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Retrieval from mixed sampling frequency: generic identifiability in the unit root VAR

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Abstract
The "retrieval from mixed frequency sampling" approach based on blocking—described e.g., in Anderson et al. (Econom Theory 32:793–826, 2016a)—is concerned with retrieving an underlying high frequency model from mixed frequency observations. In this paper, we investigate parameter-identifiability in the Johansen (Likelihood-based inference in cointegrated vector autoregressive models. Oxford University Press, Oxford, 1995) vector error correction model for mixed frequency data. We prove that from the second moments of the blocked process after taking differences at lag N (N is the slow sampling rate), the parameters of the high frequency system are generically identified. We treat the stock and the flow case.

Keywords Mixed frequency · REMIS · VAR · Cointegration · Vector error correction model · Identifiability

Mathematics Subject Classification 62M10 · 62P20

1 Introduction

Econometric analysis is often encountered with multivariate time series data sampled at mixed frequencies. Examples for treating this are Zadrozny (1988), Ghysels et al. (2007)[MIDAS-regression], Anderson et al. (2012, 2016a), Schorfheide and

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Song (2015), Ghysels (2016), and Chambers (2020). Identifiability is a prerequisite for consistent estimation (see, e.g., Deistler and Seifert 1978; Pötscher and Prucha 1997) and often is needed for economic interpretation of effects related to particular model parameters. This article investigates *identifiability* of the model parameters in a Johansen (1995) vector error correction model.

The general question is whether the internal characteristics, i.e. the model parameters θ, can be retrieved from the external characteristics—in our case "observable" population second moments (i.e. second moments which can be consistently estimated from our mixed-frequency data). Identifiability means that the mapping from the parameters to these second moments is injective. Often injectivity of this mapping can only be achieved for a certain subset of the parameterspace. Here, we prove that identifiability can be obtained for a generic subset of the parameterspace. For short we write "g-identifiability" (see Anderson et al. 2016a).

As opposed to MIDAS-regression, where the observations at high frequency are considered as additional information, we

commence from an underlying *high frequency system* (e.g., a VECM) for a multivariate process written as

$$(y_t)_{t\in\mathbb{Z}} = \left(\begin{pmatrix} y_t^f \\ y_t^s \end{pmatrix} \right)_{t\in\mathbb{Z}},$$

parameterised by θ. The dimensions are n, n_f and n_s for y_t, y_t^f (the fast variables) and y_t^s (the slow variables) respectively. Our aim is to identify and estimate the high frequency system from the observed (mixed frequency) data.

The *observational scheme* is as follows:

While the fast variables y_t^f are observed at $t \in \mathbb{Z}$, for the slow variables y_t^s we consider: 1. *Stock-Case:* y_t^s is observed only at $t \in N\mathbb{Z}$ for some sampling rate $N \geq 2$, hence we have a *missing-value problem.*
2. *Affine aggregation:* we observe an affine transformation

$$w_t := c_w + c_0 y_t^s + \cdots + c_{p_c} y_{t-p_c}^s, \tag{1}$$

where c_i are known constant matrices for $i \geq 0$, c_w is a known vector and w_t is observed at $t \in N\mathbb{Z}$. A special case of affine aggregation are flow variables: For example suppose $y_t^s = GDP_t$, the monthly gross domestic product of a country. The quarterly GDP, w_t, is the sum of three monthly GDPs. We call y_t^s *latent* whenever it is not directly observed. Hence, our aim is to retrieve the underlying high frequency parameters θ from data observed according to the observational schemes described above.[1]

With the procedure described above, we are able to retrieve the full high frequency system, whereas the MIDAS (see, e.g., Ghysels 2016) approach only covers

[1] In this example we assume that the variable considered, y_t, is integrated of order one. If by contrast $(\log y_t)$ is integrated of order one, the affine approximation of Aadland (2000) in combination with the methodology developed in this article can be applied.

relationships between observed variables. After identifying the parameters one may interpolate missing values or dis-aggregate observations in a model based way by using the retrieved parameters of the underlying high frequency system.

Estimation of continuous time models from mixed frequency data are investigated in Chambers (2003, 2016, 2020). In particular, Chambers (2003, 2020) consider co-integrating regressions and show that the scaled estimators proposed, converge in distribution to functionals of Brownian motion and to stochastic integrals. Hence, the estimators are (weakly) consistent. Then, by Gabrielsen (1978)—and for the case of strong consistency by Deistler and Seifert (1978)—the model parameters are identified.

For the stable vector auto-regessive model Anderson et al. (2012, 2016a) either used the *blocking approach* (see also Filler 2010; Ghysels 2016) or the *extended Yule-Walker equations* (see Chen and Zadrozny 1998; Anderson et al. 2016a) to show g-identifiability. For the same model class Gersing and Deistler (2021) present an alternative proof for identifiability using the so-called canonical projection form. This idea is also applied in this paper. On the other hand, Deistler et al. (2017) show that the parameters need not be identified in the auto-regressive-moving average (VARMA) case, if the order of the MA polynomial exceeds the order of the AR polynomial.

This article is organised as follows: Sect. 2 starts with the vector error correction model developed in Engle and Granger (1987) and Johansen (1995) as the underlying high frequency model.

In Sect. 2.2 we describe the observational schemes considered in detail. In particular, we introduce a stationary blocked process containing all observed variables. Section 2.3 introduces conditions, which are later shown to be sufficient for identifiability. We prove that these conditions hold generically in the underlying high frequency parameterspace. Section 3 considers the non-stationary case: Here, we use the result from Chambers (2020) that the cointegrating vectors can be identified from mixed frequency data. First, we derive a state-space representation of the blocked process that we call Canonical Projection Form (CPF). In this representation, the system matrices are simple transformations of the parameters of the underlying high frequency systems. After that we start from the unique factor of the spectrum of the blocked process (see, e.g., Deistler and Scherrer 2022, Chapter 6.2 and 7.3) to get an arbitrary minimal realisation for this factor and relate this to the CPF. From there we can retrieve the parameters of the underlying high frequency system using the structural properties of the CPF. Finally, Sect. 4 concludes.

2 Notation and Model Class

2.1 Representations and Parameterspace of the Underlying High Frequency System

In the first step, we introduce the class of underlying high frequency systems: We commence from a process which is integrated of order one and allows for cointegration. Suppose $(y_t)_{t\in\mathbb{Z}}$ is $n \times 1$ and a solution on $\mathbb{Z}$ of the vector error correction system:

$$\Delta y_t = \Pi y_{t-1} + \sum_{j=1}^{p-1} \Phi_j \Delta y_{t-j} + \nu_t, \qquad \nu_t \sim WN(\Sigma_\nu) \ , \tag{2}$$

where $(\nu_t)_{t\in\mathbb{Z}}$ is uncorrelated white noise, where Σ_ν non-singular, and Π is of rank $r > 0$ in the case of cointegrating relationships, but we also allow the case $r = 0$. Such solutions always exist and can be constructed as described in detail in Bauer and Wagner (2012). We obtain a unique factorisation of $\Pi = \alpha\beta'$ with $\alpha, \beta \in \mathbb{R}^{n\times r}$ applying the singular value decomposition to Π in the following way:

$$\begin{aligned}\Pi &= \underbrace{U}_{n\times n} \underbrace{\mathrm{diag}(d_1,\ldots,d_r,0,\ldots,0)}_{D} \underbrace{V'}_{n} \times n = \underbrace{U_1}_{n\times r} \underbrace{\mathrm{diag}(d_1,\ldots,d_r)}_{\tilde{D}} \underbrace{V_1'}_{r\times n} \\ &= U_1\tilde{D}V_1' = \underbrace{U_1 Q^{-1}}_{\alpha} \underbrace{Q\tilde{D}V_1'}_{\beta'},\end{aligned}$$

where Q is a non-singular matrix of elementary row operations that transforms $\tilde{D}V_1'$ into its reduced echelon form, such that $Q\tilde{D}V_1' = \begin{pmatrix} I_r & \beta'_{n-r}\end{pmatrix}$. We stack the parameters $\alpha, \beta, \Phi_1, \ldots, \Phi_{p-1}$ to a vector $\theta_{VECM} \in \mathbb{R}^d$, where $d = nr + (n-r)r + (p-1)n^2$.

