A NOTE ON GAMES UNDER EXPECTED UTILITY WITH RANK DEPENDENT PROBABILITIES

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Abstract. Expected utility with rank dependent probabilities is a generalization of expected utility. If such preference representations are used for the payoffs in the mixed extension of a finite game, Nash equilibrium may fail to exist. Set-valued solutions, however, do exist even for those more general utility functions. Such set-valued solutions can be shown to be robust to perturbations of the expected utility hypothesis, but may have certain conceptual shortcomings. The paper thus proposes an alternative set-valued solution concept, called fixed sets under the best-reply correspondence.

1. INTRODUCTION

Behavior patterns which are systematic, yet inconsistent with the von Neumann-Morgenstern expected utility hypothesis, have often been observed [**Allais**, 1953; **Ellsberg**, 1961; **Kahneman and Tversky**, 1979]. This has led to alternative theories of decision under risk [**Machina**, 1982; **Quiggin**, 1982; **Chew, Karni, and Safra**, 1987; **Yaari**, 1987]. Still, one of the main applications of theories of decisions under risk, namely game theory, has insisted on using the expected utility hypothesis [exceptions to this rule include: **Karni and Safra**, 1986; **Crawford**, 1990; **Dow and Werlang**, 1991; **Eichberger and Kelsey**, 1993]. This note summarizes some of the basic consequences of using expected utilities with rank dependent probabilities, henceforth EURDP, for the payoffs in the mixed extension of finite normal-form games. EURDP theory is a model of decision making under risk according to which the preference relation on the set of probability distributions is represented by the expected value of a utility function with respect to a transformation of the probability distribution on the set of outcomes [**Wakker**, 1994, provides an axiomatization].

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EURDP has gained popularity, because it allows for the modelling of experimentally observed phenomena, like the Allais- or the Ellsberg-paradox. As far as games are concerned, EURDP provides a generalization of expected utility and thus an opportunity to study the sensitivity of non-cooperative solution concepts with respect to theories of risk attitudes. After all, a prominent part of modern game-theoretic research has been motivated by decision-theoretic arguments. The main argument proposed in the present paper is that to deal with those generalized preferences in the context of games it takes set-valued solution concepts, like the set of rationalizable strategies [**Bernheim**, 1984; **Pearce**, 1984] or strategy subsets closed under rational behavior [**Basu and Weibull**, 1991].

The plan of the paper is as follows: Section 2 introduces notation and preliminary observations. Section 3 studies Nash equilibrium and gives an example of non-existence. Section 4 considers strategy subsets closed under rational behavior and demonstrates their robustness in the vicinity of expected utility. Section 5 presents and characterizes a unified set-valued solution concept for games played by players with EURDP preferences. Section 6 discusses solutions for games when payoff functions display a certain type of discontinuity. Section 7 summarizes.

2. Notation and Preliminaries

A finite n-person normal-form game Γ, played by the players $i \in \mathcal{N} = \{1, \ldots, n\}$, is a $2n$-tuple $\Gamma = (S_1, \ldots, S_n, v_1, \ldots, v_n)$, where S_i is a finite set of pure strategies $s_i \in S_i$ available to player $i \in \mathcal{N}$. The product $S = \times_{i \in \mathcal{N}} S_i$ is the set of all pure strategy combinations, with typical element $s \in S$. The set $\Delta_i = \{\sigma_i \colon S_i \rightarrow \Re_+ \mid \sum_{s_i \in S_i} \sigma_i(s_i) = 1\}$ is the set of mixed strategies of player $i \in \mathcal{N}$, and $\Delta = \times_{i \in \mathcal{N}} \Delta_i$ is the space of mixed strategy combinations. The support of a mixed strategy $\sigma_i \in \Delta_i$ is denoted by $\text{supp}(\sigma_i) = \{s_i \in S_i \mid \sigma_i(s_i) > 0\}$, and analoguously for strategy combinations $\sigma \in \Delta$. For convenience identify S_i with the set of vertices of Δ_i. For each $i \in \mathcal{N}$ the function $v_i \colon S \rightarrow \Re$ is called the (pure strategy) payoff function of player i.

To deal with EURDP takes a generalized definition of the payoffs in the mixed extension of Γ. The mixed extension of the game Γ under EURDP is the n-person normal form game $\Gamma' = (\Delta_1, \ldots, \Delta_n, V_1, \ldots, V_n)$, where $V_i \colon \Delta \rightarrow \Re$ is player i's payoff function. Given a mixed strategy combination $\sigma \in \Delta$, the probability p_s assigned to the pure strategy combination $s \in S$ is, in a non-cooperative game, obtained as

$$p_s(\sigma) = p_{(s_1, \ldots, s_n)}(\sigma) = \prod_{i \in \mathcal{N}} \sigma_i(s_i) \,, \quad \forall s \in S.$$

Clearly, $p_s \geq 0$ and $\sum_{s \in S} p_s = 1$. Let $K = |S|$ denote the number of elements in the set S and define for each player $i \in \mathcal{N}$ an indexing of the set S, $r_i\colon S \to \{1, \ldots, K\}$, with r_i onto and one-to-one, such that $v_i(r_i^{-1}(j)) \leq v_i(r_i^{-1}(j+1))$, $\forall j = 1, \ldots, K-1$. For brevity write $x_i^j = r_i^{-1}(j)$, $\forall j = 1, \ldots, K$, $\forall i \in \mathcal{N}$. Given $\sigma \in \Delta$, the payoff function V_i in the mixed extension of Γ played by players with EURDP preferences is obtained as the Choquet integral

$$
\begin{aligned}
&V_i(\sigma) = \theta_i(1) v_i(x_i^1) + \\
&+ \sum_{j=2}^{K} \theta_i\Big(\sum_{k=j}^{K} p_{x_i^k}(\sigma)\Big)[v_i(x_i^j) - v_i(x_i^{j-1})] = \theta_i(p_{x_i^K}(\sigma)) v_i(x_i^K) + \\
&+ \sum_{j=1}^{K-1} \Big[\theta_i\Big(\sum_{k=j}^{K} p_{x_i^k}(\sigma)\Big) - \theta_i\Big(\sum_{k=j+1}^{K} p_{x_i^k}(\sigma)\Big)\Big] v_i(x_i^j),
\end{aligned}
\tag{1}
$$

for all $i \in \mathcal{N}$, where $\theta_i\colon [0, 1] \to [0, 1]$ is a strictly increasing function with $\theta_i(0) = 0$ and $\theta(1) = 1$ [see e.g.: **Quiggin**, 1982; **Yaari**, 1986; **Wakker**, 1994]. The Choquet integral (1) generalizes expected utility: Given $\theta_i\colon [0, 1] \to [0, 1]$ and an indexing $r_i\colon S \to \{1, \ldots, K\}$ define for $j = 1, \ldots, K$

$$
\varphi_i^j(\sigma) = \begin{cases} \theta_i(p_{x_i^K}(\sigma)), & \text{if } j = K, \\ \theta_i(\sum_{k=j}^{K} p_{x_i^k}(\sigma)) - \theta_i(\sum_{k=j+1}^{K} p_{x_i^k}(\sigma)), & \forall j < K. \end{cases}
\tag{2}
$$

Observe that $\varphi_i^j(\sigma) \geq 0$, because θ_i is increasing, and $\sum_{j=1}^{K} \varphi_i^j(\sigma) = \theta_i(1) > 0$. Hence, V_i can be rewritten as an expectation with respect to φ_i by

$$
V_i(\sigma) = \sum_{j=1}^{K} \varphi_i^j(\sigma)\, v_i(x_i^j).
\tag{3}
$$

If θ_i happens to be the identity, then $\varphi_i^j(\sigma) = p_{x_i^j}(\sigma)$, $\forall j = 1, \ldots, K$, and von Neumann-Morgenstern expected utility results. Since no other randomizations beyond those emerging from the players' mixed strategies are considered, if there is an extensive form underlying Γ, then it is presumably one without nature. A generalization with nature as an explicit player is, however, straightforward.

