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Sustainable investment under prospect theory: 2-assets case

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Abstract

Sustainable investing, also known as ESG (environmental, social, and corporate governance) investing, is currently of intense interest in the investment world. We consider an economy with a risky and a risk-free asset. Utility is defined as the weighted sum of a prospect theory value function and an ESG score function. In addition to greenness of the risky asset, we explicitly consider the ESG score of the risk-free asset. We derive optimal investment strategies, which depend on the relative greenness of the risky to the risk-free assets. An extensive sensitivity analysis is performed that provides properties of the investor's risk taking and the indirect utility (happiness) with respect to the reference return, loss aversion, the ESG scores of the risky and the risk-free assets, and the weight of the ESG score function. Importantly, we show that the conventional positive relationship between an asset's ESG score and its demand does not hold generally: sufficiently strong ESG diminishing sensitivity can reverse the effect, depending on the relative ESG scores of the risky and the risk-free assets and the investor's portfolio leverage. If the ESG score function becomes more important, then investment into the greener asset increases. An increase in loss-aversion decreases investment into risky asset. Finally, we observe that investment into the risky asset is first decreasing and then increasing with the reference return, as also observed in ESG-neutral investment, i.e., in pure prospect theory.

Keywords: Sustainable finance, green finance, ESG investing, prospect theory, behavioral finance, portfolio optimization
JEL: G11, G41, Q56

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1. Introduction

ESG investing refers to taking environmental, social and governance (ESG) considerations into account when making investment decisions in the financial sector. Fund managers are increasingly incorporating ESG considerations into their investment process, using a large variety of different ESG strategies like negative/positive screening and thematic investing (see, e.g., Matos, 2020; and Boffo and Patalano, 2020, for reviews). This article presents optimal investment strategies for an investor with preferences including a prospect theory and an ESG component.

First, let us discuss socially responsible investment behavior and its relation to prospect theory. Van Dooren and Galema (2018) find out in their experimental study that socially responsible investors display a greater disposition effect than conventional investors, that is the tendency to sell appreciated stocks (winners) too early and hold depreciated stocks (losers) too long. Hence, one possible explanation for the disposition effect is provided by prospect theory.[1] That is, price increases (positive returns) typically imply that investors are in the domain of gains, where they are risk averse and therefore relatively insensitive to further gains. This makes them more willing to sell early. By contrast the domain of losses is typically attained by price decreases (negative returns), where investors with prospect theory based preferences are risk-seeking and therefore relatively insensitive to further losses. This makes them more willing to hold on to a stock. Prospect theory has been found to explain a number of stock market anomalies (An et al., 2020; Barberis et al., 2016; Barberis et al., 2021; Guo and Schönleber, 2020; Wang et al., 2021; Harris and Mazibas, 2022; Yu et al., 2022). In addition, a bulk of further literature supports the application of prospect theory to model investment decisions: For example, Shefrin and Statman (1985) propose a theoretical framework that links the disposition effect to Kahneman and Tversky's (1979) prospect theory and to mental accounting. Bollen (2007) proposes both prospect theory and expected mean reversion of mutual fund performance as possible explanations for the asymmetric result that cash flows into socially responsible mutual funds are more sensitive to past positive returns and less sensitive to past negative returns than those observed at conventional mutual funds. Boumda et al. (2021) investigate the disposition effect of socially responsible investment fund managers. Diaz et al. (2024) propose a portfolio construction strategy in the context of behavioral finance, where investors cannot be considered perfectly rational in the sense of traditional portfolio theory. They analyze the ESG scores and the valuation of financial assets by prospect theory value as criteria to construct portfolios of energy companies over a wide sample period. Recent literature also provides evidence that investors care about social investment including environmental aspects (see, e.g., Dangl et al, 2025; Heeb et al., 2023; Kaufmann et al., 2024; Ottoni-Wilhelm, et al., 2017; and Riedl and Smeets, 2017).

Second, literature on ESG investing either empirically investigates the impact of ESG investing on portfolio performance (see, e.g., the meta-studies of Friede et al., 2015; Busch and Friede, 2018; Hartzmark and Sussman, 2019; Matos, 2020; Krueger et al., 2020; Bolton and Kacperczyk, 2021; Bekaert et al., 2023) or models the impact of preferences including an ESG component on asset demand and asset prices. An early contribution, on green investing is Heinkel et al. (2001), where investors have CARA preferences. ESG-neutral investors consider all assets while green investors do not invest into brown assets. This

[1] An alternative way to obtain a disposition effect in an asset pricing model is via asymmetric information as pointed out in Dorn and Strobl (2023).

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results in lower asset prices and a higher cost of capital for brown assets in equilibrium. Pastor et al. (2021) examine sustainable investing for a CARA investor with an additive ESG component and solve for equilibrium. ESG preferences move asset prices and stocks of "greener firms" by lowering ex-ante CAPM alphas. Pedersen et al. (2021) consider a Markowitz type mean-variance setting augmented by an additive ESG utility component, prove a separation theorem and obtain asset prices in equilibrium. The authors provide sufficient conditions for increasing/decreasing asset returns caused by ESG investing. A major driver of this result is the proportion of ESG investors compared to non-ESG investors in their model.

This article proposes to take explicit account of sustainable investing of an investor with prospect theory preferences. We thus extend the portfolio selection model under prospect theory by the additional assumption that investors derive utility from sustainable investing. According to our knowledge, however, sustainable investing has not yet been explicitly addressed in this setup. In doing so we will be able to assess the importance of sustainable investing (ESG) in the optimal portfolio selection process and analyze several related issues. While some articles introduce ESG constraints in the portfolio selection problem (see, e.g., Qi and Li, 2020; and Drut, 2010), we follow Pastor et al. (2021) and Pedersen et al. (2021) and assume that an investor's ESG preferences can be modeled by adding an ESG utility term.

Although the majority of studies investigate relationships and impacts of ESG investing in equity, the greenness of investment opportunities is also relevant for bank lending and bonds. Houston and Shan (2022) for instance analyze the role of banks in improving ESG related behavior. Their study demonstrates that banks grant loans to firms with similar ESG characteristics, and improve ESG related performance of the corresponding borrowers. Flammer (2021) investigates green bonds, where she observes that the stock market shows a positive reaction to issuance of green bonds, while issuers decrease CO_2 emissions, and in parallel their ESG scores improve. Hence, these studies motivate the importance of taking into account the greenness of the risk-free investment alternative when considering a portfolio optimization problem that incorporates ESG preferences.

Another key innovation of this article is the explicit modelling of the ESG score of the risk-free investment. Assigning an ESG score to the risk-free asset is justified since sovereign bonds are not necessarily ESG-neutral. Governments and their bonds differ in their environmental, social, and governance practices, and investors' sustainability preferences typically extend to all asset classes. Incorporating ESG considerations for the risk-free asset therefore aligns theoretical modeling with real-world sustainable finance practices and regulatory frameworks. Thus, the specification of the score of the risk-free asset is not merely a normalization but a modelling assumption with economic consequences that should be made explicit. Pedersen et al. (2021), for example, do not explicitly consider the ESG score of the risk-free asset in their theoretical model. In their empirical application [see p. 584] the ESG score of the risk-free asset (implicitly zero) is equal to the (standardized) ESG scores of the risky assets (such that the average ESG score to the risky assets is also zero). Pastor (2021) considers an economy with n risky assets and a risk-free alternative, where the investment weights in risky assets are described by the $n-$dimensional vector $x = (x_1, \ldots, x_n)'$. W_0 is the initial wealth, where $W_0 \left(1 - \sum_{i=1}^{n} x_i\right)$ is the Dollar-amount invested into the risk-free asset. Each agent has exponential Bernoulli utility function $\exp\left(AW_0(1 + R(x)) + b'x\right)$, where $A > 0$ is the degree of risk-aversion, $R(x)$ is the portfolio return, and b is vector of non-pecuniary benefits. The benefits from investing into the risk-free alternative are zero [see page 551]. That is, also in this specification the ESG properties of the risk-free asset are not explicitly modeled. Finally, Gasser et al. (2017) who add social responsibility of an investor in a traditional mean-variance set-up, also do not

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consider any social ranking of the risk-free asset.

In this article we assign an ESG rating to the risk-free asset and assume that the ESG score function is increasing in the total amount invested. Hence, we explicitly consider the ESG rating of the risk-free asset and allow for a more general specification of the ESG score function. This, to our knowledge, was not tackled in the literature up to now. To focus on the effects arising from the ESG score of the risk-free asset, the analysis considers an economy with one risky asset and one risk-free asset. Our results show that the relationship between the ESG scores of the risky and the risk-free assets, i.e., their relative greenness, affects the optimal investment strategy.

We derive the optimal investment behavior in an implicit form and in some specific cases we also derive a closed form solution and find that optimal investment behavior depends on the relative greenness of the risky and the risk-free assets.

Our analytical results show that ESG preferences can substantially alter portfolio choices under prospect theory. When the prospect-theory reference return differs from the risk-free return, the optimal risky-asset position is generally shifted toward the greener asset relative to the prospect theory benchmark. For the bilinear value function, we additionally establish that a sufficiently brown risky asset can be optimally excluded from the portfolio, a possibility that does not arise under pure prospect theory and is also observed in our numerical simulations for more general specifications.

When the reference return coincides with the risk-free rate, the interaction between ESG preferences and loss aversion becomes particularly important. When the risky asset is greener than the risk-free asset, ESG preferences favor the risky asset, but sufficiently strong loss aversion can nevertheless lead to complete investment in the risk-free asset. Thus, the ESG attractiveness of the risky asset and the investor's aversion to prospect-theory losses work in opposite directions. When ESG preferences favor the risk-free asset, the sufficiently strong loss aversion can reinforce this preference and result in investment only in the risk-free asset. Moreover, unlike the pure prospect-theory benchmark, which yields only the two corner outcomes, namely investment only in the risk-free asset and an infinitely long risky asset position, ESG preferences can generate an interior finite risky-asset position.

Importantly, our comparative-statics results show that the conventional positive relationship between an asset's ESG score and its demand need not always hold. An improvement in the ESG score of the risky asset unambiguously increases its demand when it is less green than the risk-free asset, but when it is greener than risk-free asset, sufficiently strong diminishing sensitivity can reverse the effect. Similarly, changes in the ESG score of the risk-free asset can result in either increasing or decreasing investment, depending on the investor's portfolio composition and the strength of ESG diminishing sensitivity. These findings highlight that ESG portfolio choice is governed by the interaction between relative ESG attractiveness and diminishing sensitivity rather than by ESG scores alone. In addition, investment into the risky asset decreases with an increasing level of loss aversion.

Finally, for certain special cases (e.g., a bilinear value function), we analytically show that the relationship between optimal investment in the risky asset and the reference return is V-shaped. Specifically, when the reference return is below the risk-free rate, optimal investment in the risky asset decreases as the reference return increases. Conversely, when the reference return exceeds the risk-free rate, optimal investment increases with the reference return. This pattern was already observed in the pure prospect theory case studied by Fortin and Hlouskova (2024). We also observe this behavior in our simulation study for sufficiently loss-averse investors. In these cases, the threshold reference return – the point at which

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investment in the risky asset switches from decreasing to increasing with the reference return – coincides with the risk-free rate. For less loss-averse investors, investment in the risky asset also initially decreases and then increases with the reference return. However, the turning point occurs at a reference return below the risk-free rate. We also observe some structure with respect to the weight of the ESG score function that depends on the relative greenness of the risky and risk-free assets. In our simulations, when the risky asset is browner than the risk-free asset, we observe that the optimal investment weights show an upward jump at the corresponding threshold reference return for some parameter constellations. In economic terms this effect can be explained by counteracting effects of investments into risky assets on an ESG score function and a prospect theory utility component, respectively. Finally, happiness (indirect utility) decreases with an increasing level of ambition. This feature was observed also in pure prospect theory case (see Fortin and Hlouskova, 2024).

This article is organized as follows: Section 2 describes the model assumptions and defines preferences, that is prospect theory preferences augmented by an additive ESG score function. Section 3 obtains optimal investment weights in the case of continuously distributed returns, shows their properties as well as the properties of the indirect utility function (happiness). Section 4 provides some numerical examples to demonstrate how optimal investment changes of the reference return and the weight of the ESG score function change. Finally, Section 5 concludes. Proofs and further simulation output are provided in Appendices Appendix A and Appendix B. Supplementary material derives optimal rules, when the return of the risky asset follows a Bernoulli distribution and presents corresponding proves.

2. Problem set-up

Consider an investor whose preferences incorporate both prospect theory and ESG considerations. In a one-period model, the investor allocates her wealth between one risky asset, with portfolio weight x, and one risk-free asset, with portfolio weight $1-x$. The risky asset yields stochastic return r while the non-stochastic risk-free asset yields return r^0.[2] Throughout this paper we assume that $\mathbb{E}(r) > r^0$. The portfolio return is $R(x) = xr + (1-x)r^0 = r^0 + x(r-r^0)$. In this paper we do not allow for short selling, that is $x \geq 0$.[3]

In addition, we assign an ESG score to each asset. For the risky asset the ESG score is $s \geq 0$, while $s^0 \geq 0$ is the ESG score of the risk-free asset.[4] The ESG scores are considered to be known at the point of time the investment decision takes place.[5] The ESG score of a portfolio is $S(x) = s^0 + x(s-s^0)$.

First, we assume that the value of one monetary unit invested and resulting in the return $R(x)$ is

[2] We assume net of the dollar returns.

[3] In this article x is defined on the non-negative extended real numbers.

[4] Till now we could not find a reference where the score of the risk-free asset would be considered explicitly. In the literature, the risk-free asset is usually considered (implicitly) to be ESG-neutral, see Gasser et al. (2017), Pedersen et al. (2021), and Avramov et al. (2022). This means that investor gets no utility from investing in the risk-free asset. In our set-up it means that $s^0 = 0$.

[5] This assumption is not too binding as we assume a one period model and ESG scores – mostly available at an annual frequency – change only gradually between consecutive years.

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evaluated relative to the reference return, $\hat{r}$, by the following (prospect theory based) value function

$$V\left(R(x),\hat{r}\right)=\begin{cases}\frac{(R(x)-\hat{r})^{1-\gamma}}{1-\gamma}, & R(x)\geq\hat{r}, \quad \text{relative gains}\\ \\ -\lambda\frac{(\hat{r}-R(x))^{1-\gamma}}{1-\gamma}, & R(x)<\hat{r}, \quad \text{relative losses}\end{cases} \tag{1}$$

where $\lambda > 1$ is the loss aversion parameter. The value function $V(\cdot,\cdot)$ thus reflects the reference-dependent preferences of an investor with respect to an exogenous reference return. Thus, the investor's reduction in utility arising from a loss in her return is larger (in absolute terms) than the marginal utility from a gain in her return (of the same size) near the reference return. This is consistent with the fact that investors are more sensitive to a relative loss in their return (i.e., return below their reference return) than to a relative gain (i.e., return above their reference return). Another feature of the S-shaped value function is that investors are risk averse in the domain of gains (preference for certain smaller gains over uncertain larger gains), but become risk seekers (risk lovers) in the domain of losses (preference for uncertain large losses over small certain losses). Thus, the value function is concave in the domain of relative gains and convex in the domain of relative losses. Finally, $\gamma \in [0, 1)$ is a parameter determining the curvature of the value function for relative gains and losses.[6,7]

If W_0 denotes the initial wealth of an investor then the reference wealth becomes $W_0(1+\hat{r})$, the portfolio wealth at the end of the period becomes $W_0(1 + R(x))$. We assume that the value function has the form $W_0^{1-\gamma}V\left(R(x),\hat{r}\right)$.

Second, the investor's preferences regarding the ESG portfolio score (non-pecuniary benefits) are captured by a monotone increasing ESG score function $U_{ESG}(W_0, S(x))$. That is, we allow the non-pecuniary benefits to depend on the total amount invested, W_0, and on the ESG score of the portfolio $S(x)$. The function U_{ESG} is assumed to be monotone increasing in the first and strictly monotone increasing in the second component.[8]

Third, we assume that utility is additive in ESG score function and the utility arising from monetary payoffs. Thus, the investor maximizes the following utility function

$$\alpha\, U_{ESG}\left(W_0, S(x)\right) + (1-\alpha)W_0^{1-\gamma}\mathbb{E}\left(V\left(R(x),\hat{r}\right)\right) \tag{2}$$

which is the weighted average of her ESG (social) preference function $U_{ESG}(\cdot)$ and the expectation of her value function, by choosing the investment weight x, $x \geq 0$, optimally. Parameter $\alpha \in [0, 1]$ describes the investor's weight of her ESG (non-monetary based) preferences while $1 - \alpha$ is the weight of the investor's solely monetary prospect theory preferences. If $\alpha = 1$, then only the ESG score of the portfolio matters for an investor, while with $\alpha = 0$ we are confronted with a traditional (monetary based) portfolio optimization,

[6] It is assumed that $\gamma \in [0, 1)$ in order to be consistent with the experimental findings of Tversky and Kahneman (1992). Booij and van de Kuilen (2009) find the γ parameter to be (statistically) significantly less than unity. The curvature parameter γ may differ between domains of relative gains and relative losses; however, to ease the exposition, we assume that it is the same.

[7] We do not consider subjective probabilities in this study, which are in literature refereed as cumulative prospect theory (CPT). Ao et al. (2023) found in their empirical study that Chinese investors do not display the traits of probability weightings unlike the US investors, see Liu et al. (2022).

[8] By that specification we allow utility to depend on the total amount invested. E.g. this specification allows the non-pecuniary benefits to increase if $2W_0$ instead of W_0 is invested.

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where ESG (non-pecuniary) preferences play no role (ESG unaware investor). For $\alpha = 0$ the portfolio optimization problem (2) was solved in Fortin and Hlouskova (2024), while for $\alpha = 1$ the investor picks the asset with the higher ESG score. That is, everything is invested in the risk-free asset, i.e., $x = 0$, if $s < s^0$, while the infinite long position of the risky asset is taken, i.e. $x = +\infty$, if $s > s^0$.

In the following we assume that the ESG score function $U_{ESG}(\cdot)$ is of the form[9]

$$U_{ESG}((W_0, S(x))) = \tilde{c}\,\frac{\left(W_0(s^0 + x(s - s^0))\right)^{1-\delta}}{1-\delta}\,, \quad \text{where}$$
$$\tilde{c} > 0, \quad \delta \geq 0, \quad \text{and} \quad W_0(s^0 + x(s - s^0)) > 0 \quad \text{when} \quad \delta > 0 \tag{3}$$

Parameter $\delta \geq 0$ denotes the ESG (preference) curvature parameter or the ESG diminishing sensitivity. A low δ implies low ESG curvature or, equivalently, a low ESG diminishing sensitivity. Conversely, a high δ implies high ESG curvature or a high ESG diminishing sensitivity. The parameter $\tilde{c}$ can be viewed as a normalization parameter, which allows to rescale the ESG scores s^0 and s according to the preferences of the ESG investor. Let us briefly discuss the impact of the ESG rating of the risk-free asset on the ESG score function defined in (3). The partial derivatives of the ESG score function If $x \in [0, 1)$ the investor holds a long position in the risk-free asset and ESG score function increases if the risk-free asset (a bond) becomes greener. If $x > 1$ the investor goes short in the risk-free asset. In this case utility increases if the bond becomes browner. Note that in this case the investor wants to sell the browner bond which makes her better off.

Since the value function is of the form $W_0^{1-\gamma} V\left(R(x), \hat{r}\right)$ and the ESG score function is a power function, multiplying the expected utility function (2) by $\frac{1}{W_0^{1-\gamma}}$ simplifies the optimization problem. Let $c = \tilde{c}\,\frac{W_0^{1-\delta}}{W_0^{1-\gamma}}$, then the expected prospect theory utility maximization problem is[10]

$$\begin{aligned} \max_x \ \mathbb{E}\left(U(x)\right) \quad &= \quad \alpha c \frac{[s^0 + x(s - s^0)]^{1-\delta}}{1-\delta} + (1-\alpha)\mathbb{E}\left(V\left(r^0 + \left(r - r^0\right)x, \hat{r}\right)\right) \\ \text{such that} \quad & x \geq 0 \qquad \text{for } s > s^0 \text{ and } \delta > 0 \quad \text{or for} \quad \delta = 0 \\ & 0 \leq x \leq \frac{s^0}{s^0 - s} \quad \text{for } s < s^0 \text{ and } \delta > 0 \end{aligned} \tag{4}$$

The parameter $\alpha \in [0, 1)$ is the ESG preference weight, measuring the relative importance of ESG preferences in the investor's overall utility.[11] When $\alpha = 0$, ESG considerations do not enter the investor's preferences, and problem (4) reduces to the maximization of the expected value function, i.e., to the pure prospect theory problem. Consequently all results in Fortin and Hlouskova (2024) apply. We denote the resulting solution the ESG-neutral investment, or pure prospect theory investment, $\hat{x}$. The same applies to the case $s = s^0$, i.e., when the scores of the risky and risk-free assets are identical. Thus, in this study we focus on cases in which the ESG scores of the risky and risk-free assets differ. We also highlight, where

[9] Note that for $s, s_0 \geq 0$ and $W_0, \delta > 0$ the portfolio score needs to be non-negative, i.e., $s^0 + x(s - s^0) \geq 0$. This implies that for $s < s^0$ is $0 \leq x \leq \frac{s_0}{s_0 - s}$.

[10] Note that $\tilde{c}$ allows to rescale the term $W_0 S(x)$ to account for the fact that both ESG scores and W_0 can be measured in different units (e.g. different scales used by different data providers). To simplify the notation, we suppress the dependence of c on W_0 and the parameters δ, γ in the following.

[11] When $\alpha = 1$ we have the optimal solutions $x^* = +\infty$ for $s > s^0$ and $x^* = 0$ for $s < s^0$.

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relevant, differences between the optimal investment under combined ESG and prospect theory preferences and the ESG-neutral benchmark. We distinguish between two cases based on the relative ESG scores of the risky and risk-free assets. In the green-risky case ($s > s^0$), the risky asset is greener than the risk-free asset. Conversely, in the green-risk-free case ($s < s^0$), the risk-free asset is greener than the risky asset.

3. Optimal investment strategies

Now we assume that the risky asset return r is continuously distributed with probability density function $f(\cdot)$ and cumulative distribution function $F(\cdot)$, such that $\mathbb{E}(r^2) = \int_{-\infty}^{+\infty} r^2 f(r)dr < +\infty$, i.e., the variance exists, $0 < F(c) < 1$ for any $c \in \mathbb{R}$, $\lim_{r\to\pm\infty} |r|^{1-\gamma} f(r) = 0$, and density function $f(\cdot)$ is differentiable. These assumptions are met by a large number of distributions including the normal, the skew normal, the student t with degrees of freedom strictly larger than two, etc. Under these assumptions we can derive in some cases a closed form solution (for investment in the risky asset) when the reference return coincides with the risk-free rate, i.e., when $\hat{r} = r^0$, while we derive a solution in its implicit form when $\hat{r} \neq r^0$. However, in both cases we derive properties of investment in the risky asset (risk taking, x^*) and investor's happiness defined by utility level attained by the optimal investment strategy x^*, that is $\mathbb{E}(U(x^*)))$. A comparative static analysis shows how x^* and $\mathbb{E}(U(x^*))$ change with respect to the model parameters, such as investor's ambition, $\hat{r}$, loss aversion, λ, ESG scores, s and s^0, weight of the score function, α, etc.

