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# Venture Capital Project Selection as an Optimal Stopping Problem

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## Abstract

We present a stylized model of a venture capital firm being active in a specific market, where all possible projects compete. These projects have a payoff structure such that only the best few alternatives generate a positive return. The payoff depends on the relative ranking of a project's quality in comparison with all other projects. This discontinuous payoff structure is motivated by the internet economy, where classical price competition is often not possible, and market share is chiefly determined by the relative quality ranking of the product, its absolute quality being of second order. We assume that investment alternatives show up sequentially, and, due to competition, the investment decision cannot be postponed for too long. The information available at any period is the current project's and all prior projects' quality characteristics. Finally, after the last investment alternative has shown up, those  $n$  projects with the highest realized quality characteristics generate positive gross returns. We derive a recursive formulation of the value function. Sufficient conditions under which the value function is non-increasing in the number of periods are provided. A simulation study demonstrates how optimal investment decisions are affected by the time horizon and by attitudes towards risk. The model presented is an extension of Bruss and Ferguson (2002).

*Keywords:* Economics of superstars, Optimal stopping, High-risk investments

*JEL Classification:* C60, C81, G24

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## 1 Introduction and Motivation

Typically, the decision making process of venture capital firms is assumed to consist of the following stages (see, e.g., Fried and Hisrich, 1994): A venture capital firm receives business proposals from potential entrepreneurs. After a first screening, if the proposal looks promising to the venture capital firm, it asks for a more detailed business plan, otherwise the proposal is rejected. Endowed with the business plan, further evaluation rounds follow, where the venture capital firm has a closer look at the project, asks for additional information, and evaluates the managerial skills of the prospective entrepreneur(s). Finally, after the screening stages are completed with a positive evaluation, the venture capital firm provides (usually) equity capital, given that both parties agree on the terms and conditions. Typically, the project is evaluated after several time points stipulated in the contract. In the case of a positive evaluation, the venture capital firm often provides additional funds. Usually, the business relationship of the venture capital firm and the entrepreneur lasts for a limited time horizon. At the end of this investment horizon, the venture capitalist sells its shares either to the entrepreneur, other investors, or by means of an initial public offering (given that the venture in question successfully survives until then). The profit or loss of the venture capital firm arises from the difference between the payoff generated by selling the share in the firm and the money provided in the various investment rounds. In a survey study, Gompers et al. (2020) analyze the venture capital selection process. In particular, their article provides information on important criteria in the selection of investments and post-investment monitoring. More recently, by considering a survey of approximately 8,000 venture capital deals, Jang and Kaplan (2025) observed that venture capital has selection ability.

One important field of venture capital activities is investing in the high-tech and internet economy (see, e.g., Gompers et al., 2020). In contrast to “classical markets”, in the internet economy, a tiny number of firms acquire a substantial share of the whole world market. Most competitors fail or leave the market quite rapidly. For many markets in the internet economy, where classical price competition is often not possible. A few examples are music streaming, news websites, and internet marketplaces. These markets allow for worldwide supply at zero or very low marginal cost. In parallel, consumers can often choose the best alternative according to their taste at no higher cost than the second best, third best, etc. Typically, prices for streaming a film or listening to music on a particular platform are fixed, and hence there is no reason to stream the second best film or piece of music instead of the best, etc. How big the difference in

quality between the films, orchestras, etc., is, is of very little importance. Therefore, the relative ranking between competing artists, films, services, etc., is much more important than the absolute quality of these products. Hence, these investments have the so-called “superstar property”, where only a small number of investment projects generate a very high return, while most investment projects fail.

According to the empirical literature, a lot of venture capital investments fail even though they get financed. That is, although the project obtained a good evaluation, it need not ultimately be successful in the medium or long run (see, e.g., Moskowitz and Vissing-Jørgensen, 2002, where in their sample approximately 2/3 of the firms fail within the first 10 years).

We approximate the selection process as follows: A venture capital or private-equity firm that is highly active in a particular emerging high technology market is considered. The resources available to spend in the investment horizon are already provided by an investment fund and are considered to be exogenous. Entrepreneurs submit their business proposals. Business proposals arrive sequentially, and the venture capital firm cannot wait too long with its decision, either due to competition with other venture capitalists or time constraints of the entrepreneur (who gives up if the project is not sufficiently financed). In this article, we investigate the *full-information case*, where at every point in time, the quality of the investment opportunities observed so far is perfectly known by the investor. The case where an investor can only rank the projects observed so far, was investigated in Meier and Sögner (2023) and is called the *limited-information case*. Intuitively, compared to Meier and Sögner (2023), the investor in the current paper has much more expertise, and correctly understands the absolute quality level of a particular project proposal.

To account for the payoff structure of superstar projects discussed above, we assume that the payoffs are rank dependent. Rank dependent payoffs can be motivated by winner-takes-it-all markets discussed in the economics of superstars literature (see, e.g., Panes, 2018; Frank and Cook, 1995), where the payoff of an investment project is strongly related to the rank of the quality of a product relative to the qualities of competing products.

Since the financial resources of a venture capital investor are limited, optimal investment decisions should also take care of the opportunity cost arising from investing. For example, business proposals for a search engine hardly show up at the same time. If a good proposal for a search engine shows up, the investor also has to consider the chance that this project is replaced by a better alternative which shows up in one of the next periods. Our stylized model tries to account for this problem by considering a

sequential decision process, where the investor is aware that a better alternative shows up with a certain probability.<sup>1</sup> Thus, the optimal investment problem translates to an optimal stopping problem. To the best of our knowledge the models proposed in the Economics and Finance literature do not look at this aspect of the investment problem.

Most of the related literature on venture capital investing concentrates on the evaluation process of projects, how venture capital firms evaluate whether the proposed idea can be successful in the market, whether the project team has sufficient technical skills and management capabilities, etc. (see, e.g., Rin et al., 2013, for a survey). Criteria that are important to the venture capital firms are the entrepreneur's personality, the entrepreneur's experience, characteristics of the product or service, characteristics of the market, financial considerations (a return on investment of at least ten times the initial investment within five to ten years, investment can be made liquid easily, etc.), see, e.g., Macmillan et al. (1985). Tyebjee and Bruno (1984, Figure 2) provide a graphical description of the venture capital investment decision process. Their model consists of three steps. In Step 1, market attractiveness and product differentiation of the prospective product is evaluated, leading to an assessment of the expected return. In the same step, also the managerial capabilities and the resistance to environmental threats are analyzed, resulting in an assessment of the perceived risk. Assessing the expected return and perceived risk constitutes Step 2, leading in Step 3 to the investment decision. In our model, the investor receives a quality signal; the risk arising from the environment approximately corresponds to the event that better alternatives emerge. Further sources of risk (caused, e.g., by variations in managerial skills) are modeled by stochastic payoffs. Fried and Hisrich (1994) report a survey to construct a six-step model of the venture capital decision making process. Their Table 2 provides relative frequencies of the evaluation criteria used by venture capitalists. A more theoretical model of venture capital investing is Alperovych et al. (2026), where equilibrium contracts between investors and entrepreneurs are obtained. These contracts depend on the risk-attitude of the entrepreneur.

As stated above, the investment problem considered in this article is different from investment problems considered in Finance. To see this, when considering investment projects in Finance, in particular in portfolio choice problems, the returns of  $n$  assets, are either described by some assumptions on the mean

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<sup>1</sup>In our model the quality signals are *iid* draws from the uniform distribution on the unit interval. Suppose that the realization of a draw is  $x$ , then the probability that the next draw is better than the current draw is  $1 - x$ . If there are  $k$  investment steps to go, the probabilities of the ranks of the quality signals – given the current and all prior quality signals – are obtained in Section 2. Appendix A shows that these probabilities follow from a multinomial distribution.

and the covariance in the static case (see Markowitz, 1952), or by some assumptions on a stochastic process driving the payoffs in a discrete or continuous time dynamic model; a typical example is a geometric Brownian motion. Various extensions, for example with jumps, state dependence, learning, etc., are provided in the literature. For a brief overview, see e.g., Brandt (2009) or Back (2017). However, these models have in common that payoffs are not rank dependent.

Opp (2019) developed a macro-finance-type endogenous growth model (see, e.g., Aghion and Howitt, 1992), augmented by venture capital financing. The model allows one to derive prices and returns, and is able to explain some stylized facts related to economic growth, innovations, and venture capital investing. In more detail, the most productive entrepreneur is selected (see Section 1.3.3 and 1.3.4 in Opp, 2019), while (price taking) venture capital firms are maximizing their profits by choosing the quantity of ideas they fund (the intensity of a Poisson arrival process in more technical terms; see Section 1.4.3 in Opp, 2019). While the model of Opp (2019) has a stronger macro-finance perspective on the venture capital market, our model obtains optimal investment strategies in a stylized model which is closer to the decision process of an individual decision maker investing into risky projects with a “superstar payoff structure”.

The Mathematics literature related to our investment problem is concerned with optimal stopping problems where the payoff structure is not continuous, and which are closely related to Robbins’ problem (see, e.g., Bruss, 2005, for an overview). The article closest to the current paper is Bruss and Ferguson (2002), where winner-takes-it-all investment problems are analyzed (i.e., only the eventually best alternative is profitable), for the limited as well as for the full-information case, respectively. The limited-information case – where the eventually two best alternatives are profitable and where the investor is only able to rank the projects observed – is solved in Lebek and Szajowski (2007). The risk-neutral case for multiple winners is solved in Mucci (1973). An extension to multiple winners, with a possibly risk-averse investor, and stochastic returns, is provided in Meier and Sögner (2023). In the article at hand, we consider the full-information case, allow for stochastic returns, and extend the results obtained in Bruss and Ferguson (2002) to the case where multiple alternatives can result in a positive payoff. Compared to the limited information case investigated in Meier and Sögner (2023), we observe that the optimal strategies in the full information case become more history dependent (depending on the realizations of

the signals observed by the investor).<sup>2</sup> By that, the numerical complexity increases significantly when implementing our investment model. For example, Bruss and Ferguson (1996, Section 4.2) show that the optimal stopping rule exhibits full history dependence when the expected rank is minimized.

Summing up, in contrast to standard investment models, modeling a sequential investment process as an optimal stopping problem has received – to the best of our knowledge – no attention in the economics and finance literature. In the mathematics literature there is the notable exception of Bruss and Ferguson (2002) and Lebek and Szajowski (2007). In our framework, the investor sequentially observes investment opportunities and receives a quality signal for each project. The uncertainty faced by the investor mainly arises from the possibility that superior investment opportunities may become available later. In this article, we consider the full-information case, allow for stochastic returns, and extend the model of Bruss and Ferguson (2002) to settings in which multiple investment opportunities can generate positive payoffs. Compared with the limited-information framework developed in Meier and Sögner (2023), the optimal investment strategy under full information becomes substantially more history-dependent, as decisions condition on the realized quality signals of previously observed projects. This richer dependence on the observed history also leads to significantly greater computational complexity. In addition, we observe that a risk-neutral investor loses some investment with a surprisingly low probability in our simulation study (approximately 2.2% in an economy where 50 projects are considered). Taken together, the results in Meier and Sögner (2023) and the present article make it possible to quantify the value of information. Using simulation studies, the value of information can be measured as the difference between the certainty equivalents of a full-information investor and a limited-information investor under otherwise identical model assumptions. We derive a recursive characterization of the value function and establish sufficient conditions under which the value function is non-increasing in the remaining number of decision periods. Finally, a simulation study illustrates how optimal investment decisions depend on both the investment horizon and the investor’s degree of risk aversion.

The remainder of this article is organized as follows: Section 2 describes the investment problem in more detail and provides a recursive solution in the general case of an investor with monotone increasing and continuous Bernoulli utility function. We provide sufficient conditions for the value function to

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<sup>2</sup>To compare the limited to the full information case, Meier and Sögner (2023) obtain optimal strategies for  $n = 2$  investment alternatives and  $T = 3$  periods for the full and the limited information case, respectively. The investors are assumed to be risk-neutral. The solution for the full information case was obtained by direct calculations.

be non-increasing in the number of periods  $T$ . Section 3 investigates the risk-neutral case for  $n \geq 1$  investment alternatives and deterministic returns. Section 4 considers stochastic returns. Finally, Section 5 presents results from a simulation study to investigate how optimal investment behavior changes with the attitude towards risk and the number of time periods considered. We consider returns based on return distribution estimates for the venture capital market obtained in Cochrane (2005). The results obtained in Meier and Sögner (2023) and in this article allow to calculate (by means of simulation studies) the value of information, that is the difference between the certainty equivalents of a full-information and a limited-information investor. Proofs that do not appear in the main text can be found in Appendix A to Appendix E.

## 2 The Investment Problem

In this section we first present a stylized model of the venture capital investment process. We consider  $\tau = 1, \dots, T$  periods of time, where for notational convenience  $k := T - \tau + 1$  denotes the number of *remaining* periods (here we follow Bruss and Ferguson, 2002). An investor is endowed with initial wealth  $\underline{w} = w_T > 0$ . At each period of time, investment opportunities show up, the best of which is described by the quality characteristic  $X_k$ . That is, if  $m_k$  projects show up in period  $k$ ,  $X_k$  is the best alternative among these  $m_k$  business proposals. We assume that  $m_k > 0$ , which is reasonable given that the length of the decision period is long enough. In mathematical terms, this is modeled by means of uniformly *iid* random variables  $X_k$  on the unit interval  $[0, 1]$ . Hence, given a realization  $\{X_k = x_k\}$  and a  $v < k$ , the probability that  $X_v > x_k$  is  $1 - x_k$ , the probability that  $X_v < x_k$  is  $x_k$ , and probability that  $X_v = x_k$  is zero.

Depending on the frequency considered, a venture capital firm receives a lot of business plans in period  $k$ , while in our stylized model we consider one investment opportunity per point of time  $k$ . Note that even if more projects show up in period  $k$ , the investor only has to consider the best alternative among the potential opportunities (given that there are no diversification effects among these projects and that the best project generates the highest expected utility, the second best the second highest expected utility,

etc.).<sup>3</sup> Hence, it is sufficient to work with one project per time period and  $X_k$  is interpreted to be the best alternative observed in period  $k$ .

After observing a realization of  $X_k$ , denoted by  $x_k$  (for short  $\{X_k = x_k\}$ ), the investor can invest any amount  $b_k$ , with  $0 \leq b_k \leq w_k$  into the risky project,  $w_k$  is the amount of funds not yet invested. We assume that at time  $k$  the investor remembers past realizations  $x_j$  and past investments  $b_j$ ,  $j > k$ , that is, the history up to time  $k$ . Due to competition, or because the entrepreneur may leave the market if not financed, we assume that the venture capital firm has to decide rather quickly, that is, in period  $k$ .

By investing  $b_k$  the remaining amount to invest in future opportunities becomes  $w_{k-1} = w_k - b_k$ . The relative rank of the draw  $x_j$ , given the observations  $x_T, \dots, x_{k+1}, x_k$ , is  $\rho_k(x_j) := \sum_{m=k}^T \mathbf{1}_{\{x_m \geq x_j\}}$ , such that the best draw arising in  $T, \dots, k$  has relative rank 1 and the second best rank 2, etc.<sup>4</sup> The final or absolute rank is  $\rho_a(x_j) := \sum_{m=1}^T \mathbf{1}_{\{x_m \geq x_j\}}$ .

In the next step, we describe the returns, which are realized after the investment process is completed. In our notation, returns realize at  $k = 0$  (that is, after  $T$  on the forward counting time scale), when all proposals have been observed. The gross-returns per unit invested are described by  $\beta_k^j$ . In more detail,  $\beta_k^j$  abbreviates the return per unit invested in period  $k$ , if the draw of period  $k$  becomes the overall  $j$ -th best draw at  $k = 0$ . In this section returns are non-stochastic, while in Section 4,  $\beta_k^j$  is replaced by the random variable  $B_k^j$ ,  $B_k^j \geq 0$ . These random variables only realize at  $k = 0$ . Suppose that the best  $n$  projects,  $n \leq T$ , yield a strictly positive return. The returns per unit invested are described by  $\beta_{(n,T)} := (\beta_k^i)_{k=T, \dots, 1; i=1, \dots, n}$ , respectively  $\mathbf{B}_{(n,T)} := (B_k^i)_{k=T, \dots, 1; i=1, \dots, n}$  for the stochastic case.

The investor maximizes expected utility. The Bernoulli utility function  $u(\cdot)$  is monotone increasing and continuous.<sup>5</sup> Suppose there are  $k$  steps to go: The wealth still available to invest is  $w = w_k$ .  $z_1^k$  is the current best draw and  $z_2^k$  is the current second-best draw of the  $X_k$ ,  $\kappa = T, \dots, k+1$ , etc.<sup>6</sup>

That is,  $z_1^k$  denotes the best quality characteristic of an investment opportunity observed so far (before

<sup>3</sup>According to anecdotal evidence a venture capital firm receives around 2,000 business plans per year. 10% of them are checked more carefully and 10% of these 200 receive venture capital financing. See, e.g., <https://corporatefinanceinstitute.com/resources/knowledge/other/how-vcs-look-at-startups-and-founders/>. That is, given that an investment horizon (e.g., one year),  $T = 5, 10, 20$  assumes that within this horizon the decision maker considers the best alternative, abbreviated by  $X_k$ , within each of these blocks  $k = T, \dots, 1$ .

<sup>4</sup> $\mathbf{1}_{\{x_m \geq x_j\}}$  denotes an indicator function.

<sup>5</sup>To avoid problems associated with the domain of  $u(\cdot)$  (e.g., with the log-utility), we assume that the interval  $(\beta, \underline{w}\bar{\beta})$  is a subset of the domain of  $u(\cdot)$  if  $\underline{\beta} < \beta_k^j < \bar{\beta}$ , for all  $k$  and  $j$ , or the interval  $[\underline{\beta}, \underline{w}\bar{\beta}]$  is a subset of the domain of  $u(\cdot)$  if  $\underline{\beta} \leq \beta_k^j \leq \bar{\beta}$ , for all  $k$  and  $j$ . For half-open constraints on  $\beta_k^j$  we can proceed in the same way.

<sup>6</sup>Superscripts in "typewriter notation" like  $k$  in  $z_j^k$  (and later in  $y_{i|\ell}^k$ ) denote the number of time steps to go. If  $k$  is an exponent then we write  $(z_j^k)^k$ , this notation is only used in the Appendix.

$X_k$  realizes),  $z_2^k$  the second best, etc. The currently best, second best, etc. draw arises at  $k_1, k_2$ , etc. In more formal terms,

$$\begin{aligned} z_j^k &:= x_m \text{ such that } m \geq k+1 \text{ and } \rho_{k+1}(x_m) = j, \\ k_j &:= m \text{ such that } m \geq k+1 \text{ and } \rho_{k+1}(x_m) = j, \text{ for } j = 1, \dots, n, \\ \mathbf{z}_k &:= (z_1^k, \dots, z_n^k). \end{aligned} \tag{1}$$

In our stylized investment model, only the  $n$  best projects can generate a positive return, where  $n \leq T$ . In a so-called superstar economy  $n$  is much smaller than  $T$ .

The assumption that the venture capitalist has to invest immediately (at the end of the period in which the proposal is submitted) can be motivated as follows: Due to competition or due to the fact that the entrepreneur decides to give up if not sufficiently financed, the investor must decide whether to finance or not within the period in which the proposal is made. As already mentioned above, a positive evaluation of the project need not result in a positive return in the long run.

Typically, there are also intermediate financing rounds. In our stylized model the investor has to decide in the period in which the proposal is submitted, while the actual return of the project is realized after all investment alternatives were observed. The investment in some period is a shortcut for the first financing decision and the intermediate financing rounds, while the realization of the return at the end, approximates the fact that returns are realized after some years, when the venture capital contract expires, or earlier by bankruptcy, or the decision of the venture capitalist to stop financing.

To describe the payoffs in a mathematically precise manner, we construct the vector of potential payoffs. That is, we describe the  $n(n+1)/2$ -dimensional vector of potential payoffs  $\mathbf{y}_k$  coming from past investments.

Let  $y_{i|\ell}^k \geq 0$  denote the payoff of the current  $i$ -th best alternative (before  $X_k$  realizes) if it becomes  $\ell$ -th best in the end. For example,  $y_{1|1}^k = \beta_{k_1}^1 b_{k_1}$  is the payoff of the current best alternative (i.e.  $x_{k_1}$ , where  $k_1 > k$ ) if it remains the best in the end,  $y_{1|2}^k = \beta_{k_1}^2 b_{k_1}$  is the payoff of the current best alternative if it becomes the second best in end,  $y_{2|2}^k = \beta_{k_2}^2 b_{k_2}$  is the payoff of the current second-best alternative if it remains the second-best in the end, etc.

