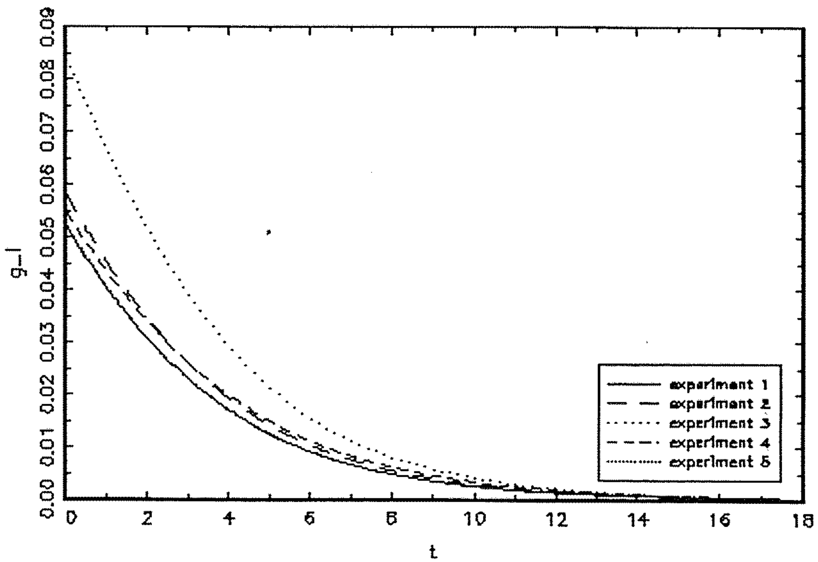
Lucas' model ($\gamma=0$) Fig.3f)



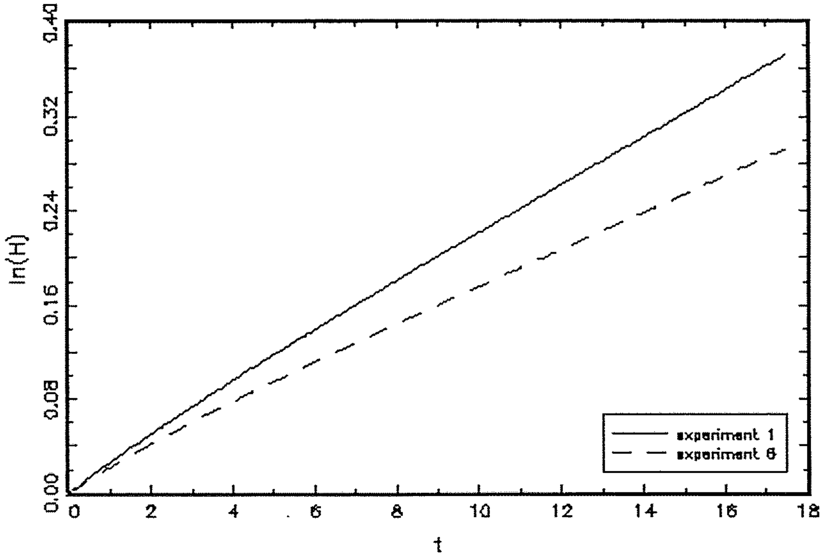
Lucas' model ($\gamma=0$) Fig.3g)



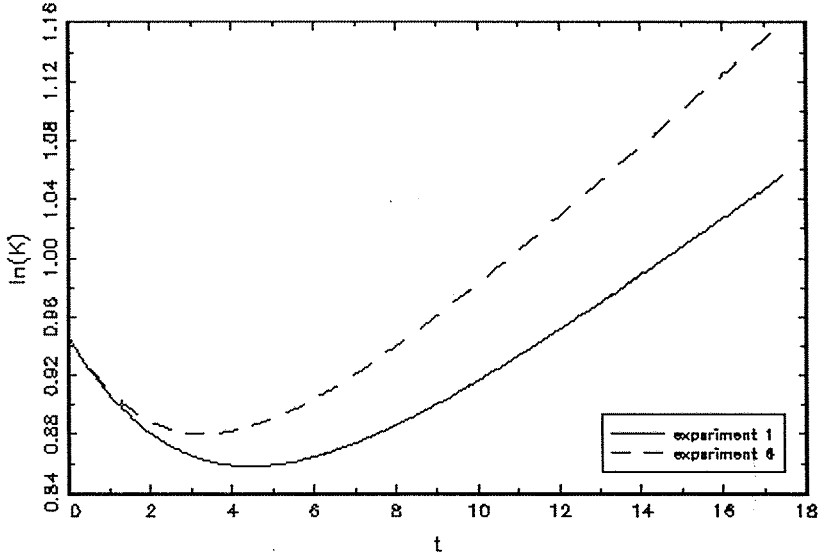
Lucas' model ($\gamma=0$) Fig.3h)