Let R be a continuous random variable describing the stochastic portfolio return and $f_p(\cdot)$ be its probability density function. Then based on (1) the expected value function with respect to return R and reference return $\hat{r}$ is

$$\mathbb{E}(V(R,\hat{r})) = \frac{1}{1-\gamma}\left(-\lambda\int_{-\infty}^{\hat{r}}(\hat{r}-R)^{1-\gamma} f_p(R)dR + \int_{\hat{r}}^{+\infty}(R-\hat{r})^{1-\gamma} f_p(R)dR\right) \tag{5}$$

and thus based on (5), the expected utility function of portfolio return
$R = R(x) = r^0 + (r - r^0)x$ is[12]

$$\begin{aligned}\mathbb{E}(U(x)) &= \alpha c\frac{[s^0 + x(s-s^0)]^{1-\delta}}{1-\delta} + (1-\alpha) \\ &\times \begin{cases} \frac{x^{1-\gamma}}{1-\gamma}\left[-\lambda\int_{-\infty}^{z(x,\hat{r})}\left(z(x,\hat{r}) - r\right)^{1-\gamma} f(r)dr + \int_{z(x,\hat{r})}^{+\infty}\left(r - z(x,\hat{r})\right)^{1-\gamma} f(r)dr\right], \\ \qquad \text{for} \quad x > 0 \\ \frac{\left(r^0-\hat{r}\right)^{1-\gamma}}{1-\gamma}, \quad \text{for} \quad x = 0 \quad \text{and} \quad \hat{r} \leq r^0 \\ -\lambda\frac{\left(\hat{r}-r^0\right)^{1-\gamma}}{1-\gamma}, \quad \text{for} \quad x = 0 \quad \text{and} \quad \hat{r} > r^0 \end{cases}\end{aligned} \tag{6}$$

[12]Note that $f_p(\cdot) = \frac{f(\cdot)}{x}$, for $x > 0$.

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where

$$z(x,\hat{r}) = r^0 + \frac{\hat{r}-r^0}{x} \tag{7}$$

Note that both integrals $\int_{-\infty}^{z}(z-r)^{1-\gamma}f(r)dr$ and[13] $\int_{z}^{+\infty}(r-z)^{1-\gamma}f(r)dr$ in (6) are non-negative and finite (see Lemma Appendix A.1) and thus also the expected value function and the expected utility function are finite, i.e. $-\infty < \mathbb{E}(V(R(x),\hat{r})) < +\infty$ and $-\infty < \mathbb{E}(U(x)) < +\infty$ for $x \in [0,+\infty)$. The proof of Proposition 3.1 shows that $\mathbb{E}(U(x))$ is continuous in x, including $x = 0$. Note in addition that the portfolio return is below reference return $\hat{r}$, i.e., $R(x) = r^0 + (r-r^0)x < \hat{r}$, if and only if $r < r^0 + \frac{\hat{r}-r^0}{x} = z(x,\hat{r})$ and portfolio return exceeds reference return $\hat{r}$, i.e., $R(x) > \hat{r}$ if and only if $r > z(x,\hat{r})$.

Let us introduce

$$K_\gamma = \frac{\int_{-\infty}^{r^0}(r^0-r)^{1-\gamma}f(r)dr}{\int_{r^0}^{+\infty}(r-r^0)^{1-\gamma}f(r)dr} \tag{8}$$

The value $1/K_\gamma$ plays an important role as, in pure prospect theory ($\alpha = 0$), it typically represents the lower bound of loss aversion, λ, which rules out infinite investment in the risky asset ($x^* = +\infty$) if $\lambda > 1/K_\gamma$. Thus, when $\lambda > 1/K_\gamma$ the investor's loss aversion is strong enough to offset the attractiveness of investing in the risky asset. Note that $1/K_0$ coincides with the Omega measure of the risky asset and thus shows similar features.[14] Based on the assumption of the risky asset's distribution, namely $\mathbb{E}(|r|) = \int_{-\infty}^{+\infty}|r|f(r)dr < +\infty$ and that $0 < F(c) < 1$ for any $c \in \mathbb{R}$, it follows that both K_γ and $\frac{1}{K_\gamma}$ are strictly positive and finite, i.e., $0 < K_\gamma, \frac{1}{K_\gamma} < +\infty$, see Lemma Appendix A.1 in Appendix Appendix A for more detail.

The following Propositions 3.1 and 3.3 present sufficient conditions for the existence of a global maximum of problem (4) for the cases $\hat{r} \neq r^0$ and $\hat{r} = r^0$, respectively. Let us at first introduce the following notation

$$\lambda_{L\gamma} = \begin{cases} \frac{1}{K_\gamma}, & \text{for } \hat{r} \neq r^0, \ \gamma < \delta \\ \frac{1}{K_\gamma} + \frac{\alpha c}{1-\alpha}\cdot\frac{(s-s^0)^{1-\gamma}}{\int_{-\infty}^{r^0}(r^0-r)^{1-\gamma}f(r)dr}, & \text{for } \hat{r} \neq r^0, \ \gamma = \delta > 0, \ s > s^0 \\ \frac{1}{K_0} + \frac{\alpha c}{1-\alpha}\cdot\frac{s-s^0}{\int_{-\infty}^{r^0}(r^0-r)f(r)dr}, & \text{for } \hat{r} \neq r^0, \ \gamma = \delta = 0 \\ \frac{1}{K_\gamma} + \frac{\alpha c}{1-\alpha}\cdot\frac{(s-s^0)^{1-\gamma}}{\int_{-\infty}^{r^0}(r^0-r)^{1-\gamma}f(r)dr}\left(\frac{\delta-\gamma}{s^0}\right)^{\delta-\gamma}\frac{\gamma^\gamma}{\delta^\delta}, & \text{for } \hat{r} = r^0, \ \gamma < \delta, \ s > s^0 \\ \frac{1}{K_0} + \frac{\alpha c}{1-\alpha}\cdot\frac{s-s^0}{\int_{-\infty}^{r^0}(r^0-r)f(r)dr}\cdot\frac{1}{(s^0)^\delta}, & \text{for } \hat{r} = r^0, \ \gamma = 0 \end{cases} \tag{9}$$

$$s_{T\gamma} = \begin{cases} s^0 - \frac{1-\alpha}{\alpha c}\cdot\frac{(s^0)^\delta}{(r^0-\hat{r})^\gamma}\left[\mathbb{E}(r) - r^0\right], & \text{for } \hat{r} < r^0 \\ s^0 - \lambda\frac{1-\alpha}{\alpha c}\cdot\frac{(s^0)^\delta}{(\hat{r}-r^0)^\gamma}\left[\mathbb{E}(r) - r^0\right], & \text{for } \hat{r} > r^0 \end{cases} \tag{10}$$

where K_γ is given by (8).[15] Based on the Leibniz integral rule, the first order condition (FOC) of (4),

[13] To simplify the notation we write $z = z(x,\hat{r})$.

[14] Harris and Mazibas (2022) refer to $1/K_\gamma$ as the hope-fear ratio, as it presents optimism (about the portfolio return exceeding a benchmark) relative to pessimism (about the portfolio return being below a benchmark), while Bernard and Ghossoub (2010) refer to $1/K_\gamma$ as the generalized Omega measure or the CPT-ratio.

[15] Lemma Appendix A.1 implies that $\lambda_{L\gamma}$ is well defined and finite, namely that $0 < \frac{1}{K_\gamma} < +\infty$ and $0 <$

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where $\mathbb{E}(U(x))$ is given by (6), is[16]

$$\alpha c \frac{(s-s^0)x^\gamma}{[s^0+x(s-s^0)]^\delta} + (1-\alpha)\left[\lambda \int_{-\infty}^{z(x,\hat{r})} \frac{r-r^0}{[z(x,\hat{r})-r]^\gamma} f(r)dr + \int_{z(x,\hat{r})}^{+\infty} \frac{r-r^0}{[r-z(x,\hat{r})]^\gamma} f(r)dr\right] = 0 \tag{11}$$

where $z(x,\hat{r})$ is given by (7).

Let $\hat{x}$ denote the ESG-neutral investment (benchmark), i.e., the solution of (4) when $\alpha = 0$ (only the pure prospect theory case is considered, see Fortin and Hlouskova, 2024).

Proposition 3.1. *Let $\mathbb{E}(r) > r^0$, $\gamma > 0$, $\hat{r} \neq r^0$, $\hat{x}$ solves (4) for $\alpha = 0$, and $\lambda_{L\gamma}$ and $s_{T\gamma}$ are given by (9) and (10). Then there exists a solution x^* of (4) such that*

$$x^* = \begin{cases} \in \left(0, \min\left\{\hat{x}, \frac{s_0}{s_0-s}\right\}\right) & \text{satisfies (11) for } s_{T\gamma} < s < s^0, \quad \delta > 0 \\ \in (0,\hat{x}) & \text{satisfies (11) for } s_{T\gamma} < s < s^0, \quad \delta = 0 \\ \in (\hat{x},+\infty) & \text{satisfies (11) for } s > s^0, \ \gamma \leq \delta, \quad \lambda > \lambda_{L\gamma} \\ +\infty, & \text{for } s > s^0, \ \gamma \leq \delta, \quad \lambda < \lambda_{L\gamma}, \ \text{or} \\ & \text{for } s > s^0, \quad \gamma > \delta \end{cases} \tag{12}$$

Proof: See Appendix Appendix A. □

The proposition characterizes how ESG preferences affect the optimal allocation relative to the ESG-neutral investment $\hat{x}$. In the green-risky case, $s > s^0$, the ESG component provides an additional incentive to invest in the risky asset, and any finite solution satisfies $x^* > \hat{x}$. In the green-risk-free case, $s < s^0$, the ESG component instead favors the risk-free asset, and any interior solution satisfies $x^* < \hat{x}$. Thus, compared with the ESG-neutral investment, ESG preferences shift the portfolio allocation toward the greener asset. This result should not, however, be interpreted as implying that an improvement in the ESG score of the risky asset necessarily increases risky-asset demand. As will be shown by our comparative-statics results (see Proposition 3.4) an increase in s generates an ESG substitution effect toward the risky asset, but this effect can be offset or even dominated by diminishing sensitivity to the portfolio's ESG level in green-risky case.[17] Thus, the proposition should be understood as describing the level effect of introducing ESG preferences relative to the ESG-neutral investment, rather than as implying a uniformly positive marginal effect of the risky asset's ESG score on its demand.

For $s > s^0$, the proposition also identifies conditions under which the ESG preferences lead to an unbounded risky-asset position. If $\gamma > \delta$, the optimal position is $x^* = +\infty$. If $\gamma \leq \delta$, the optimal position is infinite when $\lambda < \lambda_{L\gamma}$, whereas sufficiently strong loss aversion, $\lambda > \lambda_{L\gamma}$, yields a finite optimum satisfying $x^* > \hat{x}$. Thus, in the green-risky case, sufficiently strong loss aversion results in a finite position.

In the green-risk-free case, $s < s^0$, the ESG preferences instead shift the optimal allocation toward the risk-free asset. In particular, there exists a threshold $s_{T\gamma}$ such that the resulting interior solution satisfies $0 < x^* < \hat{x}$. Thus, in this case, ESG preferences reduce risky-asset investment relative to the ESG-neutral investment, but do not necessarily eliminate the risky position. The threshold $s_{T\gamma}$ has an

$\int_{-\infty}^{r^0} (r^0 - r)^{1-\gamma} f(r)dr < +\infty$.

[16] More precisely, it is the FOC pre-multiplied by x^γ.

[17] Consequently, $dx^*/ds > 0$ for sufficiently small δ, whereas dx^*/ds can become negative for sufficiently large δ. When $s < s^0$, by contrast, $dx^*/ds > 0$ for all admissible values of δ.

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intuitive interpretation as the minimum ESG score of the risky asset required for a positive risky-asset position, given the asset's expected return and the other model parameters. Equivalently, the condition $s > s_{T\gamma}$ can be expressed as a lower bound on the expected return of the risky asset:

$$\mathbb{E}(r) > \begin{cases} r^0 + \frac{\alpha c}{1-\alpha} \cdot \frac{(s^0-s)(r^0-\hat{r})^\gamma}{(s^0)^\delta}, & \text{for } \hat{r} < r^0 \\ r^0 + \frac{1}{\lambda} \cdot \frac{\alpha c}{1-\alpha} \cdot \frac{(s^0-s)(\hat{r}-r^0)^\gamma}{(s^0)^\delta}, & \text{for } \hat{r} > r^0 \end{cases} \tag{13}$$

This condition shows that the ESG disadvantage of the risky asset can be compensated by a sufficiently high expected return. Indeed, when the expected return is sufficiently large, then $s_{T\gamma} < 0$, and thus the threshold is non-binding for the economically relevant range of ESG scores. In this case ($\hat{r} \neq r^0$), the investor always maintains a strictly positive long position in the risky asset, even when the latter is less green than the risk-free asset. Hence, a relatively brown risky asset is not necessarily excluded from the portfolio; whether the investor abandons it depends on whether its expected-return advantage is sufficient to compensate for its ESG disadvantage.

If, in contrast, the risky asset is sufficiently brown, $s \leq s_{T\gamma}$, the ESG disadvantage can dominate its financial attractiveness at the margin, resulting in the corner solution $x^* = 0$.[18] This corner solution is therefore an ESG-induced exclusion of the risky asset, which does not arise in ESG-neutral investment ($\alpha = 0$), where $x^* > 0$ for $\hat{r} \neq r^0$. This is explicitly shown in the following proposition when $\gamma = 0$.

Finally, when $s^0 = 0$, which, to the best of our knowledge, is implicitly assumed in the existing literature, and ESG scores are restricted to be non-negative, only the green-risky case, $s > s^0 = 0$, is relevant. In this special case, the ESG component necessarily shifts any finite optimal allocation above the ESG-neutral investment, $x^* > \hat{x}$. However, depending on δ and λ, the overall optimum may also be unbounded.

Note finally that $\hat{x}$ is finite only when $\lambda > \frac{1}{K_\gamma}$ and for $\lambda \leq \frac{1}{K_\gamma}$ is $\hat{x} = +\infty$ (see Fortin and Hlouskova, 2024). Thus, $\hat{x}$ is an active upper bound in (12) for $s < s^0$ only when $\lambda > \frac{1}{K_\gamma}$.

The following proposition provides a complete characterization of the unique solution in the special case of a bilinear prospect-theory value function ($\gamma = 0$).[19]

Proposition 3.2. *Let $\mathbb{E}(r) > r^0$, $\hat{r} \neq r^0$, $\gamma = 0 \leq \delta$, $\hat{x}$ be the unique solution of (4) for $\alpha = 0$, and let K_0, λ_{L0}, and s_{T0} are given by (8), (9) and (10). Then there exists a unique solution $x^* \geq 0$ of problem (4) such that the following holds*

$$x^* = \begin{cases} 0, & \text{for } s < s_{T0} < s^0 \\ \in \left(0, \min\left\{\hat{x}, \frac{s_0}{s_0 - s}\right\}\right) & \text{satisfies (11) for } s_{T0} < s < s^0, \quad \delta > 0 \\ \in (0, \hat{x}) & \text{satisfies (11) for } s_{T0} < s < s^0, \quad \delta = 0, \quad \lambda > \lambda_{L0} \\ \in (\hat{x}, +\infty) & \text{satisfies (11) for } s > s^0, \quad \lambda > \lambda_{L0} \\ +\infty, & \text{for } s > s^0, \quad \lambda < \lambda_{L0} \end{cases} \tag{14}$$

Proof: See Appendix Appendix A. □

The proposition establishes that the model can generate an ESG-induced corner solution. When the risky asset is sufficiently brown, $0 \leq s < s_{T0} < s^0$, the investor invests exclusively in the risk-free asset,

[18] Our simulations indicate that this occurs for sufficiently low s.

[19] Note that the condition $s < s_{T0}$ stated in Proposition 3.2 is equivalent to $\mathbb{E}(r)$ being smaller than the expression given in (13) with $\gamma = 0$.

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$x^* = 0$.[20] Thus, when the ESG disadvantage of the risky asset is sufficiently large relative to its expected-return advantage, ESG preferences can lead to complete exclusion of the risky asset. This result requires a sufficiently low expected return of the risky asset. If, however, the expected return is sufficiently high, i.e., $s > s_{T0}$, a positive position in the risky asset can be maintained even when $s < s^0$. While the ESG-induced exclusion can be established analytically for the bilinear value function, $\gamma = 0$, we have not yet been able to establish it analytically for the general case $\gamma > 0$, although it is also observed in our numerical simulations. Note that this ESG-induced exclusion is obviously absent in the ESG-neutral investment ($\alpha = 0$), where $x^* = \hat{x} > 0$ for $\hat{r} \neq r^0$ (see Fortin and Hlouskova, 2011). The result therefore highlights a particularly sharp consequence of introducing ESG preferences into the bilinear prospect-theory framework.

For $s_{T0} < s < s^0$ the unique solution is finite and strictly positive with $x^* < \hat{x}$, while for $s > s^0$, the solution is either finite with $x^* > \hat{x}$ or unbounded, depending on the degree of loss aversion, as in the general case. The distinctive result established here is that, for sufficiently brown risky assets, $0 \leq s < s_{T0}$, the unique solution may be the corner solution in which the entire investment is allocated to the risk-free asset.

The following Proposition 3.3 investigates the case for $\hat{r} = r^0$, where, among others, also a unique closed form solution for the bilinear value function, $\gamma = 0 \leq \delta$, can be obtained.[21] In the case $\hat{r} = r^0$ the expected utility becomes

$$\mathbb{E}(U(x)) \quad = \quad \alpha c \frac{[s^0 + x(s - s^0)]^{1-\delta}}{1-\delta} - (1-\alpha)\frac{x^{1-\gamma}}{1-\gamma}\int_{-\infty}^{r^0} \left(r^0 - r\right)^{1-\gamma} f(r)dr \left(\lambda - \frac{1}{K_\gamma}\right) \tag{15}$$

The first order condition (FOC) of (4), where $\mathbb{E}(U(x))$ is given by (15), results in[22]

$$\alpha c \frac{(s - s^0)x^\gamma}{[s^0 + x^*(s - s^0)]^\delta} - (1-\alpha)\int_{-\infty}^{r^0} (r^0 - r)^{1-\gamma} f(r)dr \left(\lambda - \frac{1}{K_\gamma}\right) = 0 \tag{16}$$

Various sub-cases depending on the relationship between γ and δ, and whether $s^0 = 0$ or $s^0 > 0$ show up. In all cases but one uniqueness can be shown, and in some particular cases also closed form solutions are found. Across these sub-cases, the model generates three qualitatively different portfolio regimes: exclusive investment in the risk-free asset ($x^* = 0$), a finite positive investment in the risky asset ($0 < x^* < +\infty$) and an unbounded long position ($x^* = +\infty$). Which regime arises depends on the relative ESG scores, the curvature parameters γ and δ, and the investor's degree of loss aversion λ. A sufficiently high degree of loss aversion can make exclusive investment in the risk-free asset optimal. On the other hand, an infinite long position of the risky asset is optimal for less loss averse investors with the risky asset being greener than the risk-free asset and $\gamma < \delta$, or for investors whose shape parameter of the value function exceeds the shape parameter of the ESG score function ($\gamma > \delta$). In formal terms we obtain

Proposition 3.3. *Let $\hat{r} = r^0$, $\alpha < 1$, K_γ is given by (8), and $\lambda_{L\gamma}$ is given by (9). Then there exists a*

[20] This corner solution can arise only when $s_{T0} > 0$, so that there is a positive range of economically admissible ESG scores $0 \leq s < s_{T0}$ for which the risky asset is sufficiently brown. When $s_{T0} < 0$, this condition cannot be satisfied and the corner solution does not arise.

[21] Among other studies, also Barberis et al. (2001) choose the risk-free rate as the reference return.

[22] More precisely, it is the FOC pre-multiplied by x^γ.

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solution x^* of (4), where $\mathbb{E}(U(x))$ is given by (15), such that the following holds.

- *Let $0 < \gamma \neq \delta > 0$ and $s_0 > 0$. Then*

$$x^* = \begin{cases} 0, & s > s^0,\ \gamma < \delta,\ \lambda > \lambda_{L\gamma} \\ & s < s^0,\ \lambda > \frac{1}{K_\gamma} \\ \in (0, +\infty) \text{ and satisfies (16)}, & s > s^0,\ \gamma < \delta,\ \frac{1}{K_\gamma} < \lambda < \lambda_{L\gamma} \\ & s < s^0,\ \lambda < \frac{1}{K_\gamma} \text{ (unique solution)} \\ +\infty, & s > s^0,\ \gamma < \delta,\ \lambda < \frac{1}{K_\gamma} \\ & s > s^0,\ \gamma > \delta \end{cases} \tag{17}$$

- *Let $s^0 = 0$. Then*

$$x^* = \begin{cases} 0, & \gamma = \delta,\ \lambda > \lambda_{L\gamma} \\ \left[\frac{\alpha c}{1-\alpha} \cdot \frac{s^{1-\delta}}{\int_{-\infty}^{r^0} (r^0 - r)^{1-\gamma} f(r) dr \left(\lambda - \frac{1}{K_\gamma}\right)}\right]^{\frac{1}{\delta - \gamma}}, & \gamma < \delta,\ \lambda > \frac{1}{K_\gamma} \\ +\infty, & \gamma > \delta \\ & \lambda < \frac{1}{K_\gamma} \\ & \gamma = \delta,\ \lambda < \lambda_{L\gamma} \end{cases} \tag{18}$$

- *Let $0 < \gamma = \delta$ and $s^0 > 0$. Then*

$$x^* = \begin{cases} 0, & s > s^0,\ \lambda > \lambda_{L\gamma} \\ & s < s^0,\ \lambda > \frac{1}{K_\gamma} \\ \dfrac{s^0}{\left[\frac{\alpha c}{1-\alpha} \cdot \frac{s - s^0}{\int_{-\infty}^{r^0} (r^0 - r)^{1-\gamma} f(r) dr \left(\lambda - \frac{1}{K_\gamma}\right)}\right]^{\frac{1}{\gamma}} - s + s^0}, & s < s^0,\ \lambda < \frac{1}{K_\gamma} \\ +\infty, & s > s^0,\ \lambda < \lambda_{L\gamma} \end{cases} \tag{19}$$

- *Let $0 = \gamma < \delta$ and $s^0 > 0$. Then*

$$x^* = \begin{cases} 0, & \lambda > \lambda_{L0} \\ \frac{1}{s - s^0} \left\{ \left[\frac{\alpha c (s - s^0)}{(1-\alpha)\left(\lambda - \frac{1}{K_0}\right) \int_{-\infty}^{r^0} (r^0 - r) f(r) dr} \right]^{\frac{1}{\delta}} - s^0 \right\}, & s > s^0,\ \frac{1}{K_0} < \lambda < \lambda_{L0} \\ & s < s^0,\ \ \lambda < \lambda_{L0} \\ +\infty, & s > s^0,\ \ \lambda < \frac{1}{K_0} \end{cases} \tag{20}$$

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- *Let $0 = \gamma = \delta$. Then for $\lambda > \lambda_{L0}$ is $x^* = 0$ while for $\lambda < \lambda_{L0}$ is $x^* = +\infty$.*

Proof: See Appendix Appendix A. □

The optimal investment depends on the relative ESG scores, the curvature parameters γ and δ, and, importantly, the degree of loss aversion λ. When $s > s^0$, ESG preferences favor the risky asset, but sufficiently strong loss aversion can nevertheless result in $x^* = 0$, see (17). Thus, in this case, the ESG attractiveness of the risky asset and the investor's aversion to prospect-theory losses work in opposite directions: the former encourages investment in the risky asset, whereas the latter limits the willingness to bear its risk. This differs from the case $\hat{r} \neq r^0$, where $x^* = 0$ does not arise when the risky asset is greener than the risk-free asset; a sufficiently loss-averse investor instead holds a finite positive position satisfying $x^* > \hat{x}$. If the investor is sufficiently loss averse but not so loss averse as to choose the corner solution ($\frac{1}{K_\gamma} < \lambda < \lambda_{L\gamma}$), the resulting finite optimum satisfies $x^* > 0 = \hat{x}$ (see Proposition 2.4 in Fortin and Hlouskova, 2024). When $s < s^0$, ESG preferences favor the risk-free asset, and sufficiently strong loss aversion can reinforce this preference resulting in $x^* = 0$.

An important implication is that, when $\hat{r} = r^0$, introducing ESG preferences can generate a finite, strictly positive risky-asset position, whereas the ESG-neutral investment is either $x^* = 0$ or $x^* = +\infty$, depending on the level of loss aversion. Thus, introducing ESG preferences can create an interior portfolio allocation between the two corner outcomes of the pure prospect-theory problem.