We also have a VAR(p) representation for (y_t) of the form,

$$y_t = \mathcal{A}_1 y_{t-1} + \cdots + \mathcal{A}_p y_{t-p} + \nu_t. \tag{3}$$

Throughout this article, we assume that r and p are known a priori. For example, for the stock case y_t is observed on $N\mathbb{Z}$, then r can be determined by the Johansen rank test (see Johansen 1995, ch.6.3) (using only data on the low-frequency time grid). For the case where the slow variables are only flow, we can consider the aggregate $\sum_{j=0}^{N} y_{t-j}$, observed for $t \in N\mathbb{Z}$. By using these aggregates and the Granger representation theorem (see Johansen 1995, Theorem 4.2), we get an error correction model for $\sum_{j=0}^{N} y_{t-j}$ with the same matrix of cointegrating vectors β. This also allows to apply the Johansen rank test to estimate r. On the other hand p could be implicitly determined using estimators for the state dimension of the system in the blocked representation (13), (14); see, Pötscher (1989).

We obtain the representation in (3) by the mapping ψ:

$\psi : \theta_{VECM} \mapsto \theta_{AR}$, defined as

$$\mathcal{A}_1 = I_n + \Pi + \Phi_1, \quad \mathcal{A}_j = \Phi_j - \Phi_{j-1} \ \text{for}\ 1 < j < p, \quad \mathcal{A}_p = -\Phi_{p-1},$$

with $\theta_{AR} = \mathrm{vec}\begin{pmatrix}\mathcal{A}_1 & \cdots & \mathcal{A}_p\end{pmatrix}$. On the other hand for a θ_{AR} which has a corresponding VECM representation, we compute θ_{VECM} as follows:

$\psi^{-1} : \theta_{AR} \mapsto \theta_{VECM}$

$$\Pi = -I_n + \sum_{j=1}^{p} \mathcal{A}_j, \ \Phi_1 = -I_n + \mathcal{A}_1 + \Pi, \ \Phi_2 = \Phi_1 + \mathcal{A}_2, \ \cdots, \ \Phi_{p-1} = -\mathcal{A}_p.$$

Now, define the polynomial matrix $a(z) = I_n - \mathcal{A}_1 z - \cdots - \mathcal{A}_p z^p$ where z is a complex variable or the lag operator on $\mathbb{Z}$ depending on the context. For $\check{c} = \begin{pmatrix} I_r \\ 0 \end{pmatrix} \in \mathbb{R}^{n\times r}$ and $\check{c}_\perp = \begin{pmatrix} 0 \\ I_{n-r} \end{pmatrix} \in \mathbb{R}^{n\times(n-r)}$, $\beta_\perp := \big(I_n - \check{c}(\beta'\check{c})^{-1}\beta'\big)\check{c}_\perp$, and $\alpha_\perp$ defined analogously to $\beta_\perp$. We impose the following assumptions (Johansen 1995, Chapter4):

Assumption 1 (*Cointegrated VAR-System*)

(C1) $\operatorname{rk} \alpha\beta' = r < n$.
(C2) $\det(\alpha'_\perp(I_n - \sum_{j=1}^{p-1} \Phi_j)\beta_\perp) \neq 0$.
(C3) $\det a(z) = 0 \Rightarrow z = 1$ or $|z| > 1$.
(C4) $\Sigma_\nu = \mathbb{E}\, \nu_t \nu_t' > 0$.

We define the parameterspace as follows[2]:

$$\Theta_{VECM,1} := \psi^{-1}\left(\psi\left(\mathbb{R}^d\Big|_{C1,C2}\right)\Big|_{C3}\right), \qquad \Theta_1 := \psi\left(\Theta_{VECM,1}\right)$$

$$\text{with} \quad \Theta_1 \overset{\psi}{\leftrightarrow} \Theta_{VECM,1}$$

Note that under these assumptions ψ is a homeomorphism.

The set of vech Σ_ν with $\Sigma_\nu \in \mathbb{R}^{n\times n}$, $\Sigma_\nu = \Sigma_\nu'$ and $\Sigma_\nu > 0$ (condition (C4) in Assumption 1) is denoted by Θ_2. The overall parameterspace for the VAR(p) representation is

$$\Theta = \Theta_1 \times \Theta_2.$$

We will also need the state-space representation of $(y_t)_{t\in\mathbb{Z}}$, which follows from (3):

$$\underbrace{\begin{pmatrix} y_t \\ \vdots \\ y_{t-p+1} \end{pmatrix}}_{X_{t+1}} = \underbrace{\begin{pmatrix} \mathcal{A}_1 & \mathcal{A}_2 & \cdots & \mathcal{A}_p \\ I_n & & & 0 \\ & \ddots & & \vdots \\ & & I_n & 0 \end{pmatrix}}_{\mathcal{A}} \underbrace{\begin{pmatrix} y_{t-1} \\ \vdots \\ y_{t-p} \end{pmatrix}}_{X_t} + \underbrace{\begin{pmatrix} I_n \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{\mathcal{B}} \nu_t \tag{4}$$

$$y_t = \underbrace{\begin{pmatrix} \mathcal{A}_1 & \cdots & \mathcal{A}_p \end{pmatrix}}_{\mathcal{C}} X_t + \nu_t. \tag{5}$$

[2] We write $\mathbb{R}^d\Big|_{C1,C2}$ to denote the set of real vectors in $\mathbb{R}^d$ for which $C1$ and $C2$ hold.

Note that (4), (5) is always controllable as Σ_ν and therefore $\Gamma(t) := \mathbb{E}\left(X_{t+1}X'_{t+1}\right)$ are of full rank. The system (4), (5) is also observable whenever $\mathcal{A}_p$ is of full rank. This follows since $\mathcal{A}_p$ is then nonsingular (and therefore $\mathcal{A}$ is non-singular) from the BPH-test (see Kailath 1980, 2.4.3). Hence under Assumption 1 and if $\mathcal{A}_p$ is nonsingular the system (4), (5) is minimal. For details on controllability and observability see e.g. Deistler and Scherrer (2022), Chapter 7 or Hannan and Deistler (2012), Chapter 2.

2.2 Mixed Frequency Data: Stock and Flow Variables

A main challenge of the identifiability proof in the integrated case—as opposed to the stationary case (Anderson et al. 2016a)—is that the second moments of an integrated process (that is, $\mathbb{E}y_s y_t$, $s, t \in \mathbb{Z}$) are time dependent and cannot be estimated directly. Instead, for the sake of practical relevance of identifiability considerations, we identify from observable second moments of stationary transformations of the level process $(y_t)_{t\in\mathbb{Z}}$.

Suppose for the moment, that the matrix of cointegration vectors β is known. Our proof commences from what we call the "blocked process", where we distinguish between the Stock- and the Flow-case:

1. Stock Variables: In this case for $t \in N\mathbb{Z}$, we get the co-stationary vector $\tilde{y}_t$ of "observed" random variables. We will use $\tilde{n} := r + n + (N-1)n_f$ for the dimension of $\tilde{y}_t$ henceforth. Let $u_t^S := \beta' y_t$, $\Delta_N y_t := y_t - y_{t-N} = \sum_{j=0}^{N-1} \Delta y_{t-j}$, and

$$\tilde{y}_t = \begin{pmatrix} \beta' y_t \\ y_t - y_{t-N} \\ y^f_{t-1} - y^f_{t-N-1} \\ \vdots \\ y^f_{t-N+1} - y^f_{t-2N+1} \end{pmatrix} = \begin{pmatrix} u_t^S \\ \Delta_N y_t \\ \Delta_N y^f_{t-1} \\ \vdots \\ \Delta_N y^f_{t-N+1} \end{pmatrix}. \tag{6}$$

The blocked process $(\tilde{y}_t)$ is similar to the blocked process in Anderson et al. (2016a) with the distinction that we added the variable $\beta' y_t = u_t^S$ and take differences at lag N. Admittedly, the true β is in fact not observed, however since β can be estimated consistently (see Miller 2016; Chambers 2020 and the Supplementary Appendix of this paper, Section S-1), for the purpose of the analysis of identifiability we can assume $\beta' y_t$ to be observed.