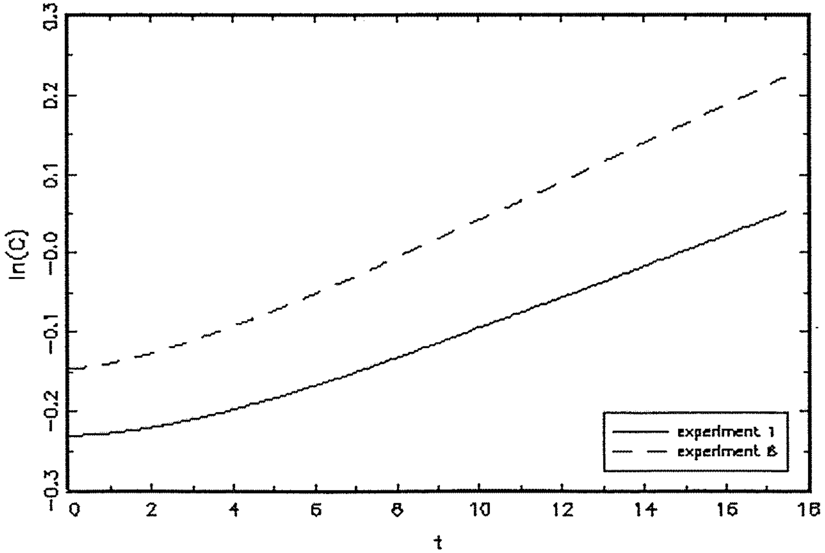
Lucas' model Fig.4a)



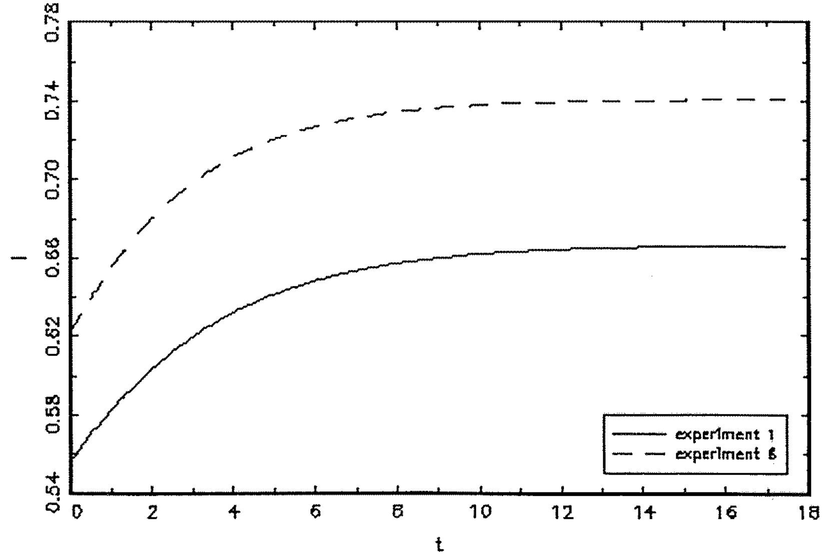
Lucas' model Fig.4b)



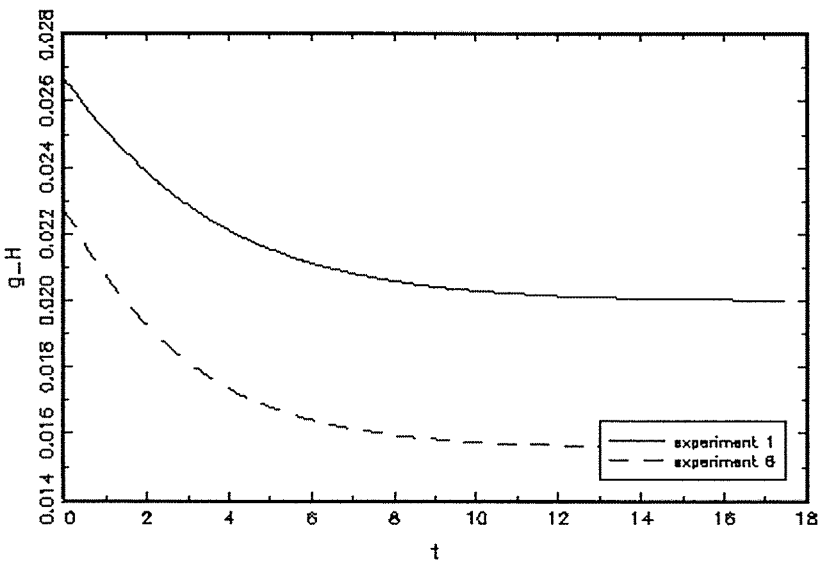
Lucas' model Fig.4c)