The following Proposition 3.4 presents some properties of the solution of (4) (optimal investment in the risky asset, x^*), namely how risk taking reacts to the change of certain parameters, such as the level of loss aversion, λ, the reference return, $\hat{r}$, ESG scores, s, s^0, and the weight of the ESG score function, α. Let us introduce the following notations

$$F(z(x^*,\hat{r})) \equiv \frac{d}{dz}\left[\lambda \int_{-\infty}^{z(x^*,\hat{r})} \frac{r - r^0}{[z(x^*,\hat{r}) - r]^\gamma} f(r)dr + \int_{z(x^*,\hat{r})}^{+\infty} \frac{r - r^0}{[r - z(x^*,\hat{r})]^\gamma} f(r)dr\right] \tag{21}$$

where $z(x,\hat{r})$ is given by (7), and

$$\delta_T = \frac{s^0 + (s - s^0)x^*}{(s - s^0)(x^* - 1)} \tag{22}$$

Proposition 3.4. *Let the assumptions of Proposition 3.1 be satisfied, let $F(z(x^*,\hat{r}))$ be given by (21, δ_T by (22) and let $\hat{x}$ be the solution of (4) for $\alpha = 0$. Then the optimal solution of (4), x^*, such that $0 < x^* < +\infty$, has the following properties*
for $s > s^0$

$$\frac{dx^*}{ds} = \begin{cases} > 0, & 0 \leq \delta < 1 + \frac{s^0}{(s-s^0)x^*} \\ = 0, & \delta = 1 + \frac{s^0}{(s-s^0)x^*} \\ < 0, & \delta > 1 + \frac{s^0}{(s-s^0)x^*} \end{cases} \tag{23}$$

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for $s < s^0$

$$\frac{dx^*}{ds} > 0 \tag{24}$$

for $s > s^0$

$$\frac{dx^*}{ds^0} \begin{cases} > 0, & x^* > 1, \quad \delta > \delta_T \\ = 0, & x^* > 1, \quad \delta = \delta_T \\ < 0, & x^* \leq 1 \\ & x^* > 1, \quad \delta < \delta_T \end{cases} \tag{25}$$

for $s < s^0$

$$\frac{dx^*}{ds^0} \begin{cases} > 0, & x^* < 1, \quad \delta > \delta_T \\ = 0, & x^* < 1, \quad \delta = \delta_T \\ < 0, & x^* \geq 1 \\ & x^* < 1, \quad \delta < \delta_T \end{cases} \tag{26}$$

$$\frac{dx^*}{d\lambda} < 0, \quad \textit{for } \hat{r} \leq r^0 \textit{ or } \hat{r} > r^0 \textit{ and } s > s^0 \tag{27}$$

$$\operatorname{sgn}\left(\frac{dx^*}{d\hat{r}}\right) = \operatorname{sgn}\left(F(z(x^*, \hat{r}))\right) \textit{ for } \hat{r} \neq r^0 \textit{ and for } \hat{r} \textit{ such that } F(z(x^*, \hat{r})) \textit{ is continuous in } \hat{r} \tag{28}$$

$$\frac{dx^*}{d\hat{r}} = \frac{x^*}{\hat{r} - r^0} \begin{cases} < 0, & \hat{r} < r^0 \\ > 0, & \hat{r} > r^0 \end{cases} \textit{for } \gamma = \delta \textit{ and } s^0 = 0, \textit{ or } \alpha = 0, \textit{ or } \gamma = \delta = 0 \tag{29}$$

$$= \frac{(\lambda - 1) f(z(x^*, \hat{r})) \frac{\hat{r} - r^0}{x^*}}{\frac{\alpha c \delta}{1-\alpha} \cdot \frac{x^*(s-s^0)^2}{[s^0 + x^*(s-s^0)]^{1+\delta}} + (\lambda - 1) f(z(x^*, \hat{r})) \left(\frac{\hat{r} - r^0}{x^*}\right)^2} \begin{cases} < 0, & \textit{for } \hat{r} < r^0 \\ > 0, & \textit{for } \hat{r} > r^0 \end{cases} \textit{for } \gamma = 0 \leq \delta \tag{30}$$

$$\frac{dx^*}{d\alpha} = \begin{cases} > 0, & \textit{for } s > s^0 \textit{ and if } \hat{r} = r^0 \textit{ then also } \lambda > \frac{1}{K_\gamma} \\ < 0, & \textit{for } s < s^0 \textit{ and if } \hat{r} = r^0 \textit{ then also } \lambda < \frac{1}{K_\gamma} \end{cases} \tag{31}$$

Proof: See Appendix Appendix A.[23] □

Proposition 3.4 shows, first, how optimal investment into the risky asset changes if the risky or the

[23] To simplify the notation we use derivative notation instead of partial derivative notation.

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risk-free asset becomes greener. These effects depend on the relative greenness of the two assets, on ESG diminishing sensitivity and on whether the portfolio is leveraged ($x^* > 1$) or not. To see the different forces at work note that the sign of dx^*/ds coincides with the sign of the sensitivity of marginal utility with respect to the risky asset's ESG score. Using Equation (A.45) in Appendix Appendix A we have

$$\operatorname{sgn}\left(\frac{dx^*}{ds}\right) = \operatorname{sgn}\left(\frac{d^2\mathbb{E}(U(x^*))}{dxds}\right) = \operatorname{sgn}\left(\underbrace{x^*}_{\text{ESG substitution effect}} - \underbrace{x^*\frac{\delta(s-s^0)x^*}{s^0+(s-s^0)x^*}}_{\text{ESG diminishing sensitivity effect}}\right) \tag{32}$$

The first part of the last term is referred to as the "ESG substitution effect," because an improvement in the ESG score of the risky asset makes it more attractive relative to the other asset, thereby creating an incentive to reallocate investment toward it. The "ESG diminishing sensitivity effect" captures the curvature of the ESG score function. Note that the ESG diminishing sensitivity effect is zero when $\delta = 0$, i.e., when the ESG score function is linear. Similarly, using Equation (A.49) in Appendix Appendix A, the sign of $\frac{dx^*}{ds^0}$ can be written as

$$\begin{aligned}\operatorname{sgn}\left(\frac{dx^*}{ds^0}\right) &= \operatorname{sgn}\left(\frac{d^2\mathbb{E}(U(x^*))}{dxds^0}\right) = \\ &= \operatorname{sgn}\left(\underbrace{-x^*}_{\text{ESG substitution effect}} - \underbrace{x^*\frac{\delta(s-s^0)(1-x^*)}{s^0+(s-s^0)x^*}}_{\text{leverage effect, ESG diminishing sensitivity effect}}\right)\end{aligned} \tag{33}$$

The decomposition now yields an "ESG substitution effect" and a "leverage effect".[24] The first effect captures the substitution of the risky asset with the risk-free asset as s^0 increases, while the second captures the effect of leveraged portfolios, i.e., when $x^* > 1$, and the effect of ESG diminishing sensitivity.

First, we consider changes in the ESG score of the risky asset. In the green-risky case, $s > s^0$, an increase in the risky asset's ESG score generates an ESG substitution effect toward the risky asset. At the same time, it generates a diminishing-sensitivity effect that can reduce the marginal ESG benefit of further investment in the asset. For sufficiently small δ, e.g., $\delta < 1$, the substitution effect dominates, yielding $dx^*/ds > 0$, whereas for sufficiently large δ, the diminishing-sensitivity effect dominates, resulting in $dx^*/ds < 0$. By contrast, in the green-risk-free case, $s < s^0$, the ESG substitution effect dominates the diminishing-sensitivity effect, irrespective of the level of δ, so that an improvement in the risky asset's ESG score unambiguously increases risky-asset demand. Thus, a negative response of risky-asset demand to an increase in its ESG score, which may seem counterintuitive at first, is possible only when the risky asset is greener than the risk-free asset. Our results thus extend the conventional positive relationship documented in the literature[25] by showing that this relationship need not be unambiguously positive.

Second, we consider changes in the ESG score of the risk-free asset and obtain similarly nuanced results. In the green-risky case, $s > s^0$, an increase in s^0 reduces the ESG advantage of the risky asset relative to

[24] This effect includes both the leverage effect and the ESG diminishing sensitivity effect. For ease of exposition, however, we refer to it simply as the leverage effect.

[25] See, e.g., Pástor et al., 2021, where asset demand increases with the ESG score of the risky asset.

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the risk-free asset and therefore generates an ESG substitution effect away from the risky asset, tending to reduce risky-asset demand. This effect is unambiguous when the investor is not leveraged, i.e., when $x^* \leq 1$. When the investor is leveraged, however, i.e., $x^* > 1$,[26] the total effect depends on the degree of diminishing sensitivity to ESG. For low diminishing sensitivity, the ESG substitution effect dominates and risky-asset demand decreases, whereas for sufficiently high diminishing sensitivity, the leverage effect dominates and an improvement in the ESG score of the risk-free asset increases risky-asset demand.

The green risk-free-case, $s < s^0$, is in several respects the mirror image of the green-risky case. An increase in s^0 strengthens the ESG advantage of the risk-free asset and hence generates an ESG substitution effect away from the risky asset. This effect dominates for $x^* \geq 1$ or when $x^* < 1$ and diminishing sensitivity is relatively weak, $\delta < \delta_T$, implying $dx^*/ds^0 < 0$. When $x^* < 1$ and diminishing sensitivity is sufficiently strong, $\delta > \delta_T$, the composition effect dominates, resulting in $dx^*/ds^0 > 0$. At $\delta = \delta_T$ the two effects exactly offset each other. Again, we find counterintuitive examples in which investment in the risky asset increases as the risk-free asset becomes greener.

In addition, Proposition 3.4 shows that in majority of cases investment in the risky asset decreases with an increasing level of loss aversion.[27] Note that when the return of the risky asset follows a Bernoulli distribution then $\frac{dx^*}{d\lambda} < 0$ only for $\hat{r} > r^0$, see Supplementary material. For $\hat{r} < r^0$ the solution does not depend on the degree of loss aversion, i.e., $\frac{dx^*}{d\lambda} = 0$, see Supplementary material. The properties of an investor's risk-taking with respect to her level of ambition, $\hat{r}$, is more difficult to analyze. If $\hat{r} \neq r^0$, then (28) holds. Note that this holds only under the assumption when $F(z(x^*, \hat{r})$ is continuous in $\hat{r}$. If there exists $\hat{r}_T$ such that $F(\cdot)$ is not continuous in this point, i.e., $\lim_{\hat{r}\to(\hat{r}_T)^-}(F(z(x^*,\hat{r})) \neq \lim_{\hat{r}\to(\hat{r}_T)^+}(F(z(x^*,\hat{r}))$, then (28) is replaced by

$$\begin{aligned} \operatorname{sgn}\left(\lim_{\hat{r}\to(\hat{r}_T)^-}\frac{dx^*}{d\hat{r}}\right) &= \operatorname{sgn}\left(\lim_{\hat{r}\to(\hat{r}_T)^-}(F(z(x^*,\hat{r}))\right) \quad \text{and} \\ \operatorname{sgn}\left(\lim_{\hat{r}\to(\hat{r}_T)^+}\frac{dx^*}{d\hat{r}}\right) &= \operatorname{sgn}\left(\lim_{\hat{r}\to(\hat{r}_T)^+}(F(z(x^*,\hat{r}))\right) \end{aligned} \tag{34}$$

We observe a V-shaped behavior when imposing further constraints provided in (29) and (30), namely when the value function is bilinear or when the ESG score of the risk-free asset is zero and the curvature coefficients of the value function and the ESG score function are the same. That is, investment into the risky asset decreases in the level of ambition if $\hat{r} < r^0$ while it increases if $\hat{r} > r^0$. As in Fortin and Hlouskova (2024), the less ambitious investor, $\hat{r} < r^0$, prefers to accept less risk and move away from the stock market when increasing her level of ambition, while the more ambitious investor, $\hat{r} > r^0$, puts more money into the risky asset when increasing her level of ambition.[28] Finally, the investor's risk taking increases with an increasing weight of the ESG score function, α, if the risky asset is greener than the risk-free asset, while investor's risk taking decreases with an increasing weight of the ESG score function, α, if the risky asset is browner than the risk-free asset. In other words, by increasing the relative importance of the ESG part in the preferences (i.e. raising α), the decision maker increases investment into the greener asset.

[26] I.e., the investor holds more than 100% in the risky asset and finances part of it by borrowing in the risk-free asset.

[27] The only case for which this was not shown analytically is when $\hat{r} > r^0$ and $s < s^0$. For this case we have found a counterexample that is available from the authors upon request.

[28] Note that the V-shape property of the investor's risk-taking with respect to her level of ambition, $\hat{r}$, namely (29), holds also for the case when risky asset's return follows a Bernoulli distribution, see Supplementary material.

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The following proposition presents properties of happiness, defined as $\mathbb{E}(U(x^*))$, namely how happiness reacts to the change of certain preference parameters, such as the level of loss aversion, λ, the reference return, $\hat{r}$, ESG scores, s, s^0, and the weight of the ESG score function, α.

Proposition 3.5. *Let assumptions of Proposition 3.1 be satisfied. Then happiness, $\mathbb{E}(U(x^*))$, has the following properties*

$$\frac{d\mathbb{E}(U(x^*))}{d\hat{r}} < 0 \tag{35}$$

$$\frac{d\mathbb{E}(U(x^*))}{d\lambda} < 0 \tag{36}$$

$$\frac{d\mathbb{E}(U(x^*))}{ds} > 0 \tag{37}$$

$$\frac{d\mathbb{E}(U(x^*))}{ds^0} \quad \begin{cases} > 0, & x^* < 1 \\ = 0, & x^* = 1 \\ < 0, & x^* > 1 \end{cases} \tag{38}$$

$$\frac{d\mathbb{E}(U(x^*))}{d\alpha} > 0 \ \textit{ for } \ \delta < 1, \ \ \hat{r} > r^0 \ \textit{ and } \ \lambda > \frac{1}{K_\gamma} \tag{39}$$

When $\hat{r} < r^0$, $s \geq s^0$, and $\delta < 1$ then a stronger sufficient condition is needed for $\frac{d\mathbb{E}(U(x^*))}{d\alpha} > 0$, namely that $\lambda > \frac{\int_{z(\hat{x},\hat{r})}^{+\infty}[r - z(\hat{x},\hat{r})]^{1-\gamma} f(r)dr}{\int_{-\infty}^{z(\hat{x},\hat{r})}[z(\hat{x},\hat{r}) - r]^{1-\gamma} f(r)dr}$, where $\hat{x}$ solves (4) for $\alpha = 0$.[29]

Proof: See Appendix Appendix A. □

Thus, happiness decreases with an increasing level of ambition ($\hat{r}$, see also Figure 1) and with the degree of loss aversion, it increases with an increasing ESG score of the risky asset (i.e., when the risky asset becomes greener), and increases in the weight of the ESG score function (α). The latter holds for a sufficiently loss averse investor. Finally, the direction of the change of happiness with respect to the ESG score of the risk-free asset depends on the optimal investment into the risky asset. Namely, happiness decreases when the risk-free asset becomes greener in the case when the portfolio is leveraged, $x^* > 1$, i.e., when the investor goes short in the risk-free asset, while the opposite occurs when the portfolio is not leveraged, $x^* < 1$.

Finally, the results obtained in Propositions 3.1–3.5 can be used to obtain additional portfolio properties, namely the sensitivities of the first four moments of the portfolio return with respect to the reference return, $\hat{r}$, the degree of loss aversion, λ, the ESG scores s and s^0, and the weight of the ESG score function, α. Let π denote an element contained in $\{\hat{r}, \lambda, s, s^0, \alpha\}$. Note that, under the assumption $\mathbb{E}(r) > r^0$ maintained throughout this paper, we have $\frac{d\mathbb{E}(R(x^*))}{d\pi} = \left[\mathbb{E}(r) - r^0\right] \frac{dx^*}{d\pi}$. Hence the effect of a change in π on the expected return of the optimal portfolio has the same sign as its effect on the optimal investment x^*.

The same applies to the effect of a change in π on the variance of the optimal portfolio. Namely, as $Var(R(x)) = Var(r^0 + (r - r^0)x) = x^2 Var(r)$, we get $\frac{Var(R(x^*))}{d\pi} = 2x^* Var(r) \frac{dx^*}{d\pi}$ such that $\operatorname{sgn}\left(\frac{dVar(R(x^*))}{d\pi}\right) =$

[29] Note that for $s \geq s^0$ is $\frac{\int_{z(\hat{x},\hat{r})}^{+\infty}[r - z(\hat{x},\hat{r})]^{1-\gamma} f(r)dr}{\int_{-\infty}^{z(\hat{x},\hat{r})}[z(\hat{x},\hat{r}) - r]^{1-\gamma} f(r)dr} > \frac{1}{K_\gamma}$. The general condition for $\frac{d\mathbb{E}(U(x^*))}{d\alpha} > 0$ is $\lambda > \frac{\int_{z(x^*,\hat{r})}^{+\infty}\left[r - z(x^*,\hat{r})\right]^{1-\gamma} f(r)dr}{\int_{-\infty}^{z(x^*,\hat{r})}[z(x^*,\hat{r}) - r]^{1-\gamma} f(r)dr}$ and $\delta < 1$.

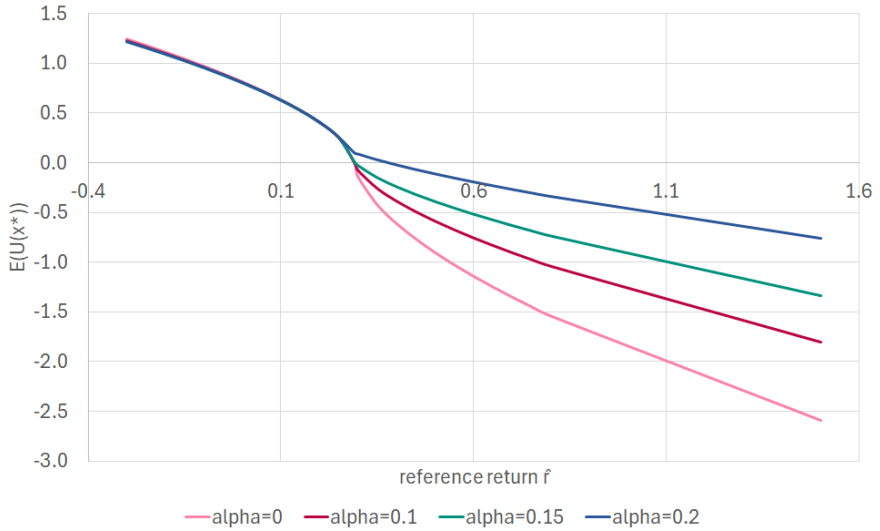


Figure 1: Happiness, $\mathbb{E}(U(x^*))$, as a function of the reference return, $\hat{r}$, for $\lambda = 2.25$, $\gamma = 0.4$, $\delta = 0.5$, $s^0 = 0$, $s = 5$, and $\alpha = 0, 0.1, 0.15, 0.2$.
The horizontal axis shows the reference return. The results are shown for normally distributed returns of the risky asset, where $\mathbb{E}(r) = 13.6\%$, $Var(r) = 18.4\%$ and $r^0 = 3.5\%$.

$\operatorname{sgn}\left(\frac{dx^*}{d\pi}\right)$.

For the third central moment we have $\mathbb{E}\left(R(x) - \mathbb{E}\left(R(x)\right)\right)^3 = x^3\mathbb{E}(r - \mathbb{E}(r))^3$ and observe that $\frac{d\mathbb{E}(R(x^*) - \mathbb{E}(R(x^*)))^3}{d\pi} = 3(x^*)^2\mathbb{E}(r - \mathbb{E}(r))^3 \frac{dx^*}{d\pi}$. This yields

$$\operatorname{sgn}\left(\frac{d\mathbb{E}\left(R(x^*) - \mathbb{E}\left(R(x^*)\right)\right)^3}{d\pi}\right) = \begin{cases} \operatorname{sgn}\left(\frac{dx^*}{d\pi}\right) & \text{for} \quad \mathbb{E}(r - \mathbb{E}(r))^3 > 0 \\ -\operatorname{sgn}\left(\frac{dx^*}{d\pi}\right) & \text{for} \quad \mathbb{E}(r - \mathbb{E}(r))^3 < 0 \end{cases} \tag{40}$$

Kwak and Pirvu (2018) observe that the skewness of the stock return distribution has a considerable impact on the distribution of the cumulative prospect theory (CPT) investor's wealth deviation, highlighting the importance of skewness.

The effect of a change in π on the fourth central moment of the optimal portfolio has again the same sign as its effect on the optimal investment x^*. Since $\mathbb{E}\left(R(x) - \mathbb{E}\left(R(x)\right)\right)^4 = x^4\mathbb{E}(r - \mathbb{E}(r))^4$ we get $\frac{d\mathbb{E}(R(x^*) - \mathbb{E}(R(x^*)))^4}{d\pi} = 4(x^*)^3\mathbb{E}(r - \mathbb{E}(r))^4 \frac{dx^*}{d\pi}$ and thus $\operatorname{sgn}\left(\frac{d\mathbb{E}(R(x^*) - \mathbb{E}(R(x^*)))^4}{d\pi}\right) = \operatorname{sgn}\left(\frac{dx^*}{d\pi}\right)$.

Finally, we observe that the pth-standardized moments are such that $\mathbb{E}\left(\frac{R(x) - \mathbb{E}(R(x))}{\sqrt{Var(R(x))}}\right)^p = \mathbb{E}\left(\frac{r - \mathbb{E}(r)}{\sqrt{Var(r)}}\right)^p$, $p = 1, 2, 3, 4$, and thus they do not depend on π, i.e., their partial derivatives with respect to the parameter $\pi \in \{\hat{r}, \lambda, s, s^0, \alpha\}$ are equal to zero. Hence, also the Sharpe ratio is insensitive with respect to π.

We refer to the case in which the expected portfolio return falls below the reference return as an expected

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loss, and to the case in which it exceeds the reference return as an expected gain. Then

$$\mathbb{E}(R(x)) \begin{cases} < \hat{r} & \text{iff } x < \frac{\hat{r}-r^0}{\mathbb{E}(r)-r^0} \quad \text{(expected portfolio loss)} \\ = \hat{r} & \text{iff } x = \frac{\hat{r}-r^0}{\mathbb{E}(r)-r^0} \\ > \hat{r} & \text{iff } x > \frac{\hat{r}-r^0}{\mathbb{E}(r)-r^0} \quad \text{(expected portfolio gain)} \end{cases} \tag{41}$$

Thus, an expected loss can occur only when investment in the risky asset is sufficiently small and when $\hat{r} > r^0$, while an expected gain can occur only when investment in the risky asset is sufficiently large. Note, in addition, that investors with $\hat{r} < r^0$ can experience only expected gains. The result obtained in (41) is the same as for the ESG-neutral investor see Fortin and Hlouskova (2024).

4. Simulations

This section provides numerical examples to demonstrate how optimal investment x^* changes if some of the model parameters change. Our main focus is to describe how investment behavior is affected by changes in the reference return $\hat{r}$ and changes in the weight α of the ESG score function in the agent's utility function, with emphasis on cases that were not shown analytically. In addition, we examine whether the assumption of an asymmetric distribution of the risky asset return has major effects on the results.

As already pointed out in Section 3 this article explicitly includes an ESG rating for the risk-free asset, s^0. The following simulations show typical optimal investments x^*, for the green-risky and the green-risk-free cases, respectively. We first consider normally distributed returns for the risky asset, corresponding to a symmetric distribution. We then consider skew-normal returns, allowing for both positive and negative skewness. This comparison enables us to assess the extent to which return asymmetry affects the results. Negative skewness is particularly relevant in the context of stock markets, where it is commonly observed. If investors assume that returns are normally distributed when they are in fact negatively skewed, they may underestimate the likelihood and severity of extreme losses. For the return of the risky asset, we consider a mean of 13.6%, a variance of 18.4% and, for the skewed distribution, a shape parameter of ± 3.1, resulting in a skewness of ± 0.68. The risk-free rate is 3.5%. These values are based on estimates using annual U.S. stock market returns over the past 30 years. In some examples we fix the degree of loss aversion at 2.25, which is the value suggested by experimental results, see, e.g., Tversky and Kahneman (1992). In the simulations we numerically compute the optimal weight of the risky asset, x^*, which is derived theoretically (in implicit and explicit forms) in Propositions 3.1 and 3.3. We apply a grid search to numerically maximize the expected utility given in (6), using the densities of the normal and skew normal distributions and numerical integration.

