CONTROL THEORY: BALANCE AND PERSPECTIVE

by

Gerhard TINTNER

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Abstract

After considering briefly the importance of deterministic control theory, some problems of econometrics are discussed and two examples of stochastic control theory in Indian planning models are presented. Stochastic control theory is presented both from the point of view of continuous variables (stochastic differential equations) and discrete random variables (stochastic difference equations). Finally, some of the problems of adaptive control theory are briefly indicated.

Résumé

Nous considérons en premier lieux l'importance de la théorie de contrôle déterministique et aussi quelques problèmes de l'économétrie. Nous donnons comme example deux modèles de la planification aux Indes pour illustrer la théorie de contrôle stochastique. Nous discutons la théorie stochastique de contrôle de point de vue des variables continues (équations différentielles stochastiques) et aussi de point de vue des variables stochastiques discrètes (équations aux différences finies stochastiques). A la fin nous indiquons quelques problèmes de la théorie de contrôle adaptive.

1. Deterministic control

Already in the 1930s some economists utilized the ideas of the classical calculus of variations for the solution of economic problems (Kalecki 1935, Frisch and Holme 1935, Evans 1930, Roos 1934). In connection with the theory of Keynes (1936) there developed an effort to apply ideas from electrical and communications engineering (Tustin 1953, Allen 1959, Fox, Sengupta and Thorbecke 1966, Shell 1967, Kuhn and Szegö 1969). This approach is of course closely related to the theory of economic policy (Theil 1958, 1964, Tinbergen 1952, 1954, 1956). We should also mention specific applications by Phillips (1954), Holt, Modigliani, Muth and Simon (1960) as well as the long time planning model of Ramsey (1928) and the corresponding literature (Koopmans 1960, 1967, Tinbergen 1960, Samuelson 1967, Hahn 1966, Uzawa 1966, Chakravarty 1969, Radner 1967, von Weizsäcker 1965, Shell 1967, Gale 1967, Mirrlees 1967, Stoleru 1965, Tintner and Rao 1968, Pugachev 1963). See also Zauberman (1967). Queuing theory and inventory problems are also relevant.

However, with the exception of some of the early work of Roos (1934) and Holt, Modigliani, Muth and Simon (1960) all these models are deterministic. Whereas perhaps in other applications of control theory the neglect of stochastic factors may be justified because errors and deviations are somehow negligible, this certainly cannot be claimed in economic applications (Tintner and Sengupta 1972; see also e.g. Konius 1964 and Pervosvanskaia 1965). In discussing balance and perspective of control theory we might notice that deterministic models have more or less exhausted their possibilities, in spite of the fact that refinements might still be possible (Beckmann 1968, Bellman 1957, 1967, 1971, Pontryagin, Boltyanskii, Gamkrelidze, Mishchenko 1962, Letov 1961; see also Burmeister and Dobell 1970, Intrilligator 1971).

2. Economic examples of stochastic control

Economic applications of stochastic control involve the application of econometric methods (Tintner 1952).

Econometricians have of course never completely neglected random variables.which enter into economic relationships. There are esentially three types of problems arising in econometric research in connection with the estimation of economic relationships: (1) Errors in the variables, similar to errors of observations. Early research of Koopmans (1937) and Frisch (1934) concentrated on this problem. There is also the related problem of multicollinearity or near multicollinearity (Tintner 1952). For an evaluation of the problems connected with errors in the variables see Morgenstern (1963). (2) Simultaneity. Since the seminal work of Haavelmo (1944) most of the efforts of econometricians have been concentrated in this area. The problem of identification has been exhaustively discussed (Fisher 1956). With the method of two stage least squares (Basmann 1957, Theil 1961, 1971) we have a method which is computationally simple and has at least under classical assumptions desirable large sample properties. (3) The time series nature of most of our data. Here fully satisfactory methods are still not available (Tintner 1968).