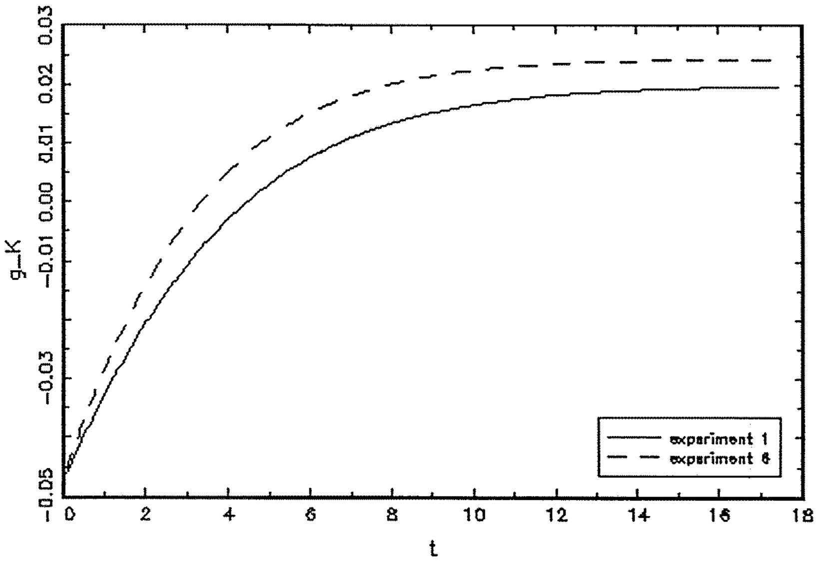
Lucas' model Fig.4d)



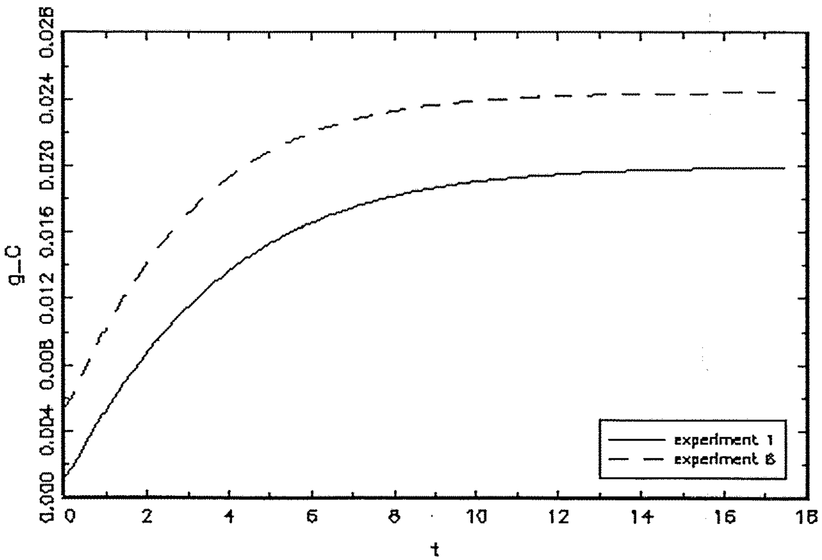
Lucas' model Fig.4e)



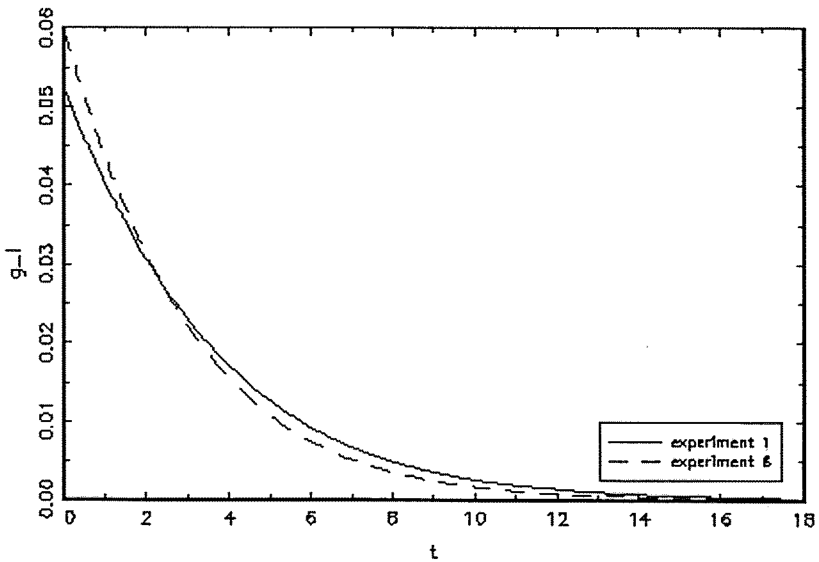
Lucas' model Fig.4f)