Figure 2 presents optimal investment strategies, x^*, as a function of the reference return, $\hat{r}$, for the green-risky case, namely $s = 5 > s^0 = 1$. The weights of the ESG score function are $\alpha = 0, 0.1, 0.15, 0.2$, that is we consider an ESG-neutral investor and investors who differ in the relative importance they assign to ESG preferences, with the shape parameters being $\gamma = 0.1$ and $\delta = 0.5$. Our simulations suggest that, when $\gamma < \delta$, the optimal investment exhibits a specific relationship with the reference return: there exists

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a threshold reference return $\hat{r}_T(\alpha)$ such that[30]

$$\frac{dx^*}{d\hat{r}} \begin{cases} < 0 \ , & \text{for } \ \hat{r} < \hat{r}_T(\alpha) \\ > 0 \ , & \text{for } \ \hat{r} > \hat{r}_T(\alpha) \end{cases} \tag{42}$$

This holds both for the green-risky case and for the green-risk-free case, discussed below. We distinguish between cases in which loss aversion λ is sufficiently large, $\lambda > \lambda_{L\gamma} \equiv \frac{1}{K_\gamma} + \frac{\alpha c}{1-\alpha} \cdot \frac{(s-s^0)^{1-\gamma}}{\int_{-\infty}^{r^0}(r^0-r)^{1-\gamma} f(r)dr} \left(\frac{\delta-\gamma}{s^0}\right)^{\delta-\gamma} \frac{\gamma^\gamma}{\delta^\delta}$, and those in which it is sufficiently small, $\frac{1}{K_\gamma} < \lambda < \lambda_{L\gamma}$. For sufficiently high loss aversion, the optimal portfolio consists entirely of the risk-free asset when $\hat{r} = r^0$.[31]

The left graph illustrates the case of higher loss aversion, with $\lambda = 2.25 > \lambda_{L\gamma}$, in which the optimal investment retains the V-shaped pattern observed in the ESG-neutral case, and risk-taking becomes zero for $\hat{r}_T(\alpha) = r^0$, see Equation (17) of Proposition 3.3. The right graph illustrates the case of lower loss aversion, with $\lambda = 1.8 \in \left(\frac{1}{K_\gamma}, \lambda_{L\gamma}\right)$, and we again observe that the optimal investment strategy x^* decreases for $\hat{r} < \hat{r}_T(\alpha)$ and increases for $\hat{r} > \hat{r}_T(\alpha)$, see (42).[32] For any fixed reference return $\hat{r}$ we observe that risk taking increases in the weight of the ESG score function α which is consistent with the findings of Proposition 3.4, see Equation (31).

Our simulations also show that for $s > s^0$, $\gamma < \delta$, and $\frac{1}{K_\gamma} < \lambda < \lambda_{L\gamma}$, the threshold reference return $\hat{r}_T(\alpha)$ decreases in α and that $\hat{r}_T(\alpha) < r^0 = \hat{r}_T(0)$ for $\alpha > 0$. Thus, in the green-risky case (when the degree of loss aversion is in a certain interval), a higher weight on the ESG score function shifts the threshold reference return $\hat{r}_T(\alpha)$ to the left and thus widens the range of reference returns over which investment in the risky asset is increasing, see Figure 2. This differs from the ESG-neutral case when x^* equals zero for $\hat{r}_T(0) = r^0$, which is in line with the corresponding graph for the larger degree of loss aversion (see the left figure in Figure 2).

Figure B.1 in Appendix Appendix B shows optimal investment strategies, x^*, as a function of the reference return, $\hat{r}$, in the green-risky case, when the risky asset's return follows a skewed rather than a normal distribution. The left panel presents the evolution of the investor's risk-taking with respect to her degree of ambition, $\hat{r}$, when her loss aversion is sufficiently large, i.e., $\lambda > \lambda_{L\gamma}$. For less ambitious investors, $\hat{r} < r^0$, risk-taking implied by the negatively skewed distribution is smaller than risk taking implied by the symmetric distribution, i.e., $x^*_{neg} < x^*_{sym}$, while for more ambitious investors, $\hat{r} > r^0$, risk taking implied by the negatively skewed distribution is larger than risk taking implied by the symmetric distribution, i.e., $x^*_{neg} > x^*_{sym}$, given all other parameters are fixed. Thus, if one falsely assumes symmetric returns when in fact they are negatively skewed, one overestimates the investment in the risky asset for less ambitious investors; and underestimates investment in the risky asset for more ambitious investors. The positive skewness of the return of risky assets has the opposite effect on risk taking. Namely, $x^*_{neg} < x^*_{sym} < x^*_{pos}$ for $\hat{r} < r^0$ and $x^*_{neg} > x^*_{sym} > x^*_{pos}$ for $\hat{r} > r^0$. Similar, though not identical, results hold for investors with

[30] As in the following discussions we compare optimal investments of investors with different ESG weights α we write $\hat{r}_T \equiv \hat{r}_T(\alpha)$.

[31] Note that $\lambda > \lambda_{L\gamma}$ is the condition for $x^* = 0$ and $\frac{1}{K_\gamma} < \lambda < \lambda_{L\gamma}$ is the condition for $x^* > 0$, when $s > s^0$ and $\hat{r} = r^0$. See (17) in Proposition 3.3.

[32] Note that the values of λ, namely 1.8 and 2.25, are in line with the estimates of loss aversion provided in Van Dolder and Vandenbroucke (2024); see their Figure 3.

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a sufficiently low degree of loss aversion, $\frac{1}{K_\gamma} < \lambda < \lambda_{L\gamma}$, see the right panel of Figure B.3.

Figure 3 presents optimal investment strategies x^* as a function of the reference return $\hat{r}$ for the green-risk-free case such that $s_{T\gamma} < s < s^0$, where $s_{T\gamma}$ is defined in (10). The weights of the ESG score function α, the distribution of the risky asset's return r, and the shape parameters δ and γ are chosen as described above. We again distinguish between the cases when loss aversion λ is sufficiently large, $\lambda > \frac{1}{K_\gamma}$, and when λ is sufficiently small, $\lambda < \frac{1}{K_\gamma}$.[33] The left panel applies to a sufficiently large loss aversion of $\lambda = 2.25 > \frac{1}{K_\gamma}$, where the optimal investment behavior remains V-shaped. Risk-taking becomes zero for $\hat{r}_T(\alpha) = r^0$ at all values of α, see Equation (17) of Proposition 3.4. The right panel applies to sufficiently small loss aversion, $\lambda = 1.3 < \frac{1}{K_\gamma}$, and we again observe property (42), i.e., the optimal investment strategy x^* decreases for $\hat{r} < \hat{r}_T(\alpha)$ and increases for $\hat{r} > \hat{r}_T(\alpha)$. For any fixed reference return $\hat{r}$ we observe that risk taking decreases in the weight of the ESG score function, α, which is consistent with the findings of Proposition 3.4, in particular Equation (31). This holds for both small and large levels of loss aversion λ and is opposite to the observation in the green-risky case, where risk taking increases in α. In addition, risk-taking can become discontinuous in $\hat{r}$. In particular, we observe that for that case $\mathbb{E}\,(U(x))$ has two local maximima in x which causes jumps in the optimal x^*. Figure B.2 in Appendix Appendix B illustrates numerically that, for sufficiently small α in this case, the expected utility $\mathbb{E}\,(U(x))$ has two local maxima. The global maximum is attained either at the local maximum associated with low risk-taking or at the one associated with high risk-taking. The former occurs at lower reference returns, whereas the latter occurs at higher reference returns.

Note finally that in the green-risk-free case, $s < s^0$, and sufficiently small loss aversion, $\lambda < \frac{1}{K_\gamma}$, the threshold reference return $\hat{r}_T(\alpha)$ becomes an increasing function in α while still $\hat{r}_T(\alpha) < \hat{r}_T(0) = r^0$ for $\alpha > 0$, see the right panel in Figure 3. Thus, in this case a larger weight on the ESG score function shifts the threshold reference return $\hat{r}_T(\alpha)$ to the right and thus narrows the range of reference returns over which investment in the risky asset is increasing.[34]

Regarding the effects of skewed versus symmetric returns in the green-risk-free case, Figure B.3 shows that the effects of skewness are similar to those observed in the green-risky case.

[33] Note that $\lambda > \frac{1}{K_\gamma}$ is the condition for $x^* = 0$ and $\lambda < \frac{1}{K_\gamma}$ is the condition for $x^* > 0$ when $s < s^0$ and $\hat{r} = r^0$. See (17) in Proposition 3.3.

[34] We conducted simulations also for $\delta > 1$ and obtained results similar to the ones presented in Figures 2 and 3. These results are presented in Figures B.4 and B.5 in Appendix Appendix B.

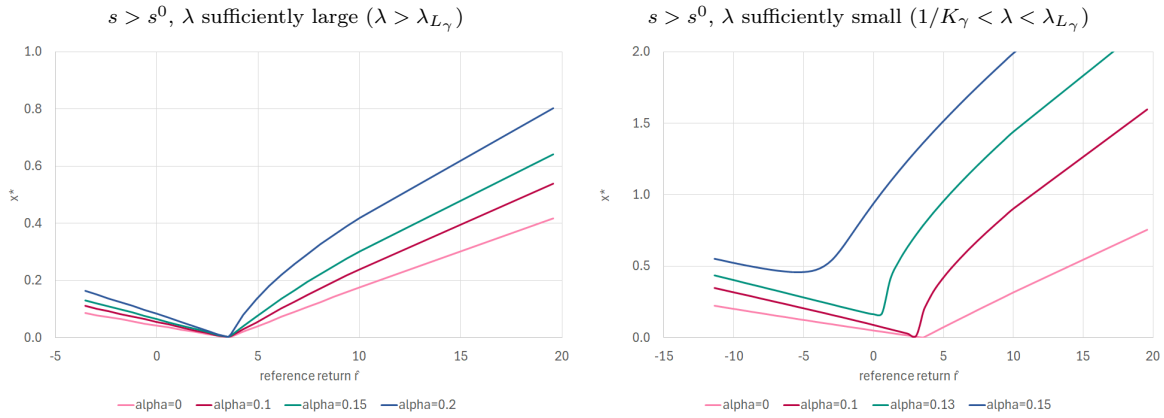


Figure 2: Risk-taking, x^*, as a function of the reference return, $\hat{r}$, for $\lambda = 2.25$ (left), $\lambda = 1.8$ (right), $\gamma = 0.1$, $\delta = 0.5$, $s^0 = 1$, $s = 5$, and $\alpha = 0, 0.1, 0.13, 0.15, 0.2$.
$\frac{1}{K_\gamma} = 1.56$ and $\lambda_{L\gamma} = 1.56, 1.81, 1.9, 1.96$ and 2.13 for $\alpha = 0$, 0.1, 0.13, 0.15, 0.2, where $\lambda_{L\gamma} = \frac{1}{K_\gamma} + \frac{\alpha c}{1-\alpha} \cdot \frac{(s-s^0)^{1-\gamma}}{\int_{-\infty}^{r^0}(r^0-r)^{1-\gamma}f(r)dr}\left(\frac{\delta-\gamma}{s^0}\right)^{\delta-\gamma}\frac{\gamma^\gamma}{\delta^\delta}$.
The horizontal axis shows the reference return $\hat{r}$ in percent. The results are shown for normally distributed returns of the risky asset, where $\mathbb{E}(r) = 13.6\%$, $Var(r) = 18.4\%$ and $r^0 = 3.5\%$.

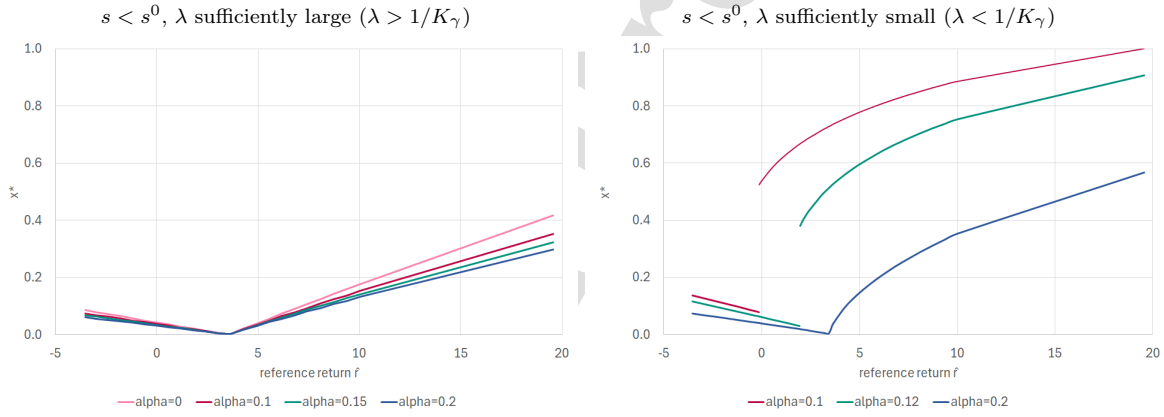


Figure 3: Risk-taking, x^*, as a function of the reference return, $\hat{r}$, for $\lambda = 2.25$ (left), $\lambda = 1.3$ (right), $\gamma = 0.1$, $\delta = 0.5$, $s^0 = 5$, $s = 1$, and $\alpha = 0, 0.1, 0.12, 0.15, 0.2$.
$\frac{1}{K_\gamma} = 1.56$, $s_{T\gamma} < 0$ and in the right graph $x^* = +\infty$ for $\alpha = 0$.
The horizontal axis shows the reference return $\hat{r}$ in percent. The results are shown for normally distributed returns of the risky asset, where $\mathbb{E}(r) = 13.6\%$, $Var(r) = 18.4\%$ and $r^0 = 3.5\%$.

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5. Conclusions

This article derives optimal investment strategies for an investor with preferences combining prospect theory and an ESG utility. We perform a comparative static analysis to show how optimal investment depends on the ESG scores of the risk-free and the risky asset, the weight of the ESG score function, loss aversion, and the reference return, respectively. In addition, this article contributes to the literature by explicitly accounting for the ESG score of the risk-free asset. We find that some results depend on wether the risky asset is greener than the risk-free asset or the risk-free asset is greener than the risky asset. For example, optimal investment in the risky asset is smaller than ESG-neutral investment when the risk-free asset is greener while it is larger than ESG-neutral investment when the risky asset is greener. Our comparative-statics results show that the effect of ESG scores on asset demand is not necessarily positive. An improvement in the risky asset's ESG score unambiguously increases its demand when the risky asset is less green than the risk-free asset, whereas, when it is greener, sufficiently strong diminishing sensitivity of the ESG score function can reverse this effect. Changes in the risk-free asset's ESG score can likewise have either sign, depending on leverage and the strength of diminishing sensitivity. Overall, our results highlight that ESG portfolio choice depends on the interaction between relative ESG attractiveness and diminishing sensitivity, rather than on ESG scores alone. In addition, investment into the greener asset increases when the relative importance of the ESG part in the decision maker's preferences increases.

Furthermore, investment into the risky asset decreases with an increasing level of loss aversion. For some special cases (e.g., a bilinear value function) a V-shaped relationship of the optimal investment in the risky asset and the reference return is derived. In the more general case, this property is also observed numerically, for sufficiently loss averse investors. The pattern in which investment in the risky asset first decreases and then increases with the references return is also observed in the simulations for less loss averse investors. A key difference from investment with sufficiently large loss aversion and from ESG-neutral investment is that, at the minimum of the optimal investment in the risky asset with respect to the reference return, the investment in the risky asset remains strictly positive. By contrast, in the V-shaped cases mentioned above, the entire investment is allocated to the risk-free asset at the minimum. Certain behavioral differences can be observed for different relative greenness. Finally, we observe that happiness decreases with increasing level of ambition.

Due to the complexity of the problem, this article considers an economy in which a risk-free asset and a single risky asset are traded. Even in this simple economy several distinct cases arise. A natural extension would be to consider an economy with multiple risk assets and investigate which of the properties established in the-two asset case continue to hold. Another extension would be to investigate the extent to which the theoretical results are supported by empirical evidence.

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References

[1] L. An, H. Wang, J. Wang, J. Yu. 2020. Lottery-related anomalies: The role of reference-dependent preferences. *Management Science*, 66, 473–501.

[2] Z. Ao, X. Ji, X. Liang. 2023. Can prospect theory explain anomalies in the Chinese stock market? *Finance Research Letters*, 58, 104466.

[3] D. Avramov, S. Cheng, A. Lioui, A. Tarelli. 2022. Sustainable investing with ESG rating uncertainty. *Journal of Financial Economics*, 145, 642–664.

[4] N. Barberis, M. Huang, J. Santos. 2001. Prospect theory and asset prices. *Quarterly Journal of Economics*, 116, 1–53.

[5] N. Barberis, A. Mukherjee, B. Wang. 2016. Prospect theory and stock returns: An empirical test. *Review of Financial Studies*, 29, 3068–3107.

[6] N. Barberis, L. Jin, B. Wang. 2021. Prospect theory and stock market anomalies. *Journal of Finance*, 76, 2639–2687.

[7] G. Bekaert, R. Rothenberg, M. Noguer. 2023. Sustainable investment - exploring the linkage between alpha, ESG, and DSGs. *Sustainable Development*, 31, 3831–3842.

[8] C. Bernard, M. Ghossoub. 2010. Static portfolio choice under cumulative prospect theory. *Mathematics and Financial Economics*, 2, 277-–306.

[9] A.S. Booij, G. van de Kuilen. 2009. A parameter-free analysis of the utility of money for the general population under prospect theory. *Journal of Economic Psychology*, 30, 651–666.

[10] R. Boffo, R. Patalano. 2020. ESG Investing: Practices, Progress and Challenges, OECD Paris. `www.oecd.org/finance/ESG-Investing-Practices-Progress-and-Challenges.pdf`

[11] N.P. Bollen. 2007. Mutual fund attributes and investor behavior. *Journal of Financial and Quantitative Analysis*, 42, 683–708.

[12] P. Bolton, M. Kacperczyk. 2021. Do investors care about carbon risk? *Journal of Financial Economics*, 142, 517–549.

[13] T. Busch, G. Friede. 2018. The robustness of the corporate social and financial performance relation: A second-order meta-analysis. *Corporate Social Responsibility and Environmental Management*, 25, 583-–608.

[14] B. Boumbda, D. Duxbury, C. Ortiz, L. Vicente. 2021. Do socially responsible investment funds sell losses and ride gains? The disposition effect in SRI funds. *Sustainability*, 13, 8142.

[15] T. Dangl, M. Halling, J. Yu, J. Zechner. 2025. Social preferences and corporate investment. *Journal of Financial Economics*, 172, 104139.

[16] A. Diaz, C. Esparcia, D. Alonso, M-T. Alonso. 2024. Portfolio management of ESG-labeled energy companies based on PTV and ESG factors. *Energy Economics*, 134, 107545.

Journal Pre-proof

[17] D. van Dolder, J. Vandenbroucke. 2024. Behavioral risk profiling: Measuring loss aversion of individual investors. *Journal of Banking and Finance*, 168, 107293.

[18] B. van Dooren, R. Galema. 2018. Socially responsible investors and the disposition effect. *Journal of Behavioral and Experimental Finance*, 17, 42–52.

[19] D. Dorn, G. Strobl. 2023. Rational disposition effects: Theory and evidence. *Journal of Banking and Finance*, 153, 106858.

[20] B. Drut. 2010. Social responsibility and mean-variance portfolio selection, CEB Working papers No. 10/002, Universite Libre de Bruxelles.

[21] C. Flammer. 2021. Corporate green bonds. *Journal of Financial Economics*, 142, 499–516.

[22] G. Friede, T. Busch, A. Bassen. 2015. ESG and financial performance: Aggregated evidence from more than 2000 empirical studies. *Journal of Sustainable Finance and Investment*, 5, 210-–233.

[23] I. Fortin, J. Hlouskova. 2011. Optimal asset allocation under linear loss aversion. *Journal of Banking and Finance*, 35, 2974–2990.

[24] I. Fortin, J. Hlouskova. 2024. Prospect theory and asset allocation. *Quarterly Review of Economics and Finance*, 94, 214–240.

[25] S.M. Gasser, M. Rammerstorfer, K. Weinmayer. 2017. Markowitz revisited: Social portfolio engineering. *European Journal of Operational Research*, 258, 1181–1190.

[26] J. Guo, L. Schönleber. 2020. Investor behavior under prospect theory: Evidence from mutual funds. Mimeo, available at SSRN:
`https://ssrn.com/abstract=3754814orhttp://dx.doi.org/10.2139/ssrn.3754814`

[27] R.D.F. Harris, M. Mazibas. 2022. Portfolio optimization with behavioural preferences and investor memory. *European Journal of Operational Research*, 296, 368–387.

[28] S.M. Hartzmark, A.B. Sussman. 2019. Do investors value sustainability? A natural experiment examining ranking and fund flows. *The Journal of Finance*, 74, 789–2837.

[29] F. Heeb, J.F. Kölbel, F. Paetzold, S. Zeisberger. 2023. Do investors care about impact? *Review of Financial Studies*, 36, 1737–1787.

[30] R. Heinkel, A. Kraus, J. Zechner. 2001. The effect of green investment on corporate behavior. *Journal of Financial and Quantitative Analysis*, 36, 431–449.

[31] J.F. Houston, H. Shan. 2022. Corporate ESG profiles and banking relationships. *The Review of Financial Studies*, 35, 3373–3417.

[32] D. Kahneman, A. Tversky. 1979. Prospect theory: An analysis of decision under risk. *Econometrica*, 47, 363–391.

[33] M. Kaufmann, P. Andre., B. Köszegi. 2024. Understanding markets with socially responsible consumers. *Quarterly Journal of Economics*, 139, 1989–2035.

Journal Pre-proof

[34] P. Krueger, Z. Sautner, L.T. Starks. 2020. The importance of climate risks for institutional investors. *The Review of Financial Studies*, 33, 1067–1111.

[35] M. Kwak, T.A. Pirvu. 2018. Cumulative prospect theory with generalized hyperbolic skewed t distribution. *SIAM Journal of Financial Mathematics*, 9, 54–89.

[36] H. Liu, C. Peng, W.A. Xiong, W. Xiong. 2022. Taming the bias zoo. *Journal of Financial Economics*, 143, 716-–741.

[37] P. Matos. 2020. *ESG and responsible institutional investing around the world: A critical review.* CFA Institute Research Foundation.

[38] M. Ottoni-Wilheim, L. Vesterlund, H. Xie. 2017. Why do people give? Testing pure and impure altruism. *American Economic Review*, 107, 3617–3633.

[39] L. Pastor, R.F. Stambaugh, L.A. Taylor. 2021. Sustainable investing in equilibrium. *Journal of Financial Economics*, 146, 403–424.

[40] L.H. Pedersen, S. Fitzgibbons, L. Pomorski. 2021. Responsible investing: The ESG-efficient frontier. *Journal of Financial Economics*, 142, 572–597.

[41] Y. Qi, X. Li. 2020. On imposing ESG constraints of portfolio selection for sustainable investment and comparing the efficient frontiers in the weight space. *SAGE Open*, 10, 2158244020975070.

[42] A. Riedl., P. Smeets. 2017. Why do investors hold socially responsible mutual funds? *Journal of Finance*, 72, 2505–2550.

[43] H. Shefrin, M. Statman. 1985. The disposition to sell winners too early and ride losers too long: Theory and evidence. *Journal of Finance*, 40, 777—790.

[44] A. Tversky, D. Kahneman. 1992. Advances in prospect theory: Cumulative representation of uncertainty. *Journal of Risk and Uncertainty*, 5, 297–323.

[45] J. Wang, C. Wu, X. Zhong. 2021. Prospect theory and stock returns: Evidence from foreign share markets. *Pacific-Basin Finance Journal*, 69, 101644.

[46] B. Yu, Y. Shen, X. Jin, Q. Xu. 2022. Does prospect theory explain mutual fund performance? Evidence from China. *Pacific-Basin Finance Journal*, 73, 101766.