Having estimated the relevant relationships of our economic model the econometrician will use certain exogenous variables as control variables, in order to achieve some goals of economic policy (Tinbergen 1952, 1954, 1956, Theil 1958, 1964, Fox, Sengupta and Thorbecke 1966). Exogenous variables are such, that they influence the development of the endogenous variables (i.e. the variables whose interaction the model is supposed to explain) but are not influenced by them.

Let us mention in this connection also two very important recent advances in econometrics: Zarembka (1968) has adapted the method of Box and Cox (1964) to the idea of estimating (in a sense) by statistical methods the very form of economic laws within the class of power transformations. Also, the recent book by Box and Jenkins (1970) makes it possible to apply, at least in a very primitive way, ideas of stochastic control to economic data. We should mention that the authors (and also Hannan 1970) advocate among other things the method of taking finite differences, a generalisation of the use of first differences (Stone 1948, 1954) which was proposed by many early workers concerned with these problems (Tintner 1940). It is remarkable that small sample distributions are available in this field (Tintner 1955, Rao and Tintner 1962, 1963) if we are willing to assume circular populations and use these results as convenient approximations. Discussion of the use of differences can also be found in: Anderson (1971), Grenander and Rosenblatt (1957), Kendall and Stuart (1966), Malinvaud (1970), Strecker (1970), Tintner (1952), Wilks (1962), Whittle (1963), Yaglom (1955).

The formulation and estimation of econometric models, however primitive, is a condition for the application of stochastic control theory. Let us mention only two more important aspects of econometrics: Recent work on size distributions (Mandelbrot 1960, 1961, 1963, Steindl 1965, Fama 1965, see also Cootner 1964) has made it likely, that classical assumptions of normality are perhaps not always fulfilled in economic data. It seems likely that the distributions involved belong to the class of infinitely divisible distributions (Feller 1966, Gnedenko and Kolmogorov 1968). If this is the case all methods related to the method of least squares would have to be abandoned, since the distributions involved have typically infinite variances. A familiar example is the famous Pareto distribution of incomes. (Pareto 1897, 1927)

Another important issue is the statistical treatment of multivariate stochastic processes, or sets of interrelated time series (Bartlett 1961, Quenouille 1957). There are still many unsolved problems in this field, especially small sample distributions are not known, even if we start from classical assumptions of normality. See also Anderson (1971).

Consider an application of the active approach in stochastic linear programming (Tintner 1960). This is a dynamic two sector planning model (Mahalanobis 1955, Lange 1960, Sengupta and Tintner 1963). Let I_t, C_t and Y_t be investment, consumption and national income in the year t. Then L_i is the control variable, the proportion of new investment allocated to the investment sector; $L_c = 1 - L_i$ the proportion allocated to the consumption sector. ($L_i \geq 0$, $L_c \geq 0$) Let also B_i and B_c be the marginal output-capital coefficients for the two sectors. In deterministic terms, the dynamic programming problem (Bellman 1957, Beckmann 1968) may be formulated as follows:

(1) Maximize $Y_T = C_T + I_T$

subject to:

(2) $I_t \leq I_{t-1} + L_i B_i I_{t-1}$

(3) $C_t \leq C_{t-1} + L_c B_c I_{t-1}$

(4) $I_t \geq 0$, $C_t \geq C_0 > 0$

(5) $\sum_{t=0}^{T} I_t \leq I_s$ $\qquad t=o,1,2...T$

(6) $L_i + L_c = 1$

Here, L_i, B_i, L_c, B_c, C_0, I_0, I_s are given constants. The decision variables (control variables) are C_t and I_t (t=1,2..T).

Applying this model to planning statistics of the third five year plan of India, the following data are given: Initial investment $I_0 = 14.40$; initial consumption $C_0 = 121.70$; total investment over t=0,1,2,3,4 = T : $I_s = 99.0$. All figures are in billions of rupees in constant 1952-3 prices. For the marginal coefficients, we use their mean values: $\bar{B}_c = 0.706$, $\bar{B}_i = 0.335$. Also, in the Indian planning system $L_i = 1-L_c = 1/3$. The simplex method (Dantzig 1963) gives the following solutions:

(7) $I_1 = 16.01, \quad C_1 = 128.48$

(8) $I_2 = 17.81, \quad C_2 = 136.02$

(9) $I_3 = 19.80, \quad C_3 = 144.00$

(10) $I_4 = 22.02, \quad C_4 = 153.72$

(11) $Y_4 = 175.74$

Now consider the stochastic version of this problem. We use the mean values of $\bar{B}_c = 0.706$, $\bar{B}_i = 0.335$ as well as the variances $\sigma_c^2 = 0.4582$, $\sigma_i^2 = 0.o319$ in order to fit the empirical gamma distributions of the marginal coefficients:

(12) $p(B_i) = (10.508)^{3.520}\ e^{-10.5088B_i}\ (B_i)^{2.520} / \Gamma(3.520)$

(13) $p(B_c) = (1.541)^{1.088}\ e^{-1.541B_c}\ (B_c)^{0.088}\ /\ \Gamma(1.088)$

The form of the distribution is determined by the method of moments and maximum likelihood methods are used for the estimation of the parameters. We also assume independence of the distributions of B_i and B_c.

Using numerical methods, we can also derive an approximation to the probability distribution of the objective function, Y_4, national income in the last year of a five year plan. Using the value of L_i (= $1-L_c$) as our control variable, we derive empirically the following table:

Table 1

Characteristic of the distribution of Y_4	Control variable		
	$L_i = 1/3$, $L_c = 2/3$	$L_i = 1/2$, $L_c = 1/2$	$L_i = 2/3$, $L_c = 1/3$
Mathematical Expectation	180.10	174.20	166.46
Lower 5% point	138.94	142.10	143.20
Mode	160.83	157.31	159.89

This table shows that the "best" policy, i.e. assignment of the control variable, depends strongly upon the objective functional. If the Indian government desires the policy which will yield the best average results in the long run (mathematical expectation) or which has the maximum probability to succeed (mode) the present policy of assigning the control variable $L_i = 1/3$, $L_c = 2/3$ is to be preferred. But if the Indian government accepts the more cautious policy of trying to accomplish the best which can be done with 95% probability (lower 5% probability point) the opposite policy of the following values of the control variables seems appropriate: $L_i = 2/3$, $L_c = 1/3$.

A lognormal diffusion process has been fitted to Indian agricultural data, 1951-64 (Tintner and Patel 1966). With a lognormal diffusion process the logarithm of the variable follows a classical diffusion process. Fitting by the method of maximum likelihood we derive for the mean value:

$$(14) \quad E\,Y(t) = x_0 \exp\,(b + \tfrac{1}{2}c)\,t$$

and for the variance:

$$(15) \quad s^2\,(t) = \left[E\,Y(t)\right]^2 \left[\exp\,(ct) - 1\right]$$

where the constants b and c have been estimated by the method of maximum likelihood and x_0 is the initial value (t=0 corresponds to 51/52). The prediction of real agricultural production in India for 1969/70 (t=18) is then 0.9215 and the 95% confidence limits are 0.708 and 1.113.

Introducing real per capita government expenditure on agriculture in India as a control variable, we derive empirically various forecasts for real per capita agricultural production in India 1969/70. Consider the following hypotheses: (a) same real per capita expenditure as in 1963/64 in each year; (b) increase of real per capita expenditure by one half; (c) doubling of expenditure; (d) two and one half the expenditure; (e) tripling the 1963/64 real per capita expenditure in each year. The results are given in the following table:

Table 2

Hypothesis	Estimated real per capita agricultural production 1969/70	95% confidence limits	
(a)	0.8541	0.5952	1.1130
(b)	0.9107	0.6347	1.1867
(c)	0.9712	0.6768	1.2656
(d)	1.0350	0.7212	1.3488
(e)	1.1040	0.7694	1.4386

3. Stochastic control theory

Consider a dynamic system characterized by n stochastic differential equations (Ito 1951):

(16) $d\,x(s,t) = f(x(s,t),\, u(s,t))dt + dz(s,t)$

where the vectors $x(s,t)$, $f(x,u)$ and $z(s,t)$ have n components; $u(s,t)$ is a control vector with m elements and $z(s,t)$ is a measurable stochastic process, e.g. Brownian motion.