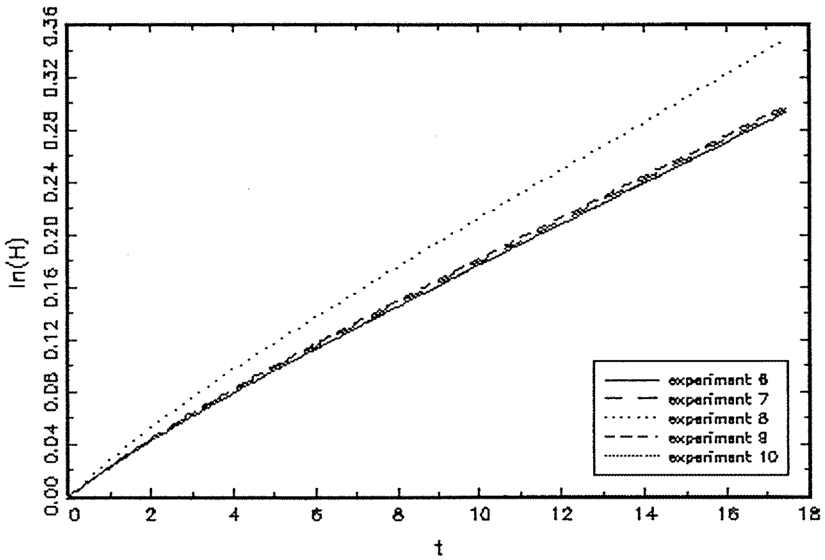
Lucas' model Fig.4g)



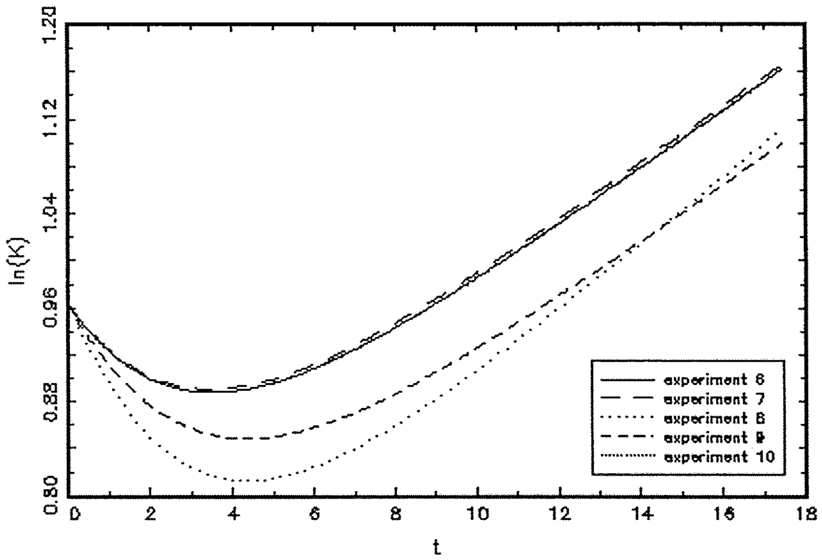
Lucas' model Fig.4h)



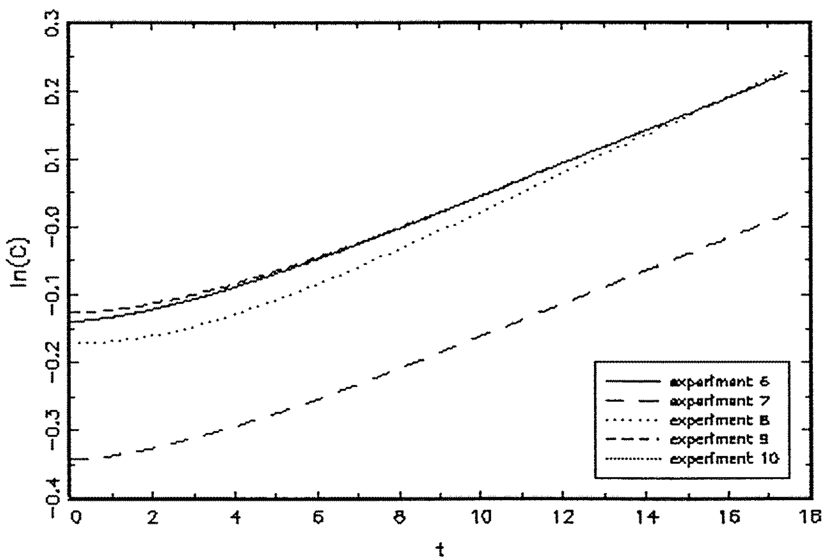
Lucas' model ($\gamma \neq 0$) Fig.5a)



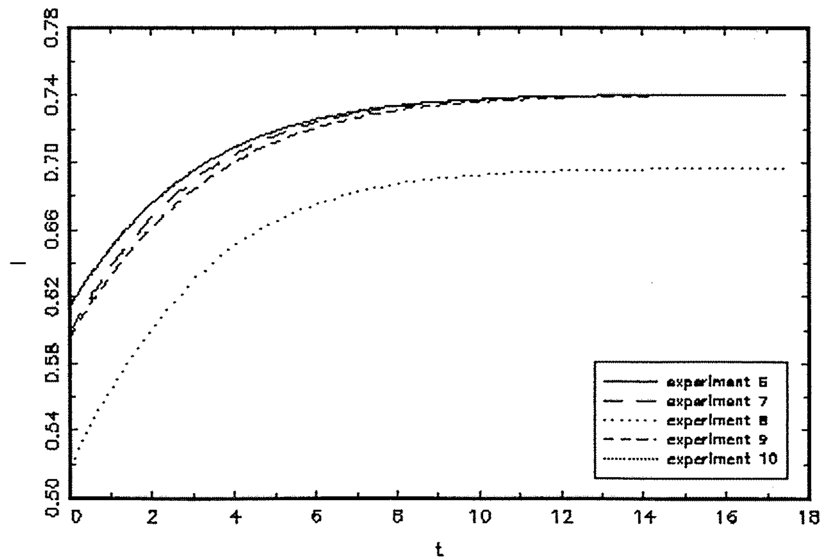
Lucas' model ($\gamma \neq 0$) Fig.5b)