Let $\sigma_{-i} \in \times_{j \neq i} \Delta_j$ denote the vector of mixed strategies of all players except for player i. Define the best-reply correspondence $\beta_i\colon \Delta \to \Delta_i$ of player $i \in \mathcal{N}$ by

$$
\beta_i(\sigma) = \{\bar{\sigma}_i \in \Delta_i \mid V_i(\sigma_{-i}, \bar{\sigma}_i) \geq V_i(\sigma_{-i}, \sigma_i'),\ \forall \sigma_i' \in \Delta_i\},
\tag{4}
$$

and let $\beta = \times_{i \in \mathcal{N}} \beta_i \colon \Delta \to \Delta$. The best-reply correspondence is of prime importance to the present approach. So its properties are now briefly discussed.

The first observation on β is simply that best-replies to pure strategy combinations are independent of risk attitudes and, therefore, of the specific form of the θ_i's.

LEMMA 1. *If $\theta_i \colon [0, 1] \to [0, 1]$ is (weakly) increasing, then*

(a) $$supp(\varphi_i(\sigma)) \subset supp(\sigma), \quad \text{and}$$

(b) $\beta_i(s) = \{\sigma_i \in \Delta_i \mid supp(\sigma_i) \subset \arg\max_{\hat{s}_i \in S_i} v_i(s_{-i}, \hat{s}_i)\}, \quad \forall s \in S.$

PROOF: (a) If θ_i is weakly increasing, one has

$$\theta_i(\sum_{k=j}^{K} p_{x_i^k}(\sigma)) > \theta_i(\sum_{k=j+1}^{K} p_{x_i^k}(\sigma)) \implies \sum_{k=j}^{K} p_{x_i^k}(\sigma) >$$
$$> \sum_{k=j+1}^{K} p_{x_i^k}(\sigma) \implies p_{x_i^j}(\sigma) > 0 \implies x_i^j \in \text{supp}(\sigma)$$

and, therefore, $\text{supp}(\sigma) \supset \text{supp}(\varphi_i(\sigma))$.

(b) Fix some $s \in S$ and let

$$L = \max\{j = 1, \ldots, K \mid \exists\, \hat{s}_i \in S_i \colon (s_{-i}, \hat{s}_i) = x_i^j\},$$

such that $v_i(x_i^L) = \max_{s_i \in S_i} v_i(s_{-i}, s_i)$, and $j > L$ implies that there exists *no* $\hat{s}_i \in S_i \colon x_i^j = (s_{-i}, \hat{s}_i)$ and, therefore, $\sum_{k \geq h} p_{x_i^k}(s_{-i}, \hat{s}_i) = 0$, $\forall \hat{s}_i \in S_i$, $\forall h \geq j$. Hence, $\text{supp}(s_{-i}, \sigma_i) \subset \{x_i^1, \ldots, x_i^L\}$ implies $\text{supp}(\varphi_i(s_{-i}, \sigma_i)) \subset \{x_i^1, \ldots, x_i^L\}$, $\forall \sigma_i \in \Delta_i$, from (a). Consider some $\sigma_i \in \beta_i(s)$. Then, from $\varphi_i^j(s_{-i}, \sigma_i) \geq 0$, $\forall j = 1, \ldots, L$, and the fact that $\sum_{j=1}^{L} \varphi_i^j(s_{-i}, \sigma_i) = \theta_i(1) \in (0, 1]$, it follows that

$$V_i(s_{-i}, \sigma_i) = \sum_{j=1}^{K} \varphi_i^j(s_{-i}, \sigma_i)\, v_i(x_i^j) \geq \theta_i(1)\, v_i(x_i^L)$$

implies from (2) and the choice of $\sigma_i \in \beta_i(s)$ that

$$\varphi_i^j(s_{-i}, \sigma_i) = \theta_i(\sum_{k=j}^{K} p_{x_i^k}(s_{-i}, \sigma_i)) - \theta_i(\sum_{k=j+1}^{K} p_{x_i^k}(s_{-i}, \sigma_i)) > 0$$
$$\implies v_i(x_i^j) \geq v_i(x_i^L), \quad \forall j \leq L.$$

Since, by the definition of L, $x_i^j \in \text{supp}(s_{-i}, \sigma_i) \implies j \leq L$, one obtains $v_i(x_i^j) = v_i(x_i^L)$, $\forall x_i^j \in \text{supp}(s_{-i}, \sigma_i)$, and, therefore,

$$s_i \in \text{supp}(\sigma_i) \implies s_i \in \arg\max_{\hat{s}_i \in S_i} v_i(s_{-i}, \hat{s}_i).$$

Conversely, if $\text{supp}(\sigma_i) \subset \arg\max_{s_i \in S_i} v_i(s_{-i}, s_i)$, then $V_i(s_{-i}, \sigma_i) = \theta_i(1)\, v_i(x_i^L)$, such that $V_i(s_{-i}, \hat{\sigma}_i) \leq \theta_i(1)\, v_i(x_i^L)$, $\forall \hat{\sigma}_i \in \Delta_i$, implies $\sigma_i \in \beta_i(s)$. This yields the statement of the Lemma. ∎

On $\Delta \setminus S$ the properties of β do depend on those of the θ_i's. In particular, Section 6 offers examples which demonstrate two facts: First, if $\theta_i \colon [0, 1] \to [0, 1]$ is increasing and satisfies $\theta_i(0) = 0$, but has a discontinuity at 0, then β_i can be *empty*-valued at some $\sigma \in \Delta$. Second, if $\theta_i \colon [0, 1] \to [0, 1]$ is increasing and satisfies $\theta_i(1) = 1$, but has a discontinuity at 1, then β_i need ***not*** be u.h.c. on the whole space Δ. So, continuity of θ_i may be necessary to guarantee that β_i is non-empty valued and u.h.c. on Δ. Since such properties may be needed at some points, the required properties of the θ_i's will be made explicit in all statements of results in the sequel.

On the other hand, the properties $\theta_i(0) = 0$ and $\theta_i(1) = 1$ (and that θ_i is *strictly* increasing) are largely redundant in what follows. The present analysis is thus compatible with strictly increasing functions $\theta_i \colon [0, 1] \to [0, 1]$ which may satisfy $\theta_i(1) < 1$, or $\theta_i(0) > 0$, which bears some resemblance with models of uncertainty [see e.g.: **Eichberger and Kelsey**, 1993]. For the present approach, however, a principal assumption is that each player faces a decision problem with risk, but without uncertainty. This is of conceptual, rather than formal significance. Uncertainty may concern the opponents' mixed strategy choices, but not the lottery induced by a player's own mixed strategy choice. This raises issues of how to reduce the compound lottery induced by a mixed strategy combination [cf. **Dekel, Safra, and Segal**, 1989] and may lead to non-additive probability distributions [see e.g.: **Schmeidler**, 1989; **Gilboa**, 1987; **Sarin and Wakker**, 1992]. No such issue arises in the present context. A mixed strategy combination $\sigma \in \Delta$ induces a single lottery over S, no part of which is ambiguous, which players *evaluate* by the Choquet integral V_i, as in (1).