2. Flow Variables:

In a similar way, we may consider the case where all slow variables are flow variables, in which case we are able to observe the temporal aggregate $w_t := \sum_{j=0}^{N-1} y^s_{t-j}$ at $t \in N\mathbb{Z}$. So

$$\Delta_N^\Sigma y_t := \sum_{j=0}^{N-1} y_{t-j} - \sum_{j=0}^{N-1} y_{t-N-j} = \Delta_N \sum_{j=0}^{N-1} \begin{pmatrix} y_t^f \\ y_t^s \end{pmatrix}.$$

If all slow variables are flow variables, we can observe $\sum_{j=0}^{N-1} y_{t-j} = \left(w_t', \sum_{j=0}^{N-1} y_{t-j}^{f\prime}\right)'$, $t \in N\mathbb{Z}$. Since $\beta' y_t$ is stationary, we have that $\left(\beta' y_t\right)_{t \in N\mathbb{Z}}$ and $u_t^{\mathcal{F}} := \beta' \sum_{j=0}^{N-1} y_{t-j} \in \mathbb{R}^r$ are integrated of order zero. For the flow case we define the co-stationary vector process

$$\tilde{y}_t = \begin{pmatrix} u_t^{\mathcal{F}} \\ \Delta_N^{\Sigma} y_t \\ \Delta_N y_{t-1}^f \\ \vdots \\ \Delta_N y_{t-N+1}^f \end{pmatrix}. \tag{7}$$

We call the autocovariance function of the (stationary) blocked process

$$\tilde{\gamma} : h \mapsto \mathbb{E}\, \tilde{y}_{t+h} \tilde{y}_t' , \qquad \text{where } h \in N\mathbb{Z} , \tag{8}$$

observed second moments, which can be consistently estimated from the data (if β is known) under standard assumptions, see also Supplementary Material S-1.

The motivation to consider this blocked process for identifiability is the following:

1. We take differences at lag N (as opposed to lag one) because these differences can be directly computed from the mixed frequency data and are stationary.
2. Note that the set of observable autocovariances given mixed frequency data is

$$\begin{aligned} \gamma_{\Delta_N y}^{ff}(h) &:= \mathbb{E}\, \Delta_N y_{t+h}^f \Delta_N y_t^{f\prime} \qquad h \in \mathbb{Z} \\ \gamma_{\Delta_N y}^{fs}(h) &:= \mathbb{E}\, \Delta_N y_{t+h}^f \Delta_N y_t^{s\prime} \qquad h \in \mathbb{Z} \\ \gamma_{\Delta_N y}^{ss}(h) &:= \mathbb{E}\, \Delta_N y_{t+h}^s \Delta_N y_t^{s\prime} \qquad h \in N\mathbb{Z} \\ \gamma_{\beta}^{\cdot}(h) &:= \mathbb{E}\, u_{t+h}^{\cdot} u_t^{\cdot\prime} \qquad h \in N\mathbb{Z} , \end{aligned}$$

where the superscript "$\cdot$" is shorthand for $\mathcal{S}$ or $\mathcal{F}$. Note that these are exactly the second moments of the autocovariance function $\tilde{\gamma}$ of the blocked process defined in Eq. (6) for the stock case. In an obvious way this is treated accordingly in the flow case (7). So the blocked process "contains the whole second moment information available" from which we can identify.

The same idea is also applied for the stationary case in Anderson et al. (2016a).

3. Our interest in the particular blocked process (6), (7) having $u_t^{\cdot}$ in the first coordinates, originates from the fact that we can obtain a minimal representation for this process (see Sect. 3), where the parameters are fairly simple functions of the parameters of the underlying high frequency system. This will finally help us to retrieve the high frequency model parameters.

Next, we define the concept of generic identifiability. Here, identifiability is concerned with the problem whether the parameters of the underlying high frequency system (4), (5) or (2) are uniquely determined from the observable second moments (defined below in this section). To be more precise, a subset $\Theta_I \subset \Theta$ is called identifiable, if the mapping attaching the observable second moments to the parameters $\theta \in \Theta_I$ is injective. In our setting identifiability for the whole set Θ cannot be obtained.

To see this, we consider a simple example where $p = 1$, $r = 1$, and $n = 2$, the first coordinate of y_t is a fast variable, denoted y_t^f, while the second coordinate, y_t^s, is a slow stock variable. We assume that the cointegrating vector $\beta = (1, \beta_s)$ is known. Recall that the observed second moments are as described in equations (6) and (8). Let σ_{ff}, $\sigma_{fs} = \sigma_{sf}$, and σ_{ss} denote the elements of the covariance matrix Σ_ν. Supplementary Appendix S-2 shows that there exist two parameter vectors $\theta^I := \left(\alpha_f^I, \alpha_s^I, 1, \beta_s, \sigma_{ff}^I, \sigma_{fs}^I, \sigma_{ss}^I\right)' \neq \theta^{II} := \left(\alpha_f^{II}, \alpha_s^{II}, 1, \beta_s, \sigma_{ff}^{II}, \sigma_{fs}^{II}, \sigma_{ss}^{II}\right)'$ such that all observable second moments are the same; hence in this case the mapping from the model parameters to observable second moments cannot be injective and the model parameters are not identified from observed second moments. In this example $\alpha_f^I = \alpha_f^{II} = 0$. This implies that the fast coordinate follows a random walk and does not provide any information on the parameter α_s, that is on how $\beta' y_t$ affects Δy_t^s, $t \in 2\mathbb{Z}$.

However, in this paper we prove that identifiability holds for a so called generic subset of Θ. Note that a set $\Theta_I \subset \Theta$ is called *generic* in Θ, if it contains a subset that is open and dense in Θ. Let $\Theta_I := (G \cap \Theta_1) \times \Theta_2$, where $G \subset \mathbb{R}^{n^2 p}$ is defined in Assumption 2 below. In this paper we show firstly that Θ_I is generic in Θ (see Sect. 2.3) and secondly that the set of high frequency systems corresponding to Θ_I is identifiable from the observable second moments (see Sect. 3). Or formally, we show that

$$\pi : \theta \mapsto \tilde{\gamma} \tag{9}$$

is injective on $\Theta_I \subset \Theta$.

Finally, in terms of identifiability, we may suppose without loss of generality that β is known. For instance Miller (2016) or Chambers (2020) propose estimators, accounting for stock and flow variables, respectively.

The estimators of β scaled by T weakly converge to a random variable bounded in probability. Hence, e.g. by White (2001), the estimator is weakly consistent. By Gabrielsen (1978) the matrix of cointegrating vectors $\beta \in \mathbb{R}^{n \times r}$ is identified from mixed frequency observations given the assumptions imposed in Chambers (2020) or Miller (2016). These assumptions are only posed on the stochastic properties of the high frequency innovations $(\nu_t)_{t \in \mathbb{Z}}$ and therefore do not restrict our results on the genericity of the identifiability conditions from Sect. 2.3. If strong consistency could be established for some estimator of β, the results of Deistler and Seifert (1978) apply and β is identified.

2.3 Generic Identifiability and Topological Properties of the Parameterspace

In this section we define the conditions that we need for identifiability and prove that these conditions result in a generic subset of the parameterspace. Define a set $G \subset \mathbb{R}^{n^2 p}$ by the following assumptions:

Assumption 2 (*g-Identifiability Assumptions*)

(I1) rk $\mathcal{A}_p = n$.
(I2) rk $\Gamma(t) = np$ where $\Gamma(t) = \mathbb{E}(X_{t+1}X'_{t+1})$.
(I3) The eigenvalues of $\mathcal{A}$ are of the form: $(1, \ldots, 1, \lambda_{n-r+1}, \ldots, \lambda_{np})$ where $|\lambda_j| < 1$ and $\lambda_i \neq \lambda_j$ for $i \neq j$ with $i, j = n - r + 1, \ldots, np$.
(I4) For non-unit eigenvalues $\lambda_i \neq \lambda_j$ it follows that $\lambda_i^N \neq \lambda_j^N$.
(I5) For all eigenvalues λ of $\mathcal{A}$ smaller than one, it holds that $1 + \lambda + \cdots + \lambda^N \neq 0$ or v_1 consisting of the first n elements of the eigenvector v of $\mathcal{A}$ corresponding to λ, it holds that $\beta' v_1 \neq 0$.
(I6) The pair $(S_{n_f}^{(1)}, A)$ is observable, where $S_{n_f}^{(1)}$ is defined in Eqs. (14), (15) and A is defined in Eq. (10).

Assumption (I1) and (I6), (I5) are needed for observability (see also Anderson et al. 2016b, proofofTheorem2). Assumption (I2) ensures controllability and already follows from $\Sigma > 0$ (e.g. Deistler and Scherrer 2022, ch.7). Finally, Assumptions (I3)-(I5) are used to uniquely retrieve the high frequency parameters from the blocked system (see proof of Theorem 3). In particular Assumption (I5) is involved to show observability for the stock and the flow case (see Lemma 6) and (I4) is used in the algorithm constructed in the proof of Theorem 4, step 2 to identify the eigenvalues of the underlying high frequency companion matrix $\mathcal{A}$.

Recall that $\Theta_I = (G \cap \Theta_1) \times \Theta_2$.