Note that  $y_{i|\ell}^k$  where  $\ell < i$  is infeasible (since “better-ranked draws remain better ranked”), while  $y_{i|\ell}^k = 0$ , for all  $\ell > n$ , by the assumption that only the best  $n$  projects can result in a positive return. If

$T - k < n$  the corresponding  $z_j^k$  and  $y_{j|}^k$ , with  $T - k < j \leq n$ , are set to zero. Let  $\ell_i$  denote the final rank of the current  $i$ -th best alternative when there are  $k$  periods to go; we set  $\ell_0 = 0$ . Observe that  $\ell_i \geq i$  and for any pair  $y_{i|\ell_i}^k, y_{j|\ell_j}^k$ , where  $i < j$ , we get  $i \leq \ell_i \leq k + i, j \leq \ell_j \leq k + j$  and  $\ell_i < \ell_j$ . By considering the current best  $n$  draws, we obtain  $\frac{n(n+1)}{2}$  terms  $y_{j|\ell_j}^k \geq 0$  and the lower triangular matrix [the remaining upper triangular part of this matrix is zero, since  $y_{j|\ell_j}^k = 0$  for all  $\ell_j < j$ ]

$$\begin{pmatrix} \sum_{j=k+1}^T \mathbf{1}_{\{\rho_{k+1}(x_j)=1\}} \beta_j^1 b_j & & & & \\ \sum_{j=k+1}^T \mathbf{1}_{\{\rho_{k+1}(x_j)=1\}} \beta_j^2 b_j & \sum_{j=k+1}^T \mathbf{1}_{\{\rho_{k+1}(x_j)=2\}} \beta_j^2 b_j & & & \\ \sum_{j=k+1}^T \mathbf{1}_{\{\rho_{k+1}(x_j)=1\}} \beta_j^3 b_j & \sum_{j=k+1}^T \mathbf{1}_{\{\rho_{k+1}(x_j)=2\}} \beta_j^3 b_j & \sum_{j=k+1}^T \mathbf{1}_{\{\rho_{k+1}(x_j)=3\}} \beta_j^3 b_j & & \\ \vdots & \vdots & \vdots & \ddots & \\ \sum_{j=k+1}^T \mathbf{1}_{\{\rho_{k+1}(x_j)=1\}} \beta_j^n b_j & \sum_{j=k+1}^T \mathbf{1}_{\{\rho_{k+1}(x_j)=2\}} \beta_j^n b_j & \sum_{j=k+1}^T \mathbf{1}_{\{\rho_{k+1}(x_j)=3\}} \beta_j^n b_j & \cdots & \sum_{j=k+1}^T \mathbf{1}_{\{\rho_{k+1}(x_j)=n\}} \beta_j^n b_j \end{pmatrix} = \begin{pmatrix} \beta_{k_1}^1 b_{k_1} & & & & \\ \beta_{k_1}^2 b_{k_1} & \beta_{k_2}^2 b_{k_2} & & & \\ \beta_{k_1}^3 b_{k_1} & \beta_{k_2}^3 b_{k_2} & \beta_{k_3}^3 b_{k_3} & & \\ \vdots & \vdots & \vdots & \ddots & \\ \beta_{k_1}^n b_{k_1} & \beta_{k_2}^n b_{k_2} & \beta_{k_3}^n b_{k_3} & \cdots & \beta_{k_n}^n b_{k_n} \end{pmatrix} = \begin{pmatrix} y_{1|1}^k & & & & \\ y_{1|2}^k & y_{2|2}^k & & & \\ y_{1|3}^k & y_{2|3}^k & y_{3|3}^k & & \\ \vdots & \vdots & \vdots & \ddots & \\ y_{1|n}^k & y_{2|n}^k & y_{3|n}^k & \cdots & y_{n|n}^k \end{pmatrix} =: \mathbf{Y}_k. \quad (2)$$

To obtain the row vector  $\mathbf{y}_k \in \mathbb{R}^{n(n+1)/2}$ , the lower triangular part of  $\mathbf{Y}_k$  is vectorized, in formal terms  $\mathbf{y}_k = (y_{1|1}^k, \dots, y_{1|n}^k, y_{2|2}^k, \dots, y_{2|n}^k, \dots, y_{n-1|n}^k, y_{n|n}^k)$ .

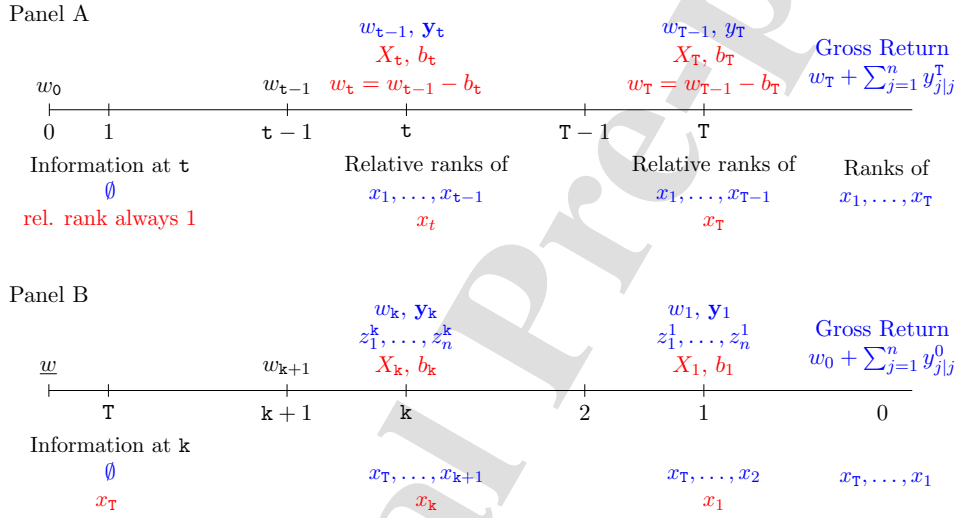
The following paragraph describes the transition of  $\mathbf{y}_k$  to  $\mathbf{y}_{k-1}$ : Depending on the realization of  $X_k$  and the investment decision  $b_k$ ,  $\mathbf{Y}_k$  and  $\mathbf{y}_k$  change to  $\mathbf{Y}_{k-1}$  and  $\mathbf{y}_{k-1}$ . To see this, suppose that at  $k$  we observe  $x_k$  with relative rank  $j$ . Consider first the case  $j \leq n$  and let  $\mathbf{v}^{j:n} = (v_j, \dots, v_n)$ . Define

$$\mathbf{Y}_k((v_j, \dots, v_n)^T) := \begin{pmatrix} y_{1|1}^k & & & & \\ y_{1|2}^k & y_{2|2}^k & & & \\ \vdots & \vdots & \ddots & & \\ y_{1|j}^k & y_{2|j}^k & \cdots & v_j & \\ y_{1|j+1}^k & y_{2|j+1}^k & \cdots & v_{j+1} & y_{j|j+1}^k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_{1|n}^k & y_{2|n}^k & \cdots & v_n & y_{j|n}^k \cdots y_{n-1|n}^k \end{pmatrix} = \begin{pmatrix} y_{1|1}^{k-1} & & & & \\ y_{1|2}^{k-1} & y_{2|2}^{k-1} & & & \\ \vdots & \vdots & \ddots & & \\ y_{1|j}^{k-1} & y_{2|j}^{k-1} & \cdots & y_{j|j}^{k-1} & \\ y_{1|j+1}^{k-1} & y_{2|j+1}^{k-1} & \cdots & y_{j|j+1}^{k-1} & y_{j+1|j+1}^{k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_{1|n}^{k-1} & y_{2|n}^{k-1} & \cdots & y_{j|n}^{k-1} & y_{j+1|n}^{k-1} \cdots y_{n|n}^{k-1} \end{pmatrix} = \mathbf{Y}_{k-1}, \quad (3)$$

where the dimension of  $\mathbf{v}^{j:n}$  determines the column  $j$  where  $\mathbf{v}^{j:n}$  is inserted. The first  $j - 1$  columns are the same as in  $\mathbf{Y}_k$ . Columns  $j$  to  $n - 1$  (of  $\mathbf{Y}_k$ ) are shifted by one position to the right and the elements of those columns that were on the main diagonal before shifting are deleted. The new entries at lower triangular positions of the columns  $j + 1$  to  $n$  are  $y_{j+1|j+1}^k = \beta_{k_j}^{j+1} b_{k_j}, \dots, y_{j|n}^k = \beta_{k_j}^n b_{k_j}, \dots$ ,

$y_{n-1|n}^k = \beta_{k_{n-1}}^n b_{k_{n-1}}$ . We obtain the new vector of payoffs  $\mathbf{y}_{k-1}$  by vectorizing  $\mathbf{Y}_k((v_j, \dots, v_n)^T)$ . For short we write  $\mathbf{y}_k(\mathbf{v}^{j:n})$ . Consider a realization  $x_k \in [z_j, z_{j-1})$  and let  $\beta_k^{j:n} := (\beta_k^j, \dots, \beta_k^n)$ . If the agent invests  $b_k$ , then  $\mathbf{v}^{j:n} = \beta_k^{j:n} b_k$  and the new vector of payoffs is  $\mathbf{y}_{k-1} = \mathbf{y}_k(\beta_k^{j:n} b_k)$  by (3). If  $x_k \leq z_n^k$ , then the relative rank of  $x_k$  is larger than  $n$  and  $\mathbf{y}_{k-1} = \mathbf{y}_k$ . It is understood that in this case our investor does not invest in period  $k$ .

Figure 2 summarizes the venture capital investment model considered in this article. Panel A presents the notation in forward time applied in Meier and Sögner (2023). Panel B presents the time-line in backward time applied in this article.



**Figure 1:** Graphical Illustration of the Investment Process. Panel A forward time scale:  $t, t = 1, \dots, T$ . Panel B - steps to go,  $k, k = T, \dots, 1, 0$ , applied in this article. Observe that  $k = T - t + 1$ . At the  $k$  considered in this figure,  $X_k = X_t$  denotes the investment signal,  $b_k = b_t$  the amount invested, and  $w_{k-1} = w_t$  the remaining wealth at the end of period  $k$ . For any investment  $b_k$ ,  $0 \leq b_k \leq w_k$ , the investor receives  $\beta_k^j$  per unit invested in the end, if the rank of  $x_k$  is  $j \leq n$  in the end. The rank dependent returns  $\beta_k^j$  and the investments  $b_k$  result in the payoffs  $y_{j|k}^k$  defined in (2);  $y_{j|k}^k$  is the payoff, if the current  $j$ -th best alternative is the  $k$ -th best in the end. The investment is  $b_k$ , where  $0 \leq b_k \leq w_k$ , the wealth after investing is  $w_{k-1} = w_k - b_k$ . The transition from  $\mathbf{z}_k$  to  $\mathbf{z}_{k-1} = \mathbf{z}(x_k)$  follows from the realization of the draw  $x_k$  and Equation (1), while the transition from  $\mathbf{y}_k$  to  $\mathbf{y}_{k-1}$  is described in Equation (3). Variables in blue ... before  $X_k$  realizes, variables in red ... at or after the realization of  $X_k$ . Typewriter  $k$  and  $t$  always denote time indices, either when they are used as subscripts or superscripts.

**Remark 1.** An alternative approach to storing  $\mathbf{y}_k$  is to work with the investments  $\mathbf{b}_k = (b_{k_1}, \dots, b_{k_n})$  in the current best projects and the periods where the current best  $n$  projects are observed, that is

$k < k_1, \dots, k_n \leq T$ . By means of  $\mathbf{b}_k$  and  $k_1, \dots, k_n$ ,  $\mathbf{y}_k$  can be obtained in a straightforward way.

In the following steps, we derive the optimal investment strategy. In more detail, we obtain a recursion which describes the optimal strategy. Let  $\mathcal{V}_T(w_T)$  denote the expected utility of the final wealth when applying an optimal investment strategy. When going into  $k = T$ ,  $\mathcal{V}_T(w_T)$  is the expected value before the first draw  $X_T$  realizes, the investor is endowed with the initial budget  $w_T = \underline{w}$ , and the initial investment decision did not take place.  $\mathcal{V}_k(w_k, \mathbf{y}_k, z_1^k, \dots, z_n^k)$  denotes the conditional expected utility of the final wealth when there are  $k$  periods to go. That is,  $\mathcal{V}_k(w_k, \mathbf{y}_k, z_1^k, \dots, z_n^k)$  is the optimal value before the random variable  $X_k$  realizes, the current budget is  $w_k$ , the previous  $n$  best draws are  $z_1^k, \dots, z_n^k$ , and the investment history is described by  $\mathbf{y}_k$ . The value functions  $\mathcal{V}_k(\cdot)$  for  $k = 0$  and  $k = 1$  are

$$\begin{aligned} \mathcal{V}_0(w_0, \mathbf{y}_0, z_1^0, \dots, z_n^0) &= u(w_0 + \sum_{j=1}^n y_{j|j}^0) \text{ and} \\ \mathcal{V}_1(w_1, \mathbf{y}_1, z_1^1, \dots, z_n^1) &= z_n^1 u \left( w_1 + \sum_{j=1}^n y_{j|j}^1 \right) \\ &\quad + \sum_{j=1}^n (z_{j-1}^1 - z_j^1) u \left( w_1 \max\{1, \beta_1^j\} + \sum_{i=1}^{j-1} y_{i|i}^1 + \sum_{i=j}^{n-1} y_{i|i+1}^1 \right), \end{aligned} \quad (4)$$

where  $z_{1-1}^1 = z_0^1 = 1$ . To see this, note that at  $k = 0$  there are no more steps to go and we obtain a wealth of  $w_0 + y_{1|1}^0 + \dots + y_{n|n}^0$ .  $w_0$  is the remaining non-invested budget and  $y_{j|j}^0$  are the payoffs of the first to  $n$ -th best alternatives when there are no further steps to go. If there is still one step to go, where  $k = 1$ , we get  $w_1 + y_{1|1}^1 + \dots + y_{n|n}^1$  if the last draw  $x_1$  is smaller than or equal to  $z_n^1$ . The probability of this event is  $z_n^1$ . With a probability of  $z_{j-1}^1 - z_j^1$  we get a draw in the interval  $[z_j^1, z_{j-1}^1)$ , in which case the current best  $j - 1$  draws remain the  $j - 1$  best,  $x_1$  becomes the  $j$ -th best, while the current  $j$ -th to  $n$ -th best draws become the  $j + 1$  to  $n + 1$ -th best. The payoff arising from investment in the now  $n + 1$ -st best draw is zero, by the model assumptions. At  $k = 1$ , the best strategy is to invest all the remaining budget  $w_1$  into this period's alternative, if and only if the relative (= absolute) rank  $i$  of  $x_1$  is smaller or equal to  $n$  and  $\beta_1^i \geq 1$ . For example, with a probability of  $1 - z_1^1$  the last draw is better than  $z_1^1$ . In this case the former best draw becomes the new second best, yielding the payoff  $y_{1|2}^1$ , while the former second best becomes third best, etc. Note that also in this case it is optimal to invest all the remaining wealth  $w_1$  into the new best alternative if  $\beta_1^1 \geq 1$ .

For the value function we get the recursion

$$\begin{aligned}
\mathcal{V}_k(w_k, \mathbf{y}_k, z_1^k, \dots, z_n^k) &= z_n^k \mathcal{V}_{k-1}(w_k, \mathbf{y}_k, z_1^k, \dots, z_n^k) \\
&+ \int_{z_1^k}^1 \max_{0 \leq b_k \leq w_k} \mathcal{V}_{k-1}(w_k - b_k, \mathbf{y}_k(\beta_k^{1:n} b_k), x, z_1^k, \dots, z_{n-1}^k) dx + \\
&+ \int_{z_2^k}^{z_1^k} \max_{0 \leq b_k \leq w_k} \mathcal{V}_{k-1}(w_k - b_k, \mathbf{y}_k(\beta_k^{2:n} b_k), z_1^k, x, z_2^k, \dots, z_{n-1}^k) dx + \\
&\quad \vdots \\
&+ \int_{z_{n-1}^k}^{z_{n-2}^k} \max_{0 \leq b_k \leq w_k} \mathcal{V}_{k-1}(w_k - b_k, \mathbf{y}_k(\beta_k^{n-1:n} b_k), z_1^k, \dots, z_{n-2}^k, x, z_{n-1}^k) dx \\
&+ \int_{z_n^k}^{z_{n-1}^k} \max_{0 \leq b_k \leq w_k} \mathcal{V}_{k-1}(w_k - b_k, \mathbf{y}_k(\beta_k^{n:n} b_k), z_1^k, \dots, z_{n-1}^k, x) dx \\
&= z_n^k \mathcal{V}_{k-1}(w_k, \mathbf{y}_k, z_1^k, \dots, z_n^k) \\
&+ \sum_{j=1}^n \int_{z_j^k}^{z_{j-1}^k} \max_{0 \leq b_k \leq w_k} \mathcal{V}_{k-1}(w_k - b_k, \mathbf{y}_k(\beta_k^{j:n} b_k), z_1^k, \dots, z_{j-1}^k, x, z_j^k, \dots, z_{n-1}^k) dx, \quad (5)
\end{aligned}$$

with the convention that  $z_0^k = 1$ .

Note that the current value described in (5) is obtained as follows: The first term accounts for draws worse than  $z_n^k$ , in which case  $\mathbf{y}_k = \mathbf{y}_{k-1}$ . The probability of this event is  $z_n^k$ . In addition, the integrals consider expected utility contributions arising if  $X_k \in [z_j^k, z_{j-1}^k)$ . The potential payoffs resulting from the period  $k$  investment,  $b_k$ , are described by  $\beta_k^{j:n} b_k$ . For  $k = T - 1$ , we have  $z_2^{T-1}, \dots, z_n^{T-1} = 0$ ,  $\mathbf{y}_{T-1} = (y_{1|1}^{T-1}, \dots, y_{1|n}^{T-1}, 0, \dots, 0)$ , and

$$\begin{aligned}
\mathcal{V}_{T-1}(w_{T-1}, y_{1|1}^{T-1}, \dots, y_{1|n}^{T-1}, 0, \dots, 0, z_1^{T-1}, 0, \dots, 0) \\
= \int_0^{z_1^{T-1}} \max_{0 \leq b_{T-1} \leq w_{T-1}} \mathcal{V}_{T-2}(w_{T-1} - b_{T-1}, y_{1|1}^{T-1}, \dots, y_{1|n}^{T-1}, \beta_{T-1}^{2:n} \cdot b_{T-1}, 0, \dots, 0, z_1^{T-1}, x, 0, \dots, 0) dx \\
+ \int_{z_1^{T-1}}^1 \max_{0 \leq b_{T-1} \leq w_{T-1}} \mathcal{V}_{T-2}(w_{T-1} - b_{T-1}, \beta_{T-1}^{1:n} \cdot b_{T-1}, y_{1|2}^{T-1}, \dots, y_{1|n}^{T-1}, 0, \dots, 0, x, z_1^{T-1}, 0, \dots, 0) dx, \quad (6)
\end{aligned}$$

while for  $k = T$  we have  $w_T = \underline{w}$  and

$$\mathcal{V}_T(w_T) := \int_0^1 \max_{0 \leq b_T \leq w_T} \mathcal{V}_{T-1}(w_T - b_T, \beta_T^{1:n} \cdot b_T, 0, \dots, 0, x, 0, \dots, 0) dx. \quad (7)$$

The value functions  $\mathcal{V}_{T-j}(w_{T-j}, \mathbf{y}_{T-j}, z_1^{T-j}, \dots, z_j^{T-j}, 0, \dots, 0)$  for  $j < n$  also follow from (5) in the same

way.

Note that for  $k = 1$  (i.e. there is one remaining step to go) the probabilities that  $X_1$  is contained in one of the  $n + 1$  intervals  $[z_1^1, 1]$ ,  $[z_2^1, z_1^1)$ ,  $\dots$ ,  $[z_n^1, z_{n-1}^1)$ ,  $[0, z_n^1)$  are given by  $z_i^1 - z_{i+1}^1$ ,  $i = 0, 1, \dots, n$ , where  $z_0^1 = 1$  and  $z_{n+1}^1 = 0$ , for all  $k$ . Suppose that there are  $k$  steps to go. Let  $\nu_i$ ,  $i = 1, \dots, n + 1$  be the number of realizations of  $X_k$ ,  $k \leq k$ , contained in the  $n + 1$  intervals  $[z_1^1, 1]$ ,  $[z_2^1, z_1^1)$ ,  $\dots$ ,  $[z_n^1, z_{n-1}^1)$ ,  $[0, z_n^1)$  in the last  $k$  draws. Note that  $\sum_{j=1}^{n+1} \nu_j = k$  and  $\nu_j \in \mathbb{N}_0$ . Specifically, the probabilities are (see, e.g., Shao, 2003, p. 98)

$$M_k(\nu_1, \dots, \nu_n, \nu_{n+1} | z_1^k, \dots, z_n^k) = \frac{k!}{\nu_1! \dots \nu_n! \nu_{n+1}!} (1 - z_1^k)^{\nu_1} (z_1^k - z_2^k)^{\nu_2} \dots (z_{n-1}^k - z_n^k)^{\nu_n} (z_n^k - 0)^{\nu_{n+1}}, \quad (8)$$

with the convention  $0^0 = 1$  [e.g., in the starting phase  $z_2^1 = \dots = z_n^1 = 0$ ]. Recall that  $\ell_i$  denotes the final rank of the current  $i$ -th best, when there are  $k$  periods to go. Given  $\nu_j$ ,  $j = 1, \dots, n + 1$ , we have  $\ell_i = i + \sum_{j=1}^i \nu_j$ , for  $i = 1, \dots, n$ . Consequently, we get  $\nu_i = \ell_i - \ell_{i-1} - 1$ , for  $i = 1, \dots, n$ . For the possible  $\ell_i$ , we have the following restriction  $0 = \ell_0 < \ell_1 < \dots < \ell_n \leq k + n$ . Observe that we always have  $k + n - \ell_n + \sum_{i=1}^n (\ell_i - \ell_{i-1} - 1) = k$ . By working with  $\ell_i$  we obtain the multinomial probabilities

$$p_k(\ell_1, \dots, \ell_n | z_1^k, \dots, z_n^k) = M_k(\ell_1 - 1, \ell_2 - \ell_1 - 1, \dots, \ell_n - \ell_{n-1} - 1, k + n - \ell_n | z_1^k, \dots, z_n^k). \quad (9)$$