The control vector $u(s,t)$ should satisfy the system of stochastic differential equations, a set of constraints:

(17) $u(t) \in U$

and minimize a risk function:

(18) $E\, c'x(s,t)$

where the end period T is fixed, c is a vector of constants with n elements and E denotes mathematical expectation.

Conditions under which solutions exist have been discussed in the literature on stochastic control theory (Kushner 1963, 1965, Ho and Newboldt 1967, Florentin 1961, Bucy and Joseph 1968).

If certain regularity conditions are fulfilled, there exists a vector of stochastic variables $p(t)$ which satisfies the canonical equations:

(19) $dp(t) = -f'_x\ (x(t), u(t))\ p(t)\ dt$

where f_x is a vector of partial derivatives. The boundary conditions are:

(20) $p(T) = c,\ u(t) \in U$

It minimizes also the conditional expected value of the Hamiltonian:

(21) $E\left[H(x(t), u(t), p(t) \mid s_u(t)\right]$

(22) $H(x(t), u(t), p(t)) = p'(t)\ f(x(t), u(t))$

Here $s_u(t)$ denotes the field over which u(t) is measurable.

Intuitively, the optimal stochastic control must minimize the conditional expectation of the Hamiltonian given the information available at time t over the class of admissible controls. Of course, these are only necessary conditions. They become sufficient if we add concavity restrictions of the functions involved.

The canonical equations are random differential equations. In the special case when there are N sample points, stochastic control can immediately be reduced to a deterministic problem. Then the functions p(s,t), x(s,t), u(s,t) can (analytically or numerically) be obtained as solutions of the 2N differential equations:

(23) $d\ p(s_i,t) = -f'_x\ (x(s_i,t), u(s_i,t))\ p(s_i,t)\ dt$

(24) $p(s_i,T) = c$

(25) $dx\ (s_i,t) = f(x(s_i,t), u(s_i,t))\ dt + dz\ (s_i,t)$

$i=1,2,\ldots N.$

The control $u(s_j,t)$ follows from:

(26) $\text{Min}\ E\left[p'(s,t)\ f(x(s,t), u(s,t) \mid S_u(t)\right] \quad u \in U$

where $S_u(t)$ is the field over which u(s,t) is measurable.

If the stochastic process z(s,t) is Brownian motion, i.e. a zero mean Gaussian process with stationary independent increments and if $S_u(t) = x(t)$, i.e. the available information at time t is x(t), we have a case of optimal feedback control. Then (Kushner 1965) we can define a scalar function V(x(t), t) as follows:

(27) $V(x(t),t) = E\left[c'x(T) \mid x(t)\right]$

(28) $V(x(T),T) = c'x(T)$

(29) $\text{grad } V(x(T),T) = c$

where grad is the gradient vector.

Kushner (1965) proves that if all admissable controls satisfy certain Lipschitz conditions, are functions of x(t) and t

(30) $\text{grad } V(x(t),t) = E\left[p(t) \mid x(t)\right]$

Hence we get results which hold for the deterministic case. Again we have certainty equivalents (Theil 1957, Simon 1956, Sengupta 1970).

In the discrete case we have difference equations instead of differential equations. (Wold 1938, Fan and Wang 1964, Rozonoer 1959, Kushner 1965).

Now let u_i be the control variables, x_i the state variables and ξ_i random variables. Then the system is:

(31) $x_{i+1} = f(x_i, u_i, \xi_i) \quad i=0,1,\ldots,m-1$

where x_0 is given; x_i, u_i, ξ_i are vectors; ξ_i is a random variable.

Scalar performance criterion:

(32) Minimize $E\, g(x_m)$

where E is mathematical expectation and the function $g(x_m)$ fulfills the necessary regularity condition.

Constraints:

(33) $h(y(\xi),\xi) = 0$

where h is a vector and we write y for $\{x_0 \ldots x_m, u_0 \ldots u_m\}$, ξ for $\{\xi_0 \ldots \xi_{m-1}\}$.