These assumptions are similar to the stationary case considered in Felsenstein (2014), Anderson et al. (2016a, b).

There, the stability condition defines an open set $\Theta' \subset \mathbb{R}^{n^2 p}$. We also have a corresponding set G' defining the identifiability conditions for the stationary case, which is generic in $\mathbb{R}^{n^2 p}$.

Then, the intersection $\Theta' \cap G'$ is generic in Θ'. However, in the integrated case, where unit roots occur, the situation is more intricate since neither Θ_1 nor G is open in $\mathbb{R}^{n^2 p}$. This follows from the fact that for a process with $n - r$ common trends, the $n - r$ eigenvalues of $\mathcal{A}$ in (4) are equal to one [note that the eigenvalues of $\mathcal{A}$ are the reciprocals of the zeros of $a(z)$]. The following Theorem 1 implies that the identifiability conditions are generically fulfilled in Θ:

Theorem 1 *Let Θ_1 be endowed with the Euclidean norm d. The set $\Theta_1 \cap G$ is open and dense in Θ_1.*

For the proof see “Appendix A”.

Since genericity is a topological property, it also holds for the homeomorphic parameterspace corresponding the vector error correction representation in (2) defined by Assumption 1.

3 Generic Identifiability

In the following paragraphs, we first define a canonical state-space representation for the blocked process running on $t \in N\mathbb{Z}$. We prove that this representation is minimal under our identifiability conditions. Then under an additional assumption on the lag order p, we show that the high frequency parameters are generically identifiabile. The proofs of minimality and identifiability make use of the canonical representation. We follow Hansen and Johansen (1999) and obtain from (2) the following state-space system for $\beta' y_t$ and first differences of y_t, that is $\Delta y_t = y_t - y_{t-1}$. Then,

$$\underbrace{\begin{pmatrix} \beta' y_t \\ \Delta y_t \\ \vdots \\ \Delta y_{t-p+2} \end{pmatrix}}_{\underline{x}_{t+1} \in \mathbb{R}^{r+n(p-1)}} = \underbrace{\begin{pmatrix} \beta'\alpha + I_r & \beta'\Phi_1 & \cdots\cdots & \beta'\Phi_{p-1} \\ \alpha & \Phi_1 & \cdots\cdots & \Phi_{p-1} \\ 0_{n\times r} & I_n & & 0_{n\times n} \\ \vdots & & \ddots & \vdots \\ & & I_n & 0 \end{pmatrix}}_{A \in \mathbb{R}^{r+n(p-1)\times r+n(p-1)}} \underbrace{\begin{pmatrix} \beta' y_{t-1} \\ \Delta y_{t-1} \\ \vdots \\ \Delta y_{t-p+1} \end{pmatrix}}_{\underline{x}_t} + \underbrace{\begin{pmatrix} \beta' \\ I_n \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{B} \nu_t \qquad (10)$$

$$\underbrace{\begin{pmatrix} \beta' y_t \\ \Delta y_t \end{pmatrix}}_{\in \mathbb{R}^{r+n}} = \underbrace{\begin{pmatrix} \beta'\alpha + I_r & \beta'\Phi_1 & \cdots\cdots & \beta'\Phi_{p-1} \\ \alpha & \Phi_1 & \cdots\cdots & \Phi_{p-1} \end{pmatrix}}_{C \in \mathbb{R}^{r+n \times r+n(p-1)}} \begin{pmatrix} \beta' y_{t-1} \\ \Delta y_{t-1} \\ \vdots \\ \Delta y_{t-p+1} \end{pmatrix} + \underbrace{\begin{pmatrix} \beta' \\ I_n \end{pmatrix}}_{D \in \mathbb{R}^{r+n \times n}} \nu_t. \qquad (11)$$

By $m := r + n(p-1)$, we denote the dimension of $\underline{x}_t$. As we will see later, given that our identifiability conditions hold, m is also the McMillan degree (see, e.g., Hannan and Deistler 2012, p. 51) of the transfer function of $(\tilde{y}_t)_{t\in N\mathbb{Z}}$.

According to the observational scheme, the slow variables y_t^s are observed only every N-th period. We derive state-space representations for the processes (6) and (7) running on $t \in N\mathbb{Z}$:

1. Case: Stock Variables: We define a new state vector x_{t+1} in the following way, with the condition that $p \geq N + 2$:

$$\underbrace{\begin{pmatrix} u_t^{\mathcal{S}} \\ \Delta_N y_t \\ \vdots \\ \Delta_N y_{t-N+1} \\ \Delta y_{t-N} \\ \vdots \\ \Delta y_{t-p+2} \end{pmatrix}}_{x_{t+1} \in \mathbb{R}^{r+n(p-1)}} = \underbrace{\begin{pmatrix} I_r & 0 & 0 & \dots & & & & \\ 0 & I_n & I_n & \dots & I_n & 0 & \dots & \\ 0 & 0 & I_n & \dots\dots & & I_n & 0 & \dots \\ \vdots & & & \ddots & & \ddots & & \\ \vdots & & & & I_n & \dots\dots & I_n & 0\dots \\ \vdots & & & \dots & 0 & I_n & 0 & \dots \\ \vdots & & & & & \ddots & & \end{pmatrix}}_{c \in \mathbb{R}^{r+n(p-1)\times r+n(p-1)}} \begin{pmatrix} u_t^{\mathcal{S}} \\ \Delta y_t \\ \vdots \\ \Delta y_{t-p+2} \end{pmatrix}. \qquad (12)$$

By iterating the system (10), (11), we get the non-miniphase system (in the sense that the transfer-function is not causally invertible as the input dimension exceeds (Nn) the output dimension $(\tilde{n})$, noting that $\Sigma_\nu > 0$):

$$x_{t+1} = \underbrace{c\, \underbrace{A^N}_{:=A_b}\, c^{-1}}_{A_{b,c}}\, x_{t-N+1} + \underbrace{c B_b}_{B_{b,c}}\, \nu_t^b \tag{13}$$

$$\tilde{y}_t = \underbrace{\underbrace{S_\zeta A^N}_{:=C_b}\, c^{-1}}_{:=C_{b,c}}\, x_{t-N+1} + D_b \nu_t^b\ , \tag{14}$$

where

$$S_\zeta := \begin{array}{r} (r\times m)\{ \\ S_{n_f}^{(1)}\ (n_f\times m)\{ \\ S_{n_s}^{(1)}\ (n_s\times m)\{ \\ S_{n_f}^{(2)}\ (n_f\times m)\{ \\ \vdots \\ S_{n_f}^{(N)}\ (n_f\times m)\{ \end{array} \begin{pmatrix} I_r & 0 & \cdots & & & & 0 \\ 0 & \left(I_{n_f},0\right) & \cdots & \left(I_{n_f},0\right) & 0 & \cdots & 0 \\ 0 & (0, I_{n_s}) & \cdots & (0, I_{n_s}) & 0 & \cdots & 0 \\ 0 & 0 & \left(I_{n_f},0\right) & \cdots & \left(I_{n_f},0\right) & & \\ & & & \ddots & & \ddots & \vdots \\ & & & \left(I_{n_f},0\right) & \cdots & \left(I_{n_f},0\right) & 0 \end{pmatrix},$$

$$C_b = \begin{pmatrix} \left(I_r\ 0 \cdots 0\right) A^N \\ S_{n_f}^{(1)} A^N \\ S_{n_s}^{(1)} A^N \\ S_{n_f}^{(2)} A^N \\ S_{n_f}^{(3)} A^N \\ \vdots \\ S_{n_f}^{(N)} A^N \end{pmatrix} = \begin{pmatrix} \left(I_r\ 0 \cdots 0\right) A^N \\ S_{n_f}^{(1)} A^N \\ S_{n_s}^{(1)} A^N \\ S_{n_f}^{(1)} A^{N-1} \\ S_{n_f}^{(1)} A^{N-2} \\ \vdots \\ S_{n_f}^{(1)} A \end{pmatrix}, \quad \nu_t^b := \begin{pmatrix} \nu_t \\ \vdots \\ \vdots \\ \nu_{t-N+1} \end{pmatrix} \in \mathbb{R}^{Nn}. \tag{15}$$

The matrices $B_{b,c} \in \mathbb{R}^{r+n(p-1)\times Nn}$ and $D_b \in \mathbb{R}^{r+n\times Nn}$ are obtained from B and A.