Computing the expected utility contribution of past investments requires the calculation of the probabilities that the current  $n$  best draws remain the  $n$  best after  $k$  steps, the current first best becomes second best and the current second to  $n - 1$ -th best become third to  $n$  best, etc. The probabilities of these events are described by (9). The payoffs follow from our construction of  $y_{i|\ell_i}^k$ , where  $y_{i|\ell_i}^k = 0$  for  $\ell_i < i$ ,  $\ell_i > n$  or  $\ell_i < \ell_j$ , whenever  $j < i$ . Hence, “if we do nothing in the remaining  $k$  periods”, we obtain a final payoff of  $w_k + y_{1|1}^k + \dots + y_{n|n}^k$  with a probability of  $z_n^k$ . In general, the payoffs of “do nothing in the remaining  $k$  periods” can be described by means of  $w_k + y_{1|\ell_1}^k + \dots + y_{n|\ell_n}^k$ . The conditional expected utility of the strategy “do nothing in the remaining  $k$  periods” is given by

$$\mathcal{W}_k(w_k, \mathbf{y}_k, z_1^k, \dots, z_n^k) := \sum_{\substack{\ell_1, \dots, \ell_n: \\ 0 < \ell_1 < \dots < \ell_n \leq k+n}} p_k(\ell_1, \dots, \ell_n | z_1^k, \dots, z_n^k) u(w_k + y_{1|\ell_1}^k + \dots + y_{n|\ell_n}^k). \quad (10)$$

Let  $\mathbf{z}(x_k) := (z_1^k, \dots, z_{j-1}^k, x_k, z_j^k, \dots, z_{n-1}^k)$ , where  $z_{j-1}^k \geq x_k > z_j^k$ , and  $\mathcal{L} := (\ell_1, \dots, \ell_n)$ ,  $\ell_i < \ell_v$  for  $i < v$ . Define  $\pi(j, w_k, b_k, \mathbf{y}_k, \mathcal{L}) := (w_k - b_k) + y_{1|\ell_1}^k + \dots + y_{j-1|\ell_{j-1}}^k + \beta_k^{\ell_j} b_k + y_{j|\ell_{j+1}}^k + \dots + y_{n-1|\ell_n}^k$ , the payoffs where  $j = \rho_k(x_k)$ , we invest  $0 \leq b_k \leq w_k$ , and the final ranks of the currently best  $n$  draws (including  $x_k$ ) are  $\ell_1, \dots, \ell_n$ . Endowed with this notation for our investment problem we obtain:

**Theorem 1.** Consider a monotone increasing and continuous utility function  $u(\cdot)$  and the conditional expected utility of the strategy “do nothing in the remaining  $k$  periods” defined in (10). For  $k = T, \dots, 1$ , the value function  $\mathcal{V}_k(w_k, \mathbf{y}_k, z_1^k, \dots, z_n^k)$  satisfies

$$\begin{aligned} \mathcal{V}_k(w_k, \mathbf{y}_k, z_1^k, \dots, z_n^k) &= \mathcal{W}_k(w_k, \mathbf{y}_k, z_1^k, \dots, z_n^k) + r_k(w_k, \mathbf{y}_k, z_1^k, \dots, z_n^k), \\ \text{where the residual term} \\ r_k(w_k, \mathbf{y}_k, z_1^k, \dots, z_n^k) &= z_n^k r_{k-1}(w_k, \mathbf{y}_k, z_1^k, \dots, z_n^k) \\ &+ \sum_{j=1}^n \int_{z_j^k}^{z_{j-1}^k} \max_{0 \leq b_k \leq w_k} \left[ \sum_{\substack{\mathcal{L}=(\ell_1, \dots, \ell_n): \\ 0 < \ell_1 < \dots < \ell_n \leq k-1+n}} p_{k-1}(\ell_1, \dots, \ell_n | \mathbf{z}(x)) [u(\pi(j, w_k, b_k, \mathbf{y}_k, \mathcal{L})) - u(\pi(j, w_k, 0, \mathbf{y}_k, \mathcal{L}))] \right. \\ &\quad \left. + r_{k-1}(w_k - b_k, \mathbf{y}_k, (\beta_k^{j:n} b_k), \mathbf{z}(x)) \right] dx \end{aligned} \quad (11)$$

and  $r_0(w_0, \mathbf{y}_0, z_1^0, \dots, z_n^0) = 0$ . The residual terms satisfy  $r_k(w_k, \mathbf{y}_k, z_1^k, \dots, z_n^k) \geq 0$ , for all  $k = T, \dots, 1, 0$ . The final realized utility is  $\mathcal{V}_0(w_0, \mathbf{y}_0, z_1^0, \dots, z_n^0) = u(w_0 + \sum_{j=1}^n y_{j|j}^0)$ .

The optimal investments  $b_k$ ,  $k = T, \dots, 1$ , are the  $b_k$  maximizing the expressions inside the integrals in (11). If  $\beta_k^j < 1$  for all  $k$  and  $j$ , then  $b_k = 0$  for all  $k$ .  $\beta_k^j \geq 1$  for some  $k$  and  $j$  is necessary but not sufficient for  $b_k > 0$ .

Theorem 1 provides a recursion describing the optimal investment strategy. For each period  $k$ , the optimal amount to be invested  $b_k$ ,  $0 \leq b_k \leq w_k$  is chosen such that conditional expected utility  $\mathcal{V}_k(w_k, \mathbf{y}_k, z_1^k, \dots, z_n^k)$  in (11) is maximized. Note that the value function obtained in (11) contains the contributions of the strategy “do nothing in the remaining  $k$  periods”, that is the expected utility term (10), and the “residual term”  $r_k(w_k, \mathbf{y}_k, z_1^k, \dots, z_n^k)$ , which is the “expected additional contribution to the value function from optimal investments in the remaining  $k$  periods”.

**Remark 2.** By skipping all the terms associated to payoffs from the second to the  $n$ -th best alternative,

we observe that the full-information models where only the best alternative generates positive payoffs are a special case of our model. Such investment problems were investigated in Bruss and Ferguson (2002), Section 5.

The following theorem compares the  $T$  period problem to the  $T + 1$  period problem, given the same initial wealth  $\underline{w} > 0$ . To do this let  $\check{\beta}_{\mathbf{k}}^j$  denote the per unit returns of the  $T + 1$  period problem;  $\check{\beta}_{(n,\mathbf{k}=T,\dots,1,0)}$  collects all the  $\check{\beta}_{\mathbf{k}}^j$ . For the  $T + 1$  period problem we apply the notation  $\mathbf{k} = T, \dots, 1, 0, -1$ .<sup>7</sup>

**Theorem 2.** Consider a monotone increasing and continuous utility function  $u(\cdot)$ ,  $T \geq n \geq 1$ , and  $\underline{w} > 0$ . Let  $\mathcal{V}_T(\underline{w}, \beta_{(n,\mathbf{k}=T,\dots,1)})$  and  $\mathcal{V}_{T+1}(\underline{w}, \check{\beta}_{(n,\mathbf{k}=T,\dots,1,0)})$  be the value functions of the  $T$  and the  $T + 1$  period problems and let

$$\beta_{\mathbf{k}}^i \geq \max\{\check{\beta}_{\mathbf{k}}^i, \check{\beta}_{\mathbf{k}-1}^i\} \quad (12)$$

for all  $i = 1, \dots, n$  and  $\mathbf{k} = T, \dots, 1$ . Then,

$$\mathcal{V}_T(\underline{w}, \beta_{(n,\mathbf{k}=T,\dots,1)}) \geq \mathcal{V}_{T+1}(\underline{w}, \check{\beta}_{(n,\mathbf{k}=T,\dots,1,0)}) . \quad (13)$$

A sufficient condition for strict inequality in (13) is a strictly monotone increasing  $u(\cdot)$  and the additional condition

$$\begin{aligned} \check{\beta}_{\mathbf{k}}^j &\geq \check{\beta}_{\mathbf{k}-1}^j, \quad \text{for all } \mathbf{k} = T, \dots, 1 \text{ as well as all } j = 1, \dots, n, \\ \check{\beta}_{\mathbf{k}}^i &\geq \check{\beta}_{\mathbf{k}}^{i+1}, \quad \check{\beta}_{\mathbf{k}}^1 > \check{\beta}_{\mathbf{k}}^2 \quad \text{for all } i = 1, \dots, n-1 \text{ and all } \mathbf{k} = T, \dots, 1, 0, \text{ and} \\ \beta_{\mathbf{k}}^i &> 1, \quad \text{for one } \mathbf{k} \text{ and one } i, \text{ such } T - \mathbf{k} \geq i - 1 . \end{aligned} \quad (14)$$

**Remark 3.** A special case of the payoff structure considered in Theorem 2 follows from discounting. That is,  $\check{\beta}_{\mathbf{k}}^i = \beta_{\mathbf{k}}^i = \delta^{T-\mathbf{k}} \beta_T^i$  and  $\delta \in (0, 1]$ . In particular, Theorem 2 also holds for  $\beta_{\mathbf{k}}^j = \check{\beta}_{\mathbf{k}}^j = \beta^j$  for all  $\mathbf{k} = T, \dots, 1, (0)$ .<sup>8</sup>

Theorem 2 provides us with a comparative static result. Given two investment problems with a comparable payoff structure but with different investment horizons, the expected utility is non-increasing in the number of investment periods.

<sup>7</sup>In the following Theorem 2 we use  $\mathbf{k} = T, \dots, 1, 0, -1$  to describe the  $T + 1$  period problem. This notation makes the comparison and the notation for the returns per unit  $\beta_{\mathbf{k}}^j$  and  $\check{\beta}_{\mathbf{k}}^j$  more intuitive.

<sup>8</sup> $T - \mathbf{k}$  is an exponent in  $\delta^{T-\mathbf{k}}$ .

### 3 Risk-Neutrality

This section considers the linear case, where  $u(\cdot)$  is the identity and the investor maximizes expected wealth. Suppose that the investor either invests nothing or the remaining budget  $w_k$  (which will turn out to be a property of an optimal strategy for a risk-neutral investor). Then, by applying such a strategy, if there are  $k$  steps to go, the remaining budget  $w_k$  is either equal to  $w_T = \underline{w}$ ,  $b_T, \dots, b_{k+1} = 0$  and by (2) all  $y_{j|\ell_j} = 0$  or the remaining budget is  $w_k = 0$ . In the second case the investor already invested  $b_v = \underline{w}$  in some period  $T \geq v \geq k+1$ . Using Equation (2),  $\mathbf{Y}_k$  contains only one non-zero column (unless the rank  $\rho_{k+1}(x_v)$  is larger than  $n$ ). It is the transpose of  $(y_{\rho_{k+1}(x_v)|\rho_{k+1}(x_v)}, \dots, y_{\rho_{k+1}(x_v)|n}) = \beta_v^{\rho_{k+1}(x_v):n} w_T$ , where  $\rho_{k+1}(x_v)$  is the relative rank of the draw  $x_v$  given the observations  $x_T, \dots, x_{k+1}$ . If there are  $k$  steps remaining and  $X_k$  has already been realized, the current rank of  $x_v$  is  $\rho_k(x_v)$  and the payoff vector becomes  $(y_{\rho_{k+1}(x_v)|\rho_k(x_v)}, \dots, y_{\rho_{k+1}(x_v)|n}) = \beta_v^{\rho_k(x_v):n} w_T$ ; this vector is contained in  $\mathbf{y}_k$  (the remaining elements of  $\mathbf{y}_k$  are zero). Hence, for the strategies with the property “do nothing in the remaining  $k$  periods” and “invest  $w_k = \underline{w}$  or 0”, we obtain

$$\begin{aligned} & \mathcal{W}_k^{\text{linear}}(w_k, \mathbf{y}_k, z_1^k, \dots, z_n^k) \\ =: & \begin{cases} w_T & , \text{ if } b_j = 0 \text{ for } j = T, T-1, \dots, k+1 \text{ and } w_k = w_T, \\ w_T \sum_{\ell=\rho_{k+1}(x_v)}^n p_k(\{\rho_a(x_v) = \ell\} | x_v, \rho_{k+1}(x_v)) \beta_v^\ell & , \text{ if } b_v = w_T \text{ for one } v \in \{T, T-1, \dots, k+1\} \text{ and } w_k = 0, \end{cases} \end{aligned} \quad (15)$$

with the convention that  $\sum_{\ell=\rho_{k+1}(x_v)}^n \cdot = 0$  if  $\ell = \rho_{k+1}(x_v) > n$ .

Since we only have to consider one investment, the multinomial probabilities considered for the general case reduce to a scenario where the draws  $x_j$ ,  $j \leq k$  are either larger than, smaller than, or equal to the draw  $x_v$ , resulting in binomial probabilities. That is, the probability that  $x_v$  has final rank  $\ell$ , given  $x_v$ , and there are  $k$  steps to go is

$$p_k(\{\rho_a(x_v) = \ell\} | x_v, \rho_{k+1}(x_v)) = p_k(\{\rho_a(x_v) = \lambda + \rho_{k+1}(x_v)\} | x_v, \rho_{k+1}(x_v)) = \binom{k}{\lambda} (1 - x_v)^\lambda x_v^{k-\lambda},$$

where  $\lambda = \ell - \rho_{k+1}(x_v) \geq 0$ . In addition, the term  $p_{k-1}(\{\rho_a(x_k) = \ell\} | x_k, \rho_k(x_k))$  denotes the  $(k-1)$ -step to go conditional probability that a realized draw  $x_k$  has absolute rank  $\ell$ . These binomial probabilities are  $p_{k-1}(\{\rho_a(x_k) = \ell\} | x_k, \rho_k(x_k)) = \binom{k-1}{\lambda} (1 - x_k)^\lambda x_k^{(k-1)-\lambda}$ , where  $\ell \geq \rho_k(x_k)$  and  $\lambda = \ell - \rho_k(x_k) \geq 0$ .

For the linear case we obtain:

**Theorem 3.** Consider a risk-neutral investor where  $u(\cdot)$  is the identity function  $id(\cdot)$ . For  $\mathbf{k} = \mathbf{T}, \dots, 1$ , the value function  $\mathcal{V}_{\mathbf{k}}(w_{\mathbf{k}}, \mathbf{y}_{\mathbf{k}}, z_1^{\mathbf{k}}, \dots, z_n^{\mathbf{k}})$  satisfies

$$\begin{aligned} \mathcal{V}_{\mathbf{k}}^{\text{linear}}(w_{\mathbf{k}}, \mathbf{y}_{\mathbf{k}}, z_1^{\mathbf{k}}, \dots, z_n^{\mathbf{k}}) &= \mathcal{W}_{\mathbf{k}}^{\text{linear}}(w_{\mathbf{k}}, \mathbf{y}_{\mathbf{k}}, z_1^{\mathbf{k}}, \dots, z_n^{\mathbf{k}}) + w_{\mathbf{k}} c_{\mathbf{k}}(z_1^{\mathbf{k}}, \dots, z_n^{\mathbf{k}}) \\ &= \begin{cases} w_{\mathbf{T}}(1 + c_{\mathbf{k}}(z_1^{\mathbf{k}}, \dots, z_n^{\mathbf{k}})) & , \text{ if } b_j = 0 \text{ for } j = \mathbf{T}, \mathbf{T} - 1, \dots, \mathbf{k} + 1 \text{ and } w_{\mathbf{k}} = w_{\mathbf{T}}, \\ w_{\mathbf{T}} \sum_{\ell=\rho_{\mathbf{k}+1}(x_v)}^n p_{\mathbf{k}}(\{\rho_a(x_v) = \ell\} | x_v, \rho_{\mathbf{k}+1}(x_v)) \beta_v^{\ell} & , \text{ if } b_v = w_{\mathbf{T}} \text{ at } v \in \{\mathbf{T}, \mathbf{T} - 1, \dots, \mathbf{k} + 1\} \text{ and } w_{\mathbf{k}} = 0, \end{cases} \\ \text{where} \\ c_{\mathbf{k}}(z_1^{\mathbf{k}}, \dots, z_n^{\mathbf{k}}) &= z_n^{\mathbf{k}} c_{\mathbf{k}-1}(z_1^{\mathbf{k}}, \dots, z_n^{\mathbf{k}}) \\ &\quad + \sum_{j=1}^n \int_{z_j^{\mathbf{k}}}^{z_{j-1}^{\mathbf{k}}} \max \left\{ -1 + \sum_{\ell=\rho_{\mathbf{k}}(x_{\mathbf{k}})}^n p_{\mathbf{k}-1}(\{\rho_a(x_{\mathbf{k}}) = \ell\} | x_{\mathbf{k}}, \rho_{\mathbf{k}}(x_{\mathbf{k}})) \beta_{\mathbf{k}}^{\ell}, c_{\mathbf{k}-1}(z(x_{\mathbf{k}})) \right\} dx_{\mathbf{k}}, \quad (16) \end{aligned}$$

and  $c_0 = 0$ . The conditional expected utility of the strategies “do nothing in the remaining  $\mathbf{k}$  periods” and “invest  $w_{\mathbf{k}}$  or 0” is defined in (15). A risk-neutral investor invests  $b_{\mathbf{k}} = w_{\mathbf{k}} = w_{\mathbf{T}}$  if and only if  $-1 + \sum_{\ell=\rho_{\mathbf{k}}(x_{\mathbf{k}})}^n p_{\mathbf{k}-1}(\{\rho_a(x_{\mathbf{k}}) = \ell\} | x_{\mathbf{k}}, \rho_{\mathbf{k}}(x_{\mathbf{k}})) \beta_{\mathbf{k}}^{\ell} \geq c_{\mathbf{k}-1}(z_1^{\mathbf{k}}, \dots, z_{j-1}^{\mathbf{k}}, x_{\mathbf{k}}, z_j^{\mathbf{k}}, \dots, z_{n-1}^{\mathbf{k}})$  for some  $x_{\mathbf{k}} \in [z_j^{\mathbf{k}}, z_{j-1}^{\mathbf{k}})$  and  $j = 1, \dots, n$ .

The residual term  $c_{\mathbf{k}}(z_1^{\mathbf{k}}, \dots, z_n^{\mathbf{k}})$  is the expected contribution per unit of remaining budget  $w$  to the value function by investing optimally instead of making no investment in the last  $\mathbf{k}$  periods. We observe that, if it is optimal to invest a positive amount, then it is also optimal to invest the whole remaining wealth  $w$  as already claimed in the first paragraph of this section. That is,  $c_{\mathbf{k}}$  can be obtained recursively by means of (16). If  $-1 + \sum_{\ell=\rho_{\mathbf{k}}(x_{\mathbf{k}})}^n p_{\mathbf{k}-1}(\{\rho_a(x_{\mathbf{k}}) = \ell\} | x_{\mathbf{k}}, \rho_{\mathbf{k}}(x_{\mathbf{k}})) \beta_{\mathbf{k}}^{\ell} \geq c_{\mathbf{k}-1}(z_1^{\mathbf{k}}, \dots, z_j^{\mathbf{k}}, x_{\mathbf{k}}, z_{j+1}^{\mathbf{k}}, \dots, z_{n-1}^{\mathbf{k}})$ , the expected value from investing is larger than or equal to deferring investment for one period and then investing optimally from that period onward. In this case it is optimal to invest the whole remaining wealth, which implies  $b_{\mathbf{k}} = w_{\mathbf{T}}$ .

**Remark 4.** Note that Robbins’ problem is a special case of the risk-neutral investment problem. Given the *iid* distributed random variables  $X_1, \dots, X_{\mathbf{T}}$  and the absolute rank of  $X_{\mathbf{k}}$  abbreviated by  $\rho_a(X_{\mathbf{k}})$ , Robbins’ problem is to find the optimal stopping rule that minimizes the expected rank of the chosen realization (see, e.g., Assaf and Samuel-Cahn, 1996; Bruss and Ferguson, 1993). With a linear utility function used in this section, Robbins’ problem for  $\mathbf{T}$  periods is a special case of the  $\mathbf{T}$ -period investment problem with initial wealth  $\underline{w} = 1$  and returns per unit  $\beta_{\mathbf{k}}^i = \mathbf{T} + 2 - i$ , for all  $i, \mathbf{k} = 1, \dots, \mathbf{T}$ . To see this, observe that all  $\beta_{\mathbf{k}}^i > 1$  and hence it is strictly better to invest all the available budget during the

periods  $1, \dots, T$ . By Theorem 3 we know that it is an optimal strategy to invest all the available budget  $w_k$ , whenever it is optimal to invest a positive amount. This implies that we can assume the that decision maker chooses exactly one draw. To maximize the expected final wealth with these payoff parameters is obviously equivalent to minimizing “ $T + 2 - \text{expected wealth}$ ”. This, in turn, is equivalent (with these payoff parameters) to minimizing the expected rank.

**Remark 5.** Robbins’ problem is known to be a very difficult optimal stopping problem (no closed form solution, full history dependence of the optimal strategy, etc.; see, e.g., Gneden, 2007). In Remark 4 we observed that Robbins’ problem is a special case of the risk-neutral case and therefore also of the more general expected utility case investigated in Section 2. That is to say, given the results for Robbins’ problem there is no hope of finding a closed-form solution for the optimal strategy and the value function.

The *Cayley–Moser problem* (see Ferguson, 2011; Moser, 1956) is to find an optimal stopping rule that maximizes the expected payoff. Suppose that the investor is risk neutral, the quality characteristics  $X_t$ ,  $t = 1, \dots, T$ , are uniformly distributed on the interval  $[0, 1]$ , and the payoff is given by  $1 + X_\tau$ , where  $\tau$  is a stopping time. Since, at time  $t$ , the utility function does not depend on the past realizations of  $X_1, \dots, X_{t-1}$ , this problem can be solved by backward induction. The solution is of the form: “stop (i.e., invest) at the first  $x_t \geq A_t$ ”, where  $A_t$  can be derived recursively. Note that, in this investment problem, the payoff is continuous in the signal  $X_t$ . Many investment problems and models considered in corporate finance assume that payoffs are continuous in some underlying signal (see, e.g., Tirole, 2010). By contrast, in the venture capital investment process, payoffs are typically more discontinuous because of the superstar property of investment projects. To account for this feature, payoffs are assumed to be rank dependent in this article. From the perspective of the optimal stopping literature, our investment problem is therefore closer to Robbins’ problem (see, e.g., Bruss, 2005). As a result, the complexity of the optimal rule increases substantially. For example, Bruss and Ferguson (1996, Section 4.2) show that the optimal rule exhibits full history dependence. That is, the optimal rule at time  $t$ , with  $t \leq T$ , depends on the realizations of  $X_1, \dots, X_t$ .