Another issue is that in descriptive contexts it is often claimed that preferences depend on the way lotteries are presented to the subjects. Since non-cooperative game theory is here understood as the theory of games with complete rules, such considerations have to remain outside the scope of the present paper.

3. Nash Equilibrium

A *Nash equilibrium* of a game under EURDP is a point $\sigma \in \Delta$ such that $\sigma \in \beta(\sigma)$. A *strict equilibrium* of a game under EURDP is a Nash equilibrium $\sigma \in \Delta$ such that $\{\sigma\} = \beta(\sigma)$. The following theorem is an instance of theorems in **Debreu** [1952], **Fan** [1952], and **Glicksberg** [1952], and implies the existence of Nash equilibria under von Neumann-Morgenstern expected utility:

THEOREM 1. *If θ_i: $[0, 1] \to [0, 1]$ is (weakly) increasing, continuous on $[0, 1)$, and (weakly) concave, for all $i \in \mathcal{N}$, then a Nash equilibrium exists.*

PROOF: Observe that for each player i the linearity of p in each $\sigma_i \in \Delta_i$ implies

$$p(\sigma_{-i}, \lambda\, \sigma_i' + (1-\lambda)\sigma_i'') = \lambda\, p(\sigma_{-i}, \sigma_i') + (1-\lambda)p(\sigma_{-i}, \sigma_i''),$$

for all $\lambda \in [0, 1]$. Consequently, from (1) it follows under concavity of θ_i that

$$V_i(\sigma_{-i}, \lambda\, \sigma_i' + (1-\lambda)\sigma_i'') \geq \lambda\, V_i(\sigma_{-i}, \sigma_i') + (1-\lambda)V_i(\sigma_{-i}, \sigma_i''),$$

for all $\lambda \in [0, 1]$, and for all $\sigma \in \Delta$. So, β_i is convex-valued. Next, assuming that θ_i is (weakly) concave and continuous on $[0, 1)$, consider a sequence $\{\varepsilon_t\}_{t=1}^{\infty}$, $\varepsilon_t \in (0, 1)$, $\forall t$, $\varepsilon_t \longrightarrow_{t\to\infty} 0$. Concavity implies $\theta_i(1-\lambda\,\varepsilon_t) \geq \lambda\,\theta_i(1-\varepsilon_t) + (1-\lambda)\theta_i(1)$, $\forall \lambda \in (0, 1]$, $\forall t$. Hence,

$$\begin{aligned} \lim_{t\to\infty} \theta_i(1-\lambda\,\varepsilon_t) = \lim_{x\nearrow 1} \theta_i(x) \geq \lambda \lim_{t\to\infty} \theta_i(1-\varepsilon_t) + \\ +(1-\lambda)\theta_i(1), \quad \forall \lambda \in (0, 1] \quad \Longrightarrow \\ \Longrightarrow (1-\lambda) \lim_{x\nearrow 1} \theta_i(x) \geq (1-\lambda)\theta_i(1), \quad \forall \lambda \in (0, 1), \end{aligned}$$

which implies that $\lim_{x\nearrow 1} \theta_i(x) = \theta_i(1)$, because θ_i is increasing, so θ_i is continuous also at 1 and thus on $[0, 1]$. But then β_i is non-empty valued by compactness of Δ and u.h.c. by the maximum theorem. Hence, by Kakutani's fixed point theorem a Nash equilibrium exists. ∎

It has been observed by several authors [see e.g. **Crawford**, 1990; **Dekel, Safra, and Segal**, 1989] that the concavity condition on the θ_i's is also necessary in the sense that without it counterexamples can be constructed. This is illustrated by the first example.

Table 1:

(v_1, v_2)	L	R
T	(0, 2)	(2, 0)
B	(3, 0)	(1, 1)

EXAMPLE 1: Table 1 gives the payoffs to pure strategy combinations, i.e. the v_i's. The payoffs to mixed strategy combinations are given by

$$\begin{aligned} V_1(\sigma) &= [p_{BR} + p_{TR} + p_{BL}]^2 + [p_{TR} + p_{BL}]^2 + [p_{BL}]^2 = \\ &= [\sigma_2(R) + \sigma_1(B)\,\sigma_2(L)]^2 + \\ &+ [\sigma_1(T)\,\sigma_2(R) + \sigma_1(B)\,\sigma_2(L)]^2 + [\sigma_1(B)\,\sigma_2(L)]^2 , \end{aligned}$$

for player 1 (the row player), while player 2 (the column player) is assumed an expected utility maximizer (such that θ_2 is the identity and, therefore, concave).

Figure 1 portrays the best-reply correspondences for player 1 (controlling the vertical axis) and player 2 (controlling the horizontal axis) and, thereby, shows that the game of Table 1 with the above payoff functions does *not* have a Nash equilibrium. Note that, since $\theta_1(q) = q^2$ is (strictly) convex, player 1 is *risk averse* in **Quiggin's** [1982] sense.

(Insert Figure 1 about here.)

A lower bound on the class of games which do not possess Nash equilibria, when played by players with EURDP preferences with strictly convex θ_i's, is provided by the following result:

PROPOSITION 1. *If $\theta_i \colon [0, 1] \to [0, 1]$ is (strictly) convex for all $i \in \mathcal{N}$, the game Γ is generic (i.e. payoffs to pure strategy combinations are all different), and $\sigma \in \Delta$ is a Nash equilibrium, then $\sigma = s \in S$ is a pure strategy combination.*

PROOF: If $\sigma \in \Delta$ is not pure, there must be at least one player $i \in \mathcal{N}$, who randomizes between at least two pure strategies at $\sigma \in \beta(\sigma)$. By the definition of p one can write

$$p(\sigma) = \sum_{s_i \in S_i} \sigma_i(s_i)\, p(\sigma_{-i}, s_i),$$

so strict convexity of θ_i and the assumption that the game is generic imply from (1) that (for convenience, normalize $v_i(x_i^1) = 0$)

$$\begin{aligned} V_i(\sigma) &= \sum_{j=2}^{K} \theta_i(\sum_{k=j}^{K} \sum_{s_i \in S_i} \sigma_i(s_i)\, p_{x_i^k}(\sigma_{-i}, s_i))[v_i(x_i^j) - v_i(x_i^{j-1})] < \\ &< \sum_{j=2}^{K} [\sum_{s_i \in S_i} \sigma_i(s_i)\, \theta_i(\sum_{k=j}^{K} p_{x_i^k}(\sigma_{-i}, s_i))][v_i(x_i^j) - v_i(x_i^{j-1})] = \qquad (5) \\ &= \sum_{s_i \in S_i} \sigma_i(s_i)\, V_i(\sigma_{-i}, s_i). \end{aligned}$$

Thus there exists some pure strategy s_i in the support of σ_i such that $V_i(\sigma) < V_i(\sigma_{-i}, s_i)$, in contradiction to the definition of a Nash equilibrium. ∎

If a Nash equilibrium does exist for (strictly) convex θ_i's, then it is essentially in pure strategies. Therefore, for many games Nash equilibrium does not exist, when the θ_i's are strictly convex. Since a large game with randomly drawn (pure strategy) payoffs has no pure Nash equilibrium with probability $1/e$ [**Dresher**, 1970], a good third of such games will not have a Nash equilibrium if the θ_i's are strictly convex. The reason why Nash equilibrium may fail to exist is that a player for whom θ_i is (strictly) convex finds mixtures of pure best-replies (strictly) less desirable.