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Appendix A. Appendix: Proofs

Lemma Appendix A.1. *Let* $\mathbb{E}(|r|) < +\infty$, $0 < F(c) < 1$, *and* $c, z \in \mathbb{R}$. *Then the following holds*

$$0 < \int_{-\infty}^{z} (z-r)^{1-\gamma} f(r)dr < +\infty, \quad \textit{and} \tag{A.1}$$

$$0 < \int_{z}^{+\infty} (r-z)^{1-\gamma} f(r)dr < +\infty. \tag{A.2}$$

Proof of Lemma Appendix A.1: The finiteness of integrals is a consequence of the assumption $\mathbb{E}(|r|) = \int_{-\infty}^{+\infty} |r| f(r)dr < +\infty$ and inequality $|r-z|^{1-\gamma} \leq C_\gamma(1+|r|+|z|)$, for any r, z, where C_γ is a constant that depends only on $\gamma \in [0,1)$. Then, $\int_{-\infty}^{z}(z-r)^{1-\gamma} f(r)dr \leq C_\gamma \int_{-\infty}^{z}(1+|r|+|z|)f(r)dr = C_\gamma \left[(1+|z|)F(z) + \int_{-\infty}^{+\infty}|r|f(r)dr\right] < +\infty$. Similarly, $\int_{z}^{+\infty}(r-z)^{1-\gamma} f(r)dr \leq C_\gamma \int_{z}^{+\infty}(1+|r|+|z|)f(r)dr =$
$C_\gamma \left[(1+|z|)(1-F(z)) + \int_{-\infty}^{+\infty}|r|f(r)dr\right] < +\infty$.

We show that the integrals are strictly positive by contradiction. Let $\Phi(z) \equiv \int_{-\infty}^{z}(z-r)^{1-\gamma} f(r)dr$ and let us assume that there exists $z_0 \in \mathbb{R}$ such that $\Phi(z_0) = 0$. This can happen only when $f(r) = 0$ for almost any $r \leq z_0$. Then $1 - F(z_0) = \int_{z_0}^{+\infty} f(r)dr = \int_{-\infty}^{+\infty} f(r)dr = 1$ which contradicts assumption that $F(c) > 0$ for any $c \in \mathbb{R}$. Similarly, let $\Phi(z) \equiv \int_{z}^{+\infty}(r-z)^{1-\gamma} f(r)dr$ and let us assume that there exists $z_0 \in \mathbb{R}$ such that $\Phi(z_0) = 0$. This can happen only when $f(r) = 0$ for any $r \geq z_0$. Then $F(z_0) = \int_{-\infty}^{z_0} f(r)dr = \int_{-\infty}^{+\infty} f(r)dr = 1$ which contradicts assumption that $F(c) < 1$ for any $c \in \mathbb{R}$. □

Proof of Proposition 3.1: Note that based on the Leibniz integral rule the derivative of $\mathbb{E}(U(x))$ with respect to x is

$$\frac{d}{dx}\mathbb{E}(U(x)) = \begin{cases} \alpha c \frac{s-s^0}{[s^0+x(s-s^0)]^\delta} + \frac{1-\alpha}{x^\gamma}\left[\lambda \int_{-\infty}^{z(x,\hat{r})} \frac{r-r^0}{[z(x,\hat{r})-r]^\gamma} f(r)dr + \int_{z(x,\hat{r})}^{+\infty} \frac{r-r^0}{[r-z(x,\hat{r})]^\gamma} f(r)dr\right] = \\ \alpha c \frac{s-s^0}{[s^0+x(s-s^0)]^\delta} + (1-\alpha)\left[\lambda \int_{-\infty}^{z(x,\hat{r})} \frac{r-r^0}{[x(r^0-r)+\hat{r}-r^0]^\gamma} f(r)dr + \int_{z(x,\hat{r})}^{+\infty} \frac{r-r^0}{[x(r-r^0)-\hat{r}+r^0]^\gamma} f(r)dr\right], \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x > 0 \\ \alpha c \frac{s-s^0}{(s^0)^\delta} + (1-\alpha)\frac{\mathbb{E}(r)-r^0}{(r^0-\hat{r})^\gamma}, \quad x = 0, \quad \hat{r} < r^0 \\ \alpha c \frac{s-s^0}{(s^0)^\delta} + (1-\alpha)\lambda\frac{\mathbb{E}(r)-r^0}{(\hat{r}-r^0)^\gamma}, \quad x = 0, \quad \hat{r} > r^0 \end{cases} \tag{A.3}$$

where $z(x,\hat{r}) = r^0 + \frac{\hat{r}-r^0}{x}$. For $s > s^0$ is $\frac{d}{dx}\mathbb{E}(U(x))\Big|_{x=0} > 0$ as $\mathbb{E}(r) > r^0$ and thus for any $\hat{r} \neq r^0$ is

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$\mathbb{E}(U(x))$ increasing in zero. Note in addition that based on (6) we obtain the following for $\gamma < \delta < 1$

$$\begin{aligned}
\lim_{x\to+\infty} \mathbb{E}(U(x)) &= \lim_{x\to+\infty} x^{1-\gamma} \left\{ \frac{\alpha c}{1-\delta} \left[\frac{s^0 + x(s-s^0)}{x^{\frac{1-\gamma}{1-\delta}}} \right]^{1-\delta} + \right. \\
&\quad \left. + \frac{1-\alpha}{1-\gamma} \left[-\lambda \int_{-\infty}^{z(x,\hat{r})} \left(z(x,\hat{r}) - r\right)^{1-\gamma} f(r)dr + \int_{z(x,\hat{r})}^{+\infty} \left(r - z(x,\hat{r})\right)^{1-\gamma} f(r)dr \right] \right\} \\
&= +\infty \times \left[0 + \frac{1-\alpha}{1-\gamma} \left(-\lambda \int_{-\infty}^{r^0} \left(r^0 - r\right)^{1-\gamma} f(r)dr + \int_{r^0}^{+\infty} \left(r - r^0\right)^{1-\gamma} f(r)dr \right) \right] \\
&= +\infty \times \underbrace{\left[-\frac{1-\alpha}{1-\gamma} \left(\lambda - \frac{1}{K_\gamma} \right) \int_{-\infty}^{r^0} \left(r^0 - r\right)^{1-\gamma} f(r)dr \right]}_{<0} \\
&= -\infty
\end{aligned} \tag{A.4}$$

as $\lambda > 1/K_\gamma$. For $\delta > 1$ the limit of ESG score function becomes

$$\lim_{x\to+\infty} -\frac{1}{(\delta-1)[s^0 + x(s-s^0)]^{\delta-1}} = 0 \tag{A.5}$$

and thus (A.4) applies also in this case. For $\delta = 1$ we obtain the following using L'Hôspital's rule

$$\lim_{x\to+\infty} \frac{\ln(s^0 + x(s-s^0))}{x^{1-\gamma}} = \frac{s-s^0}{1-\gamma} \lim_{x\to+\infty} \frac{x^\gamma}{s^0 + x(s-s^0)} = 0 \tag{A.6}$$

Thus (A.4) holds also when $\delta = 1$. For $\gamma = \delta$ we obtain the same result for a more loss averse investor, namely for an investor with

$$\lambda > \frac{1}{K_\gamma} + \frac{\alpha c}{1-\alpha} \cdot \frac{(s-s^0)^{1-\gamma}}{\int_{-\infty}^{r^0} (r^0 - r)^{1-\gamma} f(r)dr} \equiv \lambda_{L\gamma}$$

as then

$$\begin{aligned}
\lim_{x\to+\infty} \mathbb{E}(U(x)) &= +\infty \times \frac{1}{1-\gamma} \left[\alpha c (s-s^0)^{1-\gamma} - (1-\alpha) \left(\lambda - \frac{1}{K_\gamma} \right) \int_{-\infty}^{r^0} \left(r^0 - r\right)^{1-\gamma} f(r)dr \right] \\
&= -\infty
\end{aligned} \tag{A.7}$$

As, $\mathbb{E}(U(x))$ is continuous (also at zero), differentiable, increasing at zero, see (A.3) for $x = 0$, and achieving $-\infty$ in infinity, see (A.4), then it follows that the solution x^* of (4) is strictly positive and such that the first order conditions are satisfied, i.e., $\frac{d}{dx}\mathbb{E}(v(R(x^*))) = 0$ for $x^* > 0$, which coincides with (11).

Note that if $\hat{x}$ is the point where $\mathbb{E}(U(x))$ reaches its maximum when $\alpha = 0$, i.e., $\frac{d\mathbb{E}(V(R(x)))}{dx}\big|_{x=\hat{x}} = 0$ then

$$\frac{d\mathbb{E}(U(x))}{dx}\big|_{x=\hat{x}} = \alpha c \frac{s-s^0}{[s^0 + \hat{x}(s-s^0)]^\delta} + (1-\alpha)\frac{d\mathbb{E}(V(R(x)))}{dx}\big|_{x=\hat{x}} = \alpha c \frac{s-s^0}{[s^0 + \hat{x}(s-s^0)]^\delta} + 0 > 0$$

as $\alpha, c > 0$ and $s > s^0$. Then there exist $\varepsilon > 0$ such that for all $x \in (\hat{x}, \hat{x} + \varepsilon)$ is $\mathbb{E}(U(x)) > \mathbb{E}(U(\hat{x}))$ and

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thus $\mathbb{E}(U(x^*)) > \mathbb{E}(U(\hat{x}))$ and $x^* > \hat{x}$.

Note finally, that (A.4) and (A.7) imply that for $\lambda < \lambda_{L\gamma}$, where $\lambda_{L\gamma}$ is given by (9), and $\gamma \leq \delta$ is $\lim_{x\to+\infty} \mathbb{E}(U(x)) = +\infty$ and thus $x^* = +\infty$. As the expected utility function is continuous then $x^* = +\infty$ is a unique solution.

If $\gamma > \delta$ and $s > s^0$ then the following holds

$$\begin{aligned}\lim_{x\to+\infty} \mathbb{E}(U(x)) &= \lim_{x\to+\infty} x^{1-\gamma} \left\{ \frac{\alpha c \left(\frac{s^0}{x} + s - s^0\right)^{1-\delta}}{1-\delta} x^{\gamma-\delta} + \frac{1-\alpha}{1-\gamma} \left[-\lambda \int_{-\infty}^{z(x,\hat{r})} (z(x,\hat{r}) - r)^{1-\gamma} f(r) dr \right.\right. \\ &\quad \left.\left. + \int_{z(x,\hat{r})}^{+\infty} (r - z(x,\hat{r}))^{1-\gamma} f(r) dr \right] \right\} \\ &= +\infty \times \left[+\infty + \frac{1-\alpha}{1-\gamma} \left(-\lambda + \frac{1}{K_\gamma} \right) \int_{-\infty}^{r^0} \left(r^0 - r\right)^{1-\gamma} f(r) dr \right] \\ &= +\infty \end{aligned} \tag{A.8}$$

as $\int_{-\infty}^{r^0} \left(r^0 - r\right)^{1-\gamma} f(r) dr$ is finite, see Lemma Appendix A.1, and where $z(x,\hat{r}) = r^0 + \frac{\hat{r}-r^0}{x}$. Thus $x^* = +\infty$ when $\gamma > \delta$.

Note that when $s < s^0$ then the expected utility is defined on $[0, \frac{s^0}{s^0-s}]$ when $\delta > 0$.[35] The structure of the proof is as follows. At first we show that $\frac{d}{dx}\mathbb{E}(U(x))\Big|_{x=0} > 0$ and then $\lim_{x\to\left(\frac{s^0}{s^0-s}\right)^-} \frac{d}{dx}\mathbb{E}(U(x)) = -\infty < 0$ for $\delta > 0$. This implies that there is at least one global maximum of (4) in $[0, \frac{s^0}{s^0-s}]$ which satisfies the FOC (11) as $\mathbb{E}(U(x))$ is continuously differentiable. Note that (A.3) implies that for $\hat{r} < r^0$ is

$$\frac{d}{dx}\mathbb{E}(U(x))\Big|_{x=0} = \alpha c \frac{s - s^0}{(s^0)^\delta} + (1-\alpha)\frac{\mathbb{E}(r) - r^0}{(r^0 - \hat{r})^\gamma}$$

and thus $\frac{d}{dx}\mathbb{E}(U(x))\Big|_{x=0} > 0$ when $s > s^0 - \frac{1-\alpha}{\alpha c} \cdot \frac{(s^0)^\delta}{(r^0-\hat{r})^\gamma} \left[\mathbb{E}(r) - r^0\right] \equiv s_{T\gamma}$, see (10).[36] In a similar way, (A.3) implies that for $\hat{r} > r^0$ we obtain

$$\frac{d}{dx}\mathbb{E}(U(x))\Big|_{x=0} = \alpha c \frac{s - s^0}{(s^0)^\delta} + (1-\alpha)\lambda\frac{\mathbb{E}(r) - r^0}{(\hat{r} - r^0)^\gamma}$$

and thus $\frac{d}{dx}\mathbb{E}(U(x))\Big|_{x=0} > 0$ when $s > s^0 - \lambda\frac{1-\alpha}{\alpha c} \cdot \frac{(s^0)^\delta}{(\hat{r}-r^0)^\gamma} \left[\mathbb{E}(r) - r^0\right] \equiv s_{T\gamma}$, see (10).[37] Note now that (A.3) implies that for $\delta > 0$ we obtain $\lim_{x\to\left(\frac{s^0}{s^0-s}\right)^-} \frac{d}{dx}\mathbb{E}(U(x)) = -\infty < 0$ and thus there exists a positive solution $0 < x^* < \frac{s^0}{s^0-s}$ which satisfies the FOC (11).[38]

Note finally that if $\hat{x}$ is the point where $\mathbb{E}(U(x))$ reaches its maximum when $\alpha = 0$,[39] i.e., $\frac{d\mathbb{E}(V(R(x)))}{dx}\Big|_{x=\hat{x}} =$

[35] For $\delta = 0$ the expected utility is defined on $[0, +\infty)$.

[36] This is equivalent to $\mathbb{E}(r) > r^0 + \frac{\alpha c}{1-\alpha} \cdot \frac{(s^0-s)(r^0-\hat{r})^\gamma}{(s^0)^\delta}$.

[37] Note that this is equivalent to $\lambda > \frac{\alpha c}{1-\alpha} \cdot \frac{(s^0-s)(\hat{r}-r^0)^\gamma}{(s^0)^\delta(\mathbb{E}(r)-r^0)}$.

[38] This also applies for $\delta = 0$ and $\gamma > 0$ as then $\lim_{x\to+\infty} \mathbb{E}(U(x)) = -\infty$.

[39] Note that $\hat{x}$ is finite when $\lambda > \frac{1}{K_\gamma}$. See Fortin and Hlouskova (2024).

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0 then

$$\begin{aligned}\frac{d\mathbb{E}(U(x))}{dx}\Big|_{x=\hat{x}} &= \alpha c\frac{s-s^0}{[s^0+\hat{x}(s-s^0)]^\delta}+(1-\alpha)\frac{d\mathbb{E}(V(R(x)))}{dx}\Big|_{x=\hat{x}}\\ &= -\alpha c\frac{s^0-s}{[s^0+\hat{x}(s-s^0)]^\delta}+0<0\end{aligned}$$

as $\alpha, c > 0$ and $s < s^0$. Then there exist $\varepsilon > 0$ such that for all $x \in (\hat{x}-\varepsilon, \hat{x})$ is $\mathbb{E}(U(x)) > \mathbb{E}(U(\hat{x}))$ and thus $\mathbb{E}(U(x^*)) > \mathbb{E}(U(\hat{x}))$ and $x^* < \hat{x}$. □

Proof of Proposition 3.2: The following proof shows that for $\gamma = 0 \leq \delta$ is the utility function concave and thus the solution is unique. As

$$\mathbb{E}(U(x)) = \begin{cases} \alpha c\frac{[s^0+x(s-s^0)]^{1-\delta}}{1-\delta} \\ +(1-\alpha)\left\{-\lambda\int_{-\infty}^{z(x,\hat{r})}\left[x\left(r^0-r\right)+\hat{r}-r^0\right]f(r)dr+\int_{z(x,\hat{r})}^{+\infty}\left[x\left(r-r^0\right)+r^0-\hat{r}\right]f(r)dr\right\}, \\ \text{for } \ x>0 \\ \\ \alpha c\frac{(s^0)^{1-\delta}}{1-\delta}+(1-\alpha)\left(r^0-\hat{r}\right), \quad \text{for } \ x=0 \ \text{ and } \ \hat{r}\leq r^0 \\ \\ \alpha c\frac{(s^0)^{1-\delta}}{1-\delta}-(1-\alpha)\lambda\left(\hat{r}-r^0\right), \ \text{for } \ x=0 \ \text{ and } \ \hat{r}>r^0 \end{cases} \tag{A.9}$$

where $z(x,\hat{r}) = r^0 + \frac{\hat{r}-r^0}{x}$. Applying Leibniz integral rule and using the condition $\lim_{r\to\pm\infty}|r|f(r) = 0$ we obtain the following for $x > 0$

$$\frac{d\mathbb{E}(U(x))}{dx} = \alpha c\frac{s-s^0}{[s^0+x(s-s^0)]^\delta}+(1-\alpha)\left[-\lambda\int_{-\infty}^{z(x,\hat{r})}\left(r^0-r\right)f(r)dr+\int_{z(x,\hat{r})}^{+\infty}\left(r-r^0\right)f(r)dr\right] \tag{A.10}$$

After again applying the Leibniz integral rule (together with condition $\lim_{r\to\pm\infty}|r|f(r) = 0$) we obtain negative second derivative of $\mathbb{E}(U(x))$, namely

$$\frac{d^2\mathbb{E}(U(x))}{dx^2} = -\alpha c\delta\frac{(s-s^0)^2}{[s^0+x(s-s^0)]^{1+\delta}}-(1-\alpha)(\lambda-1)\frac{(r^0-\hat{r})^2}{x^3}f(z(x,\hat{r}))<0 \tag{A.11}$$

as $\lambda > 1$ which implies that the solution is unique as the expected utility function is concave.

In the next we derive the solution for the case when $\gamma = 0 \leq \delta$ and $\hat{r} \neq r^0$. The closed form solution for the case when $\hat{r} = r^0$ is dealt with in Proposition 3.3. Let

$$\lambda_{L0} = \begin{cases} \frac{1}{K_0}, & \text{for } \ \delta>0 \\ \frac{1}{K_0}+\frac{\alpha c}{1-\alpha}\cdot\frac{s-s^0}{\int_{-\infty}^{r^0}(r^0-r)f(r)dr}, & \text{for } \ \delta=0 \end{cases} \tag{A.12}$$

$$s_{T0} = \begin{cases} s^0-\frac{1-\alpha}{\alpha c}\left(s^0\right)^\delta\left[\mathbb{E}(r)-r^0\right], & \text{for } \ \hat{r}<r^0 \\ \\ s^0-\lambda\frac{1-\alpha}{\alpha c}\left(s^0\right)^\delta\left[\mathbb{E}(r)-r^0\right], & \text{for } \ \hat{r}>r^0 \end{cases} \tag{A.13}$$

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Then for $s_0 > 0$, (A.10) and (A.9) imply the following

$$\begin{aligned} \frac{d\mathbb{E}(U(x))}{dx}\Big|_{x=0} &= \alpha c \frac{s-s^0}{(s^0)^\delta} + k(1-\alpha)\int_{-\infty}^{+\infty}\left(r-r^0\right)f(r)dr \\ &= \begin{cases} >0 & \text{for } s > s_{T0} \\ <0 & \text{for } s < s_{T0} \end{cases} \end{aligned} \tag{A.14}$$

as $z(0,\hat{r}) = -\infty$ when $\hat{r} < r^0$ and $z(0,\hat{r}) = +\infty$ when $\hat{r} > r^0$, and were s_{T0} is given by (A.13). Note that for $s^0 = 0$ is $\frac{d\mathbb{E}(U(x))}{dx}\Big|_{x=0} = +\infty$. In addition for $\delta > 0$ and $s > s^0$ is[40]

$$\begin{aligned} \mathbb{E}(U(x))\Big|_{x=+\infty} &= \lim_{x\to+\infty} x\left\{\frac{\alpha c}{1-\delta}\left(\frac{s^0}{x}+s-s^0\right)^{1-\delta}\frac{1}{x^\delta} + (1-\alpha)\left[-\lambda\int_{-\infty}^{z(x,\hat{r})}\left(r^0-r+\frac{\hat{r}-r^0}{x}\right)f(r)dr\right.\right. \\ &\quad \left.\left. + \int_{z(x,\hat{r})}^{+\infty}\left(r-r^0+\frac{r^0-\hat{r}}{x}\right)f(r)dr\right]\right\} \\ &= +\infty\cdot\left[-(1-\alpha)\int_{-\infty}^{r^0}(r^0-r)f(r)dr\left(\lambda-\frac{1}{K_0}\right)\right] \\ &= \begin{cases} -\infty & \text{for } \lambda > \frac{1}{K_0} \\ +\infty & \text{for } \lambda < \frac{1}{K_0} \end{cases} \end{aligned} \tag{A.15}$$

and based on Lemma Appendix A.1 and (A.10)

$$\frac{d\mathbb{E}(U(x))}{dx}\Big|_{x=\frac{s^0-s}{s^0}} = -\infty \quad \text{for } s < s^0 \text{ and } \delta > 0 \tag{A.16}$$

For $\delta = 0$ is

$$\begin{aligned} \mathbb{E}(U(x))\Big|_{x=+\infty} &= \lim_{x\to+\infty} x\left\{\alpha c\left(\frac{s^0}{x}+s-s^0\right) + (1-\alpha)\left[-\lambda\int_{-\infty}^{z(x,\hat{r})}\left(r^0-r+\frac{\hat{r}-r^0}{x}\right)f(r)dr\right.\right. \\ &\quad \left.\left. + \int_{z(x,\hat{r})}^{+\infty}\left(r-r^0+\frac{r^0-\hat{r}}{x}\right)f(r)dr\right]\right\} \\ &= +\infty\cdot\left[\alpha c(s-s^0) - (1-\alpha)\int_{-\infty}^{r^0}(r^0-r)f(r)dr\left(\lambda-\frac{1}{K_0}\right)\right] \\ &= \begin{cases} -\infty & \text{for } \lambda > \lambda_{L0} \\ +\infty & \text{for } \lambda < \lambda_{L0} \end{cases} \end{aligned} \tag{A.17}$$

Based on proof of Proposition 3.1 it can be easily shown that $x^* > \hat{x}$ for $s > s^0$ and $x^* < \hat{x}$ for $s < s^0$. In addition, $x^* < \min\left\{\hat{x}, \frac{s_0}{s_0-s}\right\}$ for $s < s^0$ and $\delta > 0$.

As $\mathbb{E}(r) > r^0$, $\mathbb{E}(U(x))$ is a concave function, and $\hat{r} \neq r^0$ then based on (A.14)–(A.17) the following holds for the unique solution x^*:

- There exists a finite unique solution $x^* > 0$ that solves the FOC (A.10) if:

[40] For $\delta > 1$ and $\delta = 1$ the results follow from (A.5) and (A.6).

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(i) $\delta > 0$, $s > s_0$, and $\lambda > \lambda_{L0}$ (see (A.14), (A.15) and (A.17)). Then $x^* > \hat{x}$.

(ii) $\delta = 0$, $s > s_{T0}$, and $\lambda > \lambda_{L0}$ (see (A.14) and (A.17)). Then $x^* < \hat{x}$ for $s_{T0} < s < s^0$ and $x^* > \hat{x}$ for $s > s_0$.

(iii) $\delta > 0$ and $s_{T0} < s < s^0$ (see (A.14) and (A.16)). Then $x^* < \min\left\{\hat{x}, \frac{s^0}{s^0 - s}\right\}$.

- $x^* = +\infty$, if $\lambda < \lambda_{L0}$ and $\delta \geq 0$, see (see (A.15) and (A.17)).
- $x^* = 0$, if $s < s_{T0}$ (see (A.14).