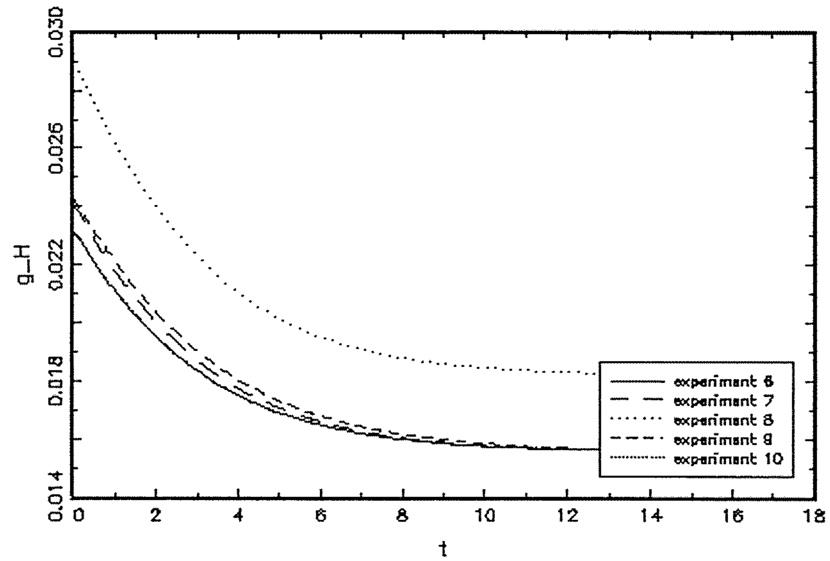
Lucas' model ($\gamma \neq 0$) Fig.5c)



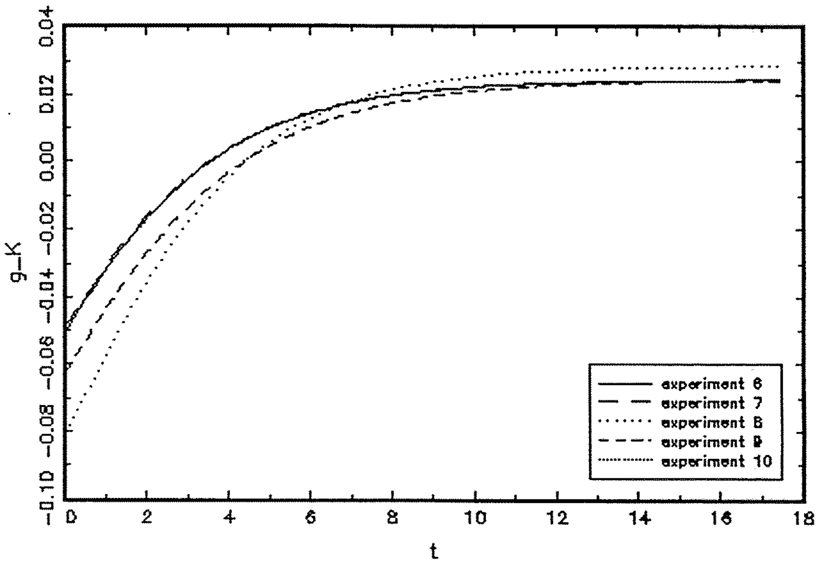
Lucas' model ($\gamma \neq 0$) Fig.5d)



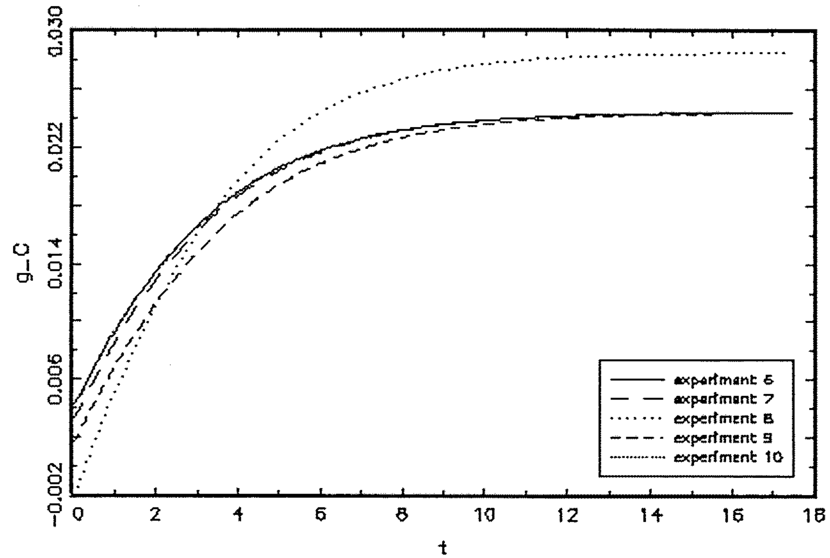
Lucas' model ($\gamma \neq 0$) Fig.5e)