On the other hand, pure equilibria are immune to risk attitudes. A simple consequence of Lemma 1(b) is that, if $s \in S$ is a pure Nash equilibrium, $s \in \beta(s)$, for *some* profile $\theta = (\theta_1, \ldots, \theta_n)$, where θ_i is strictly increasing, $\forall i \in \mathcal{N}$, then $s \in S$ is a (pure) Nash equilibrium for *any* profile $\hat{\theta} = (\hat{\theta}_1, \ldots, \hat{\theta}_n)$, where $\hat{\theta}_i$ is strictly increasing, $\forall i \in \mathcal{N}$. So it is only the mixed equilibria which are endangered by deviations from von Neumann-Morgenstern expected utility.

Mixed Nash equilibria under expected utility have the undesirable property that a player, who is instructed to mix at a Nash equilibrium, faces no cost when deviating from the recommended mix, if all other players stick to the instruction [**Harsanyi**, 1973]. If, on the other hand, deviations from a solution produce non-vanishing costs, then such a solution can at least be "locally" robust to perturbations of the expected utility hypothesis, as will be shown below. This motivates the interest in set-valued generalizations of strict equilibria which do possess the complementary property to the one defining Nash equilibrium, that deviations are costly.

4. CURB Sets

Basu and Weibull [1991] have recently offered such a set-valued generalization of strict equilibria. Define the pure best-reply correspondence $\bar{\beta}_i : \Delta \to S_i$ for player $i \in \mathcal{N}$ by

$$(6) \qquad \bar{\beta}_i(\sigma) = \{s_i \in S_i \mid V_i(\sigma_{-i}, s_i) \geq V_i(\sigma_{-i}, \sigma_i'),\ \forall \sigma_i' \in \Delta_i\},$$

where $s_i \in S_i$ is identified with the corresponding vertex of Δ_i. Again, let $\bar{\beta} = \times_{i \in \mathcal{N}} \bar{\beta}_i : \Delta \to S$, and let $\bar{\beta}(B) = \cup_{\sigma \in B} \bar{\beta}(\sigma)$, $\forall B \subset \Delta$. Let P be the set of all non-empty product sets $X \subset S$, i.e. $X = \times_{i \in \mathcal{N}} X_i$, where $X_i \subset S_i$, $\forall i \in \mathcal{N}$. For any non-empty set $X_i \subset S_i$ let $\Delta_i(X_i)$ be the set of all mixed strategies of player i with support in X_i. For any $X \in P$

let $\Delta(X) = \times_{i \in \mathcal{N}} \Delta_i(X_i)$. **Basu and Weibull** [1991] call a set $X \in P$ *closed under rational behavior (curb)*, if it contains all its pure best-replies, $\bar{\beta}(\Delta(X)) \subset X$. Clearly, for every game Γ there exists at least one *curb* set, because $\bar{\beta}(\Delta) \subset S$. The maximal set $X \in P$ which satisfies $\bar{\beta}(\Delta(X)) = X$ under von Neumann-Morgenstern expected utility is the set of *rationalizable strategies* [**Bernheim**, 1984; **Pearce**, 1984]. As a trivial consequence of the proposition below, it also exists for all games under EURDP. (The proof of the following result follows **Basu and Weibull** [1991]).

PROPOSITION 2. *For every game Γ under EURDP, where θ_i: [0, 1] $\to$ [0, 1] is (weakly) increasing, $\forall i \in \mathcal{N}$,*
(a) there exists a minimal curb set,
(b) every minimal curb set X satisfies $\bar{\beta}(\Delta(X)) = X$,
(c) if a singleton set $X = \{s\}$ is curb, then $s \in S$ is a strict equilibrium.

PROOF: (a) Observe that by Lemma 1(b) $\emptyset \neq \bar{\beta}(S) \subset \bar{\beta}(\Delta) \subset S$. Hence, the collection of *curb* sets is non-empty. It is also finite, because S is finite, and it is partially ordered by set inclusion. Thus it contains at least one minimal element.

(b) Suppose $X \in P$ is a minimal *curb* set, i.e. it does not properly contain another *curb* set, but $\bar{\beta}(\Delta(X)) \neq X$. Then for some $i \in \mathcal{N}$ and some non-empty set $Y_i \subset S_i$ one has $\bar{\beta}_i(\Delta(X)) \cup Y_i = X_i$. Set $Z_i = \bar{\beta}_i(\Delta(X))$, and $Z_j = X_j$, $\forall j \neq i$. Then $\emptyset \neq \bar{\beta}(Z) \subset \bar{\beta}(\Delta(Z)) \subset \bar{\beta}(\Delta(X)) \subset Z$, so X is not minimal - contradiction.

(c) If a singleton set $X = \{s\}$ is *curb*, then by Lemma 1(b) $\emptyset \neq \bar{\beta}(s) \subset \{s\}$, and so $\{s\} = \bar{\beta}(s)$ constitutes a strict equilibrium. ∎

The definition of *curb* sets implies that deviations from such a set are costly to the deviating player. Small perturbations of the payoff function in the mixed extension of the game will not destroy this property. Therefore, one may expect that *curb* sets are more robust to perturbations of expected utility than (mixed) Nash equilibria are. This can be made precise as follows: First recall that if θ_i is the identity, then EURDP yields expected utility. Thinking of the identity as the origin of a functional space on [0, 1], let $\|\theta_i\| = \sup_{q \in [0,1]} |\theta_i(q) - q|$ be the distance of θ_i from the identity. A perturbation θ_i of expected utility, i.e. the identity, is small if the distance of θ_i from the identity is small. With this convention the "robustness" of *curb* sets with respect to perturbations of expected utility can be stated as follows:

PROPOSITION 3. *If the set $X \in P$ is a curb set under von Neumann-Morgenstern expected utility, then there is some $\varepsilon > 0$ such that, if players have EURDP preferences with θ_i: [0, 1] $\to$ [0, 1] (weakly) in-*

creasing and $\|\theta_i\| < \varepsilon$, $\forall i \in \mathcal{N}$, *then* X *is still a curb set for the game under EURDP.*

PROOF: Suppose $X \in P$ is a *curb* set under expected utility (i.e. θ_i is the identity for all $i \in \mathcal{N}$). For each $i \in \mathcal{N}$ let $Y_i = S_i \setminus X_i$ be the complement of X_i in S_i. If $Y_i = \emptyset$, $\forall i \in \mathcal{N}$, there remains nothing to be shown, because S always contains all pure best-replies.