□

Proof of Proposition 3.3: For $\hat{r} = r^0$, problem (4) boils down to

$$\left.\begin{array}{rl} \max: & \mathbb{E}(U(x)) = \frac{\alpha c}{1-\delta}\left[s^0 + x(s - s^0)\right]^{1-\delta} - \frac{1-\alpha}{1-\gamma}\int_{-\infty}^{r^0}\left(r^0 - r\right)^{1-\gamma} f(r)dr\left(\lambda - \frac{1}{K_\gamma}\right)x^{1-\gamma} \\ \text{such that}: & x \geq 0 \end{array}\right\} \quad \text{(A.18)}$$

Let $s^0 = 0$. Then the FOC is of the form

$$\frac{d}{dx}\mathbb{E}(U(x)) = \alpha c\frac{s^{1-\delta}}{x^\delta} - (1-\alpha)\int_{-\infty}^{r^0}\left(r^0 - r\right)^{1-\gamma} f(r)dr\left(\lambda - \frac{1}{K_\gamma}\right)\frac{1}{x^\gamma} = 0 \quad \text{(A.19)}$$

The sufficient condition for $\mathbb{E}(U(x))$ to be increasing, i.e., $\frac{d}{dx}\mathbb{E}(U(x)) > 0$, see (A.35), is that $\lambda < \frac{1}{K_\gamma}$ and thus in this case $x^* = +\infty$. Let $\gamma < \delta$ and $\lambda > \frac{1}{K_\gamma}$ then

$$x^* = \left[\frac{\alpha c}{1-\alpha}\cdot\frac{s^{1-\delta}}{\int_{-\infty}^{r^0}\left(r^0 - r\right)^{1-\gamma} f(r)dr\left(\lambda - \frac{1}{K_\gamma}\right)}\right]^{\frac{1}{\delta-\gamma}} \quad \text{(A.20)}$$

satisfies the first order conditions of (A.18), namely (A.19). As

$$\begin{aligned} \lim_{x\to 0^+}\frac{d}{dx}\mathbb{E}(U(x)) &= \lim_{x\to 0^+}\frac{1}{x^\gamma}\left[\alpha c\frac{s^{1-\delta}}{x^{\delta-\gamma}} - (1-\alpha)\int_{-\infty}^{r^0}\left(r^0 - r\right)^{1-\gamma} f(r)dr\left(\lambda - \frac{1}{K_\gamma}\right)\right] \\ &= +\infty > 0 \end{aligned} \quad \text{(A.21)}$$

and $\lim_{x\to+\infty}\mathbb{E}(U(x)) = -\infty$, then $x = x^*$, given by (A.20), is a point of global maxima for $\gamma < \delta$ and $\lambda > \frac{1}{K_\gamma}$.

Let $\gamma = \delta$ and still $s^0 = 0$. Then it follows from (A.19) that necessary and sufficient condition for $\mathbb{E}(U(x))$ to be increasing, i.e., $\frac{d}{dx}\mathbb{E}(U(x)) > 0$, is

$$\alpha c s^{1-\gamma} + (1-\alpha)\int_{-\infty}^{r^0}\left(r^0 - r\right)^{1-\gamma} f(r)dr\left(\frac{1}{K_\gamma} - \lambda\right) > 0$$

This holds for $\lambda < \lambda_{L\gamma}$, where $\lambda_{L\gamma}$ is given by (9) for $s^0 = 0$ which implies that $x^* = +\infty$. Similarly, $\mathbb{E}(U(x))$ is decreasing, i.e., $\frac{d}{dx}\mathbb{E}(U(x)) < 0$, when $\lambda > \lambda_{L\gamma}$, and thus $x^* = 0$.

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Note finely that

$$\mathbb{E}(U(x)) = x^{1-\gamma}\left\{\alpha c\frac{s^{1-\delta}}{1-\delta}x^{\gamma-\delta} + \frac{1-\alpha}{1-\gamma}\left[-\lambda\int_{-\infty}^{r^0}\left(r^0-r\right)^{1-\gamma}f(r)dr + \int_{r^0}^{+\infty}\left(r-r^0\right)^{1-\gamma}f(r)dr\right]\right\}$$

This implies that for $\gamma > \delta$ is $\lim_{x\to+\infty}\mathbb{E}(U(x)) = +\infty$ and thus $x^* = +\infty$.

Now let us assume that $s^0 > 0$.

At first we show that for sufficiently loss averse investor with $0 \le \gamma = \delta$ or $0 = \gamma < \delta$ is the utility decreasing and thus $x^* = 0$. Then we continue with the case when $0 < \gamma < \delta$. At first we assume that $0 < \gamma = \delta$. Then

$$\frac{d}{dx}\mathbb{E}(U(x)) = \alpha c\frac{s-s^0}{[s^0+x(s-s^0)]^\gamma} - (1-\alpha)\int_{-\infty}^{r^0}\left(r^0-r\right)^{1-\gamma}f(r)dr\left(\lambda-\frac{1}{K_\gamma}\right)\frac{1}{x^\gamma} \tag{A.22}$$

In the following we show when the expected utility function is decreasing and thus when $x^* = 0$. Note that sufficient condition for $\frac{d}{dx}\mathbb{E}(U(x)) < 0$, when $s > s^0$, is

$$\begin{aligned} x^\gamma\frac{d}{dx}\mathbb{E}(U(x)) &= \alpha c\frac{s-s^0}{\left(\frac{s^0}{x}+s-s^0\right)^\gamma} - (1-\alpha)\int_{-\infty}^{r^0}\left(r^0-r\right)^{1-\gamma}f(r)dr\left(\lambda-\frac{1}{K_\gamma}\right) \\ &< \alpha c\frac{s-s^0}{(s-s^0)^\gamma} - (1-\alpha)\int_{-\infty}^{r^0}\left(r^0-r\right)^{1-\gamma}f(r)dr\left(\lambda-\frac{1}{K_\gamma}\right) < 0 \end{aligned} \tag{A.23}$$

which is satisfied for sufficiently loss averse investor, namely when

$$\lambda > \frac{1}{K_\gamma} + \frac{\alpha c}{1-\alpha}\cdot\frac{(s-s^0)^{1-\gamma}}{\int_{-\infty}^{r^0}\left(r^0-r\right)^{1-\gamma}f(r)dr} \equiv \lambda_{L\gamma} \tag{A.24}$$

see (9). Thus, for $\lambda > \lambda_{L\gamma}$ and $s > s^0$ is $x^* = 0$. When $s < s^0$, then, based on (A.23), sufficient condition for $\frac{d}{dx}\mathbb{E}(U(x)) < 0$ is that $\lambda > \frac{1}{K_\gamma}$ and thus $x^* = 0$. In addition, (A.22) implies that

$$\frac{d^2}{dx^2}\mathbb{E}(U(x)) = -\alpha\gamma c\frac{(s-s^0)^2}{[s^0+x(s-s^0)]^{1+\gamma}} + (1-\alpha)\gamma\int_{-\infty}^{r^0}\left(r^0-r\right)^{1-\gamma}f(r)dr\left(\lambda-\frac{1}{K_\gamma}\right)\frac{1}{x^{1+\gamma}}$$

and thus the sufficient condition for $\frac{d^2}{dx^2}\mathbb{E}(U(x)) < 0$ and thus for $\mathbb{E}(U(x))$ being concave is that $\lambda < \frac{1}{K_\gamma}$ which implies the existence of the unique solution that satisfies the FOC (A.22). It can be seen that for $s < s^0$ and $\lambda < \frac{1}{K_\gamma}$ the following satisfies the FOC (A.22)

$$x^* = \frac{s^0}{\left[\frac{\alpha c}{1-\alpha}\cdot\frac{s^0-s}{\int_{-\infty}^{r^0}(r^0-r)^{1-\gamma}f(r)dr\left(\frac{1}{K_\gamma}-\lambda\right)}\right]^{\frac{1}{\gamma}} + s^0 - s} < \frac{s^0}{s^0-s} \tag{A.25}$$

Note that x^* given by (A.25) exceeds zero also for $s > s^0$ and $\frac{1}{K_\gamma} < \lambda < \lambda_{L\gamma}$[41] but in this case it is the

[41] Sufficient condition $\lambda < \frac{1}{K_\gamma}$ for the expected utility function to be concave is not satisfied.

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point of global minima as for $x < x^*$ is $\frac{d}{dx}\mathbb{E}(U(x)) < 0$ and for $x > x^*$ is $\frac{d}{dx}\mathbb{E}(U(x)) > 0$. Thus, the maximum is reached either at $x = 0$ or at $x = +\infty$. As $\mathbb{E}(U(0)) = \frac{\alpha c}{1-\gamma}\left(s^0\right)^{1-\gamma}$ and

$$\begin{aligned}\lim_{x\to+\infty}\mathbb{E}(U(x)) &= \frac{x^{1-\gamma}}{1-\gamma}\left[\alpha c\left(\frac{s^0}{x}+s-s^0\right)^{1-\gamma}-(1-\alpha)\int_{-\infty}^{r^0}\left(r^0-r\right)^{1-\gamma}f(r)dr\left(\lambda-\frac{1}{K_\gamma}\right)\right]\\ &= +\infty\end{aligned} \tag{A.26}$$

for $s > s^0$ and $\frac{1}{K_\gamma} < \lambda < \lambda_{L\gamma}$, and thus $x^* = +\infty$. (A.22) also imply that for $s > s^0$ and $\lambda < \frac{1}{K_\gamma}$ is $x^* = +\infty$. And thus if $s > s^0$ and $\lambda < \frac{1}{K_\gamma} + \lambda_{\Delta\gamma}$ then $x^* = +\infty$.

Now we assume that $0 = \gamma < \delta$. As

$$\frac{d}{dx}\mathbb{E}(U(x)) = \alpha c\frac{s-s^0}{[s^0+x(s-s^0)]^\delta}-(1-\alpha)\int_{-\infty}^{r^0}\left(r^0-r\right)f(r)dr\left(\lambda-\frac{1}{K_0}\right) \tag{A.27}$$

then

$$\frac{d}{dx}\mathbb{E}(U(x))\Big|_{x=0} = \alpha c\frac{s-s^0}{(s^0)^\delta}-(1-\alpha)\int_{-\infty}^{r^0}\left(r^0-r\right)f(r)dr\left(\lambda-\frac{1}{K_0}\right) \tag{A.28}$$

and thus

$$\frac{d}{dx}\mathbb{E}(U(x))\Big|_{x=0} > 0 \text{ for } \lambda < \lambda_{L0} \tag{A.29}$$

$$\frac{d}{dx}\mathbb{E}(U(x))\Big|_{x=0} < 0 \text{ for } \lambda > \lambda_{L0} \tag{A.30}$$

where λ_{L0} is given by (9) for $\gamma = 0$.[42] Note that

$$\mathbb{E}(U(x)) = \alpha c\frac{[s^0+x(s-s^0)]^{1-\delta}}{1-\delta}-(1-\alpha)\int_{-\infty}^{r^0}\left(r^0-r\right)f(r)dr\left(\lambda-\frac{1}{K_0}\right)x$$

implies that for $s > s^0$ we obtain

$$\lim_{x\to+\infty}\mathbb{E}(U(x)) = \begin{cases}-\infty & \text{for } \lambda > \frac{1}{K_0}\\ +\infty & \text{for } \lambda < \frac{1}{K_0}\end{cases} \tag{A.31}$$

In addition, (A.27) implies that for $s < s^0$ we obtain

$$\lim_{x\to\left(\frac{s^0}{s^0-s}\right)^-}\frac{d}{dx}\mathbb{E}(U(x)) = -\infty \tag{A.32}$$

As

$$\frac{d^2\mathbb{E}(U(x))}{dx^2} = -\alpha\delta c\frac{(s-s^0)^2}{[s^0+x(s-s^0)]^{\delta+1}} < 0 \tag{A.33}$$

[42] Note that the make the notation concise, we assume that $0^0 = 1$.

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the maximum is unique, and equations (A.29), (A.31), and (A.32) imply that for $s > s^0$ and $\frac{1}{K_0} < \lambda < \lambda_{L0}$, or for $s < s^0$ and $\lambda < \lambda_{L0}$, the solution is unique finite and follows from $\frac{d}{dx}\mathbb{E}(U(x^*)) = 0$, where $\frac{d}{dx}\mathbb{E}(U(x))$ is given by (A.27) and thus is

$$x^* = \frac{1}{s-s^0}\left\{\left[\frac{\alpha c(s-s^0)}{(1-\alpha)\left(\lambda-\frac{1}{K_0}\right)\int_{-\infty}^{r^0}(r^0-r)\,f(r)dr}\right]^{\frac{1}{\delta}} - s^0\right\}$$

Equations (A.29) and (A.31) also imply that for $s > s^0$ and $\lambda < \frac{1}{K_0}$ is $x^* = +\infty$. Note finally, that (A.30)-(A.33) imply that $x^* = 0$ for $\lambda > \lambda_{L0}$.

Let us assume that $0 = \gamma = \delta$. Then based on

$$\frac{d}{dx}\mathbb{E}(U(x)) = \alpha c(s-s^0) - (1-\alpha)\int_{-\infty}^{r^0}(r^0-r)\,f(r)dr\left(\lambda-\frac{1}{K_0}\right)$$

it follows that $\frac{d}{dx}\mathbb{E}(U(x)) < 0$, and thus $x^* = 0$ when $\lambda > \lambda_{L0}$ where λ_{L0} is given by (9) for $\gamma = 0$. Similarly, $\mathbb{E}(U(x))$ is increasing, i.e., $\frac{d}{dx}\mathbb{E}(U(x)) > 0$, when $\lambda < \lambda_{L0}$ and thus $x^* = +\infty$.

Now we assume that $0 < \gamma \neq \delta > 0$ and $s^0 > 0$. Thus, the FOC are

$$\frac{d}{dx}\mathbb{E}(U(x)) = \alpha c\frac{s-s^0}{[s^0+x(s-s^0)]^\delta} + (1-\alpha)\left[-\lambda\int_{-\infty}^{r^0}(r^0-r)^{1-\gamma}f(r)dr + \int_{r^0}^{+\infty}(r-r^0)^{1-\gamma}f(r)dr\right]x^{-\gamma} \tag{A.34}$$

$$= \frac{1}{x^\gamma}\left\{\underbrace{\alpha c\frac{(s-s^0)x^\gamma}{[s^0+x(s-s^0)]^\delta} - (1-\alpha)\int_{-\infty}^{r^0}(r^0-r)^{1-\gamma}f(r)dr\left(\lambda-\frac{1}{K_\gamma}\right)}_{G(x)}\right\} \tag{A.35}$$

$$= 0$$

Note that

$$\mathbb{E}(U(x)) = x^{1-\delta}\left[\frac{\alpha c}{1-\delta}\left(\frac{s^0}{x}+s-s^0\right)^{1-\delta} - \frac{1-\alpha}{1-\gamma}\int_{-\infty}^{r^0}(r^0-r)^{1-\gamma}f(r)dr\left(\lambda-\frac{1}{K_\gamma}\right)x^{\delta-\gamma}\right] \tag{A.36}$$

and thus for $\gamma < \delta$ and $s > s^0$ is $\lim_{x\to+\infty}\mathbb{E}(U(x)) = -\infty$ for $\lambda > \frac{1}{K_\gamma}$, while $\lim_{x\to+\infty}\mathbb{E}(U(x)) = +\infty$ for $\lambda < \frac{1}{K_\gamma}$. Note in addition that for $\gamma > \delta$ and $s > s^0$ is $\lim_{x\to+\infty}\mathbb{E}(U(x)) = +\infty$. The structure of the rest of the proof is as follows for $s > s^0$ and $\gamma < \delta$: at first we show that the first term in $G(x)$ in (A.35), namely $\alpha c\frac{(s-s^0)x^\gamma}{[s^0+x(s-s^0)]^\delta}$, obtains its maximum for $x = \bar{x}$ where

$$\bar{x} = \frac{\gamma s^0}{(\delta-\gamma)(s-s^0)} \tag{A.37}$$

As the second part in (A.35) is a constant (namely, it does not depend on x), also $x^\gamma\frac{d}{dx}\mathbb{E}(U(x))$ obtains its maximum at $x = \bar{x}$ and if $\bar{x}^\gamma\frac{d}{dx}\mathbb{E}(U(x))\Big|_{x=\bar{x}} < 0$, i.e., it is negative in its maximum, then also $\frac{d}{dx}\mathbb{E}(U(x)) < 0$ for all $x \geq 0$ and thus $x^* = 0$. If, however, $\bar{x}^\gamma\frac{d}{dx}\mathbb{E}(U(x))\Big|_{x=\bar{x}} > 0$, then in some interval is $\mathbb{E}(U(x))$ increasing

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and as $\lim_{x\to+\infty}\mathbb{E}(U(x)) = -\infty$ for $\lambda > \frac{1}{K_\gamma}$ and $\mathbb{E}(U(x))$ is differentiable then the FOC (A.35) must be satisfied. The details of the proof are

$$\frac{d}{dx}\left[\frac{x^\gamma}{(s^0+x(s-s^0))^\delta}\right] = \frac{\gamma s^0 - (\delta-\gamma)(s-s^0)x}{x^{1-\gamma}\left[s^0+x(s-s^0)\right]^{1+\delta}} \begin{cases} >0 & \text{if } x<\bar{x} \\ <0 & \text{if } x>\bar{x} \end{cases} \tag{A.38}$$

then $x=\bar{x}$ is the point of local maximum for $\frac{x^\gamma}{[s^0+x(s-s^0)]^\delta}$. Note in addition that

$$\bar{x}^\gamma \frac{d}{dx}\mathbb{E}(U(x))\Big|_{x=\bar{x}} = \alpha c(s-s^0)^{1-\gamma}\left(\frac{\delta-\gamma}{s^0}\right)^{\delta-\gamma}\frac{\gamma^\gamma}{\delta^\delta} - (1-\alpha)\int_{-\infty}^{r^0}\left(r^0-r\right)^{1-\gamma} f(r)dr\left(\lambda-\frac{1}{K_\gamma}\right) \tag{A.39}$$

Based on (A.39), $\bar{x}^\gamma \frac{d}{dx}\mathbb{E}(U(x))\Big|_{x=\bar{x}} < 0$ for $\lambda > \lambda_{L\gamma}$, where $\lambda_{L\gamma}$ is given by (9). Then based on what was mentioned above $\mathbb{E}(U(x))$ is a decreasing function and thus $x^*=0$ when $\lambda > \lambda_{L\gamma}$. Based on (A.39), $\bar{x}^\gamma \frac{d}{dx}\mathbb{E}(U(x))\Big|_{x=\bar{x}} > 0$ for $\lambda < \lambda_{L\gamma}$ which implies that $\mathbb{E}(U(x))$ is an increasing function at some interval of x and as $\lim_{x\to+\infty}\mathbb{E}(U(x)) = -\infty$ for $\lambda > \frac{1}{K_\gamma}$, then there exists x^* that satisfies the FOC (A.35) for $\frac{1}{K_\gamma} < \lambda < \lambda_{L\gamma}$. Note, however, that this is not the unique solution as $\lim_{x\to 0^+}\frac{d}{dx}\mathbb{E}(U(x)) = -\infty$ for $\lambda > \frac{1}{K_\gamma}$ and thus there must exist also a point of a local minimum such that FOC (A.35) is satisfied (as $\mathbb{E}(U(x))$ is continuously differentiable).

Let $s < s^0$. Note that the ESG score function and thus also the whole expected utility function is defined in this case on $x \in \left[0, \frac{s^0}{s^0-s}\right]$. Based on (A.35) a sufficient condition for $\frac{d}{dx}\mathbb{E}(U(x)) < 0$ is $\lambda > \frac{1}{K_\gamma}$ and thus $x^*=0$. Let $\lambda < \frac{1}{K_\gamma}$. Let us first assume that $\gamma > \delta$. Then $\bar{x}$ given by (A.37) is such that $\bar{x} > \frac{s^0}{s^0-s}$. Based on this, (A.35), $s<s^0$, and (A.38), that holds also for $s<s^0$ and $\gamma>\delta$, it follows that $G(x)$ is decreasing in $\left[0, \frac{s^0}{s^0-s}\right]$, and as $G(0)>0$ and $G\left(\frac{s^0}{s^0-s}\right) < 0$ then thus there exists a unique x^* such that $G(x^*)=0$ which is also a unique solution of $\frac{d}{dx}\mathbb{E}(U(x))=0$, see (A.35).

Let $\gamma < \delta$. Then

$$\frac{d}{dx}\left[\frac{x^\gamma}{(s^0+x(s-s^0))^\delta}\right] = \frac{\gamma s^0 - (\delta-\gamma)(s-s^0)x}{x^{1-\gamma}\left[s^0+x(s-s^0)\right]^{1+\delta}} > 0 \quad \text{if } x > \bar{x} = -\frac{\gamma s^0}{(\delta-\gamma)(s^0-s)} \tag{A.40}$$

and based on (A.35), $s<s^0$, and (A.40) it follows that $G(x)$ is decreasing for $x>\bar{x}$. As, however, in this case is $\bar{x}<0$ then it implies that $G(x)$ is decreasing in its domain. The rest follows from the case when $\gamma>\delta$ and thus there exist a unique x^* that satisfies the FOC (A.35). □

Proof of Proposition 3.4: The proof is based on implicit function differentiation and Equation (11). Let $x^*>0$ be a solution of

$$\frac{d}{dx}\mathbb{E}(U(x)) = 0$$

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Thus, the first order conditions are satisfied and

$$G(\lambda,\hat{r},s,\alpha,x) \equiv \alpha c\frac{(s-s^0)x^\gamma}{[s^0+x(s-s^0)]^\delta} +(1-\alpha)\left[\lambda\int_{-\infty}^{z(x,\hat{r})}\frac{r-r^0}{(z(x,\hat{r})-r)^\gamma}f(r)dr+\int_{z(x,\hat{r})}^{+\infty}\frac{r-r^0}{(r-z(x,\hat{r}))^\gamma}f(r)dr\right]=0 \tag{A.41}$$

where $z(x,\hat{r})=r^0+\frac{\hat{r}-r^0}{x}$. Then

$$\frac{dx}{d\lambda}=-\frac{\frac{dG}{d\lambda}}{\frac{dG}{dx}},\quad \frac{dx}{d\hat{r}}=-\frac{\frac{dG}{d\hat{r}}}{\frac{dG}{dx}},\quad \frac{dx}{ds}=-\frac{\frac{dG}{ds}}{\frac{dG}{dx}},\quad \text{and}\quad \frac{dx}{d\alpha}=-\frac{\frac{dG}{d\alpha}}{\frac{dG}{dx}} \tag{A.42}$$

where $\hat{r}$, s, α are fixed in the first case, λ, s α are fixed in the second case, λ, $\hat{r}$, α are fixed in the third case, and λ, $\hat{r}$, s are fixed in the last case. Note that $\frac{G(\lambda,\hat{r},s,\alpha,x)}{x^\gamma}=\frac{d}{dx}\mathbb{E}(U(x))$, see (A.3), and as x^* is the point of local maximum then $\frac{d^2\mathbb{E}(U(x))}{dx^2}<0$ for $x=x^*$ which implies then that $\frac{dG(\lambda,\hat{r},s,\alpha,x)}{dx}<0$. Namely

$$\begin{aligned}\frac{d^2\mathbb{E}(U(x))}{dx^2} &= \frac{d}{dx}\left(x^{-\gamma}G(\lambda,\hat{r},s,\alpha,x)\right)=-x^{-1-\gamma}\gamma\,G(\lambda,\hat{r},s,\alpha,x)+x^{-\gamma}\frac{dG(\lambda,\hat{r},s,\alpha,x)}{dx}\\ &= -\frac{\gamma}{x}\cdot\frac{d}{dx}\mathbb{E}(U(x))+x^{-\gamma}\frac{dG(\lambda,\hat{r},s,\alpha,x)}{dx}=x^{-\gamma}\frac{dG(\lambda,\hat{r},x)}{dx}<0\end{aligned}$$

as $\frac{d}{dx}\mathbb{E}(U(x))=0$ when $x=x^*$. Thus, based on this and (A.42), the sign of $\frac{dx}{d\lambda}$ coincides with the sign of $\frac{dG(\lambda,\hat{r},s,\alpha,x)}{d\lambda}$. (A.41) implies that

$$\frac{dG(\lambda,\hat{r},s,\alpha,x)}{d\lambda}=(1-\alpha)\int_{-\infty}^{z(x,\hat{r})}\frac{r-r^0}{[z(x,\hat{r})-r]^\gamma}f(r)dr \tag{A.43}$$

For $\hat{r}<r^0$ and $x>0$ is $z(x,\hat{r})<r^0$ and thus based on (A.43) is $\frac{dG(\lambda,\hat{r},s,\alpha,x)}{d\lambda}<0$. On the other hand, for $\hat{r}>r^0$ and $x>0$ is $z(x,\hat{r})>r^0$ and thus $\int_{z(x,\hat{r})}^{+\infty}\frac{r-r^0}{[r-z(x,\hat{r})]^\gamma}f(r)dr>0$. This and Equation (A.41) imply that for $x=x^*$ is

$$\alpha c\frac{(s-s^0)x^\gamma}{[s^0+x(s-s^0)]^\delta}+(1-\alpha)\lambda\int_{-\infty}^{z(x,\hat{r})}\frac{r-r^0}{[z(x,\hat{r})-r]^\gamma}f(r)dr<0$$

which gives that for $s>s^0$ is $(1-\alpha)\lambda\int_{-\infty}^{z(x,\hat{r})}\frac{r-r^0}{[z(x,\hat{r})-r]^\gamma}f(r)dr=\lambda\frac{dG(\lambda,\hat{r},s,\alpha,x)}{d\lambda}<0$ and thus $\frac{dG(\lambda,\hat{r},s,\alpha,x)}{d\lambda}<0$. This gives $\frac{dx}{d\lambda}|_{x=x^*}<0$ for $\hat{r}<r^0$ and for $\hat{r}>r^0$ when $s>s^0$. Note that for $\hat{r}=r^0$ we obtain

$$G(\lambda,\hat{r},s,\alpha,x)\equiv\alpha c\frac{(s-s^0)x^\gamma}{[s^0+x(s-s^0)]^\delta}-(1-\alpha)\left(\lambda-\frac{1}{K_\gamma}\right)\int_{-\infty}^{r^0}\left(r^0-r\right)^{1-\gamma}f(r)dr=0 \tag{A.44}$$

which implies that $\frac{dG}{d\lambda}=-(1-\alpha)\left(\lambda-\frac{1}{K_\gamma}\right)\int_{-\infty}^{r^0}\left(r^0-r\right)^{1-\gamma}f(r)dr<0$ and thus $\frac{dx^*}{d\lambda}<0$ when $\hat{r}=r^0$.