Since, when comparing two draws  $X_t$  and  $X_s$ , choosing the one with the higher realization also means choosing the one with the lower rank, one might be tempted to think that the optimal strategy for the Cayley-Moser problem (the strategy that maximizes the expectation of the chosen  $X_\tau$ ) might also be

the optimal strategy for Robbins' problem (of minimizing the expected rank). That this is not so, is illustrated by the following example (notation here is in forward time): Suppose that the decision maker has not chosen any realization  $x_1, x_2, \dots, x_{98}$ , we are in period 99 with  $x_{99} = 0.1$ , and all the other past realizations are very near to 0.1, but strictly lower than 0.1 (so that  $x_{99}$  has a relative rank of 1). The decision maker now contemplates whether to choose the current draw  $x_{99} = 0.1$ . In the Cayley-Moser problem, the decision maker would clearly continue and choose  $X_{100}$  as she would expect  $\mathbb{E}(X_{100}) = 0.5$  on average, which is much better than 0.1. In Robbins' problem, continuing would give the decision maker with approximately 90% a rank of 1, but with approximately 10% a rank of 100, which gives an expected rank for continuing of  $0.9 \times 1 + 0.1 \times 100 = 10.9$ . Stopping would give her with approximately 90% a rank of 2, and with 10% a rank of 1, which results in an expected rank for stopping of  $0.9 \times 2 + 0.1 \times 1 = 1.9$ . Hence, in Robbins' problem, it is optimal to stop, while in the Cayley-Moser problem it is optimal to continue.

## 4 Stochastic Returns

Suppose that the returns per unit are random variables. These random variables are  $B_T^j, \dots, B_1^j$ ,  $j = 1, \dots, n$ , the realizations are abbreviated  $\beta_k^j$ . The stochastic analogons of  $\beta_k^{j:n}$  is  $\mathbf{B}_k^{j:n}$ . All the returns are realized and known ex-post by the investor, that is at  $\mathbf{k} = 0$ . We assume  $\mathbb{P}(B_k^j > 0) > 0$  and  $\mathbb{P}(B_k^j \geq 0) = 1$ , for all  $j = 1, \dots, n$  and  $\mathbf{k} = T, \dots, 1$ . We assume that the returns  $B_k^j$  only depend on the rank of the corresponding signal, that is the absolute rank of  $X_k$ , and on the period  $\mathbf{k}$ .

$$\mathbb{W}_k(w_k, \tilde{\mathbf{y}}_k, z_1^k, \dots, z_n^k) := \sum_{\substack{\ell_1, \dots, \ell_n: \\ 0 < \ell_1 < \dots < \ell_n \leq k+n}} p_k(\ell_1, \dots, \ell_n | z_1^k, \dots, z_n^k) \mathbb{E} \left( u \left( w_k + \tilde{y}_{1|\ell_1}^k + \dots + \tilde{y}_{n|\ell_n}^k \right) \right), \quad (17)$$

denote the utility contribution of the strategy "do nothing in the remaining  $\mathbf{k}$  periods". Note that in  $\mathbb{W}_k(w_k, \tilde{\mathbf{y}}_k, z_1^k, \dots, z_n^k)$  we take the conditional expectation of random payoffs  $\tilde{\mathbf{y}}_k$ .  $\mathbb{E}(u(\cdot))$  denotes expected utility. Existence of  $\mathbb{E}(\cdot)$  is assumed. Note that in this stochastic setup the former payoffs  $\mathbf{y}_k$  become random, therefore the notation  $\tilde{\mathbf{y}}_k$  and  $\tilde{y}_{j|\ell}$  is used. In addition, the stochastic payoffs when investing  $b$  are  $\pi(j, w_k, b_k, \tilde{\mathbf{y}}_k, \mathcal{L}) := (w_k - b_k) + \tilde{y}_{1|\ell_1}^k + \dots + \tilde{y}_{j-1|\ell_{j-1}}^k + B_{k-1}^{\ell_j} b_k + \tilde{y}_{j|\ell_j+1}^k + \dots + \tilde{y}_{n-1|\ell_n}^k$ , while the stochastic payoffs for the strategy  $b_k = 0$  in the remaining  $\mathbf{k}$  periods are  $\pi(j, w_k, 0, \tilde{\mathbf{y}}_k, \mathcal{L}) := w_k + \tilde{y}_{1|\ell_1}^k + \dots + \tilde{y}_{j-1|\ell_{j-1}}^k + B_{k-1}^{\ell_j} 0 + \tilde{y}_{j|\ell_j+1}^k + \dots + \tilde{y}_{n-1|\ell_n}^k$ . Endowed with this notation for our investment

problem we obtain:

**Theorem 4.** Consider the investment problem investigated in Theorem 1, with stochastic returns  $B_k^j$  replacing the deterministic returns  $\beta_k^j$ . The value contribution of “doing nothing in the remaining  $k$  periods” follows from (17). For  $k = T, \dots, 1$ , the value function  $\mathbb{V}_k(w_k, \tilde{\mathbf{y}}_k, z_1, \dots, z_n)$  satisfies

$$\begin{aligned} \mathbb{V}_k(w_k, \tilde{\mathbf{y}}_k, z_1^k, \dots, z_n^k) &= \mathbb{W}_k(w_k, \tilde{\mathbf{y}}_k, z_1^k, \dots, z_n^k) + R_k(w_k, \tilde{\mathbf{y}}_k, z_1^k, \dots, z_n^k) , \\ \text{where the residual term} \\ R_k(w_k, \tilde{\mathbf{y}}_k, z_1^k, \dots, z_n^k) \\ &= z_n^k R_{k-1}(w_k, \tilde{\mathbf{y}}_k, z_1^k, \dots, z_n^k) \\ &\quad + \sum_{j=1}^n \int_{z_j^k}^{z_{j-1}^k} \max_{0 \leq b_k \leq w_k} \left( \sum_{\substack{\ell_1, \dots, \ell_n: \\ 0 < \ell_1 < \dots < \ell_n \leq k-1+n}} p_{k-1}(\ell_1, \dots, \ell_n | \mathbf{z}(x)) \mathbb{E} [u(\pi(j, w_k, b_k, \tilde{\mathbf{y}}_k, \mathcal{L})) - u(\pi(j, w_k, 0, \tilde{\mathbf{y}}_k, \mathcal{L}))] \right. \\ &\quad \left. + R_{k-1}(w_k - b_k, \tilde{\mathbf{y}}_k(\mathbf{B}_k^{j:n} b_{k-1}), z_1^k, \dots, z_{j-1}^k, x, z_j^k, \dots, z_{n-1}^k) \right) dx . \end{aligned} \quad (18)$$

$\mathbb{V}_0(w_0, \tilde{\mathbf{y}}_0, z_1^0, \dots, z_n^0) = \mathbb{E} \left( u(w_0 + \sum_{j=1}^n \tilde{y}_{j|j}^0) \right)$ .  $R_k(w_k, \tilde{\mathbf{y}}_k, z_1^k, \dots, z_n^k) \geq 0$  for all  $k = T, \dots, 1, 0$  and  $R_0(w_k, \tilde{\mathbf{y}}_0, z_1^k, \dots, z_n^k) = 0$ . The optimal investments  $b_k$ ,  $k = T, \dots, 1$ , are the  $b_k$  maximizing the expressions inside the integrals in (18).

For a risk-neutral investor, Theorem 4 implies:

**Corollary 1.** Consider a risk-neutral investor where  $u(\cdot)$  is the identity. Let  $\beta_k^j = \mathbb{E}(B_k^j)$ . Then, we can follow the deterministic case investigated in Theorem 3. This reduces the numerical burden in the simulation analysis, where we need not approximate the conditional expectation terms by means of numerical integration.

**Proposition 1.** Theorem 1 and Theorem 4 also hold for a risk-loving investor. If the utility function  $u$  is strictly increasing, convex, and twice continuously differentiable, and the differentiation Lemma applies<sup>9</sup>, then an optimal  $b_k > 0$  implies that also  $b_k = w_k$  is optimal. That is, all the wealth is invested in the corresponding investment alternative.

We suspect that Proposition 1 holds for a general monotone increasing, and convex utility function  $u$ . Note that in the risk-neutral case investigated in Theorem 3, the strategy “invest all the remaining wealth

<sup>9</sup>See, e.g., Klenke (2008), Theorem 6.28.

if it is optimal to invest a positive amount” is also optimal. Proposition 1 shows that this also applies to a risk-loving investor.

Theorem 4 and Corollary 1 recursively define optimal investment strategies when the returns  $B_t^j$  are stochastic. For the risk-neutral investor the optimal strategy with stochastic returns is already provided by the optimal strategies with deterministic returns, only the deterministic payoff  $\beta_k^j$  has to be replaced by the expected value  $\mathbb{E}(B_k^j)$ . This does not hold for a non-risk-neutral investor. Although the residual terms  $r_{k-1}$  and  $R_{k-1}$  are obtained recursively for deterministic and stochastic returns, respectively, stochastic returns generate additional uncertainty. The risk arising from the model assumption that only  $n$  projects can generate a positive return, affects the decisions of all the decision makers considered in this article. The riskiness of the final returns only plays a role for non-risk-neutral agents. The risk arising during the time span after the investment process is completed is important for non-risk-neutral investors only.

The results obtained in the Theorems 1 and 3 allow to investigate further economic problems. For example, the assumption that the investor obtains a payoff of zero if the final rank of an investment is larger than  $n$  can be modified to a scenario where an arbitrary fixed gross-return per unit invested is obtained. In this case  $n = T$  and  $\beta_k^i = \beta^0 \in \mathbb{R}$  for any  $i \geq n$ .

## 5 Simulation Analysis and Optimal Investment

This section provides results obtained from a simulation study for risk-neutral investors and risk-averse investors with Bernoulli utility function  $u(w) = \ln(w)$ . The initial wealth is  $\underline{w} = w_T = 1$ , the number of periods/projects considered is  $T = 5$  with a grid size of  $15 + T + 1$  for the risk-averse case, and  $T \in \{5, 10, 20, 50\}$  with a grid size of  $100 + T + 1$  for the risk-neutral case. To perform this simulation study, a large number of integrals have to be solved numerically to calculate the residual terms  $c(\cdot)$ ,  $r(\cdot)$  and especially  $R(\cdot)$ . Hence, the number of Monte-Carlo replications is  $M = 2,000$  for the risk-neutral case, and relatively small, with  $M = 500$  replications for the risk-averse case. Note, however, that running multiple replications constitutes an embarrassingly parallel task which can be trivially sped up when using multi-core computers or even clusters of workstations. For prototyping the code and obtaining the results presented in the current paper, the software package **Matlab** was used.<sup>10</sup> Approximate runtimes per replication on a single core of an Apple M1 chip are 0.34s ( $T = 5$ ), 1.4s ( $T = 10$ ), 4.9s ( $T = 20$ ), and

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<sup>10</sup>Code is available on request.

35s ( $T = 50$ ) for the risk-neutral case, and 300s ( $T = 5$ ) for the risk-averse case. In addition, experiments with `C` have shown that a speed-up of up to 50x is possible when using a compiled language, in particular for larger values of  $T$ . This is due to the recursive nature of the problem at hand, where repetitive code interpretation renders interpreted languages such as `Matlab` rather unsuitable for production code.

Since the investment problems investigated in Section 2 to 4 relate to the decision process observed on the venture capital market, we use estimates provided in Cochrane (2005). In particular, we use estimates of “round-to-round” log-returns, where the sample mean of the log-returns is 0.53 and the estimated standard deviation is 0.85 (see Cochrane, 2005, Table 6). Since we are interested in the returns of a project with a high evaluation, we use these *non-bias-corrected* estimates of the returns.

The log-normal distribution arises naturally in discrete-time approximations of the Black-Scholes model, implies that the underlying return model corresponds to a geometric Brownian motion, and provides a reasonable approximation of the return distribution, more precisely, the return distribution conditional on the investment project having final rank at most  $n = 2$ . Thus, in what is to follow, we assume that  $B_k^1$  is log-normally distributed with mean parameter  $\mu_1 = 0.6$  and variance parameter  $\sigma^2 = 0.85^2$ , while  $B_k^2$  is log-normally distributed with mean parameter  $\mu_2 = 0.5$  and variance parameter  $\sigma^2 = 0.85^2$ , for all  $k = T, \dots, 1$ . The arithmetic gross-returns  $B_k^1$  have an expectation of  $\exp(\mu_1 + 1/2\sigma^2) = 2.6150$  and variance  $(\exp(\sigma^2) - 1) \exp(2\mu_1 + \sigma^2) = 7.2454$  while  $B_k^2$  has expectation  $\exp(\mu_2 + 1/2\sigma^2) = 2.3661$  and variance  $(\exp(\sigma^2) - 1) \exp(2\mu_2 + \sigma^2) = 5.9320$ , respectively. The probability  $\mathbb{P}(B_k^1 < 1) \approx 24.01\%$ , while  $\mathbb{P}(B_k^2 < 1) \approx 27.82\%$ . For the deterministic returns we set  $\beta_k^1 = 2.6150$  and  $\beta_k^2 = 2.3661$  for all  $k = T, \dots, 1$ . Note, however, that other assumptions about the return distribution can easily be accommodated by our proposed methodology, as long as random draws are available.

Table 1 presents the average amounts invested in the interval  $[z_1, 1]$ , denoted  $\bar{b}_{I1}$  and its sample standard deviation  $SD_{b_{I1}}$ , as well as the mean amounts invested in the interval  $[z_2^k, z_1^k]$ , denoted by  $\bar{b}_{I2}$  and its sample standard deviation  $SD_{b_{I2}}$ . Note that the intervals  $[z_1^k, 1]$  and  $[z_2^k, z_1^k]$  are neither equal for the various simulation runs (they are based on realizations of *iid.* uniformly distributed random numbers) nor for one particular draw (the  $z_1^k$  and  $z_2^k$ , obtained from  $x_T, \dots, x_{k+1}$ , are monotone increasing when the “number of the steps to go”,  $k$ , declines, or in other words when  $t$  increases). Since an investor hardly invests before  $t = 10$  (of course for  $T > 10$ ), we start to present statistics at  $t = 10$ , the second column summarizes the largest investments made before  $t < 10$ . Although the amounts invested are small, under

full-information a risk-neutral agent already invests into the first best alternative if the realization of  $X_k$  is sufficiently large. This is in contrast to the limited-information case investigated in Meier and Sögner (2023), where no investments in the first periods are observed (already for  $T = 5$ , see Table 2 for  $T = 5$  and log-utility). The investor also invests if the realized relative rank of  $X_k$  is two, however this is only observed when the remaining number of steps to go is small.

For  $T = 5$ , stochastic returns, full information, and risk-neutrality, the mean non-invested wealth is approximately 0.2065 (sample standard deviation 0.4049), while the mean final wealth is 2.4025 (standard deviation 0.8360), whereas for  $T = 10$ , the mean non-invested wealth is approximately 0.2155 (sample standard deviation 0.4113) and the mean final wealth is 2.3255 (standard deviation 0.9016). For  $T = 20$  and  $T = 50$ , the sample mean of the non-invested wealth is approximately 0.2415 and 0.2430, respectively (the sample standard deviations 0.4281 and 0.4290). The mean final wealth is 2.2690 (standard deviation 0.9127) for  $T = 20$  and 2.2585 (standard deviation 0.9215) for  $T = 50$ .

For the risk-neutral case, we also obtain the relative frequencies when (i) a risk-neutral investor does not invest, (ii) invests and the final rank is  $\leq n = 2$  (investor receives log-normally distributed return), and (iii) invests and loses all money invested. In addition, for (ii) we obtain the relative frequency that the investor gets the best or the second best investment alternative. For  $n \in \{5, 10, 20, 50\}$ , these corresponding relative frequencies are provided in Table 5. Thus, we obtain estimates of the probabilities that the investor receives the initial wealth  $\underline{w}$ , a log-normally distributed return, or zero in the end. Of particular interest is that even with  $T = 50$  periods, the probability of investing and loosing it all (the realized final rank of the chosen project is at least 3 is only around 2.2%). For  $T = 50$ , the investor does not invest with a probability of roughly 25%. In the case where the agent invests, the chosen project has final rank two much more often than rank one.

As already stated at the beginning of this section, due to computational constraints for the log-utility investor, only the case  $T = 5$  is considered. Table 2 presents the average amounts invested at  $t = 1, \dots, 5$  and the corresponding standard deviations. Since the optimal investment rule is not affected by the variance of  $B_k^j$ , we only investigate stochastic returns for the risk-neutral case. By contrast, the riskiness of the returns has an additional impact on how much is invested optimally for a risk-averse investor. The first rows of Table 2 show the average amounts invested and its standard deviations for stochastic and deterministic returns. In addition, the remaining rows present the amounts invested under limited-

information as investigated in Meier and Sögner (2023). The main difference is that a full-information investor already invests in  $\mathfrak{t} = 1$  (where are  $\mathbf{k} = 5$  steps to go). This happens if the realization of  $X_{\mathbf{k}}$ ,  $\mathbf{k} = 5$ , is large. By contrast, for the limited-information investor, the realized relative rank for  $X_5$  is always one. Table 4 also presents the average and the sample standard deviation of the remaining budget  $w_0$ . We observe that the non-invested budget is higher with stochastic returns, for full and limited-information, respectively. For a risk-averse investor, this effect follows from the additional risk arising for the riskiness of the returns. In addition, the remaining budget is higher with limited-information. As can be expected, the utility levels (and therefore the certainty equivalents) increase when the returns are deterministic. In addition, by pairwise comparison, we observe that the estimates of the certainty equivalent improve when moving from limited to full-information. This effect is quite strong for stochastic returns, where the certainty equivalent increases approximately by 0.37 percentage points on average ( $w_T = 1$  in our simulation studies). For deterministic returns, this effect is less pronounced. Although the agent maximizes expected utility, we observe that the final wealth  $w_0 + y_{1|1}^0 + y_{2|2}^0$  significantly increases with full-information compared to limited-information. Under full-information, the average final wealth turns out to be higher with stochastic returns compared to deterministic returns. However, this effect is not statistically significant at the 5% level.

Finally, we seek to link our model to more standard asset pricing models. To do so, suppose that an investor provides funds to a venture capital fund and considers the returns generated by venture capital investments. The fund is assumed to follow an optimal strategy derived in this article. For simplicity, suppose that all agents involved are risk-neutral. The fund only considers the returns generated during the investment cycle, that is,  $\tilde{\mathbf{y}}_0$  and  $w_0$  arising from one “investment cycle” (in our T-period model). If the final rank of the investment project is  $\leq n$ , then the investor receives a payoff following a mixture of log-normal distributions. Such a distribution can also be obtained from a mixture of diffusion processes. With a probability estimated by the relative frequencies reported in Table 5 (for the case  $n = 2$ ), the investor receives zero. Thus, continuous-time asset pricing models that allow for a default state can be applied to approximately value such a venture capital investment fund (see, e.g., Linetsky, 2006; Carr and Linetsky, 2006, where valuation formulas are also available).

As already stated at the beginning of this section, due to computational constraints, only the case  $T = 5$  was considered for the log-utility investor. Table 2 presents the average amounts invested at  $\mathfrak{t} = 1, \dots, 5$

and the corresponding standard deviations. Since the optimal investment rule is not affected by the variance of  $B_k^j$ , we only consider stochastic returns for the risk-neutral case. By contrast, the riskiness of the returns has an additional impact on the optimal investment amounts for a risk-averse investor. The first rows of Table 2 report the average amounts invested and their standard deviations for stochastic and deterministic returns.

In addition, the remaining rows present the amounts invested under limited information, as investigated by Meier and Sögner (2023). The main difference is that a full-information investor already invests at  $t = 1$  (when there are  $k = 5$  steps remaining). This occurs if the realization of  $X_k$ , with  $k = 5$ , is sufficiently large. By contrast, for the limited-information investor, the realized relative rank of  $X_5$  is always one. Table 4 also reports the average and the sample standard deviation of the remaining budget  $w_0$ . We observe that the non-invested budget is higher with stochastic returns, under both full and limited information. For a risk-averse investor, this effect results from the additional uncertainty arising from the stochastic nature of the returns. Moreover, the remaining budget is higher under limited information.