Thus, assume that there is some $i \in \mathcal{N}$ with $Y_i \neq \emptyset$. Since under expected utility the best-reply correspondence is convex valued, by the *curb* property under expected utility for every $\sigma \in \Delta(X)$ and $s_i' \in Y_i$ there exists a pure best-reply $s \in X$ such that

$$v_i(x_i^1) + \sum_{j=2}^{K}\sum_{k=j}^{K} p_{x_i^k}(\sigma_{-i}, s_i)[v_i(x_i^j) - v_i(x_i^{j-1})] >$$
$$> v_i(x_i^1) + \sum_{j=2}^{K}\sum_{k=j}^{K} p_{x_i^k}(\sigma_{-i}, s_i')[v_i(x_i^j) - v_i(x_i^{j-1})],$$

from (1). Consequently, there exists $\varepsilon_i(\sigma, s_i') > 0$ such that for all δ which satisfy $0 < \delta \leq \varepsilon_i(\sigma, s_i')$ the following inequality holds at $\sigma_{-i} \in \Delta_{-i}(X_{-i}) = \times_{j \in \mathcal{N}\setminus\{i\}} \Delta_j(X_j)$:

$$v_i(x_i^1) + \sum_{j=2}^{K}\sum_{k=j}^{K} p_{x_i^k}(\sigma_{-i}, s_i)[v_i(x_i^j) - v_i(x_i^{j-1})] - \delta\, v_i(x_i^K) =$$
$$= [1-\delta]v_i(x_i^1) + \sum_{j=2}^{K}[\sum_{k=j}^{K} p_{x_i^k}(\sigma_{-i}, s_i) - \delta][v_i(x_i^j) - v_i(x_i^{j-1})] \geq$$
$$\geq v_i(x_i^1) + \sum_{j=2}^{K}\sum_{k=j}^{K} p_{x_i^k}(\sigma_{-i}, s_i')[v_i(x_i^j) - v_i(x_i^{j-1})] + \delta\, v_i(x_i^K) =$$
$$= [1+\delta]v_i(x_i^1) + \sum_{j=2}^{K}[\sum_{k=j}^{K} p_{x_i^k}(\sigma_{-i}, s_i') + \delta][v_i(x_i^j) - v_i(x_i^{j-1})].$$

Since under expected utility the payoff function in the mixed extension is continuous, there exists $\varepsilon > 0$ such that

$$0 < \varepsilon \leq \varepsilon_i(\sigma, s_i'), \quad \forall \sigma \in \Delta(X),\ \forall s_i' \in Y_i,\ \forall i \in \mathcal{N}.$$

Consider now any weakly increasing function $\theta_i \colon [0, 1] \to [0, 1]$ which satisfies $\|\theta_i\| = \sup_{q \in [0,1]} |\theta_i(q) - q| < \varepsilon$. Then for every $q \in [0, 1]$ one has $q+\varepsilon > \theta_i(q) > q-\varepsilon$ and, therefore,

$$
\begin{aligned}
&\theta_i(1)v_i(x_i^1) + \sum_{j=2}^{K} \theta_i(\sum_{k=j}^{K} p_{x_i^k}(\sigma_{-i}, s_i))[v_i(x_i^j) - v_i(x_i^{j-1})] > \\
>&(1-\varepsilon)v_i(x_i^1) + \sum_{j=2}^{K}[\sum_{k=j}^{K} p_{x_i^k}(\sigma_{-i}, s_i) - \varepsilon][v_i(x_i^j) - v_i(x_i^{j-1})] \geq \\
\geq&(1+\varepsilon)v_i(x_i^1) + \sum_{j=2}^{K}[\sum_{k=j}^{K} p_{x_i^k}(\sigma_{-i}, s_i') + \varepsilon][v_i(x_i^j) - v_i(x_i^{j-1})] > \\
>&\theta_i(1)v_i(x_i^1) + \sum_{j=2}^{K} \theta_i(\sum_{k=j}^{K} p_{x_i^k}(\sigma_{-i}, s_i'))[v_i(x_i^j) - v_i(x_i^{j-1})].
\end{aligned}
$$

Thus $s_i' \in Y_i \implies s_i' \notin \bar{\beta}_i(\sigma)$, $\forall \sigma \in \Delta(X)$, where $\bar{\beta}_i$ denotes the pure best-reply correspondence with respect to θ_i. This is equivalent to $\bar{\beta}_i(\Delta(X)) \subset X_i$, $\forall i \in \mathcal{N}$, so X is a *curb* set for the game played by players with EURDP preferences which satisfy that θ_i is increasing and $\|\theta_i\| < \varepsilon$, $\forall i \in \mathcal{N}$. ∎

Proposition 3 shows that *curb* sets are not only a solution concept that yields non-empty solutions for all games, but one that is locally robust in the vicinity of expected utility.

In the context of EURDP there are, however, conceptual difficulties with *curb* sets. It is a solution concept that portrays players as basically choosing *pure* strategies. Mixed strategy combinations $\sigma \in \Delta \setminus S$ are only thought of as representing beliefs. But if players choose pure strategies, why should they consider beliefs that correspond to (non-degenerate) randomizations? Under expected utility the justification is that if a player has several pure best-replies, then there is nothing that keeps her from randomizing among those. But consider EURDP with strictly convex θ_i's - the case where Nash equilibrium may fail to exist. As the proof of Proposition 1, in particular (5), reveals, this is a case where for any generic game *all* best-replies are pure strategies, so the player will never randomize. Thus it is a case where the justification for mixed beliefs is not entirely clear. Also one might argue that a necessary condition for self-enforcingness is that all relevant beliefs are justified as best-replies, $\beta(A) \supset A$. For, if there is some $\sigma \in A \setminus \beta(A)$, then players should attach zero probability to it. In this sense *curb* sets may be considered "too large".

An even more basic point is that, with EURDP and strictly *concave* θ_i's, the pure best-reply correspondence $\bar{\beta}(\sigma)$ may be empty-valued for some $\sigma \in \Delta \setminus S$. Formally this does not constitute a problem for the definition of *curb* sets, because by Lemma 1(b) the best-reply to a pure strategy combination always is the convex hull of the pure best-replies. But conceptionally it is unclear, why players choose pure strategies, when they could improve by randomizing - as it may occur for strictly concave θ_i's. For, in the latter case, there may be (mixed) strategies which are better than any pure strategy in the *curb* set, i.e. $\beta(\Delta(X)) \not\subset X$, despite the *curb* property $\bar{\beta}(\Delta(X)) \subset X$. (Strictly concave θ_i's, however, have rather undesirable behavorial implications, see: **Green** [1987].)

To circumvent these conceptual difficulties with *curb* sets in the context of EURDP, the next section explores a tighter set-valued solution concept.

5. Fixed Sets

In a sense, up to this point two different solution concepts for two different classes of preferences have been considered. Nash equilibrium is guaranteed to exist only when all θ_i's are (weakly) concave and continuous on [0, 1]. *Curb* sets always exist, but are convincing only when all θ_i's are (strictly) convex. A synthesis is provided by an alternative set-valued solution concept which once again is a generalization of strict equilibrium.

Formally: Let $\beta\colon \Delta \to \Delta$ be the (mixed) best-reply correspondence defined in (4) and for every subset $A \subset \Delta$ let $\beta(A) = \cup_{\sigma \in A}\beta(\sigma)$. A closed subset $A \subset \Delta$ is called a *fixed set* under the best-reply correspondence β, if $\beta(A) = A$. The terminology is motivated by the fact that a fixed set is a fixed point of β, when the best-reply correspondence is viewed as a function from the power set of Δ into itself. Fixed sets under the best-reply correspondence have both the property $\beta(A) \subset A$ that every best-reply to a belief in A is in A, *and* the property $\beta(A) \supset A$ that every belief in A is justified as a best-reply to some strategy combination in A. Hence, fixed sets under the best-reply correspondence avoid the conceptual difficulties of *curb* sets, but, moreover, exist for all games, provided β is u.h.c.

THEOREM 2. *If* $\theta_i\colon [0, 1] \to [0, 1]$ *is (weakly) increasing and continuous on* [0, 1], *then for every game* Γ *there exists a compact non-empty set* $A \subset \Delta$ *such that* $\beta(A) = A$.