In the special case where (Kushner 1965) the multivariate probability distribution of ξ has continuous probability density, the functions $h_j(y(\xi),\xi)$, $g(y(\xi),\xi)$ and their derivatives are continuous, also certain expectations exist (are not infinite).

Case I. All $u_0 \ldots u_{m-1}$ are to be determined at time $i=0$. Here x_i is a function of $\xi_1 \ldots \xi_{i-1}$ and there are constraints on the u_i in the function $h(u_0 \ldots u_{m-1}) = 0$.

Case II. Both u_i and x_i are functions of $\xi_0 \ldots \xi_{i-k}$ for $k \geqslant 0$. The constraints $h(u_i,\xi)$ are stochastic. Let $\xi = \{\xi_0 \ldots \xi_{m-1}\}$ and $\xi^i = \{\xi_0 \ldots \xi_i\}$:
There must exist piecewise continuous random Lagrange multipliers $\alpha_i(\xi)$ and $\beta(\xi)$ such that the mathematical expectation of their squares exists (is not infinite).
Construct the function:

$$(34) \quad G = Eg(x_m(\xi)) + E\sum_{i=1}^{m} \alpha'_i(\xi)\left[f(x_{i-1}(\xi^{i-2}), u_{i-1}, \xi_{i-1}) - x_i(\xi^{i-1})\right] + E\ \beta(\xi)'\ h(u)$$

The necessary conditions for case I are:

$$(35) \quad E\left[f_{u_i}(x_i(\xi^{i-1}), u_i, \xi^i) + h_{u_i}(u)\beta(\xi)\right] = 0$$

$$(36) \quad \alpha_i(\xi) = f_{x_i}(x_i(\xi^{i-1}), u_i, \xi_i)\ \alpha_{i+1}(\xi) \qquad i \leqslant m-1$$

$$(37) \quad \alpha_m(\xi) = g_{x_m}(x_m(\xi))$$

These conditions can also be written in terms of the expected Hamiltonian:

$$(38) \quad E\ H = Eg(x_m(\xi)) + E\sum_{i=1}^{m} \alpha_i(\xi).$$

We have the condition for minimisation of the modified Hamiltonian E H the canonical equations and the boundary condition. If we have inequality constraints, we get conditions analogous to the Kuhn-Tucker (1951) conditions. See Sengupta 1970.

For case II we have stochastic constraints of the form:
$h(u_i(\xi^{i-2}) = 0$; $u_i(\xi^{i-1})$ and $x_i(\xi^{i-1})$ are stochastic.
The necessary conditions are in this case:

$$(39) \quad E\left[f_{u_i}(x_i(\xi^{i-1}),\xi_i)\alpha_{i+1}(\xi) + h_{u_i}(u_i(\xi^{i-1})\beta_i(\xi) \,\middle|\, \xi^{i-1}\right] = 0$$

$$(40) \quad \alpha_i(\xi) = f_{x_i}(x_i(\xi^{i-1}),u_i(\xi^{i-1})\alpha_{i+1}(\xi) \quad i \leq m-1$$

$$(41) \quad \alpha_m(\xi) = g_{x_m}(x_m(\xi))$$

See also Box and Jenkins (1962), Kalaba (1962).

Adaptive control methods refer to a situation in which the decision maker can modify the control action in order to make it more acceptable. There may be parameter adjustment through a reference model (McGrath, Rajaraman and Rideout 1961, Aizerman 1963, Feldbaum 1960). Another possibility is adjustment through initial conditions and the planning horizon (Merriam 1960, Wonham and Johnson 1964, Balakrishnan and Neustadt 1967, Leitman 1967). Sensitivity analysis is another possibility (Tomovich 1963, Balakrishnan and Neustadt 1967, Sengupta and Fox 1969). Bayesian methods could be used (Feldbaum 1960, Schultz 1965, Chernoff 1968, Bellman 1961, Zellner 1971). Another related field is games against nature (Isaacs 1965, Ho 1965, Pontryagin 1965, Kornai and Liptak 1965, Luce and Raiffa 1967).

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