2. Case: Flow Variables:

Next, we obtain the state vector x_{t+1} for the flow case. Note that $y_{t-j} = y_t - \sum_{\ell=1}^{j} \Delta y_{t-\ell}$, such that $\sum_{j=0}^{N-1} y_{t-j} = \sum_{j=0}^{N-1} \left(y_t - \sum_{\ell=1}^{j} \Delta y_{t-\ell} \right) = N y_t - (N-1)\Delta y_{t-1} - \cdots - \Delta y_{t-N+1}$. Analogously to Eq. (12), this yields for $p \geq 2N+1$ that

$$\underbrace{\begin{pmatrix} \beta' \sum_{j=0}^{N-1} y_{t-j} \\ \Delta_N^{\Sigma} y_t \\ \Delta_N y_{t-1} \\ \vdots \\ \Delta_N y_{t-N+1} \\ \Delta y_{t-N} \\ \vdots \\ \Delta y_{t-p+2} \end{pmatrix}}_{x_{t+1} \in \mathbb{R}^{r+n(p-1)}} = \underbrace{\begin{pmatrix} N I_r & -(N-1)\beta' & -(N-2)\beta' & \cdots & -\beta' & 0 & \cdots & \\ 0 & I_n & \cdots & I_n & -I_n & \cdots & -I_n & 0 \\ 0 & 0 & I_n & \cdots & I_n & 0 & \cdots & \\ \vdots & & & \ddots & & \ddots & & \\ \vdots & & & & I_n & \cdots & I_n & 0 \\ \vdots & & & \cdots & 0 & I_n & 0 & \cdots \\ \vdots & & & & & & \ddots & \end{pmatrix}}_{c \in \mathbb{R}^{r+n(p-1)\times r+n(p-1)}} \begin{pmatrix} u_t^{\mathcal{S}} \\ \Delta y_t \\ \vdots \\ \Delta y_{t-p+2} \end{pmatrix}. \tag{16}$$

We use the same notation for $\tilde{y}_t, x_t, c$ for both cases. With this notation, we obtain the following state-space representation for blocked process in the flow case:

$$\begin{aligned} x_{t+1} &= \underbrace{cA_b c^{-1}}_{A_{b,c}} x_{t-N+1} + \underbrace{cB_b}_{B_{b,c}} v_t^b \\ \tilde{y}_t &= \underbrace{\underbrace{S_\zeta A^N}_{C_b \in \mathbb{R}^{\tilde{n}\times m}} \; c^{-1}}_{C_{b,c} \in \mathbb{R}^{\tilde{n}\times m}} x_{t-N+1} + D_{b,c} v_t^b \end{aligned} \tag{17}$$

where

$$S_\zeta = \begin{pmatrix} NI_r & -(N-1)\beta' & -(N-2)\beta' & \cdots & -\beta' & 0 & \cdots & & \\ 0 & I_n & \cdots & I_n & -I_n & \cdots & -I_n & 0 & \cdots \\ 0 & 0 & (I_{n_f}, 0) & \cdots & (I_{n_f}, 0) & 0 & \cdots & & \\ & & & & \ddots & & & & \\ 0 & & 0 & (I_{n_f}, 0) & \cdots & (I_{n_f}, 0) & 0 & \cdots & \end{pmatrix}. \tag{18}$$

The matrix $D_{b,c} \in \mathbb{R}^{\tilde{n}\times Nn}$ follows from D_b, the matrix c and the selection of the corresponding rows resulting in $\tilde{y}_t$.

3. Case: Mixed Case: Consider the case where we have slow stock as well as slow flow variables:

For example, if (y_t) is a three-dimensional process, where $n_f = 1, n_s = 2, N = 2$, $c_1 = I_2$, and $c_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ in Eq. (1). Then $\beta' \left(y_t^{f\prime}, w_t' \right)'$ is (in general) not stationary.

However, in special cases, such as separate cointegrating relationships among the slow flow variables only, or among the slow stock and fast variables only, etc. we can proceed similarly to the flow case. In the following we only consider the stock or the flow case.

The problem with the systems considered above is that the inputs v_t^b are not the innovations of $\tilde{y}_t$. However, from the stable miniphase spectral factorisation, we only obtain transfer functions corresponding to systems in innovation form (see, e.g., Deistler and Scherrer 2022, Chapter 7). The following Theorem 2 is the first step for obtaining a canonical state-space representation for the blocked process. A minimal state-space representation is called "canonical" if its parameters are uniquely determined from the transfer function. We introduce the following notation for specific subspaces of $L^2(\Omega, \mathcal{A}, P)$, the space of square integrable random variables on the underlying probability space $(\Omega, \mathcal{A}, P)$:

$$\begin{aligned} \mathbb{H}(y) &:= \overline{\mathrm{sp}}(y_{it} \mid t \in \mathbb{Z}, i = 1, \ldots, n) \\ \mathbb{H}_t(y) &:= \overline{\mathrm{sp}}(y_{is} \mid s \leq t, i = 1, \ldots, n) \\ {}_N\mathbb{H}(y) &:= \overline{\mathrm{sp}}(y_{it} \mid t \in N\mathbb{Z}, i = 1, \ldots, n) \\ {}_N\mathbb{H}_t(y) &:= \overline{\mathrm{sp}}(y_{is} \mid s \leq t \text{ and } s \in N\mathbb{Z}, i = 1, \ldots, n)\ , \end{aligned}$$

where $\overline{\mathrm{sp}}(\cdot)$ denotes the closed span and $\mathrm{proj}(v \mid U)$ the projection of v on a closed subspace U of L^2.

Theorem 2 *Suppose that* Assumption 1 *holds. Consider the blocked process* $(\tilde{y}_t)_{t \in N\mathbb{Z}}$ *and set*

$$\begin{aligned} s_{t-N+1} &:= \operatorname{proj}(x_{t-N+1} \mid {}_N\mathbb{H}_{t-N}(\tilde{y})) \\ \tilde{\nu}_t &:= \tilde{y}_t - \operatorname{proj}(\tilde{y}_t \mid {}_N\mathbb{H}_{t-N}(\tilde{y})). \end{aligned}$$

Then there exists $\tilde{B}_c \in \mathbb{R}^{np \times \tilde{n}}$ *such that*

$$s_{t+1} = A_{b,c} s_{t-N+1} + \tilde{B}_c \tilde{\nu}_t \tag{19}$$

$$\tilde{y}_t = C_{b,c} s_{t-N+1} + \tilde{\nu}_t \tag{20}$$

is a miniphase and stable state-space representation of $(\tilde{y}_t)_{t \in N\mathbb{Z}}$*, i.e. it is in innovation form.*

For the proof see "Appendix B".

We call the representation in (19), (20) *canonical projection form* (CPF) of $\tilde{y}_t$. Note that the CPF provides an algorithm for computing the transfer function $\tilde{k}(\tilde{z})$ of $(\tilde{y}_t)_{t \in N\mathbb{Z}}$ which corresponds to the Wold representation, where $\tilde{z} := z^N$. Next we show that the system (19) and (20) is observable and controllable and therefore minimal (see, e.g., Hannan and Deistler 2012, Theorem 2.3.3) for all $\theta \in \Theta_I$.

Theorem 3 *For* $\theta \in \Theta_I$*, the system* (19) *and* (20) *is minimal.*

For the proof see "Appendix C".

By Theorem 3, we know that the McMillan degree of the transfer function of the blocked process $(\tilde{y}_t)_{t \in N\mathbb{Z}}$ corresponding to an underlying high frequency VECM is $m = r + n(p-1)$. This will be used in the proof of the subsequent Theorem 4, where we can relate an arbitrary minimal realisation $(\bar{A}_{b,c}, \bar{B}_{b,c}, \bar{C}_{b,c})$ of the transfer function $\tilde{k}(\tilde{z}) = \left(\bar{C}_{b,c}\left(I_m \tilde{z}^{-1} - \bar{A}_{b,c}\right)\bar{B}_{b,c} + I_{\tilde{n}}\right)$ (where $\tilde{z} := z^N$) to the CPF $(A_{b,c}, \tilde{B}_c, C_{b,c})$. The minimal realisation $(\bar{A}_{b,c}, \bar{B}_{b,c}, \bar{C}_{b,c})$ can be either obtained by the spectral factorisation and e.g. the echelon realisation from the Hankel matrix of the transfer function (see e.g. Hannan and Deistler 2012, Theorem 2.6.2) or directly from the Hankel matrix of the observed second moments (see, e.g. Anderson et al. 2016a, Proof of Theorem 8). In the next step we relate the CPF to the underlying VECM/VAR—exploiting the fact that the parameters θ of the underlying VECM reappear in the CPF. Finally, we show that the parameters of the high frequency system are generically identifiable from the observed second moments, i.e. from $\tilde{\gamma}$ (see Eq. (8)):

Theorem 4 (Generic-Identifiability: Flow or Stock Case) *Let* $p \geq N+2$ *for stock case or* $p \geq 2N+1$ *for the flow case. Then,*

1. *The mapping,* π *in Eq.* (9) *which attaches the second moments of* $(\tilde{y}_t)_{t \in N\mathbb{Z}}$ *to the high frequency parameters* θ *is injective on* Θ_I.
2. *Its inverse,* π^{-1}*, is continuous on* $\pi(\Theta_I)$.