PROOF: If θ_i is continuous on [0, 1], $\forall i \in \mathcal{N}$, then $\beta\colon \Delta \to \Delta$ is non-empty valued by compactness of Δ and u.h.c. by the maximum theorem. The statement thus follows from Theorem 8 in **Berge** [1963, p.113]. ∎

Clearly, if a fixed set under β is a singleton set, $A = \{\sigma\}$, then $\beta(\sigma) = \{\sigma\}$ shows that it constitutes a strict equilibrium. Unlike strict equilibria under expected utility, strict equilibria under EURDP need *not* be the only best-reply to a neighbourhood of themselves. Still all fixed sets under β do satisfy a weaker criterion: If A is a fixed set under β, then *all* best-replies to nearby strategy combinations are close to A. Formally: For compact subsets B and C of Δ denote by $d_H(B, C)$ their Hausdorff distance, i.e. $d_H(B, C) = \min_{(\sigma^1, \sigma^2) \in B \times C} \|\sigma^1, \sigma^2\|$, where $\|\,.\,,\,.\,\|$ denotes euclidean distance.

COROLLARY 1. *If a closed set $A \subset \Delta$ satisfies $\beta(A) \subset A$ and $\theta_i\colon [0, 1] \to [0, 1]$ is (weakly) increasing and continuous on $[0, 1]$, $\forall i \in \mathcal{N}$, then for every $\varepsilon > 0$ there exists some $\delta > 0$ such that $d_H(\{\sigma\}, A) < \delta$ implies $d_H(\{\sigma'\}), A) < \varepsilon$, $\forall \sigma' \in \beta(\sigma)$.*

PROOF: Because β is u.h.c. by the continuity assumption on the θ_i's and the maximum theorem, β is a closed mapping, i.e. whenever $\sigma^1 \in \Delta$, $\sigma^2 \in \Delta$, and $\sigma^2 \notin \beta(\sigma^1)$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that $\|\sigma, \sigma^1\| < \delta_1 \implies d_H(\beta(\sigma), \sigma^2) \geq \delta_2$ [cf. **Berge**, 1963, p.111], where $\|\,.\,,\,.\,\|$ denotes euclidean distance. Let the closed set $A \subset \Delta$ satisfy $\beta(A) \subset A$ and choose some $\varepsilon > 0$. Consider some $\sigma' \in \Delta$ which satisfies $d_H(\{\sigma'\}, A) \geq \varepsilon$. Since $\beta(A) \subset A$, this implies $\sigma' \notin \beta(\sigma)$, $\forall \sigma \in A$. Consequently, for each $\sigma \in A$ there exists $\delta_\varepsilon(\sigma, \sigma') > 0$ such that $\|\hat{\sigma}, \sigma\| < \delta_\varepsilon(\sigma, \sigma') \implies \sigma' \notin \beta(\hat{\sigma})$. Since both A and $\{\sigma' \in \Delta \mid d_H(\{\sigma'\}, A) \geq \varepsilon\}$ are closed sets, there exists $\delta_\varepsilon > 0$ such that $d_H(\{\sigma'\}, A) \geq \varepsilon$ and $d_H(\{\hat{\sigma}\}, A) < \delta_\varepsilon$ imply $\sigma' \notin \beta(\hat{\sigma})$. This is equivalent to $d_H(\{\hat{\sigma}\}, A) < \delta_\varepsilon \implies d_H(\{\sigma'\}, A) < \varepsilon$, $\forall \sigma' \in \beta(\hat{\sigma})$. ∎

The hypothesis on A in Corollary 1 only requires $\beta(A) \subset A$, rather than a fixed set. Thus the statement is applicable to a wider range of sets. Still fixed sets, or even *minimal* fixed sets (which do not contain further fixed sets), may be the natural focus of interest. That the latter indeed exist is guaranteed by the next result.

COROLLARY 2. *If $\theta_i\colon [0, 1] \to [0, 1]$ is (weakly) increasing and continuous on $[0, 1]$, $\forall i \in \mathcal{N}$, then there exists at least one minimal fixed set under β.*

PROOF: By Theorem 2 the collection of fixed sets is non-empty and it is partially ordered by set inclusion. If A_1 and A_2 are fixed sets, then by $A_1 \cap A_2 = \beta(A_1) \cap \beta(A_2) = \beta(A_1 \cap A_2)$ their intersection is a fixed set. Hence every completely ordered chain of fixed sets has a lower bound. By Zorn's Lemma there exists a minimal fixed set. ∎

Beyond this property of fixed sets under β one may wonder what the structure of such fixed sets is. Again generic normal-form games and the two polar cases of (strictly) convex or concave θ_i's provide some insight.

PROPOSITION 4. *If* θ_i: [0, 1] $\rightarrow$ [0, 1] *is strictly increasing,* $\forall i \in \mathcal{N}$, Γ *is a generic normal-form game, and*
(a) θ_i *is strictly convex,* $\forall i \in \mathcal{N}$, *then every curb set* $X \in P$ *contains a fixed set under* β, *and every fixed set under* β *belongs to* S;
(b) θ_i *is strictly concave,* $\forall i \in \mathcal{N}$, *then every Nash equilibrium is a fixed set under* β *and, hence, constitutes a strict equilibrium.*

PROOF: (a) By the argument in the proof of Proposition 1, in particular (5), if Γ is generic and all θ_i's are strictly convex, then $\beta(\sigma) \subset S, \forall \sigma \in \Delta$, such that $\bar{\beta} = \beta$. So the second claim is established by $\beta(A) = A \implies A \subset S$. To see the first claim, consider $X \in P$ such that $\bar{\beta}(\Delta(X)) \subset X$. From $\bar{\beta} = \beta$ it follows that $\beta(X) \subset \bar{\beta}(\Delta(X)) \subset X$. Since the collection of subsets Y of X which satisfy $\beta(Y) \subset Y$ is non-empty, partially ordered by set inclusion, and finite, it must contain a minimal element. By the argument in the proof of Proposition 2(b) such a minimal element satisfies $\beta(Y) = Y \subset X$.

(b) If Γ is generic, then by Lemma 1(b) for all $s \in S$ one has $\beta(s) = \{s'\}$, for some $s' \in S$. If all θ_i's are strictly concave, then for all $\sigma \in \Delta \backslash S$ one has $\beta(\sigma) = \{\sigma'\}$, for some $\sigma' \in \Delta$, because otherwise, $\{\sigma', \sigma'') \subset \beta(\sigma)$, $\sigma' \neq \sigma''$, implies that there is some $i \in \mathcal{N}$ such that $\sigma'_i \neq \sigma''_i$ and for all $\lambda \in (0, 1)$

$$V_i(\sigma_{-i}, \lambda\, \sigma'_i + (1-\lambda)\sigma''_i) = \theta_i(1)v_i(x_i^1) + \sum_{j=2}^{K} \theta_i(\lambda \sum_{k=j}^{K} p_{x_i^k}(\sigma_{-i}, \sigma'_i) +$$

$$+(1-\lambda)\sum_{k=j}^{K} p_{x_i^k}(\sigma_{-i}, \sigma''_i))[v_i(x_i^j) - v_i(x_i^{j-1})] >$$

$$> \lambda\, V_i(\sigma_{-i}, \sigma'_i) + (1-\lambda)V_i(\sigma_{-i}, \sigma''_i),$$

because Γ is generic, which contradicts $\{\sigma'_i, \sigma''_i\} \subset \beta_i(\sigma)$. Hence, for all $\sigma \in \Delta$ one has $\beta(\sigma) = \{\sigma'\}$, for some $\sigma' \in \Delta$, i.e. β is single-valued. If σ is a Nash equilibrium, then $\sigma \in \beta(\sigma) \implies \beta(\sigma) = \{\sigma\}$ such that every Nash equilibrium is a fixed set under β and, hence, a strict equilibrium. ▮

More intuitively what Proposition 4 says can be rephrased as follows: For generic games and strictly convex θ_i's any fixed set under β is a "purified" *curb* set that only considers "justified" (and, hence, pure) beliefs. It is in fact a fixed set of the pure best-reply correspondence $\bar{\beta}$ on S. On the other hand, for generic games and strictly concave θ_i's the notions of Nash equilibrium, fixed sets under the best-reply correspondence, and strict equilibrium agree. In this case, however, a strict equilibrium need *not* be in pure strategies, but is simply a strategy combination that constitutes the only best-reply to itself.