As expected, the utility levels and, consequently, the certainty equivalents increase when the returns are deterministic. Furthermore, pairwise comparisons show that the estimated certainty equivalents improve when moving from limited to full information. This effect is particularly pronounced for stochastic returns, where the certainty equivalent increases by approximately 0.37 percentage points on average ( $w_T = 1$  in our simulation study). For deterministic returns, this effect is less pronounced. Although the agent maximizes expected utility, we observe that the final wealth  $w_0 + \tilde{y}_{1|1}^0 + \tilde{y}_{2|2}^0$  increases substantially under full information compared with the limited-information case. Under full information, the average final wealth is higher with stochastic returns than with deterministic returns. However, this difference is not statistically significant at the 5% level.

| k                              | > 10   | 10     | 9      | 8      | 7      | 6      | 5      | 4      | 3      | 2      | 1      |
|--------------------------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| Risk-neutral, T = 5            |        |        |        |        |        |        |        |        |        |        |        |
| t                              |        |        |        |        |        |        | 1      | 2      | 3      | 4      | 5      |
| $\bar{b}_{I1}$                 |        |        |        |        |        |        | 0.1440 | 0.1410 | 0.1515 | 0.1665 | 0.1390 |
| $SD_{b_{I1}}$                  |        |        |        |        |        |        | 0.3512 | 0.3481 | 0.3580 | 0.3726 | 0.3460 |
| $\bar{b}_{I2}$                 |        |        |        |        |        |        | 0.0000 | 0.0000 | 0.0000 | 0.0005 | 0.0510 |
| $SD_{b_{I2}}$                  |        |        |        |        |        |        | 0.0000 | 0.0000 | 0.0000 | 0.0224 | 0.2201 |
| Risk-averse, T = 5 see Table 2 |        |        |        |        |        |        |        |        |        |        |        |
| Risk-neutral, T = 10           |        |        |        |        |        |        |        |        |        |        |        |
| t                              |        | 1      | 2      | 3      | 4      | 5      | 6      | 7      | 8      | 9      | 10     |
| $\bar{b}_{I1}$                 |        | 0.0635 | 0.0650 | 0.0680 | 0.0670 | 0.0880 | 0.0825 | 0.0820 | 0.0940 | 0.0675 | 0.0610 |
| $SD_{b_{I1}}$                  |        | 0.0595 | 0.0608 | 0.0634 | 0.0625 | 0.0803 | 0.0757 | 0.0753 | 0.0852 | 0.0630 | 0.0573 |
| $\bar{b}_{I2}$                 |        | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0005 | 0.0030 | 0.0135 | 0.0290 |
| $SD_{b_{I2}}$                  |        | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0005 | 0.0030 | 0.0133 | 0.0282 |
| Risk-neutral, T = 20           |        |        |        |        |        |        |        |        |        |        |        |
| $\bar{b}_{I1}$                 | 0.0450 | 0.0405 | 0.0435 | 0.0385 | 0.0445 | 0.0470 | 0.0490 | 0.0330 | 0.0315 | 0.0290 | 0.0225 |
| $SD_{b_{I1}}$                  | 0.0430 | 0.0389 | 0.0416 | 0.0370 | 0.0425 | 0.0448 | 0.0466 | 0.0319 | 0.0305 | 0.0282 | 0.0220 |
| $\bar{b}_{I2}$                 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0005 | 0.0005 | 0.0015 | 0.0030 | 0.0115 | 0.0125 | 0.0135 |
| $SD_{b_{I2}}$                  | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0005 | 0.0005 | 0.0015 | 0.0030 | 0.0114 | 0.0123 | 0.0133 |
| Risk-neutral, T = 50           |        |        |        |        |        |        |        |        |        |        |        |
| $\bar{b}_{I1}$                 | 0.0225 | 0.0205 | 0.0135 | 0.0170 | 0.0155 | 0.0145 | 0.0125 | 0.0115 | 0.0125 | 0.0135 | 0.0065 |
| $SD_{b_{I1}}$                  | 0.0220 | 0.0201 | 0.0133 | 0.0167 | 0.0153 | 0.0143 | 0.0123 | 0.0114 | 0.0123 | 0.0133 | 0.0065 |
| $\bar{b}_{I2}$                 | 0.0080 | 0.0010 | 0.0020 | 0.0020 | 0.0055 | 0.0055 | 0.0055 | 0.0070 | 0.0080 | 0.0040 | 0.0060 |
| $SD_{b_{I2}}$                  | 0.0079 | 0.0010 | 0.0020 | 0.0020 | 0.0055 | 0.0055 | 0.0055 | 0.0070 | 0.0079 | 0.0040 | 0.0060 |

**Table 1:** Sample means and sample standard deviations of the amounts invested if there are  $k$  steps to go:  $\bar{b}_{I2}$  is the average amount invested when  $x_k \in [z_2^k, z_1^k]$ ,  $SD_{b_{I2}}$  is the sample standard deviation of these investments;  $\bar{b}_{I1}$  is the sample mean of the amounts invested for  $x_k \in [z_1, 1]$ ,  $SD_{b_{I1}}$  is the corresponding sample standard deviation. All results are obtained from  $M = 2000$  simulation runs.

| t              | Log-utility, Full-Information, T = 5    |        |        |        |        |                       |        |        |        |        |
|----------------|-----------------------------------------|--------|--------|--------|--------|-----------------------|--------|--------|--------|--------|
|                | Stochastic Returns                      |        |        |        |        | Deterministic Returns |        |        |        |        |
|                | 1                                       | 2      | 3      | 4      | 5      | 1                     | 2      | 3      | 4      | 5      |
| $\bar{b}_{I1}$ | 0.2309                                  | 0.1825 | 0.1681 | 0.0714 | 0.1031 | 0.2846                | 0.1923 | 0.1377 | 0.0810 | 0.0722 |
| $SD_{b_{I1}}$  | 0.3187                                  | 0.3005 | 0.2886 | 0.1662 | 0.2382 | 0.3570                | 0.3080 | 0.2580 | 0.2211 | 0.1998 |
| $\bar{b}_{I2}$ | 0.0000                                  | 0.0006 | 0.0037 | 0.0075 | 0.0615 | 0.0000                | 0.0166 | 0.0571 | 0.0068 | 0.0410 |
| $SD_{b_{I2}}$  | 0.0000                                  | 0.0061 | 0.0227 | 0.0296 | 0.1627 | 0.0000                | 0.0640 | 0.1735 | 0.0267 | 0.1239 |
|                | Log-utility, Limited-Information, T = 5 |        |        |        |        |                       |        |        |        |        |
|                | Stochastic Returns                      |        |        |        |        | Deterministic Returns |        |        |        |        |
|                | 1                                       | 2      | 3      | 4      | 5      | 1                     | 2      | 3      | 4      | 5      |
| $\bar{b}_{I1}$ | 0.0000                                  | 0.1337 | 0.1902 | 0.1972 | 0.1855 | 0.0000                | 0.1560 | 0.2489 | 0.2394 | 0.1900 |
| $SD_{b_{I1}}$  | 0.0000                                  | 0.1317 | 0.2895 | 0.3585 | 0.3775 | 0.0000                | 0.1580 | 0.3551 | 0.4053 | 0.3927 |
| $\bar{b}_{I2}$ | 0.0000                                  | 0.0259 | 0.0535 | 0.1063 | 0.1982 | 0.0000                | 0.0000 | 0.1037 | 0.1454 | 0.2180 |
| $SD_{b_{I2}}$  | 0.0000                                  | 0.1609 | 0.2312 | 0.3261 | 0.4452 | 0.0000                | 0.0000 | 0.3220 | 0.3814 | 0.4669 |

**Table 2:** Sample means and sample standard deviations of the amounts invested: For the risk-averse investor,  $u(w) = \ln(w)$  and  $T = 5$ .  $\bar{b}_{I2}$  is the average amount invested when  $x_k \in [z_2^k, z_1^k]$ ,  $SD_{b_{I2}}$  is the sample standard deviation of these investments, while  $\bar{b}_{I1}$  is the sample mean of the amounts invested for  $x_k \in [z_1^k, 1]$ ,  $SD_{b_{I1}}$  is the corresponding sample standard deviation. Results obtained from  $M = 500$  simulation runs.

| T                   | 5      | 10     | 20     | 50     |
|---------------------|--------|--------|--------|--------|
| $\hat{H}_w$         | 0.1940 | 0.2285 | 0.2350 | 0.2545 |
| $\hat{H}_y$         | 0.7975 | 0.7580 | 0.7430 | 0.7235 |
| $\hat{H}_{y1}$      | 0.1890 | 0.1810 | 0.1480 | 0.1620 |
| $\hat{H}_{y2}$      | 0.6085 | 0.5770 | 0.5950 | 0.5615 |
| $\hat{H}_\emptyset$ | 0.0085 | 0.0135 | 0.0220 | 0.0220 |

**Table 3:** Relative frequencies: not investing,  $\hat{H}_w$ , invest and receive stochastic return,  $\hat{H}_y$ , invest and receive stochastic return where the project has final rank 1,  $\hat{H}_{y1}$ , invest and receive stochastic return where the project has final rank 2,  $\hat{H}_{y2}$ , invest and receive zero,  $\hat{H}_\emptyset$ . A risk-neutral investor is considered,  $n = 2$ ,  $T = 5, 10, 20, 50$ ;  $M = 2,000$  simulation runs.

|      | Log-utility, Full-Information, T = 5    |              |                               |        |                       |               |                               |        |
|------|-----------------------------------------|--------------|-------------------------------|--------|-----------------------|---------------|-------------------------------|--------|
|      | Stochastic Returns                      |              |                               |        | Deterministic Returns |               |                               |        |
|      | CI                                      | $\mathbb{V}$ | $w_0 + y_{1 1}^0 + y_{2 2}^0$ | $w_0$  | CI                    | $\mathcal{V}$ | $w_0 + y_{1 1}^0 + y_{2 2}^0$ | $w_0$  |
| mean | 1.6859                                  | 0.5223       | 2.1842                        | 0.1706 | 1.8313                | 0.6050        | 2.0124                        | 0.1107 |
| SD   | 0.0608                                  | 0.7100       | 1.8530                        | 0.2439 | 0.0395                | 0.5339        | 0.6380                        | 0.2015 |
|      | Log-utility, Limited-Information, T = 5 |              |                               |        |                       |               |                               |        |
|      | Stochastic Returns                      |              |                               |        | Deterministic Returns |               |                               |        |
|      | CI                                      | $\mathbb{V}$ | $w_0 + y_{1 1}^0 + y_{2 2}^0$ | $w_0$  | CI                    | $\mathcal{V}$ | $w_0 + y_{1 1}^0 + y_{2 2}^0$ | $w_0$  |
| mean | 1.3171                                  | 0.2754       | 1.5783                        | 0.2796 | 1.7236                | 0.5444        | 1.8268                        | 0.1776 |
| SD   | 0.0954                                  | 0.3455       | 1.2257                        | 0.0954 | 0.0311                | 0.1434        | 0.2825                        | 0.1776 |

**Table 4:** Sample means and sample standard deviations of the certainty equivalent  $CI$ , expected utility, final payoff  $w_0 + y_{1|1}^0 + y_{2|2}^0$ , and the remaining budget  $w_0$  for the risk-averse investor  $u(w) = \ln(w)$  and  $T = 5$ . For the certainty equivalent we estimated the standard error by applying the delta-method, an estimate of the certainty equivalent follows from  $\exp \hat{\mathbb{V}}$  or  $\exp \hat{\mathcal{V}}$ , the estimates of expected utility are the corresponding sample means. Results obtained from  $M = 500$  simulation runs.

The above simulation analysis allows us to estimate the value of information. That is, for the risk-neutral case, we consider the difference between the expected values generated by the optimal strategies under full information (the realization of  $X_t$  is observed) and limited information (only the realized relative rank of  $X_t$  is observed).<sup>11</sup> For both strategies, we also estimate the standard deviation. Table 5 presents estimates of the expected final wealth under the optimal strategies for full and limited information, respectively. In both cases, the returns are stochastic with the parameters provided in the preceding paragraphs. The initial wealth is  $w_T = 1$ . We observe that the expected payoff decreases slowly as  $T$  increases. The sample standard deviations under full information are smaller than those under limited information. Comparing the two information structures, we find that the difference in expected final wealth is approximately between 0.34 and 0.41, corresponding to between 34% and 41% of the initial wealth.

For the risk-averse case, the value of information is measured by the difference in certainty equivalents. For  $T = 5$  and log utility, the certainty equivalents are reported in Table 4. As already observed above, the difference in certainty equivalents (full information minus limited information) is approximately 0.37 for stochastic returns and 0.11 for deterministic returns.

<sup>11</sup>The optimal strategy for the limited-information case is obtained in Meier and Sögner (2023).

| T                                   | 5      | 10     | 20     | 50     |
|-------------------------------------|--------|--------|--------|--------|
| Risk-neutral, Full-Information      |        |        |        |        |
| $mean(w_0 + y_{1 1}^0 + y_{2 2}^0)$ | 2.4025 | 2.3255 | 2.2690 | 2.2585 |
| $SD(w_0 + y_{1 1}^0 + y_{2 2}^0)$   | 0.8360 | 0.9016 | 0.9127 | 0.9215 |
| Risk-neutral, Limited-Information   |        |        |        |        |
| $mean(w_0 + y_{1 1}^0 + y_{2 2}^0)$ | 2.0678 | 1.9540 | 1.8945 | 1.8411 |
| $SD(w_0 + y_{1 1}^0 + y_{2 2}^0)$   | 1.2654 | 1.2349 | 1.2411 | 1.2544 |

**Table 5:** Estimates of the expected final wealth (sample mean of  $w_0 + y_{1|1}^0 + y_{2|2}^0$ ) of the optimal strategy under full and limited-information and sample standard deviation  $SD(w_0 + y_{1|1}^0 + y_{2|2}^0)$ .

## 6 Conclusions

This article provides a stylized model of the venture capital investment process. Investment alternatives arise sequentially, and, due to competition, the investor has to decide in that period on whether and how much to invest. An investor observes a sequence of  $T$  investment alternatives, and only the best  $n$  opportunities can generate positive returns, while lower-ranked alternatives have a payoff of zero. Deterministic and stochastic payoffs are considered. While in the limited information case only the relative rank of the corresponding project is observed in any period (see, e.g., Meier and Sögner, 2023), this article considers the full information case, where the information available to an investor in any period are the current and all prior quality characteristics. For this investment problem, a recursive formulation of the value function for risk-neutral and risk-averse investors is obtained. The value function is a sum of expected utilities arising from past investments and a residual term. The first part is closed-form, while the residual term has to be obtained recursively. We also show that the value function is non-increasing in the number of investment periods  $T$ . For the risk-neutral investor, we show that a strategy of the “all-in-or-nothing”-type is optimal. We conduct a simulation study to show how investment behavior changes with the number of periods and risk-attitude. Finally, simulations for the full- and limited-information cases allow us to estimate the value of information. For the simulation designs considered, we observe that under stochastic returns the certainty equivalents increase by 34 to 41% when passing from the limited to the full information setting.

Standard asset pricing literature often considers assets with given expected returns and covariance matrix  $\Sigma$ . If  $\Sigma$  is diagonal with the same entries on the main diagonal, risk – measured in terms of portfolio variance – is minimized by choosing an equally weighted portfolio. Even if these idealized assumptions are

not satisfied, the equally weighted portfolio often performs quite well in empirical applications (see, e.g., DeMiguel et al., 2007). By contrast, in our investment problem, this diversification effect is not available. Due to the superstar property, only a small set of investment projects is profitable.

In this paper, we also obtain results on the value of information. In particular, we consider the full-information case, in which an investor observes the value of a quality signal, and the limited-information case, in which the investor knows only the rank of the investment project. In our simulations, we find that the expected payoff is approximately 20% higher under full information, while the variance of payoffs is approximately 30% lower for the fully informed investor. Although the model investigates the investment process at an abstract level, it demonstrates the importance of information. Thus, the expertise of venture capital investors is crucial. This finding is in line with the literature, e.g., Jang and Kaplan (2025) empirically verify that venture capitalists are able to distinguish promising projects from lower-quality ones.

The optimal strategies obtained in this article are rather complex. For Robbins' problem, it is well known that simple threshold strategies perform quite well. A threshold strategy is defined as the stopping rule: "choose the first  $x_{\mathbf{k}}$  such that  $x_{\mathbf{k}} \geq C_{\mathbf{k}}$ ", where  $C_{\mathbf{k}}$  denotes a threshold (depending on  $\mathbf{k}$ ). Examples of such strategies are provided by Gnedin (2007), Bruss (2005), Meier and Sögner (2017) among others. A question for further research is whether it is possible to find a closed form formula that is optimal (or approximately optimal) within the class of threshold strategies. Simulations could then be used to verify that the resulting values are not too far from the optimal values (when using the optimal strategies obtained in this article).

## A Probabilities

In the main text we obtained the probabilities  $p_{\mathbf{k}}(\ell_1, \dots, \ell_n | z_1^{\mathbf{k}}, \dots, z_n^{\mathbf{k}})$  as follows:<sup>12</sup> Let  $\nu_i, i = 1, \dots, n+1$ , count the number of realizations of  $X_{\kappa}, \kappa \leq \mathbf{k}$ , contained in the  $n+1$  intervals  $[z_1, 1], [z_2, z_1), \dots, [z_n, z_{n-1}), [0, z_n)$  in the last  $\mathbf{k}$  draws. Note that  $\sum_{j=1}^{n+1} \nu_j = \mathbf{k}$  and  $\nu_j \in \mathbb{N}_0$ . By our assumptions on  $X_{\mathbf{k}}$ , these probabilities follow from a multinomial distribution (see, e.g., Shao, 2003, p. 98), specifically the

<sup>12</sup>Since the results obtained in this section hold for any  $z_1^{\mathbf{k}}, \dots, z_n^{\mathbf{k}}$ , within this section we simply write  $z_1, \dots, z_n$  instead of  $z_1^{\mathbf{k}}, \dots, z_n^{\mathbf{k}}$ .

probabilities are

$$M_{\mathbf{k}}(\nu_1, \dots, \nu_n, \nu_{n+1} | z_1, \dots, z_n) = \frac{\mathbf{k}!}{\nu_1! \dots \nu_n! \nu_{n+1}!} (1 - z_1)^{\nu_1} (z_1 - z_2)^{\nu_2} \dots (z_{n-1} - z_n)^{\nu_n} (z_n - 0)^{\nu_{n+1}}, \quad (19)$$

with the convention  $0^0 = 1$  [e.g., in the starting phase  $z_2 = \dots = z_n = 0$ ]. Recall that  $\ell_i$  denotes the final rank of the current  $i$ -th best when there are  $\mathbf{k}$  periods to go. Given  $\nu_j$ ,  $j = 1, \dots, n+1$ , we have  $\ell_i = i + \sum_{j=1}^i \nu_j$ . Consequently, we get  $\nu_i = \ell_i - \ell_{i-1} - 1$ , for  $i = 1, \dots, n$ . For the possible  $\ell_i$  we have the following restriction  $0 = \ell_0 < \ell_1 < \dots < \ell_n \leq \mathbf{k} + n$ . Observe that we always have  $\mathbf{k} + n - \ell_n + \sum_{i=1}^n (\ell_i - \ell_{i-1} - 1) = \mathbf{k}$ . By working with  $\ell_i$  we obtain the multinomial probabilities

$$p_{\mathbf{k}}(\ell_1, \dots, \ell_n | z_1, \dots, z_n) = M_{\mathbf{k}}(\ell_1 - 1, \ell_2 - \ell_1 - 1, \dots, \ell_n - \ell_{n-1} - 1, \mathbf{k} + n - \ell_n | z_1, \dots, z_n). \quad (20)$$

In the following step, we link the probabilities at period  $\mathbf{k}$  to those at period  $\mathbf{k} - 1$ . To do this, we consider the following notation.  $\ell_1, \dots, \ell_n$  are the final ranks of draws with current ranks  $m_1, m_2, \dots, m_n$ . The probability of  $\ell_1, \dots, \ell_n$  given  $m_1, \dots, m_n$  and  $z_1, \dots, z_n$  are denoted by

$$p_{\mathbf{k}} \left( \begin{array}{cccc} \ell_1 & \ell_2 & \dots & \ell_n \\ m_1 & m_2 & \dots & m_n \end{array} \middle| z_1, \dots, z_n \right).$$

For  $m_1, \dots, m_n = 1, \dots, n$  we write

$$p_{\mathbf{k}} \left( \begin{array}{cccc} \ell_1 & \ell_2 & \dots & \ell_n \\ m_1 & m_2 & \dots & m_n \end{array} \middle| z_1, \dots, z_n \right) =: p_{\mathbf{k}}(\ell_1, \dots, \ell_n | z_1, \dots, z_n).$$

For any pair  $(\ell_1, \dots, \ell_n), (m_1, \dots, m_n)$ , where  $\ell_i < m_i$  for some  $i$ , the probability  $p_{\mathbf{k}}(\cdot) = 0$ . In addition, if  $\ell_{i+1} - \ell_i < m_{i+1} - m_i$ , then  $p_{\mathbf{k}}(\cdot) = 0$ . As well, if  $\ell_i - m_i > \mathbf{k}$ , then  $p_{\mathbf{k}}(\cdot) = 0$ . Finally, if  $\ell_n > \mathbf{k} - 1 + n$ , then  $p_{\mathbf{k}-1}(\cdot) = 0$ .