Journal Pre-proof

As

$$x^{\gamma}\frac{d^2\mathbb{E}(U(x))}{dxds}=\frac{dG}{ds} = \alpha c x^{\gamma}\cdot\frac{s^0+(1-\delta)(s-s^0)x}{[s^0+x(s-s^0)]^{1+\delta}}$$
$$= \frac{\alpha c x^{\gamma}}{[s^0+(s-s^0)x]^{\delta}}\left[\underbrace{1}_{\text{ESG substitution effect}} - \underbrace{\frac{\delta(s-s^0)x}{s^0+(s-s^0)x}}_{\text{ESG diminishing sensitivity effect}}\right] \tag{A.45}$$

Let $s>s^0$, then based on (A.45) is $\frac{dG}{ds}>0$ if and only if

$$s^0+(1-\delta)(s-s^0)x>0 \tag{A.46}$$

Note that this holds when $0\leq\delta<1+\frac{s^0}{(s-s^0)x}$. Similarly, $\frac{dG}{ds}=0$ if and only if $\delta=1+\frac{s^0}{(s-s^0)x}$ and $\frac{dG}{ds}<0$ if and only if $\delta>1+\frac{s^0}{(s-s^0)x}$. For $s<s^0$ (A.45) implies that $\frac{dG}{ds}>0$. As the sign of $\frac{dG}{ds}$ coincides with the sign of $\frac{dx^*}{ds}$ then the following holds for $s>s^0$

$$\frac{dx}{ds}\Big|_{x=x^*}\begin{cases} >0, & 0\leq\delta<1+\frac{s^0}{(s-s^0)x} \\ =0, & \delta=1+\frac{s^0}{(s-s^0)x} \\ <0, & \delta>1+\frac{s^0}{(s-s^0)x}\end{cases} \tag{A.47}$$

while for $s<s^0$ the following holds

$$\frac{dx}{ds}\Big|_{x=x^*}>0 \tag{A.48}$$

Thus, for $s>s^0$ the investment in the risky asset increases with increasing its ESG score s for sufficiently small diminishing sensitivity of the ESG score function (δ), while it decreases for sufficiently large δ. However for $s<s^0$ the investment in the risky asset can only increase when its ESG score s is increased.

Note that

$$x^{\gamma}\frac{d^2\mathbb{E}(U(x))}{dxds^0}=\frac{dG}{ds^0} = -\alpha c x^{\gamma}\cdot\frac{s^0+[\delta+(1-\delta)x](s-s^0)}{[s^0+x(s-s^0)]^{1+\delta}}$$
$$= \frac{\alpha c x^{\gamma}}{[s^0+(s-s^0)x]^{\delta}}\left[\underbrace{-1}_{\text{ESG substitution effect}} - \underbrace{\frac{\delta(s-s^0)(1-x)}{s^0+(s-s^0)x}}_{\text{leverage effect, ESG diminishing sensitivity effect}}\right] \tag{A.49}$$

Let $s>s^0$. Then in follows from (A.49) that

$$\frac{dG}{ds^0}<0 \quad \text{for} \quad x\leq 1 \tag{A.50}$$

Journal Pre-proof

Let $x > 1$. Then (A.49) implies that $\frac{dG}{ds^0} \leq 0$ when

$$-\delta(x-1)(s-s^0) \geq -s^0 - x(s-s^0)$$

and thus when

$$\delta \leq \frac{s^0 + x(s-s^0)}{(x-1)(s-s^0)} \equiv \delta_T$$

i.e.,

$$\frac{dG}{ds^0} \leq 0 \quad \text{for} \quad x > 1 \quad \text{and} \quad \delta \leq \delta_T \tag{A.51}$$

In addition, (A.49) implies that $\frac{dG}{ds^0} > 0$ when

$$-\delta(x-1)(s-s^0) < -s^0 - x(s-s^0)$$

which does not hold for $x \leq 1$ and thus $x > 1$ and the last inequality holds for $\delta > \delta_T$. We obtain

$$\frac{dG}{ds^0} > 0 \quad \text{for} \quad x > 1 \quad \text{and} \quad \delta > \delta_T \tag{A.52}$$

Let $s < s^0$. Then (A.49) implies that

$$\frac{dG}{ds^0} < 0 \quad \text{for} \quad x \geq 1 \tag{A.53}$$

Let us assume that $x < 1$. Then (A.49) implies that $\frac{dG}{ds^0} \leq 0$ when

$$-\delta(1-x)(s^0-s) \geq -s^0 + x(s^0-s)$$

and thus when

$$\delta \leq \frac{s^0 - x(s^0-s)}{(1-x)(s^0-s)} = \delta_T$$

i.e.,

$$\frac{dG}{ds^0} \leq 0 \quad \text{for} \quad x < 1 \quad \text{and} \quad \delta \leq \delta_T \tag{A.54}$$

In addition, (A.49) implies that $\frac{dG}{ds^0} > 0$ when

$$\delta(1-x)(s^0-s) > s^0 - x(s^0-s)$$

which does not hold for $x \geq 1$ as $s^0 - x(s^0-s) > 0$[43] and thus $x < 1$ and the last inequality holds for $\delta > \delta_T$. We obtain

$$\frac{dG}{ds^0} > 0 \quad \text{for} \quad x < 1 \quad \text{and} \quad \delta > \delta_T \tag{A.55}$$

[43]This follows from the constraint $x \leq \frac{s^0}{s^0-s}$ when $s < s^0$.

Journal Pre-proof

As the sign of $\frac{dG}{ds^0}$ coincides with the sign of $\frac{dx^*}{ds^0}$ then the following holds for $s > s^0$ based on (A.50)-(A.52)

$$\frac{dx}{ds^0}\Big|_{x=x^*} \begin{cases} > 0, & x > 1, \quad \delta > \delta_T \\ = 0, & x > 1, \quad \delta = \delta_T \\ < 0, & x \leq 1 \\ & x > 1, \quad \delta < \delta_T \end{cases} \tag{A.56}$$

while for $s < s^0$ the following holds based on (A.53)-(A.55)

$$\frac{dx}{ds^0}\Big|_{x=x^*} \begin{cases} > 0, & x < 1, \quad \delta > \delta_T \\ = 0, & x < 1, \quad \delta = \delta_T \\ < 0, & x \geq 1 \\ & x < 1, \quad \delta < \delta_T \end{cases} \tag{A.57}$$

Equation (A.41) can be written, after applying integration by parts on second part of (A.41), by using assumption $\lim_{r\to\pm\infty} |r|^{2-\gamma} f(r) = 0$, as follows[44]

$$\begin{aligned} G(\lambda, \hat{r}, x) \quad = \quad & \alpha c \frac{(s-s^0)x^\gamma}{[s^0 + x(s-s^0)]^\delta} \\ & + (1-\alpha) \left\{ - \int_{z(x,\hat{r})}^{+\infty} \left[\frac{(r - z(x,\hat{r}))^{2-\gamma}}{2-\gamma} + \frac{(r - z(x,\hat{r}))^{1-\gamma}(z(x,\hat{r}) - r^0)}{1-\gamma} \right] \frac{df(r)}{dr} dr \right. \\ & \left. + \lambda \int_{-\infty}^{z(x,\hat{r})} \left[- \frac{(z(x,\hat{r}) - r)^{2-\gamma}}{2-\gamma} + \frac{(z(x,\hat{r}) - r)^{1-\gamma}(z(x,\hat{r}) - r^0)}{1-\gamma} \right] \frac{df(r)}{dr} dr \right\} = 0 \end{aligned} \tag{A.58}$$

After applying the Leibniz integral rule we obtain

$$\frac{dG}{dx}\Big|_{x=x^*} = \alpha c H(x, s, s^0, \gamma, \delta)\Big|_{x=x^*} + (1-\alpha) \frac{dz(x,\hat{r})}{dx} F(z(x,\hat{r}))\Big|_{x=x^*} \tag{A.59}$$

$$\frac{dG}{d\hat{r}}\Big|_{x=x^*} = (1-\alpha) \frac{dz(x,\hat{r})}{d\hat{r}} F(z(x,\hat{r}))\Big|_{x=x^*} \tag{A.60}$$

where $H(x)$ is defined by

$$H(x) \quad \equiv \quad \frac{d}{dx}\left[\frac{(s-s^0)x^\gamma}{(s^0 + x(s-s^0))^\delta} \right] = \frac{(s-s^0)x^\gamma}{[s^0 + x(s-s^0)]^\delta} \left[\frac{\gamma}{x} - \frac{\delta(s-s^0)}{s^0 + x(s-s^0)} \right] \tag{A.61}$$

[44] See (A.18) in Fortin and Hlouskova (2024) for more detail.

Journal Pre-proof

and $F(z(x,\hat{r}))$ is given by[45]

$$\begin{aligned} F(z(x,\hat{r})) &\equiv \lambda\int_{-\infty}^{z(x,\hat{r})}\left[\frac{\gamma}{1-\gamma}(z(x,\hat{r})-r)^{1-\gamma}+\frac{z(x,\hat{r})-r^0}{(z(x,\hat{r})-r)^{\gamma}}\right]\frac{df(r)}{dr}dr \\ &\quad -\int_{z(x,\hat{r})}^{+\infty}\left[\frac{\gamma}{1-\gamma}(r-z(x,\hat{r}))^{1-\gamma}-\frac{z(x,\hat{r})-r^0}{(r-z(x,\hat{r}))^{\gamma}}\right]\frac{df(r)}{dr}dr \\ &= \lambda\int_{-\infty}^{z(x,\hat{r})}\frac{\gamma(r^0-r)+z(x,\hat{r})-r^0}{(1-\gamma)(z(x,\hat{r})-r)^{\gamma}}\frac{df(r)}{dr}dr \\ &\quad -\int_{z(x,\hat{r})}^{+\infty}\frac{\gamma(r-r^0)+r^0-z(x,\hat{r})}{(1-\gamma)(r-z(x,\hat{r}))^{\gamma}}\frac{df(r)}{dr}dr \\ &= \frac{d}{dz}\left[\lambda\int_{-\infty}^{z(x,\hat{r})}\frac{r-r^0}{(z(x,\hat{r})-r)^{\gamma}}f(r)dr+\int_{z(x,\hat{r})}^{+\infty}\frac{r-r^0}{(r-z(x,\hat{r}))^{\gamma}}f(r)dr\right] \end{aligned}$$

$\frac{dz(x,\hat{r})}{dx}=-\frac{\hat{r}-r^0}{(x^*)^2}$ and $\frac{dz(x,\hat{r})}{d\hat{r}}=\frac{1}{x^*}$. Thus, if $F(z(x^*,\hat{r}))$ is continuous in $\hat{r}$ then it follows from (A.60) that

$$\operatorname{sgn}\left(\frac{dx}{d\hat{r}}\Big|_{x=x^*}\right)=\operatorname{sgn}\left(\frac{dG}{d\hat{r}}\Big|_{x=x^*}\right)=\operatorname{sgn}\left(F(z(x^*,\hat{r}))\right) \tag{A.62}$$

If there exists $\hat{r}_T$ such that $F(\cdot)$ is not continuous in this point, i.e., $\lim_{\hat{r}\to^-\hat{r}_T}(F(z(x^*,\hat{r}))\neq\lim_{\hat{r}\to^+\hat{r}_T}(F(z(x^*,\hat{r}))$ then

$$\begin{aligned} \operatorname{sgn}\left(\lim_{\hat{r}\to^-\hat{r}_T}\frac{dx}{d\hat{r}}\Big|_{x=x^*}\right) &= \operatorname{sgn}\left(\lim_{\hat{r}\to^-\hat{r}_T}(F(z(x^*,\hat{r})))\right) \quad \text{and} \\ \operatorname{sgn}\left(\lim_{\hat{r}\to^+\hat{r}_T}\frac{dx}{d\hat{r}}\Big|_{x=x^*}\right) &= \operatorname{sgn}\left(\lim_{\hat{r}\to^+\hat{r}_T}(F(z(x^*,\hat{r})))\right) \end{aligned} \tag{A.63}$$

After applying implicit function theorem we obtain

$$\begin{aligned} \frac{dx}{d\hat{r}}\Big|_{x=x^*} &= -\frac{\frac{dG}{d\hat{r}}}{\frac{dG}{dx}}\Big|_{x=x^*}=-\frac{(1-\alpha)\frac{dz(x,\hat{r})}{d\hat{r}}F(z(x,\hat{r}))}{\alpha cH(x)+(1-\alpha)\frac{dz(x,\hat{r})}{dx}F(z(x,\hat{r}))}\Big|_{x=x^*} \\ &= \frac{1}{-\frac{\alpha c}{1-\alpha}\frac{x^*H(x^*)}{F(z(x^*,\hat{r}))}+\frac{\hat{r}-r^0}{x^*}} \end{aligned} \tag{A.64}$$

Note that for $\hat{r}=r^0$ function $F(z(x,r^0))$ does not depend on x and only on parameters λ, r^0, and γ, as

$$F(z(x,r^0))=F(r^0)=\frac{\gamma}{1-\gamma}\left[\lambda\int_{-\infty}^{r^0}(r^0-r)^{1-\gamma}\frac{df(r)}{dr}dr-\int_{r^0}^{+\infty}(r-r^0)^{1-\gamma}\frac{df(r)}{dr}dr\right] \tag{A.65}$$

Note that if $\int_{-\infty}^{r^0}(r^0-r)^{1-\gamma}\frac{df(r)}{dr}dr>0$ then $F(z(x,r^0))>0$ for $\lambda>\frac{\int_{r^0}^{+\infty}(r-r^0)^{1-\gamma}\frac{df(r)}{dr}dr}{\int_{-\infty}^{r^0}(r^0-r)^{1-\gamma}\frac{df(r)}{dr}dr}$. Condition $\int_{-\infty}^{r^0}(r^0-r)^{1-\gamma}\frac{df(r)}{dr}dr>0$ is satisfied for unimodal symmetric distributions as $r^0<\mathbb{E}(r)$ and $\frac{df(r)}{dr}>0$ for $r<r^0$.

[45] Note that function $H(x)$ depends also on s, s^0, γ, and δ, and function $F(\cdot)$ depends also on r^0, λ, and γ, but for the sake of brevity we write only $H(x)$ and $F(z(x,\hat{r}))$.

Journal Pre-proof

Note that for $\delta = \gamma$ is

$$H(x) = \frac{\gamma s^0 (s - s^0)}{x^{1-\gamma}[s^0 + x(s - s^0)]^{1+\gamma}}$$

and thus for $s^0 = 0$ or $\gamma = 0$ is $H(x) = 0$. For these cases and based on (A.64) is

$$\frac{dx^*}{d\hat{r}} = \frac{x^*}{\hat{r} - r^0} \begin{cases} < 0, & \text{if } \hat{r} < r^0 \\ > 0, & \text{if } \hat{r} > r^0 \end{cases} \tag{A.66}$$

For $\gamma = 0 \leq \delta$ we define $G(\lambda, \hat{r}, s, \alpha, x) = \frac{d\mathbb{E}(U(x))}{dx}$ as given by (A.10). Then using (A.11)

$$\begin{aligned} \frac{dG}{dx} &= -\alpha c \delta \frac{(s - s^0)^2}{[s^0 + x(s - s^0)]^{1+\delta}} - (1 - \alpha)(\lambda - 1) f(z(x, \hat{r})) \frac{(\hat{r} - r^0)^2}{x^3} \\ \frac{dG}{d\hat{r}} &= (1 - \alpha)(\lambda - 1) f(z(x, \hat{r})) \frac{\hat{r} - r^0}{x^2} \end{aligned}$$

and thus

$$\begin{aligned} \frac{dx^*}{d\hat{r}} &= -\frac{\frac{dG}{d\hat{r}}}{\frac{dG}{dx}}\Big|_{x=x^*} \\ &= \frac{(\lambda - 1) f(z(x^*, \hat{r})) \frac{\hat{r} - r^0}{x^*}}{\frac{\alpha c \delta}{1-\alpha} \frac{x^*(s - s^0)^2}{[s^0 + x(s - s^0)]^{1+\delta}} + (\lambda - 1) f(z(x^*, \hat{r})) \left(\frac{\hat{r} - r^0}{x^*}\right)^2} \begin{cases} < 0, & \text{for } \hat{r} < r^0 \\ > 0, & \text{for } \hat{r} > r^0 \end{cases} \end{aligned}$$

Note that $\frac{dG}{d\alpha}$ has the following form

$$\frac{dG}{d\alpha} = \frac{c(s - s^0)x^\gamma}{[s^0 + x(s - s^0)]^\delta} - \lambda \int_{-\infty}^{z(x,\hat{r})} \frac{r - r^0}{(z(x, \hat{r}) - r)^\gamma} f(r) dr - \int_{z(x,\hat{r})}^{+\infty} \frac{r - r^0}{(r - z(x, \hat{r}))^\gamma} f(r) dr \tag{A.67}$$

Based on (A.41) and for $s > s^0$ the term

$$\lambda \int_{-\infty}^{z(x,\hat{r})} \frac{r - r^0}{(z(x, \hat{r}) - r)^\gamma} f(r) dr + \int_{z(x,\hat{r})}^{+\infty} \frac{r - r^0}{(r - z(x, \hat{r}))^\gamma} f(r) dr \tag{A.68}$$

is negative, while for $s < s^0$ expression (A.68) is positive. This and (A.67) imply that $\frac{dG}{d\alpha} > 0$ for $s > s^0$ and $\frac{dG}{d\alpha} < 0$ for $s < s^0$ and thus $\frac{dx^*}{d\alpha} > 0$ for $s > s^0$ and $\frac{dx^*}{d\alpha} < 0$ for $s < s^0$. Note that for $\hat{r} = r^0$ is

$$\frac{dG}{d\alpha} = \frac{c(s - s^0)x^\gamma}{[s^0 + x(s - s^0)]^\delta} + \left(\lambda - \frac{1}{K_\gamma}\right) \int_{-\infty}^{r^0} \left(r^0 - r\right)^{1-\gamma} f(r) dr \tag{A.69}$$

and thus $\frac{dx^*}{d\alpha} > 0$ for $s > s^0$ and $\lambda > \frac{1}{K_\gamma}$ while $\frac{dx^*}{d\alpha} < 0$ for $s < s^0$ and $\lambda < \frac{1}{K_\gamma}$. These are sufficient conditions.

As $\frac{dG}{dc} = \alpha \frac{(s - s^0)x^\gamma}{[s^0 + x(s - s^0)]^\delta}$ then $\frac{dx^*}{dc} > 0$ for $s > s^0$ and $\frac{dx^*}{dc} < 0$ for $s < s^0$.

Journal Pre-proof

Proof of Proposition 3.5: Based on (6), (11), and the Leibniz integral rule we obtain

$$\begin{aligned}
\frac{d\mathbb{E}(U(x^*))}{d\hat{r}} &= \alpha c \frac{s-s^0}{[s^0+x^*(s-s^0)]^\delta}\frac{dx^*}{d\hat{r}} + (1-\alpha)\left\{-\lambda\int_{-\infty}^{z(x^*,\hat{r})}\frac{(r^0-r)\frac{dx^*}{d\hat{r}}+1}{[x^*(r^0-r)+\hat{r}-r^0]^\gamma}f(r)dr\right. \\
&\quad \left. + \int_{z(x^*,\hat{r})}^{\infty}\frac{(r-r^0)\frac{dx^*}{d\hat{r}}-1}{[x^*(r-r^0)-\hat{r}+r^0]^\gamma}f(r)dr\right\} \\
&= \frac{dx^*}{d\hat{r}}\left\{\alpha c\frac{s-s^0}{[s^0+x^*(s-s^0)]^\delta} + (1-\alpha)\left[-\lambda\int_{-\infty}^{z(x^*,\hat{r})}\frac{r^0-r}{(x^*(r^0-r)+\hat{r}-r^0)^\gamma}f(r)dr\right.\right. \\
&\quad \left.\left. + \int_{z(x^*,\hat{r})}^{\infty}\frac{r-r^0}{(x^*(r-r^0)-\hat{r}+r^0)^\gamma}f(r)dr\right]\right\} \\
&\quad -(1-\alpha)\left[\int_{-\infty}^{z(x^*,\hat{r})}\frac{\lambda}{(x^*(r^0-r)+\hat{r}-r^0)^\gamma}f(r)dr + \int_{z(x^*,\hat{r})}^{\infty}\frac{1}{(x^*(r-r^0)-\hat{r}+r^0)^\gamma}f(r)dr\right] \\
&= 0-(1-\alpha)\left[\int_{-\infty}^{z(x^*,\hat{r})}\frac{\lambda}{(x^*(r^0-r)+\hat{r}-r^0)^\gamma}f(r)dr + \int_{z(x^*,\hat{r})}^{\infty}\frac{1}{(x^*(r-r^0)-\hat{r}+r^0)^\gamma}f(r)dr\right] \\
&< 0 \qquad \text{(A.70)}
\end{aligned}$$

Thus, happiness decreases with increasing reference return $\hat{r}$.