For the proof see "Appendix D".

Since by Theorem 1, Θ_I is a generic subset of Θ, we say that θ is generically identifiable from the observed autocovariance function $\tilde{\gamma}$. Theorems 3 and 4 imply that the representation (19), (20) is indeed canonical on $\pi(\Theta_I)$. Since the second moments of $(\tilde{y}_t)$ can be consistently estimated from the data under mild conditions, by the continuity of π^{-1} it follows that we have a consistent estimator for θ.

Finally, we consider the question whether $\pi^{-1}(\pi(\Theta_I)) = \Theta_I$. This is important to ensure that outside of the identified parameter set Θ_I there are no elements, say $\theta_{\neg I}$, which result in the same observable second moments as some $\theta \in \Theta_I$:

Theorem 5 *For all $\theta_{\neg I} \in \Theta \backslash \Theta_I$ there exists no $\theta \in \Theta_I$ such that $\pi(\theta_{\neg I}) = \pi(\theta) = \tilde{\gamma}$.*

For the proof see "Appendix E".

4 Conclusion

In this paper, we generalise the results on identifiability from mixed frequency data in Anderson et al. (2016a, b) obtained for stationary VAR-systems to the case of unit-roots and cointegrating relationships. As is well known these systems have also a *vector error correction representation*. The corresponding parameterspaces are homeomorphic. We commence from a solution of the (unstable) VAR system on the integers $\mathbb{Z}$ (see Bauer and Wagner 2012, for the existence of such a solution). Then we take differences at lag N (which is the sampling rate of the slow/aggregated process) and stack these to what we call the "blocked process". In addition, the blocked process also contains the stationary process $\beta' y_t$, where β is the matrix of cointegrating relationships. This matrix is identified from mixed frequency data as already shown in Chambers (2020). This blocked process is stationary and contains all relevant differences of the observations. The contribution of this paper can be seen as an extension of the results in Chambers (2020), by proving that also the remaining parameters of the vector error correction model (i.e. besides β) are (generically) identified from mixed frequency observations.

Finally, we show that all common cases of deterministic terms in the VECM can be reduced to the case of non-deterministic terms (see Supplementary Appendix S-3).

A Proof of Theorem 1

1. ($G \cap \Theta_1$ is dense.)
Suppose that $\theta_0 \in \Theta_1$ does not satisfy at least one of the identifiability conditions. Let $\varepsilon > 0$, we show that there exists $\theta \in G \cap \Theta_1$ such that $\|\theta - \theta_0\| < \varepsilon$ by perturbing the eigenvalues / eigenvectors of the companion matrix A corresponding to θ_0.
For this we define a mapping f_{θ_0} that maps $\mathcal{A}$ to a companion matrix $\mathcal{A}^*$ with perturbed eigenvalues and eigenvectors such that $\theta = \text{vec}\left(\mathcal{A}_1^* \cdots \mathcal{A}_p^*\right)$ is in $G \cap \Theta_1$:

1. Compute the Jordan decomposition of $\mathcal{A} = Q \Lambda Q^{-1}$.

2. Perturb the eigenvalues:

$$\bar{\mathcal{A}}^* = Q \underbrace{\left(\Lambda + \mathrm{diag}(\underbrace{0, \ldots, 0}_{n} - r - \text{times}, \xi_1, \ldots, \xi_{np-(n-r)})\right)}_{\tilde{\Lambda}} Q^{-1}. \tag{A.1}$$

3. We transform $\bar{\mathcal{A}}^*$ to a similar matrix $\mathcal{A}^*$ that has the companion structure by using the procedure from Anderson et al. (2016a):

$$\mathcal{A}^* = T \bar{\mathcal{A}}^* T^{-1}, \qquad \text{hence} \quad \mathcal{A}^* T = \begin{pmatrix} \mathcal{A}_1^* & \cdots & & \mathcal{A}_p^* \\ I_n & & & 0 \\ & \ddots & & \vdots \\ & & I_n & 0 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ \vdots \\ T_p \end{pmatrix} = T \bar{\mathcal{A}}^* \,,$$

where T_j for $j = 1, \ldots, p$ are the $n \times np$ rowblocks of T. Now we set $T_1 = \begin{bmatrix} I_n & 0 & \cdots & 0 \end{bmatrix}$ and solve the equation above:

$$\mathcal{A}_1^* T_1 + \cdots + \mathcal{A}_p^* T_p = T_1 \bar{\mathcal{A}}^*, \qquad T_1 = T_2 \bar{\mathcal{A}}^*, \qquad \ldots, \qquad T_{p-1} = T_p \bar{\mathcal{A}}^*,$$

which yields

$$T_j = T_{j-1} \bar{\mathcal{A}}^{*-1} \quad \text{for } j = 2, \ldots, p.$$

Clearly, the mapping $f_{\theta_0} : \xi \mapsto \mathcal{A}^* \mapsto \theta$ for $\xi = (\xi_{n-r+1}, \ldots, \xi_{np})' \in \mathbb{R}^{np-(n-r)}$ is continuous at θ_0 and $f_{\theta_0}(0) = \theta_0$ (as in this case $T = I_{np}$). So for the ε-neighborhood around θ_0 denoted by $U_\varepsilon(\theta_0)$ there exists a $\delta > 0$, such that for all $\xi \in U_\delta(0)$ we have $f_{\theta_0}(\xi) \in U_\varepsilon(\theta_0)$, where $U_\delta(0)$ is the open δ-neighborhood around 0 in $\mathbb{R}^{np-(n-r)}$.

Now, $\lambda^* := (1, \ldots, 1, \lambda_{n-r+1} + \xi_1, \ldots, \lambda_{np} + \xi_{np-n-r})$ are the eigenvalues of $\mathcal{A}^*$ because they are the zeros of the characteristic polynomial of $\tilde{\Lambda}$ in Eq. (A.1) from which we obtain $\mathcal{A}^*$ by similarity transformation with TQ. For any $\delta > 0$, we can find a $\xi \in U_\delta(0)$ such that the corresponding eigenvalues λ^* of $\mathcal{A}^*$ satisfy the conditions **(I1)**, **(I3)**, **(I4)** and **(I5)**. Analogously to Eq. (A.1), we can perturb the eigenvalues and eigenvectors of A to ensure conditions **(I5)** (second part) and **(I6)**.

We have to ensure that the image $f_{\theta_0}(\xi)$ is real valued: Since $\mathcal{A}$ is real valued, for any complex eigenvalue $z = a + ib \in \mathbb{C} \setminus \mathbb{R}$, the conjugate $\bar{z} = a - ib$ is also an eigenvalue of $\mathcal{A}$. If the algebraic multiplicity of z is larger than 1, z has to be perturbed. As is easily shown, if we add to z and $\bar{z}$ the same small real number, the resulting $\bar{\mathcal{A}}^*$ (and therefore also $\mathcal{A}^*$) is again real valued.
Thus, we found $\theta \in G$ close to θ_0 and are left with checking whether θ is also in Θ_1. **(C3)** is trivial.

For **(C1)**, note that, still $n - r$ eigenvalues of $\mathcal{A}^*$ equal unity which ensures that $\mathrm{rk}\, \Pi = r$ (see Bauer and Wagner 2012). Applying the procedures described above, we obtain the vector error correction representation corresponding to $f_{\theta_0}(\xi) = \theta$, say $(\alpha(\xi), \beta(\xi), \Phi_1(\xi), \ldots, \Phi_{p-1}(\xi))$, and see that

$$g : \theta \mapsto \det \alpha_\perp(\xi)'\big(I_n - \sum_{j=1}^{p-1} \Phi_j(\xi)\big)\beta_\perp(\xi)$$

is continuous at $\theta = \theta_0$. We know that $g(\theta_0) \neq 0$ since $\theta_0 \in \Theta_1$. So there exists $\varepsilon_3 > 0$ such that the neighbourhood $U_{\varepsilon_3}\big(g(\theta_0)\big)$ is bounded away from zero. By continuity there exists $\varepsilon_2 > 0$ such that for all $\theta \in U_{\varepsilon_3}(\theta_0)$, we have $g(\theta) \in U_{\varepsilon_2}\big(g(\theta_0)\big)$. For the same reasons as above we can find suitable ξ such that $f_{\theta_0}(\xi) = \theta \in U_\varepsilon(\theta_0) \cap U_{\varepsilon_3}(\theta_0)$. Hence $\Theta \cap G$ is dense in Θ.