Note that  $\nu_{n+1} = \mathbf{k} - \sum_{i=1}^n \nu_i = \mathbf{k} - (\ell_n - m_n)$ . Then, by means of (19) we obtain

$$\begin{aligned} & p_{\mathbf{k}} \left( \begin{array}{cccc} \ell_1 & \ell_2 & \dots & \ell_n \\ m_1 & m_2 & \dots & m_n \end{array} \middle| z_1, \dots, z_n \right) \\ &= M_{\mathbf{k}}(\ell_1 - m_1, \ell_2 - \ell_1 - (m_2 - m_1), \dots, \ell_n - \ell_{n-1} - (m_n - m_{n-1}), \mathbf{k} + m_n - \ell_n | z_1, \dots, z_n). \quad (21) \end{aligned}$$

Let  $z_0 = 1$  and  $z_{n+1} = 0$ . The following Lemma is used in the proofs of the Theorems 1 and 4:

**Lemma 1.**

$$p_{\mathbf{k}} \left( \begin{array}{cccc} \ell_1 & \ell_2 & \dots & \ell_n \\ m_1 & m_2 & \dots & m_n \end{array} \middle| z_1, \dots, z_n \right) = \sum_{i=1}^{n+1} (z_{i-1} - z_i) p_{\mathbf{k}-1} \left( \begin{array}{ccccccc} \ell_1 & \dots & \ell_{i-1} & \ell_i & \ell_{i+1} & \dots & \ell_n \\ m_1 & \dots & m_{i-1} & m_i + 1 & m_{i+1} + 1 & \dots & m_n + 1 \end{array} \middle| z_1, \dots, z_n \right). \quad (22)$$

*Proof.* Consider  $M_{\mathbf{k}}(\cdot)$  defined in (19). Observe that

$$\begin{aligned} & \sum_{i=1}^{n+1} (z_{i-1} - z_i) M_{\mathbf{k}-1}(\nu_1, \dots, \nu_{i-1}, \nu_i - 1, \nu_{i+1}, \dots, \nu_{n+1} | z_1, \dots, z_n) \\ &= \sum_{i=1}^{n+1} (z_{i-1} - z_i) \frac{(\mathbf{k}-1)!}{\nu_1! \dots \nu_{i-1}! (\nu_i - 1)! \nu_{i+1}! \dots \nu_n! \nu_{n+1}!} \\ & \quad \cdot (1 - z_1)^{\nu_1} (z_1 - z_2)^{\nu_2} \dots (z_{i-2} - z_{i-1})^{\nu_{i-1}} (z_{i-1} - z_i)^{\nu_i - 1} (z_i - z_{i+1})^{\nu_{i+1}} \dots (z_{n-1} - z_n)^{\nu_n} (z_n - 0)^{\nu_{n+1}} \\ &= \sum_{i=1}^{n+1} \frac{(\mathbf{k}-1)!}{\nu_1! \dots \nu_{i-1}! (\nu_i - 1)! \nu_{i+1}! \dots \nu_n! \nu_{n+1}!} \\ & \quad \cdot (1 - z_1)^{\nu_1} (z_1 - z_2)^{\nu_2} \dots (z_{i-2} - z_{i-1})^{\nu_{i-1}} (z_{i-1} - z_i)^{\nu_i} (z_i - z_{i+1})^{\nu_{i+1}} \dots (z_{n-1} - z_n)^{\nu_n} (z_n - 0)^{\nu_{n+1}} \\ &= \sum_{i=1}^{n+1} \frac{\nu_i}{\mathbf{k}} \frac{\mathbf{k}!}{\nu_1! \dots \nu_{i-1}! \nu_i! \nu_{i+1}! \dots \nu_n! \nu_{n+1}!} \\ & \quad \cdot (1 - z_1)^{\nu_1} (z_1 - z_2)^{\nu_2} \dots (z_{i-2} - z_{i-1})^{\nu_{i-1}} (z_{i-1} - z_i)^{\nu_i} (z_i - z_{i+1})^{\nu_{i+1}} \dots (z_{n-1} - z_n)^{\nu_n} (z_n - 0)^{\nu_{n+1}} \\ &= \frac{1}{\mathbf{k}} \left( \frac{\mathbf{k}!}{\nu_1! \dots \nu_{i-1}! \nu_i! \nu_{i+1}! \dots \nu_n! \nu_{n+1}!} (1 - z_1)^{\nu_1} (z_1 - z_2)^{\nu_2} \dots (z_{n-1} - z_n)^{\nu_n} (z_n - 0)^{\nu_{n+1}} \right) \underbrace{\sum_{i=1}^{n+1} \nu_i}_{=\mathbf{k}} \\ &= \frac{\mathbf{k}!}{\nu_1! \dots \nu_{i-1}! \nu_i! \nu_{i+1}! \dots \nu_n! \nu_{n+1}!} (1 - z_1)^{\nu_1} (z_1 - z_2)^{\nu_2} \dots (z_{n-1} - z_n)^{\nu_n} (z_n - 0)^{\nu_{n+1}} \\ &= M_{\mathbf{k}}(\nu_1, \dots, \nu_{n+1} | z_1, \dots, z_n). \end{aligned} \quad (23)$$

Since  $p_{\mathbf{k}-1} \left( \begin{array}{ccccccc} \ell_1 & \dots & \ell_{i-1} & \ell_i & \ell_{i+1} & \dots & \ell_n \\ m_1 & \dots & m_{i-1} & m_i + 1 & m_{i+1} + 1 & \dots & m_n + 1 \end{array} \middle| z_1, \dots, z_n \right)$  follows from  $M_{\mathbf{k}-1}(\ell_1 - m_1, \ell_2 - \ell_1 - (m_2 - m_1), \dots, \ell_i - \ell_{i-1} - (m_i + 1 - m_{i-1}), \ell_{i+1} - \ell_i - (m_{i+1} + 1 - (m_i + 1)), \dots, \ell_n - \ell_{n-1} - (m_n + 1 - m_{n-1} - 1), \mathbf{k} + m_n - \ell_n | z_1, \dots, z_n)$  we finally get Equation (22).  $\square$

## B Proof of Theorem 1

We consider the recursion (5) and start with the following “Ansatz”:

$$\begin{aligned}\mathcal{V}_{\mathbf{k}}(w_{\mathbf{k}}, \mathbf{y}_{\mathbf{k}}, z_1^{\mathbf{k}}, \dots, z_n^{\mathbf{k}}) &= \mathcal{W}_{\mathbf{k}}(w_{\mathbf{k}}, \mathbf{y}_{\mathbf{k}}, z_1^{\mathbf{k}}, \dots, z_n^{\mathbf{k}}) + r_{\mathbf{k}}(w_{\mathbf{k}}, \mathbf{y}_{\mathbf{k}}, z_1^{\mathbf{k}}, \dots, z_n^{\mathbf{k}}) \\ &= \sum_{\substack{\ell_1, \dots, \ell_n: \\ 0 < \ell_1 < \dots < \ell_n \leq \mathbf{k}+n}} p_{\mathbf{k}}(\ell_1, \dots, \ell_n | z_1^{\mathbf{k}}, \dots, z_n^{\mathbf{k}}) u\left(w_{\mathbf{k}} + y_{1|\ell_1}^{\mathbf{k}} + \dots + y_{n|\ell_n}^{\mathbf{k}}\right) \\ &\quad + r_{\mathbf{k}}(w_{\mathbf{k}}, \mathbf{y}_{\mathbf{k}}, z_1^{\mathbf{k}}, \dots, z_n^{\mathbf{k}}),\end{aligned}\tag{24}$$

where  $r_{\mathbf{k}}(w_{\mathbf{k}}, \mathbf{y}_{\mathbf{k}}, z_1^{\mathbf{k}}, \dots, z_n^{\mathbf{k}})$  is assumed to be nonnegative.  $\mathcal{W}_{\mathbf{k}}(w_{\mathbf{k}}, \mathbf{y}_{\mathbf{k}}, z_1^{\mathbf{k}}, \dots, z_n^{\mathbf{k}})$  was defined in (10). In the following steps, we verify that the value function  $\mathcal{V}_{\mathbf{k}}(w_{\mathbf{k}}, \mathbf{y}_{\mathbf{k}}, z_1^{\mathbf{k}}, \dots, z_n^{\mathbf{k}})$  is of the structure imposed in (24) and that the residual term  $r_{\mathbf{k}}(w_{\mathbf{k}}, \mathbf{y}_{\mathbf{k}}, z_1^{\mathbf{k}}, \dots, z_n^{\mathbf{k}})$  is non-negative.

For  $\mathbf{k} = 0$ , equation (4) results in  $\mathcal{V}_0(w, \mathbf{y}_0, z_1^0, \dots, z_n^0) = u(w_0 + \sum_{j=1}^n y_{j|j}^0)$ , such that (24) holds with  $r_0(w_0, \mathbf{y}_0, z_1^0, \dots, z_n^0) = 0$ . To see this, note that  $p_0(1, \dots, n | z_1^0, \dots, z_n^0) = 1$ . For  $\mathbf{k} = 1$ , the recursion (4) implies

$$\begin{aligned}\mathcal{V}_1(w_1, \mathbf{y}_1, z_1^1, \dots, z_n^1) &= z_n^1 u\left(w_1 + \sum_{j=1}^n y_{j|j}^1\right) + \sum_{j=1}^n (z_{j-1}^1 - z_j^1) u\left(w_1 + \sum_{i=1}^{j-1} y_{i|i}^1 + \sum_{i=j}^{n-1} y_{i|i+1}^1\right) + r_1(w_1, \mathbf{y}_1, z_1^1, \dots, z_n^1) \\ &\quad [\text{where, with } r_0(w_0, \mathbf{y}_0, z_1^0, \dots, z_n^0) = 0,] \\ r_1(w_1, \mathbf{y}_1, z_1^1, \dots, z_n^1) &= z_n^1 r_0(w_1, \mathbf{y}_1, z_1^1, \dots, z_n^1) \\ &\quad + \sum_{j=1}^n (z_{j-1}^1 - z_j^1) \left[ \max_{0 \leq b_1 \leq w_1} u\left(w_1 - b_1 + \sum_{i=1}^{j-1} y_{i|i}^1 + b_1 \beta_1^j + \sum_{i=j}^{n-1} y_{i|i+1}^1\right) - u\left(w_1 + \sum_{i=1}^{j-1} y_{i|i}^1 + \sum_{i=j}^{n-1} y_{i|i+1}^1\right) \right].\end{aligned}\tag{25}$$

Hence, for  $\mathbf{k} = 1$  equation (24) holds as well. To see this, for any  $x_1 \geq z_n^1$  consider the term  $\left[ \max_{0 \leq b_1 \leq w_1} u\left(w_1 - b_1 + \sum_{i=1}^{j-1} y_{i|i}^1 + b_1 \beta_1^j + \sum_{i=j}^{n-1} y_{i|i+1}^1\right) - u\left(w_1 + \sum_{i=1}^{j-1} y_{i|i}^1 + 0 + \sum_{i=j}^{n-1} y_{i|i+1}^1\right) \right]$ . The second term denotes the utility arising from  $b_1 = 0$ . The first term takes the optimal  $b_1$ . Since  $b_1 \in [0, w_1]$

in the above optimization problem, we observe that

$$\begin{aligned} & \max_{0 \leq b_1 \leq w_1} u \left( w_1 - b_1 + \sum_{i=1}^{j-1} y_{i|i}^1 + b_1 \beta_1^j + \sum_{i=j}^{n-1} y_{i|i+1}^1 \right) \\ & \geq u \left( w_1 + \sum_{i=1}^{j-1} y_{i|i}^1 + 0 + \sum_{i=j}^{n-1} y_{i|i+1}^1 \right) \geq u(w_1). \end{aligned} \quad (26)$$

Hence,  $r_1(w_1, \mathbf{y}_1, z_1^1, \dots, z_n^1) \geq 0$ . If  $w_1 > 0$  and  $\beta_1^j > 1$  for some  $j$ , we get  $r_1(w_1, \mathbf{y}_1, z_1^1, \dots, z_n^1) > 0$ . In addition, we observe that for any  $\beta_1^j \geq 1$  the investor chooses  $b_1 = w_1$ , while  $\beta_1^j < 1$  for all  $j$  implies that  $b_1 = 0$ . For  $x_1 < z_n^1$ , the return from investing in period 1 is zero. Hence,  $b_1 = 0$  and  $\mathbf{y}_1 = \mathbf{y}_0$ .

Next, consider some  $\mathbf{k} \leq \mathbf{T}$ . Using (5) and the “Ansatz” (24) we derive:

$$\begin{aligned} & \gamma_{\mathbf{k}}(w_{\mathbf{k}}, \mathbf{y}_{\mathbf{k}}, z_1^{\mathbf{k}}, \dots, z_n^{\mathbf{k}}) \\ &= z_n^{\mathbf{k}} \gamma_{\mathbf{k}-1}(w_{\mathbf{k}}, \mathbf{y}_{\mathbf{k}}, z_1^{\mathbf{k}}, \dots, z_n^{\mathbf{k}}) \\ & \quad + \sum_{j=1}^n \int_{z_j^{\mathbf{k}}}^{z_{j-1}^{\mathbf{k}}} \max_{0 \leq b_{\mathbf{k}} \leq w_{\mathbf{k}}} \gamma_{\mathbf{k}-1} \left( w_{\mathbf{k}} - b_{\mathbf{k}}, \mathbf{y}_{\mathbf{k}} \left( \beta_{\mathbf{k}}^{j:n} b_{\mathbf{k}} \right), z_1^{\mathbf{k}}, \dots, z_{j-1}^{\mathbf{k}}, x, z_j^{\mathbf{k}}, \dots, z_{n-1}^{\mathbf{k}} \right) dx \\ &= z_n^{\mathbf{k}} \left( \sum_{\substack{\ell_1, \dots, \ell_n: \\ 0 < \ell_1 < \dots < \ell_n \leq \mathbf{k}-1+n}} p_{\mathbf{k}-1}(\ell_1, \dots, \ell_n | z_1^{\mathbf{k}}, \dots, z_n^{\mathbf{k}}) u \left( w_{\mathbf{k}} + y_{1|\ell_1}^{\mathbf{k}} + \dots + y_{n|\ell_n}^{\mathbf{k}} \right) + r_{\mathbf{k}-1}(w_{\mathbf{k}}, \mathbf{y}_{\mathbf{k}}, z_1^{\mathbf{k}}, \dots, z_n^{\mathbf{k}}) \right) \\ & \quad + \sum_{j=1}^n \int_{z_j^{\mathbf{k}}}^{z_{j-1}^{\mathbf{k}}} \max_{0 \leq b_{\mathbf{k}} \leq w_{\mathbf{k}}} \gamma_{\mathbf{k}-1} \left( w_{\mathbf{k}} - b_{\mathbf{k}}, \mathbf{y}_{\mathbf{k}} \left( \beta_{\mathbf{k}}^{j:n} b_{\mathbf{k}} \right), z_1^{\mathbf{k}}, \dots, z_{j-1}^{\mathbf{k}}, x, z_j^{\mathbf{k}}, \dots, z_{n-1}^{\mathbf{k}} \right) dx \\ &= z_n^{\mathbf{k}} \left( \sum_{\substack{\ell_1, \dots, \ell_n: \\ 0 < \ell_1 < \dots < \ell_n \leq \mathbf{k}-1+n}} p_{\mathbf{k}-1}(\ell_1, \dots, \ell_n | z_1^{\mathbf{k}}, \dots, z_n^{\mathbf{k}}) u \left( w_{\mathbf{k}} + y_{1|\ell_1}^{\mathbf{k}} + \dots + y_{n|\ell_n}^{\mathbf{k}} \right) + r_{\mathbf{k}-1}(w_{\mathbf{k}}, \mathbf{y}_{\mathbf{k}}, z_1^{\mathbf{k}}, \dots, z_n^{\mathbf{k}}) \right) \\ & \quad + \sum_{j=1}^n \int_{z_j^{\mathbf{k}}}^{z_{j-1}^{\mathbf{k}}} \max_{0 \leq b_{\mathbf{k}} \leq w_{\mathbf{k}}} \gamma_{\mathbf{k}-1} \left( w_{\mathbf{k}} - b_{\mathbf{k}}, \mathbf{y}_{\mathbf{k}} \left( \beta_{\mathbf{k}}^{j:n} b_{\mathbf{k}} \right), z_1^{\mathbf{k}}, \dots, z_{j-1}^{\mathbf{k}}, x, z_j^{\mathbf{k}}, \dots, z_{n-1}^{\mathbf{k}} \right) dx \\ & \quad - \sum_{j=1}^n \int_{z_j^{\mathbf{k}}}^{z_{j-1}^{\mathbf{k}}} \gamma_{\mathbf{k}-1} \left( w_{\mathbf{k}}, \mathbf{y}_{\mathbf{k}} \left( \beta_{\mathbf{k}}^{j:n} 0 \right), z_1^{\mathbf{k}}, \dots, z_{j-1}^{\mathbf{k}}, x, z_j^{\mathbf{k}}, \dots, z_{n-1}^{\mathbf{k}} \right) dx \\ & \quad + \sum_{j=1}^n \int_{z_j^{\mathbf{k}}}^{z_{j-1}^{\mathbf{k}}} \gamma_{\mathbf{k}-1} \left( w_{\mathbf{k}}, \mathbf{y}_{\mathbf{k}} \left( \underbrace{\beta_{\mathbf{k}}^{j:n} 0}_{=0} \right), z_1^{\mathbf{k}}, \dots, z_{j-1}^{\mathbf{k}}, x, z_j^{\mathbf{k}}, \dots, z_{n-1}^{\mathbf{k}} \right) dx. \end{aligned} \quad (27)$$

Note that if  $x_{\mathbf{k}} < z_n^{\mathbf{k}}$ ,  $\mathbf{y}_{\mathbf{k}} = \mathbf{y}_{\mathbf{k}-1}$ . In addition,  $x_{\mathbf{k}} < z_n^{\mathbf{k}}$  also implies that  $r_{\mathbf{k}-1}(w_{\mathbf{k}-1}, \mathbf{y}_{\mathbf{k}}, z_1^{\mathbf{k}-1}, \dots, z_n^{\mathbf{k}-1}) = r_{\mathbf{k}-1}(w_{\mathbf{k}}, \mathbf{y}_{\mathbf{k}-1}, z_1^{\mathbf{k}}, \dots, z_n^{\mathbf{k}})$ . The probability that all remaining draws  $X_{\kappa}$ ,  $\kappa = \mathbf{k}-1, \dots, 1$ , are smaller than

$z_n^{\mathbf{k}}$  is  $(z_n^{\mathbf{k}})^{\mathbf{k}-1}$ , while for  $\mathbf{k}$  steps to go the probability becomes  $(z_n^{\mathbf{k}})^{\mathbf{k}}$ .<sup>13</sup> Further multinomial probabilities  $p_{\mathbf{k}}(\ell_1, \dots, \ell_n; z_1^{\mathbf{k}}, \dots, z_n^{\mathbf{k}})$  follow by means of some algebra from sums containing products of  $(z_{j-1}^{\mathbf{k}} - z_j^{\mathbf{k}})$  and  $p_{\mathbf{k}-1}(\ell_1, \dots, \ell_n; z_1^{\mathbf{k}-1}, \dots, z_n^{\mathbf{k}-1})$ , see equation (9).

Let  $y_{j|\ell}^{\mathbf{k}}$  denote an element of the row vector  $\mathbf{y}_{\mathbf{k}}$ . For  $\mathcal{W}_{\mathbf{k}}(w_{\mathbf{k}}, \mathbf{y}_{\mathbf{k}}, z_1^{\mathbf{k}}, \dots, z_n^{\mathbf{k}})$  we obtain the following recursion:

$$\begin{aligned}
& z_n^{\mathbf{k}} \mathcal{W}_{\mathbf{k}-1}(w_{\mathbf{k}}, \mathbf{y}_{\mathbf{k}}, z_1^{\mathbf{k}}, \dots, z_n^{\mathbf{k}}) + \sum_{j=1}^n \int_{z_j^{\mathbf{k}}}^{z_{j-1}^{\mathbf{k}}} \mathcal{W}_{\mathbf{k}-1}(w_{\mathbf{k}}, \mathbf{y}_{\mathbf{k}}(\beta_{\mathbf{k}}^{j:n} \cdot 0), z_1^{\mathbf{k}}, \dots, z_{j-1}^{\mathbf{k}}, x, z_j^{\mathbf{k}}, \dots, z_{n-1}^{\mathbf{k}}) dx \\
&= z_n^{\mathbf{k}} \left[ \sum_{\substack{\ell_1, \dots, \ell_n: \\ 0 < \ell_1 < \dots < \ell_n \leq \mathbf{k}-1+n}} p_{\mathbf{k}-1} \left( \begin{array}{cccc} \ell_1 & \dots & \dots & \ell_n \\ 1 & \dots & \dots & n \end{array} \middle| z_1^{\mathbf{k}}, \dots, z_n^{\mathbf{k}} \right) u(w_{\mathbf{k}} + y_{1|\ell_1}^{\mathbf{k}} + \dots + y_{n|\ell_n}^{\mathbf{k}}) \right] \\
&+ \sum_{j=1}^n \int_{z_j^{\mathbf{k}}}^{z_{j-1}^{\mathbf{k}}} \sum_{\substack{\ell_1, \dots, \ell_n: \\ 0 < \ell_1 < \dots < \ell_n \leq \mathbf{k}+n}} p_{\mathbf{k}-1} \left( \begin{array}{ccccccc} \ell_1 & \dots & \ell_{j-1} & \ell_j & \ell_{j+1} & \dots & \ell_n \\ 1 & \dots & j-1 & j+1 & (j+1)+1 & \dots & n+1 \end{array} \middle| z_1^{\mathbf{k}}, \dots, z_n^{\mathbf{k}} \right) \\
&\quad \cdot u(w_{\mathbf{k}} + y_{1|\ell_1}^{\mathbf{k}} + \dots + y_{j-1|\ell_{j-1}}^{\mathbf{k}} + 0 + y_{j|\ell_j}^{\mathbf{k}} + \dots + y_{n-1|\ell_{n-1}}^{\mathbf{k}} + \underbrace{y_{n|\ell_n}^{\mathbf{k}}}_{=0}) dx \\
&= \sum_{j=1}^{n+1} (z_{j-1}^{\mathbf{k}} - z_j^{\mathbf{k}}) \sum_{\substack{\ell_1, \dots, \ell_n: \\ 0 < \ell_1 < \dots < \ell_n \leq \mathbf{k}+n}} p_{\mathbf{k}-1} \left( \begin{array}{ccccccc} \ell_1 & \dots & \ell_{j-1} & \ell_j & \ell_{j+1} & \dots & \ell_n \\ 1 & \dots & j-1 & j+1 & (j+1)+1 & \dots & n+1 \end{array} \middle| z_1^{\mathbf{k}}, \dots, z_n^{\mathbf{k}} \right) \\
&\quad \cdot u(w_{\mathbf{k}} + y_{1|\ell_1}^{\mathbf{k}} + \dots + y_{n|\ell_n}^{\mathbf{k}}) \\
&= \sum_{\substack{\ell_1, \dots, \ell_n: \\ 0 < \ell_1 < \dots < \ell_n \leq \mathbf{k}+n}} p_{\mathbf{k}}(\ell_1, \dots, \ell_n | z_1^{\mathbf{k}}, \dots, z_n^{\mathbf{k}}) u(w_{\mathbf{k}} + y_{1|\ell_1}^{\mathbf{k}} + \dots + y_{n|\ell_n}^{\mathbf{k}}) = \mathcal{W}_{\mathbf{k}}(w_{\mathbf{k}}, \mathbf{y}_{\mathbf{k}}, z_1^{\mathbf{k}}, \dots, z_n^{\mathbf{k}}) . \quad (28)
\end{aligned}$$

The last equation follows from Lemma 1 provided in the Appendix A.

<sup>13</sup>  $(z_n^{\mathbf{k}})^{\mathbf{k}-1}$  denotes  $z_n^{\mathbf{k}}$  raised by  $\mathbf{k}-1$ .