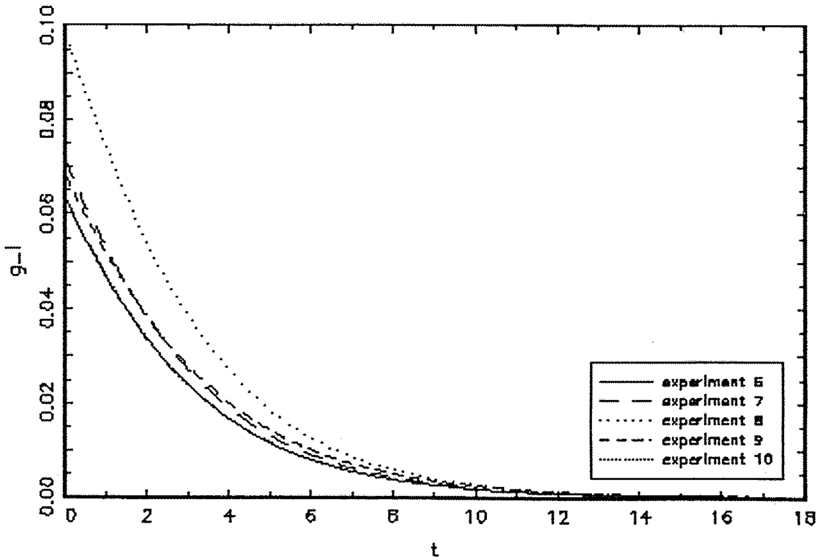
Lucas' model ($\gamma \neq 0$) Fig.5f)



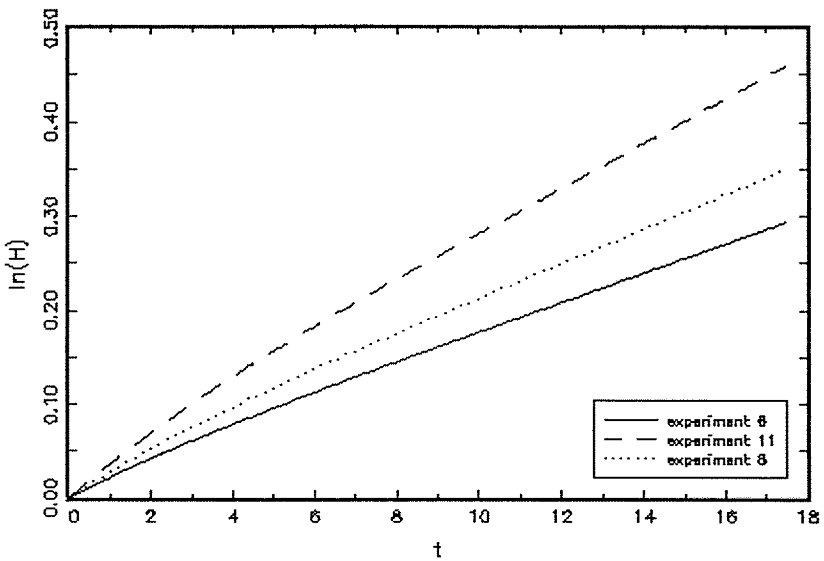
Lucas' model ($\gamma \neq 0$) Fig.5g)



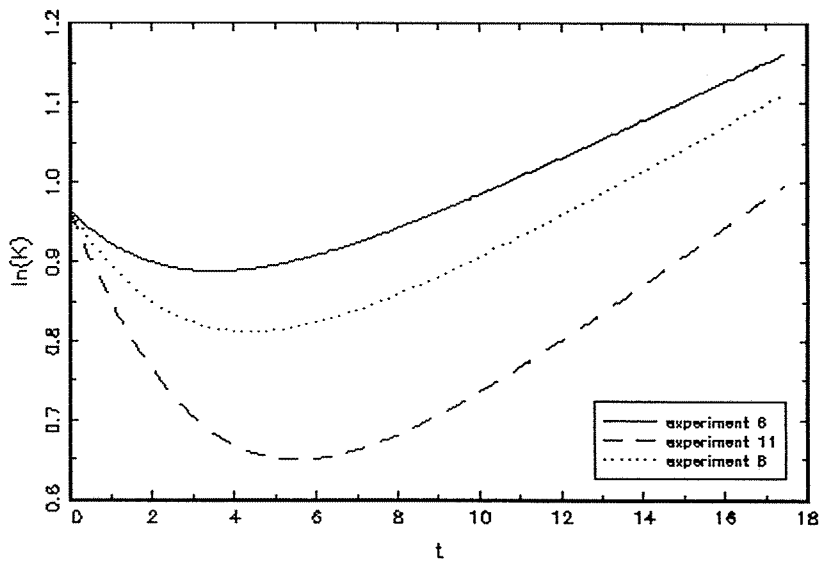
Lucas' model ($\gamma \neq 0$) Fig.5h)