In this sense fixed sets under the best-reply correspondence provide, for continuous θ_i's, a synthesis between the two solution concepts "Nash equilibrium" (for concave θ_i's) and "*curb* sets" (for strictly convex θ_i's). This is so, because fixed sets under β are a set-valued generalization of a solution concept which is both, a Nash equilibrium *and* a *curb* set, namely strict equilibrium.

6. APPROXIMATE FIXED SETS

A major ingredient of the arguments in Section 5 is the assumption of continuity of θ_i on [0, 1]. Still there is empirical interest in discontinuities of θ_i, in particular at 0 and 1. A discontinuity of θ_i at 1 is sometimes referred to as the "certainty effect". Such a discontinuity at 1 can be handled within the present framework, as this Section will demonstrate.

This is *not* so for a discontinuity of θ_i at 0. The latter may actually lead to an empty-valued best-reply correspondence, as the following example shows.

Table 2:

v_1	L	R
T	0	2
B	3	1

EXAMPLE 2: Table 2 gives the (pure strategy) payoffs to the row player 1, whose probability transformation is derived from

$$\theta_1(x) = \begin{cases} 0, & \text{if } x = 0, \\ \varepsilon + (1-\varepsilon)\,x\,, & \text{if } x \in (0,\, 1], \end{cases}$$

for some $\varepsilon \in (0,\, 1)$. Let $q = \sigma_2(L)$ denote the probability with which the column player 2 chooses the first column and consider some $q \in (0,\, 1/4)$. Playing the first row yields player 1 a payoff of $V_1(1,\, q) = 2 - 2(1-\varepsilon)q$ and playing the second row yields her $V_1(0,\, q) = 1 + 2\varepsilon + 2(1-\varepsilon)q$. On the other hand, playing the first row with probability $p = \sigma_1(T) \in (0,\, 1)$ yields

$$V_1(p,\, q) = 1 + 2\varepsilon + (1-\varepsilon)[p + 2q - 4pq] \longrightarrow_{p\nearrow 1} 2 + \varepsilon - 2(1-\varepsilon)q.$$

Since $2 + \varepsilon - 2(1-\varepsilon)q > 1 + 2\varepsilon + 2(1-\varepsilon)q$ if and only if $q < 1/4$ and $2 + \varepsilon - 2(1-\varepsilon)q > 2 - 2(1-\varepsilon)q$ if and only if $\varepsilon > 0$, one concludes $\beta_1(q) = \emptyset$, $\forall q \in (0,\, 1/4)$.

A discontinuity of θ_i at 1 has a different effect. It leads to a failure of the u.h.c. property of β_i, as the following example shows.

Table 3:

v_1	L	M	R
T	0	2	3
B	4	1	5

EXAMPLE 3: Table 3 gives the (pure strategy) payoffs for the row player 1, whose θ_1 is

$$\theta_1(x) = \begin{cases} 1, & \text{if } x = 1, \\ (1-\delta)\,x\,, & \text{if } x \in [0,\,1), \end{cases}$$

for some $\delta \in (0,\,1)$. Denote by $q_1 = \sigma_2(L)$ resp. $q_2 = \sigma_2(M)$ the probability that the column player 2 assigns to the first resp. second column and consider a sequence $\{q^t\}_{t=1}^{\infty}$ with $q_1^t = 1/(3+t)$ and $q_2^t = 2/3$. Along this sequence player 1 obtains $V_1(1,\,q^t) = (1-\delta)(12+7t)/(9+3t)$ from choosing the first row and $V_1(0,\,q^t) = 1 + (1-\delta)(9+4t)/(9+3t)$, if she chooses the second row. If player 1 randomizes with probability $p = \sigma_1(T) \in (0,\,1)$ on the first row, she obtains

$$V_1(p,\,q^t) = (1-\delta)\frac{18+7t-6p}{9+3t}.$$

It is easily verified that $V_1(0,\,q^t) > V_1(p,\,q^t)$, $\forall p \in (0,\,1)$, and also $V_1(0,\,q^t) > V_1(1,\,q^t)$ for all t. So the (constant) best-reply for player 1 along the sequence is to choose the second row ($p = 0$). However, the sequence $\{q^t\}_{t=1}^{\infty}$ converges to $q^o = (0,\,2/3)$, where

$$V_1(p,\,q^o) = \begin{cases} 2-\delta+(1-\delta)\frac{p}{3}, & \forall p \in [0,\,1), \\ 2+(1-\delta)\frac{1}{3}, & \text{if } p = 1. \end{cases}$$

Consequently, choosing the second row ($p = 0$) is *not* a best-reply against q^o; instead the only best-reply against q^o is to choose the first row ($p = 1$). Thus β_1 is not u.h.c.

Though β may fail to be u.h.c., if θ_i has a discontinuity at 1, it will still be non-empty valued in this case.

LEMMA 2. *If* $\theta_i\colon [0,\,1] \to [0,\,1]$ *is (weakly) increasing and continuous on* $[0,\,1)$, *then* β_i *is non-empty valued for all* $\sigma \in \Delta$.

PROOF: If θ_i is continuous on $[0,\,1]$, standard arguments apply. So assume that θ_i has a discontinuity at 1. Since θ_i is increasing, $\theta_i(1) > \lim_{x \nearrow 1} \theta_i(x)$.

For $\sigma_{-i} \in \Delta_{-i}$ and any non-empty set $X_i \subset S_i$ define

$$L_i(\sigma_{-i}, X_i) = \min\{j = 1, \ldots, K \mid x_i^j \in \text{supp}(\sigma_{-i}) \times X_i\}.$$

This definition implies that for all $\sigma_i \in \text{int}(\Delta_i(X_i)) = \{\sigma_i \in \Delta_i(X_i) \mid \sigma_i(s_i) > 0,\ \forall s_i \in X_i\}$ and fixed $\sigma_{-i} \in \Delta_{-i}$ one has

$$\sum_{k=j}^{K} p_{x_i^k}(\sigma_{-i}, \sigma_i) = 1, \quad \forall j \leq L_i(\sigma_{-i}, X_i),$$

$$\sum_{k=j}^{K} p_{x_i^k}(\sigma_{-i}, \sigma_i) < 1, \quad \forall j > L_i(\sigma_{-i}, X_i).$$