2. ($G \cap \Theta_1$ is open in (Θ_1, d)),
where d denotes the Euclidean metric. Suppose now for $\theta^* \in G \cap \Theta_1$, we have to show that there exists $\varepsilon > 0$ such that $U_\varepsilon(\theta_0) \subset G \cap \Theta_1$. The eigenvalues are the zeros of the characteristic polynomial of A and therefore continuous functions at θ^* (since as is well known, the zeros of any polynomial are continuous function of its coefficients).

So the mapping

$$e : \theta \mapsto \mathcal{A} \mapsto \big(\lambda_{n-r+1} \cdots \lambda_{np}\big) = \lambda$$

is continuous in θ^*. Clearly there is an open neigbourhood $U \subset \mathbb{C}^{np-(n-r)}$ of $\lambda^* = e(\theta^*)$ such that for all $\lambda \in U$ the corresponding spectrum $\big(1 \cdots 1\ \lambda'\big)'$ satisfies the identifiability conditions. The pre-image $e^{-1}(U) \subset G$ is an open neighborhood of θ_0. Analogously to the arguments applied above, we can establish **(C2)**.
(I2) follows from **(C4)** and **(I1)** which completes the proof.

B Proof of Theorem 2

This follows from transforming a state-space system into prediction error form. See Hannan and Deistler (2012)[Chapter 1] and Gersing and Deistler (2021). From Johansen (1995)[Proof of Theorem 4.2] it follows that the largest eigenvalue of A is in modulus smaller than one. Hence the system is stable. The linear expansion of the transfer function for a stable system is already the Wold representation as the inputs $\tilde{\nu}_t$ are the innovations. Hence, the system is also miniphase (see, e.g., Deistler and Scherrer 2022, Chapters 2 and 7.3).

C Proof of Theorem 3

By Johansen (1995)[Proof of Theorem 4.2] it follows that the eigenvalues of modulus smaller than 1 are the same, for $\mathcal{A}$ and A.

1.1 *Observability for the Stock Case:* We use the PBH-Test (see, e.g., Kailath 1980, Section 2.4.3) to prove that the pair (A_b, C_b) is generically observable (note that the observability of (A_b, C_b) also implies the observability of $(A_{b,c}, C_{b,c})$ since c is non-singular). For this, note that the eigenvectors of A_b are the same as the eigenvectors of A. Let λ be an eigenvalue of A and $q = \left(q'_\beta \; q'_1 \; \cdots \; q'_{p-1}\right)'$ the corresponding eigenvector. We write

$$Aq = \begin{bmatrix} \beta'\alpha + I_r & \beta'\Phi_1 & \cdots & \beta'\Phi_{p-1} \\ \alpha & \Phi_1 & \cdots & \Phi_{p-1} \\ 0 & I_n & & 0 \\ \vdots & & \ddots & \\ 0 & & I_n & 0 \end{bmatrix} \begin{pmatrix} q_\beta \\ q_1 \\ \vdots \\ q_{p-1} \end{pmatrix} = \lambda \begin{pmatrix} q_\beta \\ q_1 \\ \vdots \\ q_{p-1} \end{pmatrix},$$

where q_β is $r \times 1$ and q_i is $n \times 1$ for $i = 1, \ldots, p-1$. From this, we obtain the relations

$$(\beta'\alpha + I_r)q_\beta + \sum_{i=1}^{p-1} \beta'\Phi_i q_i = \lambda q_\beta \tag{A.2}$$

$$\alpha q_\beta + \sum_{i=1}^{p-1} \Phi_i q_i = \lambda q_1 \tag{A.3}$$

$$q_i = \lambda q_{i+1}\,, \qquad i = 1, \ldots, p-2. \tag{A.4}$$

Since A is of full rank, $\lambda \neq 0$ and $q_1 = 0$ imply $q = 0$, which is a contradiction (noting that α has rank r). Now we look at

$$C_b q = \begin{pmatrix} I_r & 0 & \cdots & & & & 0 \\ 0 & I_n & \cdots & I_n & 0 & \cdots & 0 \\ 0 & 0 & (I_{n_f}, 0) & \cdots & (I_{n_f}, 0) & & \\ & & & \ddots & & \ddots & \vdots \\ & & & (I_{n_f}, 0) & \cdots & (I_{n_f}, 0) & 0 \end{pmatrix} \begin{pmatrix} \lambda^N q_\beta \\ \lambda^N q_1 \\ \vdots \\ \lambda^N q_{p-1} \end{pmatrix}, \tag{A.5}$$

which is not equal to zero. If, for example,

$$\lambda^N q_1 + \cdots + \lambda^N q_N = \lambda^N q_1 + \lambda^{N-1} q_1 + \cdots + q_1 = (1 + \lambda + \cdots + \lambda^N) q_1 \neq 0$$
$$\Leftrightarrow (1 + \lambda + \cdots + \lambda^N) \neq 0,$$

which is generically the case (see Assumption 2).

Recall that by q we denote eigenvectors of A and by v eigenvectors of $\mathcal{A}$, where both correspond to the same eigenvalue $|\lambda| < 1$. In Lemma 6, we show that

$$q_\beta = \frac{\lambda}{\lambda - 1} \beta' q_1 = \beta' v_1, \tag{A.6}$$

so if we suppose that v_1 is not in the right kernel of β', we also get $C_b q \neq 0$.

1.2 *Observability for the Flow Case:*

The first part of the proof is analogous to the stock case. It remains to show that there exists no eigenvector that is in the right kernel of C_b, where C_b is now defined in (18).

Now, analogously to the procedure in (A.5) we obtain, that an eigenvector of A_b is not in the rightkernel of C_b if e.g.

$$\begin{aligned}
&\lambda^N q_1 + \cdots + \lambda^N q_N - \lambda^N q_{N+1} - \cdots - \lambda^N q_{2N} \\
&= \lambda^N q_1 + \lambda^{N-1} q_1 + \cdots + \lambda q_1 - q_1 - \cdots - \lambda^{-N+1} q_1 \\
&= \lambda^{N-1}(-1 - \lambda - \cdots - \lambda^{N-1} + \lambda^N + \cdots + \lambda^{2N-1}) q_1 \neq 0 \\
&\Leftrightarrow (-1 - \lambda - \cdots - \lambda^{N-1} + \lambda^N + \cdots + \lambda^{2N-1}) \neq 0,
\end{aligned}$$

Also the second part is similar to the stock case: By Lemma 6, $q_\beta = \frac{\lambda}{\lambda-1}\beta' q_1 = \beta' v_1$. Assume that v_1 is not in the right kernel of β' (as already done in the stock case). In addition, by considering the first r rows of the matrix c for the flow case, provided in (16), we get

$$\begin{aligned}
&N I_r q_\beta - (N-1)\beta' q_1 - (N-2)\beta' q_2 - \cdots - 2\beta' q_{N-2} - \beta' q_{N-1} \\
&= N I_r \frac{\lambda}{\lambda-1}\beta' q_1 - \frac{(N-1)}{\lambda^0}\beta' q_1 - \frac{(N-2)}{\lambda}\beta' q_1 - \cdots - -\frac{2}{\lambda^{N-2}}\beta' q_1 - \frac{1}{\lambda^{N-1}}\beta' q_1 \\
&= \left(N\frac{\lambda}{\lambda-1} - \frac{(N-1)}{\lambda^0} - \frac{(N-2)}{\lambda} - \cdots - \frac{2}{\lambda^{N-2}} - \frac{1}{\lambda^{N-1}}\right)\beta' q_1 \\
&= \frac{1}{\lambda^{N-1}}\left(N\frac{\lambda^N}{\lambda-1} - (N-1)\lambda^{N-1} - (N-2)\lambda^{N-2} - \cdots - 2\lambda - 1\right)\beta' q_1.
\end{aligned}$$

Note that $\lambda \neq 1$ and $\lambda \neq 0$ by the model assumptions (recall that by Johansen (1995)[Proof of Theorem 4.2] it follows that the eigenvalues of modolus smaller than 1 are the same, for $\mathcal{A}$ and A). Hence, if v_1 is not in the right kernel of β' and $N\frac{\lambda^N}{\lambda-1} - (N-1)\lambda^{N-1} - (N-2)\lambda^{N-2} - \cdots - 2\lambda - 1 \neq 0$ we also get that $\tilde{C}_b q \neq 0$ for the flow case.