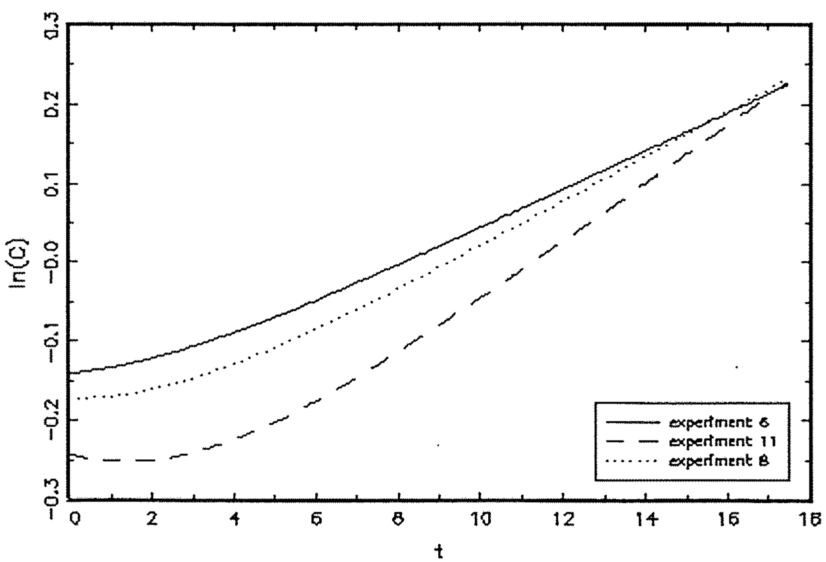
Lucas' model ($\gamma \neq 0$) Fig.6a)



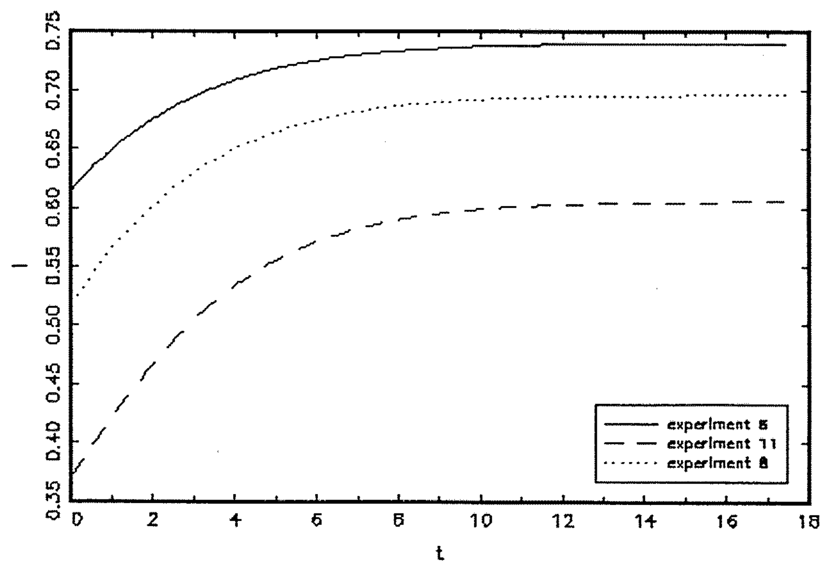
Lucas' model ($\gamma \neq 0$) Fig.6b)



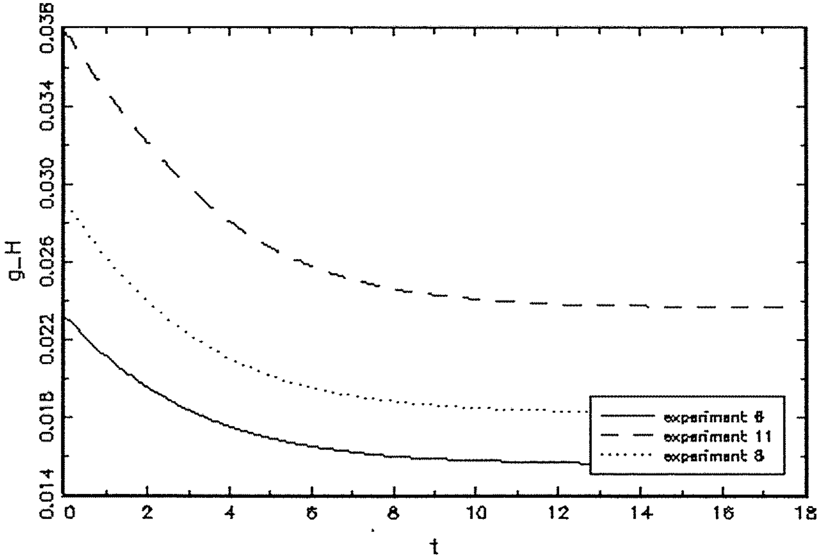
Lucas' model ($\gamma \neq 0$) Fig.6c)