(6), (11), and the Leibniz integral rule also imply

$$\begin{aligned}
\frac{d\mathbb{E}(U(x^*))}{d\lambda} &= \alpha c \frac{s-s^0}{[s^0+x^*(s-s^0)]^\delta}\frac{dx^*}{d\lambda} + (1-\alpha)\left\{-\lambda\int_{-\infty}^{z(x^*,\hat{r})}\frac{(r^0-r)\frac{dx^*}{d\lambda}}{[x^*(r^0-r)+\hat{r}-r^0]^\gamma}f(r)dr\right. \\
&\quad \left. - \int_{-\infty}^{z(x^*,\hat{r})}\left[x^*(r^0-r)+\hat{r}-r^0\right]^{1-\gamma}f(r)dr + \int_{z(x^*,\hat{r})}^{\infty}\frac{(r-r^0)\frac{dx^*}{d\lambda}}{[x^*(r-r^0)-\hat{r}+r^0]^\gamma}f(r)dr\right\} \\
&= \frac{dx^*}{d\lambda}\left\{\alpha c\frac{s-s^0}{[s^0+x^*(s-s^0)]^\delta} + (1-\alpha)\left[-\lambda\int_{-\infty}^{z(x^*,\hat{r})}\frac{r^0-r}{(x^*(r^0-r)+\hat{r}-r^0)^\gamma}f(r)dr\right.\right. \\
&\quad \left.\left. + \int_{z(x^*,\hat{r})}^{\infty}\frac{r-r^0}{(x^*(r-r^0)-\hat{r}+r^0)^\gamma}f(r)dr\right]\right\} \\
&\quad -(1-\alpha)\int_{-\infty}^{z(x^*,\hat{r})}\left[x^*(r^0-r)+\hat{r}-r^0\right]^{1-\gamma}f(r)dr \\
&= 0-(1-\alpha)\int_{-\infty}^{z(x^*,\hat{r})}\left[x^*(r^0-r)+\hat{r}-r^0\right]^{1-\gamma}f(r)dr \\
&< 0 \qquad \text{(A.71)}
\end{aligned}$$

Journal Pre-proof

Similarly

$$\begin{aligned}
\frac{d\mathbb{E}(U(x^*))}{ds} &= \alpha c\frac{s-s^0}{[s^0+x^*(s-s^0)]^\delta}\frac{dx^*}{ds}+\alpha c\frac{x^*}{[s^0+x^*(s-s^0)]^\delta} \\
&\quad +(1-\alpha)\left\{-\lambda\int_{-\infty}^{z(x^*,\hat{r})}\frac{(r^0-r)\frac{dx^*}{ds}}{[x^*(r^0-r)+\hat{r}-r^0]^\gamma}f(r)dr+\int_{z(x^*,\hat{r})}^{\infty}\frac{(r-r^0)\frac{dx^*}{ds}}{[x^*(r-r^0)-\hat{r}+r^0]^\gamma}f(r)dr\right\} \\
&= \frac{dx^*}{ds}\left\{\alpha c\frac{s-s^0}{[s^0+x^*(s-s^0)]^\delta}+(1-\alpha)\left[-\lambda\int_{-\infty}^{z(x^*,\hat{r})}\frac{r^0-r}{(x^*(r^0-r)+\hat{r}-r^0)^\gamma}f(r)dr\right.\right. \\
&\quad \left.\left.+\int_{z(x^*,\hat{r})}^{\infty}\frac{r-r^0}{(x^*(r-r^0)-\hat{r}+r^0)^\gamma}f(r)dr\right]\right\}+\alpha c\frac{x^*}{[s^0+x^*(s-s^0)]^\delta} \\
&= 0+\alpha c\frac{x^*}{[s^0+x^*(s-s^0)]^\delta} \\
&> 0
\end{aligned} \tag{A.72}$$

And in the same way

$$\begin{aligned}
\frac{d\mathbb{E}(U(x^*))}{ds^0} &= \alpha c\frac{s-s^0}{[s^0+x^*(s-s^0)]^\delta}\frac{dx^*}{ds^0}+\alpha c\frac{1-x^*}{[s^0+x^*(s-s^0)]^\delta} \\
&\quad +(1-\alpha)\left\{-\lambda\int_{-\infty}^{z(x^*,\hat{r})}\frac{(r^0-r)\frac{dx^*}{ds^0}}{[x^*(r^0-r)+\hat{r}-r^0]^\gamma}f(r)dr+\int_{z(x^*,\hat{r})}^{\infty}\frac{(r-r^0)\frac{dx^*}{ds^0}}{[x^*(r-r^0)-\hat{r}+r^0]^\gamma}f(r)dr\right\} \\
&= \frac{dx^*}{ds^0}\left\{\alpha c\frac{s-s^0}{[s^0+x^*(s-s^0)]^\delta}+(1-\alpha)\left[-\lambda\int_{-\infty}^{z(x^*,\hat{r})}\frac{r^0-r}{(x^*(r^0-r)+\hat{r}-r^0)^\gamma}f(r)dr\right.\right. \\
&\quad \left.\left.+\int_{z(x^*,\hat{r})}^{\infty}\frac{r-r^0}{(x^*(r-r^0)-\hat{r}+r^0)^\gamma}f(r)dr\right]\right\}+\alpha c\frac{1-x^*}{[s^0+x^*(s-s^0)]^\delta} \\
&= 0+\alpha c\frac{1-x^*}{[s^0+x^*(s-s^0)]^\delta} \\
&= \begin{cases} >0, & x^*<1 \\ =0, & x^*=1 \\ <0, & x^*>1 \end{cases}
\end{aligned} \tag{A.73}$$

Journal Pre-proof

Finally, (6), (11), and the Leibniz integral rule

$$\begin{aligned}
\frac{d\mathbb{E}(U(x^*))}{d\alpha} &= \alpha c \frac{s-s^0}{[s^0+x^*(s-s^0)]^\delta}\frac{dx^*}{d\alpha} + c\frac{[s^0+x^*(s-s^0)]^{1-\delta}}{1-\delta} \\
&+(1-\alpha)\left\{-\lambda\int_{-\infty}^{z(x^*,\hat{r})}\frac{(r^0-r)\frac{dx^*}{d\alpha}}{[x^*(r^0-r)+\hat{r}-r^0]^\gamma}f(r)dr + \int_{z(x^*,\hat{r})}^{+\infty}\frac{(r-r^0)\frac{dx^*}{d\alpha}}{[x^*(r-r^0)-\hat{r}+r^0]^\gamma}f(r)dr\right\} \\
&-\frac{1}{1-\gamma}\left[-\lambda\int_{-\infty}^{z(x^*,\hat{r})}\left[x^*(r^0-r)+\hat{r}-r^0\right]^{1-\gamma}f(r)dr + \int_{z(x^*,\hat{r})}^{+\infty}\left(x^*(r-r^0)-\hat{r}+r^0\right)^{1-\gamma}f(r)dr\right] \\
&= \frac{dx^*}{d\alpha}\left\{\alpha c\frac{s-s^0}{[s^0+x^*(s-s^0)]^\delta} + (1-\alpha)\left[-\lambda\int_{-\infty}^{z(x^*,\hat{r})}\frac{r^0-r}{(x^*(r^0-r)+\hat{r}-r^0)^\gamma}f(r)dr\right.\right. \\
&\left.\left.+\int_{z(x^*,\hat{r})}^{+\infty}\frac{r-r^0}{(x^*(r-r^0)-\hat{r}+r^0)^\gamma}f(r)dr\right]\right\} + c\frac{[s^0+x^*(s-s^0)]^{1-\delta}}{1-\delta} \\
&-\frac{1}{1-\gamma}\left[-\lambda\int_{-\infty}^{z(x^*,\hat{r})}\left[x^*(r^0-r)+\hat{r}-r^0\right]^{1-\gamma}f(r)dr + \int_{z(x^*,\hat{r})}^{+\infty}\left(x^*(r-r^0)-\hat{r}+r^0\right)^{1-\gamma}f(r)dr\right] \\
&= c\frac{[s^0+x^*(s-s^0)]^{1-\delta}}{1-\delta} \\
&+\frac{1}{1-\gamma}\int_{-\infty}^{z(x^*,\hat{r})}\left[x^*(r^0-r)+\hat{r}-r^0\right]^{1-\gamma}f(r)dr\left[\lambda - \frac{\int_{z(x^*,\hat{r})}^{+\infty}[r-z(x^*,\hat{r})]^{1-\gamma}f(r)dr}{\int_{-\infty}^{z(x^*,\hat{r})}[z(x^*,\hat{r})-r]^{1-\gamma}f(r)dr}\right]
\end{aligned} \tag{A.74}$$

Let $\delta < 1$. Note that a sufficient condition for $\frac{d\mathbb{E}(U(x^*))}{d\alpha} > 0$ is that the last term in (A.74) is positive, i.e., when

$$\lambda > \frac{\int_{z(x^*,\hat{r})}^{+\infty}[r-z(x^*,\hat{r})]^{1-\gamma}f(r)dr}{\int_{-\infty}^{z(x^*,\hat{r})}[z(x^*,\hat{r})-r]^{1-\gamma}f(r)dr} \tag{A.75}$$

Let $\hat{r} > r^0$. Then it is easy to show that[46]

$$\frac{\int_{z(x^*,\hat{r})}^{+\infty}[r-z(x^*,\hat{r})]^{1-\gamma}f(r)dr}{\int_{-\infty}^{z(x^*,\hat{r})}[z(x^*,\hat{r})-r]^{1-\gamma}f(r)dr} < \frac{\int_{r^0}^{+\infty}\left[r-r^0\right]^{1-\gamma}f(r)dr}{\int_{-\infty}^{r^0}[r^0-r]^{1-\gamma}f(r)dr} = \frac{1}{K_\gamma}$$

As we assume that $\lambda > \frac{1}{K_\gamma}$ then for $\hat{r} > r^0$ is $\frac{d\mathbb{E}(U(x^*))}{d\alpha} > 0$. Let $\hat{r} < r^0$. Then

$$\frac{\int_{z(x^*,\hat{r})}^{+\infty}[r-z(x^*,\hat{r})]^{1-\gamma}f(r)dr}{\int_{-\infty}^{z(x^*,\hat{r})}[z(x^*,\hat{r})-r]^{1-\gamma}f(r)dr} > \frac{\int_{r^0}^{+\infty}\left[r-r^0\right]^{1-\gamma}f(r)dr}{\int_{-\infty}^{r^0}[r^0-r]^{1-\gamma}f(r)dr} = \frac{1}{K_\gamma}$$

so in this case condition $\lambda > \frac{1}{K_\gamma}$ does not guarantee (A.75). However, for $s > s^0$ is $x^* > \hat{x}$, where $\hat{x}$ is the

[46]The following is implied from $z(x^*,\hat{r}) > r^0$, as $\hat{r} > r^0$, and from the fact that function $\int_z^\infty (r-z)^{1-\gamma}f(r)dr$ is decreasing in z.

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solution of (4) for $\alpha = 0$, then $z(x^*, \hat{r}) = r^0 - \frac{r^0 - \hat{r}}{x^*} > r^0 - \frac{r^0 - \hat{r}}{\hat{x}} = z(\hat{x}, \hat{r})$ and thus

$$\frac{\int_{z(x^*,\hat{r})}^{+\infty} [r - z(x^*, \hat{r})]^{1-\gamma} f(r)dr}{\int_{-\infty}^{z(x^*,\hat{r})} [z(x^*, \hat{r}) - r]^{1-\gamma} f(r)dr} < \frac{\int_{z(\hat{x},\hat{r})}^{+\infty} [r - z(\hat{x}, \hat{r})]^{1-\gamma} f(r)dr}{\int_{-\infty}^{z(\hat{x},\hat{r})} [z(\hat{x}, \hat{r}) - r]^{1-\gamma} f(r)dr}$$

Thus

$$\lambda > \frac{\int_{z(\hat{x},\hat{r})}^{+\infty} [r - z(\hat{x}, \hat{r})]^{1-\gamma} f(r)dr}{\int_{-\infty}^{z(\hat{x},\hat{r})} [z(\hat{x}, \hat{r}) - r]^{1-\gamma} f(r)dr}$$

is a sufficient condition for $\frac{d\mathbb{E}(U(x^*))}{d\alpha} > 0$ when $\hat{r} < r^0$. □

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Appendix B. Appendix: Further simulation results

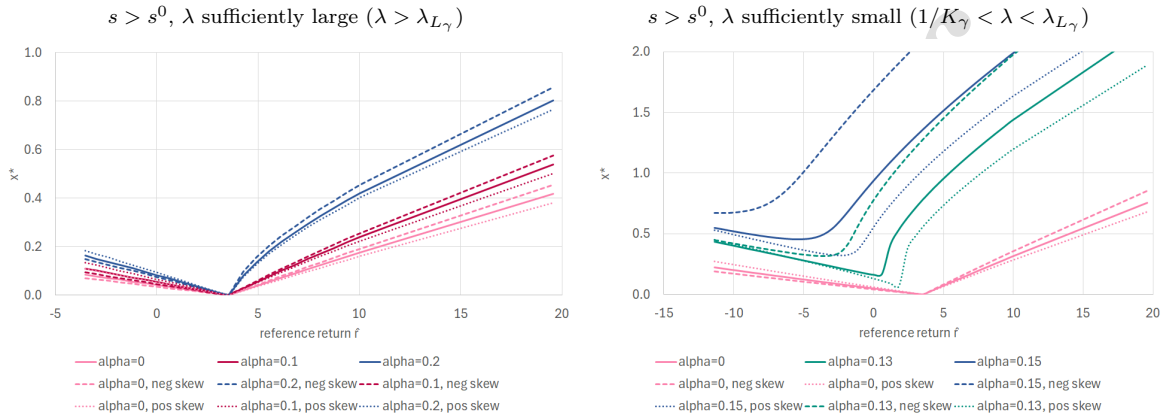


Figure B.1: Risk-taking, x^*, as a function of the reference return, $\hat{r}$, for $\lambda = 2.25$, $\gamma = 0.1$, $\delta = 0.5$, $s^0 = 0$, $s = 5$, and $\alpha = 0, 0.1, 0.15, 0.2$.
The horizontal axis shows the reference return in percent.
The results are shown for returns of the risky asset following the the normal and the (negative/positive) skew normal distributions, where $\mathbb{E}(r) = 13.6\%$, $Var(r) = 18.4\%$, and the shape parameter is ± 3.1, resulting in a skewness of ± 0.68. The risk-free rate $r^0 = 3.5\%$.

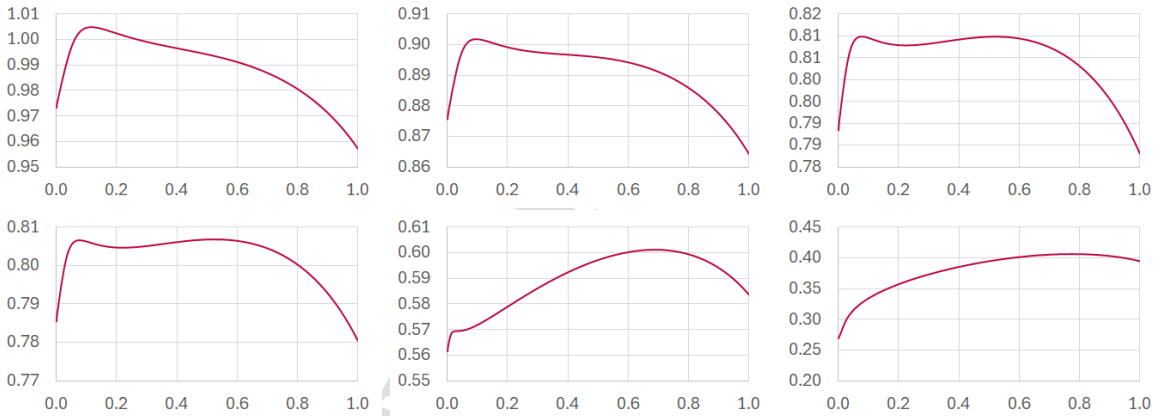


Figure B.2: Expected utilities showing two local maxima for certain reference returns $\hat{r}$.
$\lambda = 1.3$, $\gamma = 0.1$, $\delta = 0.5$, $r^0 = 0.29$, $s = 1$, $s^0 = 5$, $\alpha = 0.1$.
$\hat{r} = -0.2, -0.1, -0.013, -0.01, 0.2, 0.4$.
Horizontal axis presents risk-taking x, the vertical axis presents expected utility $\mathbb{E}(U(x))$.

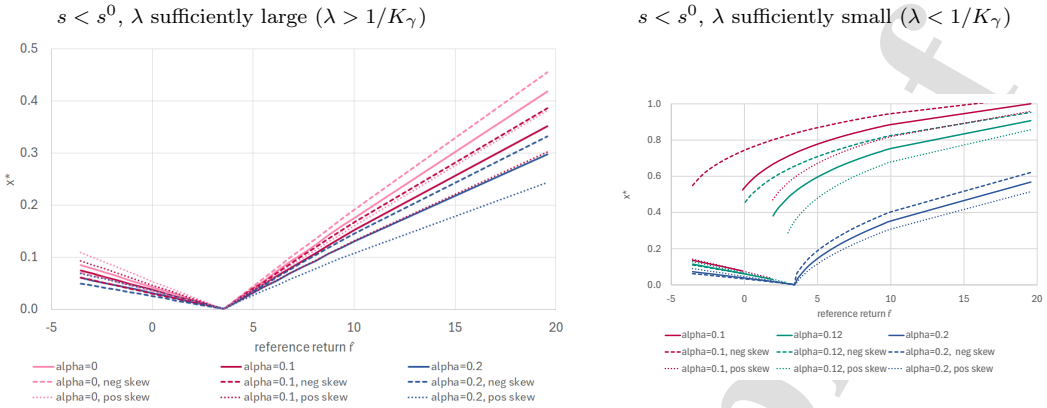


Figure B.3: Risk-taking, x^*, as a function of the reference return, $\hat{r}$, for $\lambda = 2.25$, $\gamma = 0.1$, $\delta = 0.5$, $s^0 = 5$, $s = 1$, and $\alpha = 0, 0.1, 0.15, 0.2$. Note that in the right graph $x^* = +\infty$ for $\alpha = 0$.
The horizontal axis shows the reference return $\hat{r}$ in percent.
The results are shown for returns of the risky asset following the the normal and the (negative/positive) skew normal distributions, where $\mathbb{E}(r) = 13.6\%$, $Var(r) = 18.4\%$, and the shape parameter is ± 3.1, resulting in a skewness of ± 0.68. The risk-free rate $r^0 = 3.5\%$.

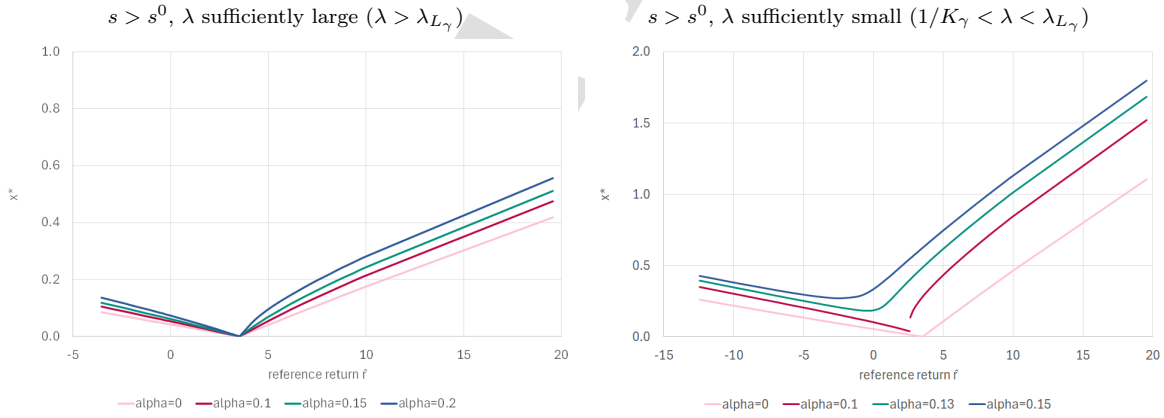


Figure B.4: Risk-taking, x^*, as a function of the reference return, $\hat{r}$, for $\lambda = 2.25$ (left), $\lambda = 1.7$ (right), $\gamma = 0.1$, $\delta = 1.1$, $s^0 = 1$, $s = 5$, and $\alpha = 0, 0.1, 0.13, 0.15, 0.2$. Note that $\frac{1}{K_\gamma} = 1.56$ and $\lambda_{L_\gamma} = 1.56, 1.79, 1.87, 1.93$ and 2.09 for $\alpha = 0$, 0.1, 0.13, 0.15, and 0.2, where $\lambda_{L_\gamma} = \frac{1}{K_\gamma} + \frac{\alpha c}{1-\alpha} \cdot \frac{(s-s^0)^{1-\gamma}}{\int_{-\infty}^{r^0}(r^0-r)^{1-\gamma} f(r)dr} \left(\frac{\delta-\gamma}{s^0}\right)^{\delta-\gamma} \frac{\gamma^\gamma}{\delta^\delta}$.
The horizontal axis shows the reference return $\hat{r}$ in percent. The results are shown for normally distributed returns of the risky asset, where $\mathbb{E}(r) = 13.6\%$, $Var(r) = 18.4\%$ and $r^0 = 3.5\%$.

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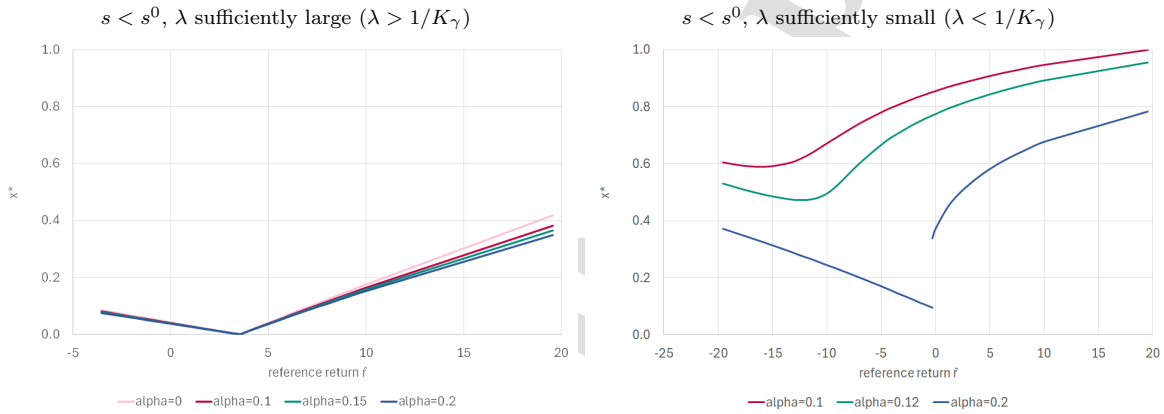


Figure B.5: Risk-taking, x^*, as a function of the reference return, $\hat{r}$, for $\lambda = 2.25$ (left), $\lambda = 1.3$ (right), $\gamma = 0.1$, $\delta = 1.1$, $s^0 = 5$, $s = 1$, and $\alpha = 0, 0.1, 0.15, 0.2$. Note that $\frac{1}{K_\gamma} = 1.56$ and in the right graph $x^* = +\infty$ for $\alpha = 0$.
The horizontal axis shows the reference return $\hat{r}$ in percent. The results are shown for normally distributed returns of the risky asset, where $\mathbb{E}(r) = 13.6\%$, $Var(r) = 18.4\%$ and $r^0 = 3.5\%$.

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Sustainable investment under prospect theory: 2-assets case*

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Abstract

Sustainable investing, also known as ESG (environmental, social, and corporate governance) investing, is currently of intense interest in the investment world. We consider an economy with a risky and a risk-free asset. Utility is defined as the weighted sum of a prospect theory value function and an ESG score function. In addition to greenness of the risky asset, we explicitly consider the ESG score of the risk-free asset. We derive optimal investment strategies, which depend on the relative greenness of the risky to the risk-free assets. An extensive sensitivity analysis is performed that provides properties of the investor's risk taking and the indirect utility (happiness) with respect to the reference return, loss aversion, the ESG scores of the risky and the risk-free assets, and the weight of the ESG score function. Importantly, we show that the conventional positive relationship between an asset's ESG score and its demand does not hold generally: sufficiently strong ESG diminishing sensitivity can reverse the effect, depending on the relative ESG scores of the risky and the risk-free assets and the investor's portfolio leverage. If the ESG score function becomes more important, then investment into the greener asset increases. An increase in loss-aversion decreases investment into risky asset. Finally, we observe that investment into the risky asset is first decreasing and then increasing with the reference return, as also observed in ESG-neutral investment, i.e., in pure prospect theory.

Keywords: Sustainable finance, green finance, ESG investing, prospect theory, behavioral finance, portfolio optimization
JEL: G11, G41, Q56

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Highlights

- Theoretical investment decision analysis of sustainable investment through the lens of prospect theory for one risky and one risk-free asset
- The link between the ESG scores of the risk-free and risky assets affects optimal investment strategy
- Investment into the greener asset increases with the weights of ESG preferences
- V-shaped relationship between risky asset investment and reference return, consistent with pure prospect theory
- Investor decreases her exposures to risky asset with increasing loss aversion

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Declaration of Interest Statement

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