Consider  $k \leq T$ . By using (5) and the “Ansatz” (24) we derive:

$$\begin{aligned}
& \mathcal{V}_k(w_k, \mathbf{y}_k, z_1^k, \dots, z_n^k) + r_k(w_k, \mathbf{y}_k, z_1^k, \dots, z_n^k) = \mathcal{V}_k(w_k, \mathbf{y}_k, z_1^k, \dots, z_n^k) \\
&= z_n^k \mathcal{V}_{k-1}(w_k, \mathbf{y}_k, z_1^k, \dots, z_n^k) \\
&\quad + \sum_{j=1}^n \int_{z_j^k}^{z_{j-1}^k} \max_{0 \leq b_k \leq w_k} \mathcal{V}_{k-1}\left(w_k - b_k, \mathbf{y}_k\left(\beta_k^{j:n} b_k\right), z_1^k, \dots, z_{j-1}^k, x, z_j^k, \dots, z_{n-1}^k\right) dx \\
&= z_n^k \left( \mathcal{V}_{k-1}(w_k, \mathbf{y}_k, z_1^k, \dots, z_n^k) + r_{k-1}(w_k, \mathbf{y}_k, z_1^k, \dots, z_n^k) \right) \\
&\quad + \sum_{j=1}^n \int_{z_j^k}^{z_{j-1}^k} \max_{0 \leq b_k \leq w_k} \left( \mathcal{V}_{k-1}\left(w_k - b_k, \mathbf{y}_k\left(\beta_k^{j:n} b_k\right), z_1^k, \dots, z_{j-1}^k, x, z_j^k, \dots, z_{n-1}^k\right) \right. \\
&\quad \left. + r_{k-1}\left(w_k - b_k, \mathbf{y}_k\left(\beta_k^{j:n} b_k\right), z_1^k, \dots, z_{j-1}^k, x, z_j^k, \dots, z_{n-1}^k\right) \right) dx \\
&= z_n^k \mathcal{V}_{k-1}(w_k, \mathbf{y}_k, z_1^k, \dots, z_n^k) + z_n^k r_{k-1}(w_k, \mathbf{y}_k, z_1^k, \dots, z_n^k) \\
&\quad + \sum_{j=1}^n \int_{z_j^k}^{z_{j-1}^k} \max_{0 \leq b_k \leq w_k} \left( \mathcal{V}_{k-1}\left(w_k - b_k, \mathbf{y}_k\left(\beta_k^{j:n} b_k\right), z_1^k, \dots, z_{j-1}^k, x, z_j^k, \dots, z_{n-1}^k\right) \right. \\
&\quad - \mathcal{V}_{k-1}\left(w_k - 0, \mathbf{y}_k\left(\beta_k^{j:n} 0\right), z_1^k, \dots, z_{j-1}^k, x, z_j^k, \dots, z_{n-1}^k\right) \\
&\quad + \mathcal{V}_{k-1}\left(w_k - 0, \mathbf{y}_k\left(\beta_k^{j:n} 0\right), z_1^k, \dots, z_{j-1}^k, x, z_j^k, \dots, z_{n-1}^k\right) \\
&\quad \left. + r_{k-1}\left(w_k - b_k, \mathbf{y}_k\left(\beta_k^{j:n} b_k\right), z_1^k, \dots, z_{j-1}^k, x, z_j^k, \dots, z_{n-1}^k\right) \right) dx \\
&= z_n^k \mathcal{V}_{k-1}(w_k, \mathbf{y}_k, z_1^k, \dots, z_n^k) + \sum_{j=1}^n \int_{z_j^k}^{z_{j-1}^k} \mathcal{V}_{k-1}\left(w_k - 0, \mathbf{y}_k\left(\beta_k^{j:n} 0\right), z_1^k, \dots, z_{j-1}^k, x, z_j^k, \dots, z_{n-1}^k\right) dx \\
&\quad + z_n^k r_{k-1}(w_k, \mathbf{y}_k, z_1^k, \dots, z_n^k) \\
&\quad + \sum_{j=1}^n \int_{z_j^k}^{z_{j-1}^k} \max_{0 \leq b_k \leq w_k} \left( \mathcal{V}_{k-1}\left(w_k - b_k, \mathbf{y}_k\left(\beta_k^{j:n} b_k\right), z_1^k, \dots, z_{j-1}^k, x, z_j^k, \dots, z_{n-1}^k\right) \right. \\
&\quad - \mathcal{V}_{k-1}\left(w_k - 0, \mathbf{y}_k\left(\beta_k^{j:n} 0\right), z_1^k, \dots, z_{j-1}^k, x, z_j^k, \dots, z_{n-1}^k\right) \\
&\quad \left. + r_{k-1}\left(w_k - b_k, \mathbf{y}_k\left(\beta_k^{j:n} b_k\right), z_1^k, \dots, z_{j-1}^k, x, z_j^k, \dots, z_{n-1}^k\right) \right) dx \\
&= \mathcal{V}_k(w_k, \mathbf{y}_k, z_1^k, \dots, z_n^k) \\
&\quad + z_n^k r_{k-1}(w_k, \mathbf{y}_k, z_1^k, \dots, z_n^k) \\
&\quad + \sum_{j=1}^n \int_{z_j^k}^{z_{j-1}^k} \max_{0 \leq b_k \leq w_k} \left( \sum_{\substack{\ell_1, \dots, \ell_n: \\ 0 < \ell_1 < \dots < \ell_n \leq k-1+n}} p_{k-1}(\ell_1, \dots, \ell_n | \mathbf{z}(x)) [u(\pi(j, w_k, b_k, \mathbf{y}_k, \mathcal{L})) - u(\pi(j, w_k, 0, \mathbf{y}_k, \mathcal{L}))] \right. \\
&\quad \left. + r_{k-1}\left(w_k - b_k, \mathbf{y}_k\left(\beta_k^{j:n} b_k\right), z_1^k, \dots, z_{j-1}^k, x, z_j^k, \dots, z_{n-1}^k\right) \right) dx. \tag{29}
\end{aligned}$$

To obtain (29) we proceeded as follows: In the first step we used the recursion for the value function (5). Then we plugged in the decomposition of  $\mathcal{V}_k$  into  $\mathcal{V}_k$  and  $r_k$ . For the terms  $\mathcal{V}_k$  the recursion (28) applies. The remaining terms are equal to the residual term defined in Theorem 1.

## C Proof of Theorem 2

The following arguments are based on Chow et al. (1964) and Assaf and Samuel-Cahn (1996). Consider the  $T + 1$  period problem, but where the investor is told the period and value of the worst realization of the random variable  $X_k$ .

Obviously the value of this modified problem is at least as high as the original  $T + 1$  period problem. Conditional on the worst value being  $\eta \geq 0$  and being drawn in period  $\hat{k}$ , as we assumed  $T + 1 > n$ , investing in period  $\hat{k}$  would for sure result in a complete loss of the investment. Hence, the investor will avoid “investing in period  $\hat{k}$ ” given his information about the worst value. Given the information about the worst value, the other  $X_k$  are iid uniformly distributed in the interval  $[\eta, 1]$  and the investor faces a problem equivalent to the modified  $T$ -period problem with return rates  $\gamma_k^i = \check{\beta}_k^i$  for  $k > \hat{k}$  and  $\gamma_k^i = \check{\beta}_{k-1}^i$  for  $k \leq \hat{k}$  following from (12).

By (12) the return rates in the original  $T$ -period are at least as high as in the modified  $T$ -period problem (i.e.,  $\beta_k^i \geq \gamma_k^i$ ). Hence, conditional on any worst draw in the  $T + 1$ -period problem, the modified  $T$ -period problem (where the investor is informed about the value of the worst draw and the corresponding period  $\hat{k}$ ), has a value at most as high as the original  $T$ -period problem. Hence, the original  $T + 1$ -period problem has a value no better than the  $T$ -period problem.

For the strict inequality we additionally impose (14). First, we suppose that  $\check{\beta}_k^i \leq 1$ , for all  $i = 1, \dots, n$  and  $k = T, \dots, 1, 0$ . In this case “never invest” is optimal in the  $T + 1$  period problem and the value function  $\mathcal{V}_{T+1} \left( \underline{w}, \left( \check{\beta}_k^i \right)_{k=T, \dots, 1, 0; i=1, \dots, n} \right) = u(\underline{w})$ .

In the following step, we obtain a simple investment strategy (which need not be optimal) for the  $T$  period problem, such that the expected utility is strictly larger than  $u(w)$  for  $w = \underline{w} > 0$ . Since this expected utility is smaller than or equal to the optimal value  $\mathcal{V}_T \left( w, \left( \beta_k^i \right)_{k=T, \dots, 1; i=1, \dots, n} \right)$ , we obtain the inequality  $\mathcal{V}_{T+1} \left( w, \left( \check{\beta}_k^i \right)_{k=T, \dots, 1, 0; i=1, \dots, n} \right) = u(w) < \mathcal{V}_T \left( w, \left( \beta_k^i \right)_{k=T, \dots, 1; i=1, \dots, n} \right)$ .

As already assumed,  $u(\cdot)$  is strictly monotone increasing and  $w > 0$ . Given the additional assumptions in (14), there exists  $\beta_k^i > 1$ , where  $T - k \geq i - 1$ . Observe the following: Since  $w > 0$  and  $\beta_k^i > 1$  we have

$$u \left( \beta_k^i \frac{w}{2} + \frac{w}{2} \right) > u(w) > u \left( \frac{w}{2} \right). \quad (30)$$

Note that there exists a  $\widehat{\delta} \in (0, 1)$  [e.g.  $\widehat{\delta}$  sufficiently close to 1] such that

$$\delta^{k-1} u\left(\beta_k^i \frac{w}{2} + \frac{w}{2}\right) + (1 - \delta^{k-1}) u\left(\frac{w}{2}\right) > u(w) , \quad (31)$$

for all  $\delta \in [\widehat{\delta}, 1]$ .

Consider a  $\delta$  satisfying inequality (31) and there are  $k$  steps to go. Note that there exists a strictly positive probability that the in event, denoted  $\mathcal{E}$ , the following happens: Exactly  $i - 1$  draws of the past draws  $X_j$ ,  $j > k$ , are in the interval  $(\frac{1+\delta}{2}, 1]$  and the remaining  $T - k - i + 1$  past draws are in the interval  $[0, \delta)$ , and the draw  $X_k$  is in the interval  $[\delta, \frac{1+\delta}{2}]$ . The probability of this event is abbreviated by  $\mathbb{P}(\mathcal{E}) > 0$ .

Consider the following investment strategy: Never invest before and after period  $k$ . Invest  $\frac{w_k}{2}$  in period  $k$  if and only if  $\mathcal{E}$  realizes. The expected utility of this strategy is at least

$$\mathbb{P}(\mathcal{E}) \left[ \delta^{k-1} u\left(\beta_k^i \frac{w}{2} + \frac{w}{2}\right) + (1 - \delta^{k-1}) u\left(\frac{w}{2}\right) \right] + (1 - \mathbb{P}(\mathcal{E})) u(w) > u(w) . \quad (32)$$

Hence, the expected utility in (32) is larger than  $u(w) = \mathcal{V}_{T+1}\left(w, (\check{\beta}_k^i)_{k=T, \dots, 1, 0; i=1, \dots, n}\right)$ , which implies that  $\mathcal{V}_{T+1}\left(w, (\check{\beta}_k^i)_{k=T, \dots, 1, 0; i=1, \dots, n}\right) = u(w) < \mathcal{V}_T\left(w, (\beta_k^i)_{k=T, \dots, 1; i=1, \dots, n}\right)$ .

Recall that by assumption  $w > 0$ .

Now, suppose that  $\check{\beta}_k^i > 1$  for some  $i$  and  $k$ . Let  $\widehat{\beta} := \max\{1, \max_{i=2, \dots, n, k=T-1, \dots, 0} \{\check{\beta}_k^i\}\} \geq 1$ . Our conditions in Theorem 3 in Equation (14) imply  $\check{\beta}_T^1 > \widehat{\beta}$ . Then, there exists  $\eta$ , with  $0 < \eta < w$  such that  $\check{\beta}_T^1(w - \eta) > \widehat{\beta}w$ . This implies, that there is  $\widehat{\varepsilon} > 0$  with

$$\begin{aligned} \check{\beta}_T^1(w - \eta) &> \widehat{\beta}(w - \widehat{\varepsilon}) + \check{\beta}_T^1 \widehat{\varepsilon} \quad \text{and} \\ \widehat{\beta}(w - \widehat{\varepsilon}) + \check{\beta}_T^1 \widehat{\varepsilon} &\geq \widehat{\beta}(w - \varepsilon) + \check{\beta}_T^1 \varepsilon , \end{aligned} \quad (33)$$

for all  $0 \leq \varepsilon \leq \widehat{\varepsilon}$ . This implies

$$u\left(\check{\beta}_T^1(w - \eta)\right) > u\left(\widehat{\beta}(w - \widehat{\varepsilon}) + \check{\beta}_T^1 \widehat{\varepsilon}\right) \geq u\left(\widehat{\beta}(w - \varepsilon) + \check{\beta}_T^1 \varepsilon\right) , \quad (34)$$

for all  $0 \leq \varepsilon \leq \widehat{\varepsilon}$ . Since  $+\infty > u\left(\check{\beta}_T^1 w\right) > u(\eta) > -\infty$ , there exists  $\widehat{\delta} \in (0, 1)$ , such that investing  $\widehat{\varepsilon}$  in the

first period gives at most

$$\widehat{\delta}^{T-1} u \left( \widehat{\beta} (w - \widehat{\varepsilon}) + \check{\beta}_T^1 \widehat{\varepsilon} \right) + \left( 1 - \widehat{\delta}^{T-1} \right) u \left( \check{\beta}_{T-1}^1 (w - \widehat{\varepsilon}) + \check{\beta}_T^2 \widehat{\varepsilon} \right) < \widehat{\delta}^{T-1} u \left( \check{\beta}_T^1 (w - \eta) \right) + \left( 1 - \widehat{\delta}^{T-1} \right) u (\eta) . \quad (35)$$

Equation (35) holds for all  $\delta$ , where  $\widehat{\delta} \leq \delta \leq 1$ . To see this, use (35) and the equality  $u(\eta) \leq u(w) \leq u(\check{\beta}_T^1 w)$ . Hence, by equation (35) the optimal  $T+1$ -period investor invests more than  $\widehat{\varepsilon}$  with strictly positive probability  $1 - \widehat{\delta}$ . With probability  $\frac{1-\widehat{\delta}}{2}$  the first draw is in the interval  $\left[ \widehat{\delta}, \widehat{\delta} + \frac{1-\widehat{\delta}}{2} \right]$ . With probability  $\left( \frac{1-\widehat{\delta}}{2} \right)^T$  all subsequent draws  $X_{T-1}, \dots, X_1, X_0$  are in the interval  $\left( \widehat{\delta} + \frac{1-\widehat{\delta}}{2}, 1 \right]$ . Hence, the uninformed optimal  $T+1$ -period investor invests with probability  $\left( \frac{1-\widehat{\delta}}{2} \right)^{T+1}$  at least  $\widehat{\varepsilon}$  (or even more) in a draw, that turns out to be the worst. The informed  $T+1$ -period investor avoids this loss. Hence, she has a strictly higher expected utility. By the argument used in the first part of the proof, the  $T$ -period investor achieves an expected utility at least as high as the informed  $T+1$ -period investor.

## D Proof of Theorem 3

From Theorem 1 and  $u(\cdot) = id(\cdot)$  we get

$$\begin{aligned} \mathcal{V}_k(w_k, \mathbf{y}_k, z_1^k, \dots, z_n^k) &= \mathcal{W}_k(w_k, \mathbf{y}_k, z_1^k, \dots, z_n^k) + r_k(w_k, \mathbf{y}_k, z_1^k, \dots, z_n^k) \\ &= \sum_{\substack{\ell_1, \dots, \ell_n: \\ 0 < \ell_1 < \dots < \ell_n \leq k+n}} p_k(\ell_1, \dots, \ell_n | z_1^k, \dots, z_n^k) \left( w_k + y_{1|\ell_1}^k + \dots + y_{n|\ell_n}^k \right) \\ &\quad + r_k(w_k, \mathbf{y}_k, z_1^k, \dots, z_n^k) , \end{aligned} \quad (36)$$

where  $\mathcal{W}_k(w_k, \mathbf{y}_k, z_1^k, \dots, z_n^k)$  is an affine function in  $w_k$ . In addition, we assume that  $r_k(w_k, \mathbf{y}_k, z_1^k, \dots, z_n^k) = w_k c_k(z_1^k, \dots, z_n^k)$ . The following paragraphs will show that the residual term is of this structure for the risk-neutral case.

For  $k=0$ , equation (4) and (11) result in  $r_0(w_0, \mathbf{y}_0, z_1^0, \dots, z_n^0) = c_0(z_1^0, \dots, z_n^0) = 0$ . For  $k=1$ , from

(25) we obtain

$$\begin{aligned}
& \mathcal{V}_1(w_1, \mathbf{y}_1, z_1^1, \dots, z_n^1) \\
&= z_n^1 \left( w_1 + \sum_{j=1}^n y_{j|j}^1 \right) + \sum_{j=1}^n (z_{j-1}^1 - z_j^1) \left( w_1 + \sum_{i=1}^{j-1} y_{i|i}^1 + \sum_{i=j}^{n-1} y_{i|i+1}^1 \right) + r_1(w_1, \mathbf{y}_1, z_1^1, \dots, z_n^1) \\
& \quad [\text{where, with } c_0 = 0, ] \\
& r_1(w_1, \mathbf{y}_1, z_1^1, \dots, z_n^1) = z_n^1 w_1 c_0 \\
& \quad + \sum_{j=1}^n (z_{j-1}^1 - z_j^1) \left[ \max_{0 \leq b_1 \leq w_1} \left( w_1 - b_1 + \sum_{i=1}^{j-1} y_{i|i}^1 + b_1 \beta_1^j + \sum_{i=j}^{n-1} y_{i|i+1}^1 \right) - \left( w_1 + \sum_{i=1}^{j-1} y_{i|i}^1 + \sum_{i=j}^{n-1} y_{i|i+1}^1 \right) \right]. \quad (37)
\end{aligned}$$

The investor invests all the remaining wealth, if  $\beta_1^j \geq 1$  and  $x_1$  is contained in some interval  $[z_j^1, z_{j-1}^1]$ ,  $j = 1, \dots, n$  otherwise  $b_1 = 0$  [in the notation of Theorem 3  $b_1 = w_1$  if and only if  $\beta_1^j - 1 \geq c_0 = 0$ ]. That is,  $b_1 = w_1$  or  $b_1 = 0$ , such that, for  $k = 1$ , the residual

$$r_1(w_1, \mathbf{y}_1, z_1^1, \dots, z_n^1) = w_1 \sum_{j: \beta_1^j \geq 1} (z_{j-1}^1 - z_j^1)(\beta_1^j - 1) \quad (38)$$

Therefore,  $r_1(w_1, \mathbf{y}_1, z_1^1, \dots, z_n^1) = w_1 c_1(z_1^1, \dots, z_n^1)$ , hence, linear in  $w_1$ .

In the next step, we use an induction argument to show that the residual term is linear in  $w_k$ . First, we use (11) as well as  $u(\cdot) = id(\cdot)$  and impose the *induction assumption*  $r_{k-1}(w_{k-1}, \mathbf{y}_{k-1}, z_1^{k-1}, \dots, z_n^{k-1}) =$

$w_{k-1}c_{k-1}(z_1^{k-1}, \dots, z_n^{k-1})$ , resulting in

$$\begin{aligned}
& r_k(w_k, \mathbf{y}_k, z_1^k, \dots, z_n^k) \\
&= z_n^k w_k c_{k-1}(z_1^k, \dots, z_n^k) \\
&+ \sum_{j=1}^n \int_{z_j^k}^{z_{j-1}^k} \max_{0 \leq b_k \leq w_k} \left[ \sum_{\substack{\ell_1, \dots, \ell_n: \\ 0 < \ell_1 < \dots < \ell_n \leq k-1+n}} p_{k-1}(\ell_1, \dots, \ell_n | z_1^k, \dots, z_{j-1}^k, x, z_j^k, \dots, z_{n-1}^k) \right. \\
&\quad \cdot \left( (w_k - b_k) + y_{1|\ell_1}^k + \dots + y_{j-1|\ell_{j-1}}^k + \beta_k^{\ell_j} b_k + y_{j|\ell_j+1}^k + \dots + y_{n|\ell_n+1}^k \right) \\
&\quad + (w_k - b_k) c_{k-1}(z_1^k, \dots, z_{j-1}^k, x, z_j^k, \dots, z_{n-1}^k) \\
&\quad - \sum_{\substack{\ell_1, \dots, \ell_n: \\ 0 < \ell_1 < \dots < \ell_n \leq k-1+n}} p_{k-1}(\ell_1, \dots, \ell_n | z_1^k, \dots, z_{j-1}^k, x, z_j^k, \dots, z_{n-1}^k) \\
&\quad \cdot \left( w_k + y_{1|\ell_1}^k + \dots + y_{j-1|\ell_{j-1}}^k + 0 + y_{j|\ell_j+1}^k + \dots + y_{n|\ell_n+1}^k \right) \Big] dx \\
&= z_n^k w_k c_{k-1}(z_1^k, \dots, z_n^k) \\
&+ \sum_{j=1}^n \int_{z_j^k}^{z_{j-1}^k} \max_{0 \leq b_k \leq w_k} \left[ \sum_{\substack{\ell_1, \dots, \ell_n: \\ 0 < \ell_1 < \dots < \ell_n \leq k-1+n}} p_{k-1}(\ell_1, \dots, \ell_n | z_1^k, \dots, z_{j-1}^k, x, z_j^k, \dots, z_{n-1}^k) b_k (\beta_k^{\ell_j} - 1) \right. \\
&\quad \left. + (w_k - b_k) c_{k-1}(z_1^k, \dots, z_{j-1}^k, x, z_j^k, \dots, z_{n-1}^k) \right] dx. \tag{39}
\end{aligned}$$

Note that the terms inside the squared brackets of (39) are affine in  $b_k$ . Hence, we obtain an optimal investment strategy:  $b_k = w_k$  if and only if

$$\sum_{\substack{\ell_1, \dots, \ell_n: \\ 0 < \ell_1 < \dots < \ell_n \leq k-1+n}} p_{k-1}(\ell_1, \dots, \ell_n | z_1^k, \dots, z_{j-1}^k, x, z_j^k, \dots, z_{n-1}^k) (\beta_k^{\ell_j} - 1) \geq c_{k-1}(z_1^k, \dots, z_{j-1}^k, x, z_j^k, \dots, z_{n-1}^k),$$

otherwise  $b_k = 0$ , conditional on  $x \in [z_j^k, z_{j-1}^k)$ . For any  $x \leq z_n^k$ ,  $b_k = 0$ . Therefore, given  $x \in [z_j^k, z_{j-1}^k)$ , maximizing gives either  $w_k \sum_{\substack{\ell_1, \dots, \ell_n: \\ 0 < \ell_1 < \dots < \ell_n \leq k-1+n}} p_{k-1}(\ell_1, \dots, \ell_n | z_1^k, \dots, z_{j-1}^k, x, z_j^k, \dots, z_{n-1}^k) (\beta_k^{\ell_j} - 1)$  or  $w_k c_{k-1}(z_1^k, \dots, z_{j-1}^k, x, z_j^k, \dots, z_{n-1}^k)$ . In either case, the solution is linear in  $w_k$ . Therefore,  $r_k(w_k, \mathbf{y}_k, z_1^k, \dots, z_n^k) = w_k c_k(z_1^k, \dots, z_n^k)$ .