Therefore, on $\text{int}(\Delta_i(X_i))$ the Choquet integral

$$V_i(\sigma_{-i}, \sigma_i) = \theta_i(1)\, v_i(x_i^{L_i(\sigma_{-i}, X_i)}) + \\ + \sum_{j=L_i(\sigma_{-i}, X_i)+1}^{K} \theta_i(\sum_{k=j}^{K} p_{x_i^k}(\sigma_{-i}, \sigma_i))[v_i(x_i^j) - v_i(x_i^{j-1})]$$

is continuous in $\sigma_i \in \text{int}(\Delta_i(X_i))$. For given $\sigma_{-i} \in \Delta_{-i}$ and $X_i \subset S_i$ consider some $\bar{\sigma}_i \in \arg\sup_{\sigma_i \in \text{int}(\Delta_i(X_i))} V_i(\sigma_{-i}, \sigma_i)$. If $\text{supp}(\bar{\sigma}_i) = X_i$, then $V_i(\sigma_{-i}, \bar{\sigma}_i) = \sup_{\sigma_i \in \text{int}(\Delta_i(X_i))} V_i(\sigma_{-i}, \sigma_i)$. If $\text{supp}(\bar{\sigma}_i) = Y_i \subset_{\neq} X_i$, then $L_i(\sigma_{-i}, Y_i) > L_i(\sigma_{-i}, X_i)$ and

$$\sum_{j=L_i(\sigma_{-i}, X_i)+1}^{L_i(\sigma_{-i}, Y_i)} \theta_i(\sum_{k=j}^{K} p_{x_i^k}(\sigma_{-i}, \bar{\sigma}_i))[v_i(x_i^j) - v_i(x_i^{j-1})] \leq \\ \leq \theta_i(1)[v_i(x_i^{L_i(\sigma_{-i}, Y_i)}) - v_i(x_i^{L_i(\sigma_{-i}, X_i)})]$$

(by the property that θ_i is increasing) imply

$$V_i(\sigma_{-i}, \bar{\sigma}_i) = \theta_i(1)\, v_i(x_i^{L_i(\sigma_{-i}, Y_i)}) + \\ + \sum_{j=L_i(\sigma_{-i}, Y_i)+1}^{K} \theta_i(\sum_{k=j}^{K} p_{x_i^k}(\sigma_{-i}, \bar{\sigma}_i))[v_i(x_i^j) - v_i(x_i^{j-1})] \geq \\ \geq \sup_{\sigma_i' \in \text{int}(\Delta_i(X_i))} V_i(\sigma_{-i}, \sigma_i') = \theta_i(1)\, v_i(x_i^{L_i(\sigma_{-i}, X_i)}) + \\ + \sum_{j=L_i(\sigma_{-i}, X_i)+1}^{K} \theta_i(\sum_{k=j}^{K} p_{x_i^k}(\sigma_{-i}, \bar{\sigma}_i))[v_i(x_i^j) - v_i(x_i^{j-1})].$$

In both cases the set

$$\{\sigma_i \in \Delta_i \mid V_i(\sigma_{-i}, \sigma_i) \geq \sup_{\sigma'_i \in \text{int}(\Delta_i(X_i))} V_i(\sigma_{-i}, \sigma'_i)\}$$

is non-empty. Since the collection on non-empty subsets X_i of S_i is finite, the correspondence

$$\beta_i(\sigma) = \{\sigma_i \in \Delta_i \mid V_i(\sigma_{-i}, \sigma_i) \geq$$
$$\geq \max_{\emptyset \neq X_i \subset S_i} \sup_{\sigma'_i \in \text{int}(\Delta_i(X_i))} V_i(\sigma_{-i}, \sigma'_i)\}$$

is non-empty valued for all $\sigma_{-i} \in \Delta_{-i}$.

Since for any $\sigma_i \in \Delta_i$ the choice $X_i = \text{supp}(\sigma_i)$ implies

$$V_i(\sigma_{-i}, \sigma_i) \leq \sup_{\sigma'_i \in \text{int}(\Delta_i(X_i))} V_i(\sigma_{-i}, \sigma'_i) \leq$$
$$\leq \max_{\emptyset \neq Y_i \subset S_i} \sup_{\sigma'_i \in \text{int}(\Delta_i(Y_i))} V_i(\sigma_{-i}, \sigma'_i),$$

β_i is indeed the best-reply correspondence. ∎

If θ_i has a discontinuity only at 1, β_i is non-empty valued but may fail to be u.h.c. by Example 3. Hence one cannot hope for the existence of fixed sets under β. What is feasible as a solution concept, however, is a closed subset $A \subset \Delta$ of beliefs which contains all its best-replies, $\beta(A) \subset A$, such that every belief in A is justified by being arbitrary close to a best-reply against some strategy combination in A. Formally: An *approximate fixed set under the best-reply correspondence* is a closed subset $A \subset \Delta$ such that $\beta(A) \subset A$ and $\beta(A)$ is dense in A.

THEOREM 3. *If* $\theta_i\colon [0, 1] \to [0, 1]$ *is (weakly) increasing and continuous on* $[0, 1)$, $\forall i \in \mathcal{N}$, *then for every game* Γ *there exists a closed set* $A \subset \Delta$ *such that* $\beta(A) \subset A$ *and closure*$(\beta(A)) = A$.

PROOF: Let $\mathcal{G}(\beta) = \{(\sigma, \sigma') \in \Delta \times \Delta \mid \sigma' \in \beta(\sigma)\}$ be the graph of the best-reply correspondence which is non-empty valued by Lemma 2. Define the correspondence $\gamma\colon \Delta \to \Delta$ by

$$\gamma(\sigma) = \{\sigma' \in \Delta \mid (\sigma, \sigma') \in \text{closure}(\mathcal{G}(\beta))\}.$$

Direct verification shows that $\beta(\sigma) \subset \gamma(\sigma)$, $\forall \sigma \in \Delta$. By definition γ has a closed graph and is, therefore, u.h.c.; so by the arguments in the proof of Theorem 2 there exists a closed subset $A \subset \Delta$ with $\gamma(A) = \cup_{\sigma \in A} \gamma(\sigma) = A$ and, hence, $\beta(A) \subset \gamma(A) = A$.

Consider now some $\sigma \in A$. Since $A = \gamma(A)$, there exists some $\sigma^o \in A$ with $\sigma \in \gamma(\sigma^o)$ such that either $\sigma \in \beta(\sigma^o)$ or for every $\varepsilon > 0$ there exists a pair $(\sigma^1, \sigma^2) \in \mathcal{G}(\beta) \cap (A \times A)$ with $\|(\sigma^1, \sigma^2), (\sigma^o, \sigma)\| < \varepsilon$.

Since $\|(\sigma^1, \sigma^2), (\sigma^o, \sigma)\| \geq \| \sigma^2, \sigma \|$, this yields $\sigma^2 \in \beta(\sigma^1)$, $\sigma^1 \in A$, and $\| \sigma^2, \sigma \| < \varepsilon$. Hence, for every $\varepsilon > 0$ there exists $\sigma^2 \in \beta(A)$ with $\| \sigma^2, \sigma \| < \varepsilon$, which is equivalent to $\sigma \in$ closure($\beta(A)$). Therefore, $A \subset$ closure($\beta(A)$) yields the statement of the theorem. ∎

If θ_i has a discontinuity at 1, and is otherwise continuous, the condition $\beta(A) \supset A$ that all beliefs are justfied as best-replies is thus possibly lost. Still Theorem 3 shows that this necessary condition for a solution to be self-enforcing can "almost" be satisfied.

7. Conclusions

This paper has studied possible solution concepts for games where players have EURDP preferences. Since Nash equilibrium may fail to exist in such a case, one has to turn to set-valued solutions. Set-valued solutions relying on pure strategy choices by the players always exist, but can have certain conceptual shortcomings. If payoff functions in the mixed extension are continuous, a tighter solution concept with mixed strategy choices, avoiding these conceptual problems, is available: *fixed sets under the best-reply correpondence*. If continuity fails, the best one can hope for is an approximate solution set.

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Figure 1