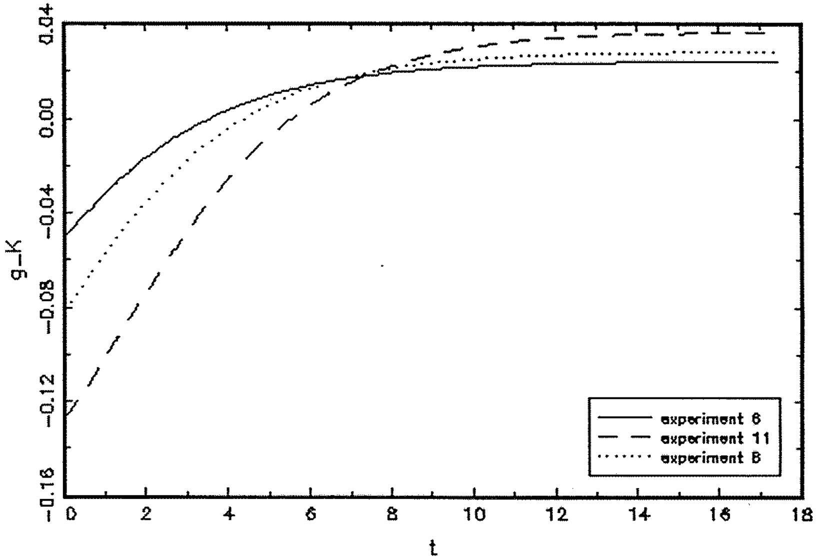
Lucas' model ($\gamma \neq 0$) Fig.6d)



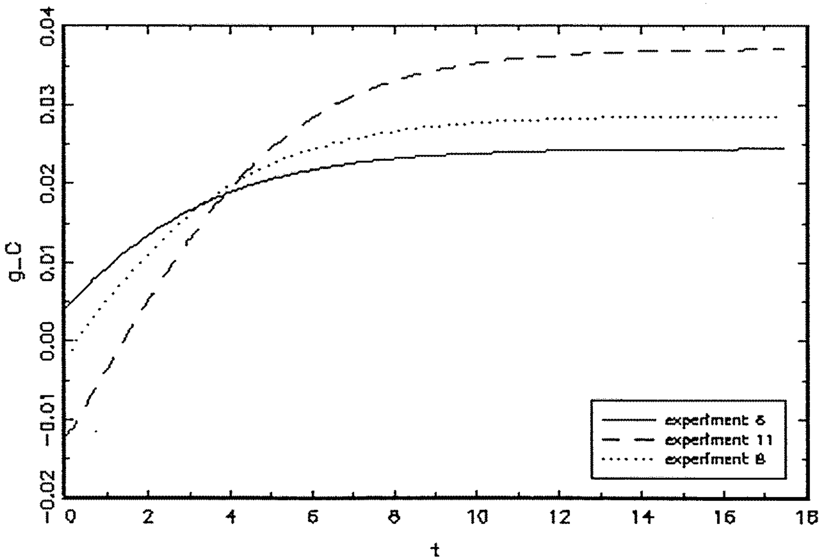
Lucas' model ($\gamma \neq 0$) Fig.6e)



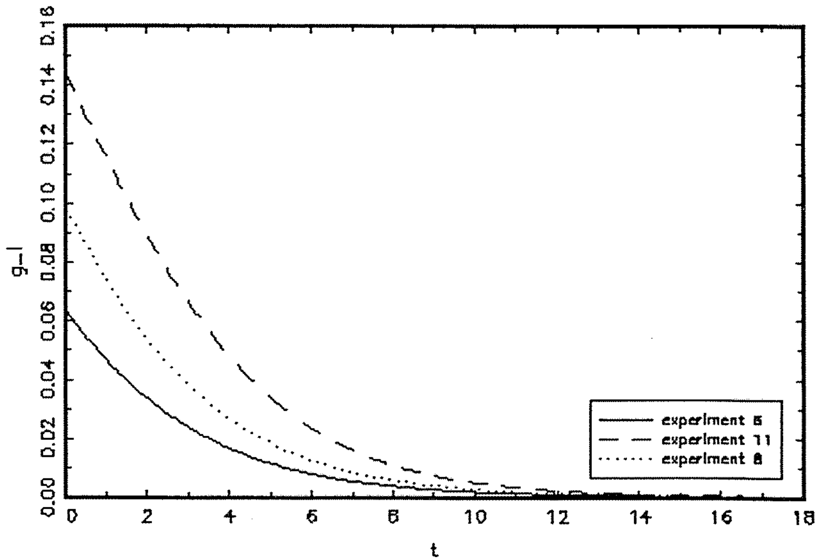
Lucas' model ($\gamma \neq 0$) Fig.6f)



Lucas' model ($\gamma \neq 0$) Fig.6g)



Lucas' model ($\gamma \neq 0$) Fig.6h)



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