Since the decision maker invests at most once, the multinomial probabilities in (39) can be simplified:

Recall that  $\rho_a(x_v)$  denotes the final rank of  $x_v$ . Let  $v \geq k+1$ .  $\rho_{k+1}(x_v)$  is the current rank of  $x_v$  before  $X_k$  realizes. Consider  $\ell = \rho_{k+1}(x_v) + \lambda$ . The goal is to obtain the conditional probabilities  $p_k(\{\rho_a(x_v) = \ell\} | x_v)$ , which is the probability that  $\lambda = 0, 1, \dots, k$  of the remaining  $k$  draws  $X_k, \dots, X_1$

are better than  $x_v$ . Based on our model assumptions, we consider  $k$  independent draws without replacement. The number of “successes”  $\lambda$ , where  $\lambda = 0, 1, \dots, k$ , follows from a Binomial distribution with “success probability”  $1 - x_v$ . Hence,  $p_k(\{\rho_a(x_v) = \ell\} | x_v) = p_k(\{\rho_a(x_v) = \lambda + \rho_{k+1}(x_v)\} | x_v) = \binom{k}{\lambda} (1 - x_v)^\lambda x_v^{k-\lambda}$ , where  $\ell = \lambda + \rho_{k+1}(x_v)$ .

Therefore the optimal investment strategy of a risk-neutral investor is to invest all the remaining wealth  $w_k$  if and only if  $-1 + \sum_{\ell=\rho_k(x_k)}^n p_{k-1}(\{\rho_a(x_k) = \ell\} | x_k) \beta_k^\ell \geq c_{k-1}(z_1^k, \dots, z_{j-1}^k, x_k, z_j^k, \dots, z_{n-1}^k)$ , and zero otherwise. Plugging in the above results into (39) yields:

$$c_k(z_1^k, \dots, z_n^k) = z_n^k c_{k-1}(z_1^k, \dots, z_n^k) + \sum_{j=1}^n \int_{z_j^k}^{z_{j-1}^k} \max \left\{ -1 + \sum_{\ell=\rho_k(x_k)}^n p_{k-1}(\{\rho_a(x) = \ell\} | x) \beta_k^\ell, c_{k-1}(z_1^k, \dots, z_j^k, x, z_{j+1}^k, \dots, z_{n-1}^k) \right\} dx, \quad (40)$$

where  $c_0 = 0$ .

## E Proof of Theorem 4

The proof of Theorem 4, works in almost the same way as proof of Theorem 1, where deterministic returns were considered. In the following steps, we assume that the conditional expectations, considered in this section, are well defined.  $B_k^j$  and  $\mathbf{B}_k^{j:n}$  are random variables and stochastic analogues to  $\beta_k^j$  and  $\beta_k^{j:n}$ .  $\mathbb{E}_k(\cdot)$  abbreviates the corresponding conditional expectation, when there are  $k$  steps to go. Since  $X_T, \dots, X_k$  are  $\mathcal{F}_k$ -measurable, the current  $z_1^k, \dots, z_n^k$  are  $\mathcal{F}_k$ -measurable.  $\tilde{\mathbf{y}}_k$  is now a random vector [here  $\sim$  denotes that this object is a random variable or vector]. We consider the recursion (5). Since the expectation is a linear operator, the recursion used for deterministic returns can be extended to the case of stochastic returns (utilities  $u(\cdot)$  in the deterministic case have to be replaced by expected utilities). Hence, start with the following “Ansatz”:

$$\begin{aligned} \mathbb{V}_k(w_k, \tilde{\mathbf{y}}_k, z_1^k, \dots, z_n^k) &= \mathbb{W}_k(w_k, \tilde{\mathbf{y}}_k, z_1^k, \dots, z_n^k) + R_k(w_k, \tilde{\mathbf{y}}_k, z_1^k, \dots, z_n^k) \\ &= \mathbb{E}_k \left( \sum_{\substack{\ell_1, \dots, \ell_n: \\ 0 < \ell_1 < \dots < \ell_n \leq k+n}} p_k(\ell_1, \dots, \ell_n | z_1^k, \dots, z_n^k) u(w_k + \tilde{y}_{1|\ell_1}^k + \dots + \tilde{y}_{n|\ell_n}^k) \right) \\ &\quad + R_k(w_k, \tilde{\mathbf{y}}_k, z_1^k, \dots, z_n^k) \\ &= \sum_{\substack{\ell_1, \dots, \ell_n: \\ 0 < \ell_1 < \dots < \ell_n \leq k+n}} p_k(\ell_1, \dots, \ell_n | z_1^k, \dots, z_n^k) \mathbb{E} \left( u(w_k + \tilde{y}_{1|\ell_1}^k + \dots + \tilde{y}_{n|\ell_n}^k) \right) \\ &\quad + R_k(w_k, \tilde{\mathbf{y}}_k, z_1^k, \dots, z_n^k), \end{aligned} \quad (41)$$

where  $\mathbb{W}_{\mathbf{k}}(w_{\mathbf{k}}, \tilde{\mathbf{y}}_k, z_1^{\mathbf{k}}, \dots, z_n^{\mathbf{k}})$  was defined in (17). We get

$$\begin{aligned}
& z_n^{\mathbf{k}} \mathbb{W}_{\mathbf{k}-1}(w_{\mathbf{k}}, \mathbf{y}_{\mathbf{k}}, z_1^{\mathbf{k}}, \dots, z_n^{\mathbf{k}}) + \sum_{j=1}^n \int_{z_j^{\mathbf{k}}}^{z_{j-1}^{\mathbf{k}}} \mathbb{W}_{\mathbf{k}-1}(w_{\mathbf{k}}, \tilde{\mathbf{y}}_{\mathbf{k}}(\beta_{\mathbf{k}}^{j:n} \cdot 0), z_1^{\mathbf{k}}, \dots, z_{j-1}^{\mathbf{k}}, x, z_j^{\mathbf{k}}, \dots, z_{n-1}^{\mathbf{k}}) dx \\
&= z_n^{\mathbf{k}} \left[ \sum_{\substack{\ell_1, \dots, \ell_n: \\ 0 < \ell_1 < \dots < \ell_n \leq \mathbf{k}-1+n}} p_{\mathbf{k}-1} \left( \begin{matrix} \ell_1 & \dots & \dots & \ell_n \\ 1 & \dots & \dots & n \end{matrix} \middle| z_1^{\mathbf{k}}, \dots, z_n^{\mathbf{k}} \right) \mathbb{E} \left( u \left( w_{\mathbf{k}} + \tilde{y}_{1|\ell_1}^{\mathbf{k}} + \dots + \tilde{y}_{n|\ell_n}^{\mathbf{k}} \right) \right) \right] \\
&+ \sum_{j=1}^n \int_{z_j^{\mathbf{k}}}^{z_{j-1}^{\mathbf{k}}} \sum_{\substack{\ell_1, \dots, \ell_n: \\ 0 < \ell_1 < \dots < \ell_n \leq \mathbf{k}+n}} p_{\mathbf{k}-1} \left( \begin{matrix} \ell_1 & \dots & \ell_{j-1} & \ell_j & \ell_{j+1} & \dots & \ell_n \\ 1 & \dots & j-1 & j+1 & (j+1)+1 & \dots & n+1 \end{matrix} \middle| z_1^{\mathbf{k}}, \dots, z_n^{\mathbf{k}} \right) \\
&\quad \cdot \mathbb{E} \left( u \left( w_{\mathbf{k}} + \tilde{y}_{1|\ell_1}^{\mathbf{k}} + \dots + \tilde{y}_{j-1|\ell_{j-1}}^{\mathbf{k}} + 0 + \tilde{y}_{j|\ell_j}^{\mathbf{k}} + \dots + \tilde{y}_{n-1|\ell_{n-1}}^{\mathbf{k}} + \underbrace{\tilde{y}_{n|\ell_n}^{\mathbf{k}}}_{=0} \right) dx \right) \\
&= \sum_{j=1}^{n+1} (z_{j-1}^{\mathbf{k}} - z_j^{\mathbf{k}}) \sum_{\substack{\ell_1, \dots, \ell_n: \\ 0 < \ell_1 < \dots < \ell_n \leq \mathbf{k}+n}} p_{\mathbf{k}-1} \left( \begin{matrix} \ell_1 & \dots & \ell_{j-1} & \ell_j & \ell_{j+1} & \dots & \ell_n \\ 1 & \dots & j-1 & j+1 & (j+1)+1 & \dots & n+1 \end{matrix} \middle| z_1^{\mathbf{k}}, \dots, z_n^{\mathbf{k}} \right) \\
&\quad \cdot \mathbb{E} \left( u \left( w_{\mathbf{k}} + \tilde{y}_{1|\ell_1}^{\mathbf{k}} + \dots + \tilde{y}_{n|\ell_n}^{\mathbf{k}} \right) \right) \\
&= \sum_{\substack{\ell_1, \dots, \ell_n: \\ 0 < \ell_1 < \dots < \ell_n \leq \mathbf{k}+n}} p_{\mathbf{k}}(\ell_1, \dots, \ell_n | z_1^{\mathbf{k}}, \dots, z_n^{\mathbf{k}}) \mathbb{E} \left( u \left( w_{\mathbf{k}} + \tilde{y}_{1|\ell_1}^{\mathbf{k}} + \dots + \tilde{y}_{n|\ell_n}^{\mathbf{k}} \right) \right) = \mathbb{W}_{\mathbf{k}}(w_{\mathbf{k}}, \tilde{\mathbf{y}}_k, z_1^{\mathbf{k}}, \dots, z_n^{\mathbf{k}}) . \quad (42)
\end{aligned}$$

To obtain the recursion (42), we applied Lemma 1. The distribution of the payoffs  $\tilde{\mathbf{y}}_{\mathbf{k}}$  depends on the ranks of  $X_{\kappa}$ ,  $\kappa = \mathbf{T}, \dots, \mathbf{k}$ , by our model assumptions.

By using (42) and the “Ansatz (41)” we derive:

$$\begin{aligned}
& \mathbb{W}_k(w_k, \tilde{\mathbf{y}}_k, z_1^k, \dots, z_n^k) + R_k(w_k, \tilde{\mathbf{y}}_k, z_1^k, \dots, z_n^k) = \mathbb{V}_k(w_k, \tilde{\mathbf{y}}_k, z_1^k, \dots, z_n^k) \\
&= z_n^k \mathbb{V}_{k-1}(w_k, \tilde{\mathbf{y}}_k, z_1^k, \dots, z_n^k) \\
&\quad + \sum_{j=1}^n \int_{z_j^k}^{z_{j-1}^k} \max_{0 \leq b_k \leq w_k} \mathbb{V}_{k-1} \left( w_k - b_k, \tilde{\mathbf{y}}_k \left( \mathbf{B}_k^{j:n} b_k \right), z_1^k, \dots, z_{j-1}^k, x, z_j^k, \dots, z_{n-1}^k \right) dx \\
&= z_n^k (\mathbb{W}_{k-1}(w_k, \tilde{\mathbf{y}}_k, z_1^k, \dots, z_n^k) + R_{k-1}(w_k, \tilde{\mathbf{y}}_k, z_1^k, \dots, z_n^k)) \\
&\quad + \sum_{j=1}^n \int_{z_j^k}^{z_{j-1}^k} \max_{0 \leq b_k \leq w_k} \left( \mathbb{W}_{k-1} \left( w_k - b_k, \tilde{\mathbf{y}}_k \left( \mathbf{B}_k^{j:n} b_k \right), z_1^k, \dots, z_{j-1}^k, x, z_j^k, \dots, z_{n-1}^k \right) \right. \\
&\quad \left. + R_{k-1} \left( w_k - b_k, \tilde{\mathbf{y}}_k \left( \mathbf{B}_k^{j:n} b_k \right), z_1^k, \dots, z_{j-1}^k, x, z_j^k, \dots, z_{n-1}^k \right) \right) dx \\
&= z_n^k \mathbb{W}_{k-1}(w_k, \tilde{\mathbf{y}}_k, z_1^k, \dots, z_n^k) + z_n^k R_{k-1}(w_k, \tilde{\mathbf{y}}_k, z_1^k, \dots, z_n^k) \\
&\quad + \sum_{j=1}^n \int_{z_j^k}^{z_{j-1}^k} \max_{0 \leq b_k \leq w_k} \left( \mathbb{W}_{k-1} \left( w_k - b_k, \tilde{\mathbf{y}}_k \left( \mathbf{B}_k^{j:n} b_k \right), z_1^k, \dots, z_{j-1}^k, x, z_j^k, \dots, z_{n-1}^k \right) \right. \\
&\quad - \mathbb{W}_{k-1} \left( w_k - 0, \tilde{\mathbf{y}}_k \left( \mathbf{B}_k^{j:n} 0 \right), z_1^k, \dots, z_{j-1}^k, x, z_j^k, \dots, z_{n-1}^k \right) \\
&\quad + \mathbb{W}_{k-1} \left( w_k - 0, \tilde{\mathbf{y}}_k \left( \mathbf{B}_k^{j:n} 0 \right), z_1^k, \dots, z_{j-1}^k, x, z_j^k, \dots, z_{n-1}^k \right) \\
&\quad \left. + R_{k-1} \left( w_k - b_k, \tilde{\mathbf{y}}_k \left( \mathbf{B}_k^{j:n} b_k \right), z_1^k, \dots, z_{j-1}^k, x, z_j^k, \dots, z_{n-1}^k \right) \right) dx \\
&= z_n^k \mathbb{W}_{k-1}(w_k, \tilde{\mathbf{y}}_k, z_1^k, \dots, z_n^k) + \sum_{j=1}^n \int_{z_j^k}^{z_{j-1}^k} \mathbb{W}_{k-1} \left( w_k - 0, \tilde{\mathbf{y}}_k \left( \mathbf{B}_k^{j:n} 0 \right), z_1^k, \dots, z_{j-1}^k, x, z_j^k, \dots, z_{n-1}^k \right) dx \\
&\quad + z_n^k R_{k-1}(w_k, \tilde{\mathbf{y}}_k, z_1^k, \dots, z_n^k) \\
&\quad + \sum_{j=1}^n \int_{z_j^k}^{z_{j-1}^k} \max_{0 \leq b_k \leq w_k} \left( \mathbb{W}_{k-1} \left( w_k - b_k, \tilde{\mathbf{y}}_k \left( \mathbf{B}_k^{j:n} b_k \right), z_1^k, \dots, z_{j-1}^k, x, z_j^k, \dots, z_{n-1}^k \right) \right. \\
&\quad - \mathbb{W}_{k-1} \left( w_k - 0, \tilde{\mathbf{y}}_k \left( \mathbf{B}_k^{j:n} 0 \right), z_1^k, \dots, z_{j-1}^k, x, z_j^k, \dots, z_{n-1}^k \right) \\
&\quad \left. + R_{k-1} \left( w_k - b_k, \tilde{\mathbf{y}}_k \left( \mathbf{B}_k^{j:n} b_k \right), z_1^k, \dots, z_{j-1}^k, x, z_j^k, \dots, z_{n-1}^k \right) \right) dx \\
&= \mathbb{W}_k(w_k, \tilde{\mathbf{y}}_k, z_1^k, \dots, z_n^k) \\
&\quad + z_n^k R_{k-1}(w_k, \tilde{\mathbf{y}}_k, z_1^k, \dots, z_n^k) \\
&\quad + \sum_{j=1}^n \int_{z_j^k}^{z_{j-1}^k} \max_{0 \leq b_k \leq w_k} \left( \sum_{\substack{\ell_1, \dots, \ell_n: \\ 0 < \ell_1 < \dots < \ell_n \leq k-1+n}} p_{k-1}(\ell_1, \dots, \ell_n | \mathbf{z}(x)) \mathbb{E} [u(\pi(j, w_k, b_k, \tilde{\mathbf{y}}_k, \mathcal{L})) - u(\pi(j, w_k, 0, \tilde{\mathbf{y}}_k, \mathcal{L}))] \right. \\
&\quad \left. + R_{k-1} \left( w_k - b_k, \tilde{\mathbf{y}}_k \left( \mathbf{B}_k^{j:n} b_k \right), z_1^k, \dots, z_{j-1}^k, x, z_j^k, \dots, z_{n-1}^k \right) \right) dx. \tag{43}
\end{aligned}$$

To obtain (43) we proceeded as follows: In the first step, we used the “Ansatz” (41) and the stochastic version of the value function (5). Then we used the decomposition of  $\mathbb{V}_{k-1}$  into  $\mathbb{W}_{k-1}$  and  $R_{k-1}$  according to the “Ansatz” (41). For the terms  $\mathbb{W}_k$  the recursion (42) applies, and the remaining terms are equal to

the residual term defined in Theorem 4.

## F Sketch of the Proof of Proposition 1

Consider Theorem 1. The residual term  $r_k(w_k, \mathbf{y}_k, z_1, \dots, z_n)$  follows from

$$\begin{aligned}
 & r_k(w_{k+1} - b, \mathbf{y}_k, z_1^k, \dots, z_n^k) \\
 = & z_n r_{k-1}(w_{k+1} - b, \mathbf{y}_k, z_1^k, \dots, z_n^k) \\
 & + \sum_{j=1}^n \int_{z_j^k}^{z_j^{k-1}} \max_{0 \leq b_k \leq w_{k+1} - b} \left[ \sum_{\substack{\ell_1, \dots, \ell_n: \\ 0 < \ell_1 < \dots < \ell_n \leq k-1+n}} p_{k-1}(\ell_1, \dots, \ell_n | \mathbf{z}(x)) [u(\pi(j, w_{k+1} - b, b_k, \mathbf{y}_k, \mathcal{L})) - u(\pi(j, w_{k+1} - b, 0, \mathbf{y}_k, \mathcal{L}))] \right. \\
 & \left. + r_{k-1}(w_{k+1} - b - b_k, \mathbf{y}_k, \beta_{k-1}^{j:n} b_k, \mathbf{z}(x)) \right] dx
 \end{aligned} \tag{44}$$

The residual terms satisfy  $r_k(w_k, \mathbf{y}_k, z_1^k, \dots, z_n^k) \geq 0$  for all  $k = T, \dots, 1, 0$ . The final realized utility is  $\mathcal{V}_0(w_0, \mathbf{y}_0, z_1^0, \dots, z_n^0) = u(w_0 + \sum_{j=1}^n y_{j|j})$  and  $r_0 = 0$ . For the stochastic case considered in Theorem 4, only the terms  $u(\pi(\cdot, \cdot, \cdot, \cdot))$  have to be replaced by their expectation. To simplify notation, we proceed with deterministic returns.

For  $k = 0$  we obtained  $r_0 = 0$ . Therefore it has second derivative 0. Next, suppose that  $r_{k-1}(w_k - b', \mathbf{y}_{k-1}, z_1^{k-1}, \dots, z_n^{k-1})$  has non negative second derivative with respect to  $b'$  (and therefore it is convex in  $b'$ ). Taking the first and second derivatives of (44) with respect to  $b$ , using that the utility function  $u$  and  $r_{k-1}$  have non-negative second derivatives, shows that the residual term  $r_k(w_{k+1} - b, \mathbf{y}_k, z_1^k, \dots, z_n^k)$  is convex in  $b$ . By that, the investor either invests all the remaining wealth or zero (of depending in  $X_k$  and  $k$ ).

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Title:

Venture Capital Project Selection as an Optimal Stopping Problem

Ms. No.:

QUAECO-D-24-00225

Highlights:

Stylized model of a venture capital investment process.

Consider a discontinuous payoff structure where price competition is not possible and only few firms (superstars) generate profits.

Consider the full information case where a quality characteristic is observed by the investor.

Derive a recursive formulation of the value function for risk-neutral and risk-averse investors.

The value function is a sum of expected utilities arising from past investments and a residual term that describes the expected utility contribution from investing the remaining wealth optimally.

Show that the value function is non-increasing in the number of periods.

Provide a simulation study to observe how the time horizon and the attitudes towards risk affect investment behavior.

Obtain the value of information.

Title:

Venture Capital Project Selection as an Optimal Stopping Problem

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Conflict of Interest Statement:

All authors declare that there is no conflict of interest.

Kind regards,  
Martin Meier, Leopold Sögner, and Gregor Kastner