2. Controllability: It is enough to show that the matrix $\mathbb{E}\, x_{t+1}\left(\tilde{y}_t' \; \tilde{y}_{t-N}' \; \cdots\right)'$ has full rank. For k sufficiently large, we have

$$\begin{aligned}
&x_{t+N-1} = A_{b,c}^{k-1} x_{t-kN+1} + \sum_{j=0}^{k-1} A_{b,c}^{j} B_{b,c} \nu_{t-N-jN}^{b} \\
&\Delta_N y_{t-kN} = \underbrace{\left[0_{n\times r} \;\; I_n \;\; 0 \cdots 0\right]}_{S_{\Delta_N y}} x_{t-kN+1} \\
&\mathbb{E}\, \Delta_N y_{t-kN} x_{t-N+1}' \\
&\quad = \mathbb{E}\left\{ S_{\Delta_N y} x_{t-kN+1} x_{t-kN+1}' A_{b,c}^{k-1\prime} + S_{\Delta_N y} x_{t-kN+1}\left(\sum_{j=0}^{k-2} A_{b,c}^{j} B_{b,c} \nu_{t-N-jN}^{b}\right)' \right\} \\
&\quad = S_{\Delta_N y} \underbrace{c\Gamma_{rp} c'}_{\Gamma_{rp,c}} A_{b,c}^{k-1\prime}.
\end{aligned}$$

Therefore

$$\begin{aligned}&\mathbb{E}\, x_{t-N+1} \left(\Delta_N y'_{t-kN} \;\; \Delta_N y'_{t-(k+1)N} \;\cdots\; \Delta_N y'_{t-(k+p-1)N}\right)\\&\quad= A_{b,c}^{k-1}\left[\Gamma_{rp,c}S'_{\Delta_N y} \;\; A_{b,c}\Gamma_{rp,c}S'_{\Delta_N y} \;\cdots\; A_{b,c}^{p-1}\Gamma_{rp,c}S'_{\Delta_N y}\right],\end{aligned}$$

which has full rank if $\Gamma_{rp} > 0$ as follows from the proof of Theorem 7 in Anderson et al. (2016a).

By Hannan and Deistler (2012)[Theorem 2.3.3] controllability and observability imply that the system is minimal.

Lemma 6 *Suppose the Assumptions 1 and 2 hold. Then Eq. (A.6) holds.*

Proof Substracting β' times (A.3) from (A.2), we obtain

$$q_\beta = \lambda q_\beta - \lambda\beta' q_1 \text{ such that } q_\beta = \frac{\lambda}{\lambda-1}\beta' q_1.$$

Next, we consider the eigenvector $v = \left(v_1' \cdots v_p'\right)'$ of $\mathcal{A}$ corresponding to λ (recall that eigenvalues in modulus smaller that one of A and $\mathcal{A}$ are the same). By using the relations of the parameters between the VECM and VAR representation, we get

$$\begin{aligned}\lambda v_1 &= (I_n + \alpha\beta')v_1 + \Phi_1(v_1 - v_2) + \Phi_2(v_2 - v_3) + \cdots + \Phi_{p-1}(v_{p-1} - v_p)\\ \alpha\beta' v_1 &+ \Phi_1\frac{\lambda-1}{\lambda}v_1 + \Phi_2\frac{\lambda-1}{\lambda^2}v_1 + \ldots \frac{\lambda-1}{\lambda^{p-1}}v_1 = (\lambda-1)v_1\ ,\end{aligned}$$

where the last relation follows from $v_i = \lambda v_{i+1}$ for $i = 1, \ldots, p-1$, which results by the companion structure of A. Now, we see that $q_1 = ((\lambda-1)/\lambda)v_1$ solves (A.2) and (A.3). ☐

D Proof of Theorem 4

Consider the stable, miniphase spectral factor $\tilde{k}(\tilde{z})$, $\tilde{z} := z^N$, corresponding to the Wold representation of $(\tilde{y}_t)_{t\in N\mathbb{Z}}$.

Step 1: We obtain an arbitrary minimal realisation $(\bar{A}_{b,c}, \bar{B}_{b,c}, \bar{C}_{b,c})$ of $\tilde{k}(\tilde{z})$, e.g. by taking the echelon form, see Hannan and Deistler (2012)[Thm 2.5.2.].

Step 2: (Obtain eigenvalues $\Lambda = diag\left(\lambda_1, \ldots, \lambda_{r+n(p-1)}\right)$ and a linear combination of the eigenvectors of A, denoted q_i, from $\bar{A}_{b,c}$).

By, e.g., Hannan and Deistler (2012)[Theorem 2.3.4] the parameter matrices of minimal systems relate via $\bar{A}_{b,c} = T^{-1}A_{b,c}T$, $\bar{C}_{b,c} = C_{b,c}T$ and $\bar{B}_{b,c} = T^{-1}\tilde{B}_c$, where T is a non-singular matrix.

Since $\mathcal{A}$ (see Eq. (4)) is assumed to be diagonalizable (Assumption 2), the matrix A (see Eq. (10); recall that by Johansen (1995)[Proof of Theorem 4.2] the of modulus smaller than 1 are the same, for $\mathcal{A}$ and A) can be expressed by means of $A = Q\Lambda Q^{-1}$, where $\Lambda = diag(\lambda_1, \ldots, \lambda_{r+n(p-1)})$ is the diagonal matrix of eigenvalues of A and $Q = \left(q_1, \ldots, q_{r+n(p-1)}\right)$ contains the eigenvectors. $A_b = A^N$, $C_{b,c} = C_b c^{-1}$ and $A_{b,c} = cA_b c^{-1}$, such that $\bar{A}_{b,c} = T^{-1}A_{b,c}T = T^{-1}cA_b c^{-1}T = \left(T^{-1}cQ\right)\Lambda^N \left(T^{-1}cQ\right)^{-1}$, $\bar{C}_{b,c} = C_{b,c}T = C_b c^{-1}T$.

By the eigen-decomposition of $\bar{A}_{b,c}$, we obtain $\left(T^{-1}cQ\right)$ and Λ^N. In addition, $(0_{n\times r}, 0_{n\times n}, I_n\, 0\ldots 0)\, A^2 = (0_{n\times r}, I_n\, 0\ldots 0)\, A$ by the companion structure of A. Hence by (15), we have

$$\bar{C}_{b,c}T^{-1}cQ = C_b c^{-1}TT^{-1}cQ = C_b Q = \begin{pmatrix} (I_r \ 0 \cdots 0)\, A^N \\ S^{(1)}_{n_f} A^N \\ S^{(1)}_{n_s} A^N \\ S^{(1)}_{n_f} A^{N-1} \\ S^{(1)}_{n_f} A^{N-2} \\ \vdots \\ S^{(1)}_{n_f} A \end{pmatrix} Q. \tag{A.7}$$

Now we look at the last two rowblocks of C_b with the eigenvectors q_i, $1 \le i \le m$. From assumption 2 **(I4)**, it follows that the eigenvectors of A are the same as the eigenvectors of A^2 (also as A^N) (see Felsenstein 2014, Lemma3.2.1), therefore we have

$$\begin{aligned} S^{(1)}_{n_f} A^2 q_i &= S^{(1)}_{n_f} \lambda_i^2 q_i \\ S^{(1)}_{n_f} A q_i &= S^{(1)}_{n_f} \lambda_i q_i, \end{aligned} \tag{A.8}$$

and we can compute all eigenvalues not equal to one since $S^{(1)}_{n_f} q_i \neq 0$ by Assumption 2 **(I6)**. The flow-case is analogous.

Summing up, from $\bar{A}_{b,c}$ we are able to obtain $T^{-1}cQ$, $\Lambda^N = diag(\lambda_1^N, \ldots, \lambda_{r+n(p-1)}^N)$ and $\Lambda = diag(\lambda_1, \ldots, \lambda_{r+n(p-1)})$.

Step 3: (relate $c^{-1}T$ to T) To jointly treat the stock and the flow case, we write

$$c = \begin{pmatrix} c_{\beta\beta} & c_{\beta 1} & c_{\beta 2} & \cdots & c_{\beta N-1} & 0 & \cdots & \\ 0 & c_{11} & c_{12} & \cdots & \cdots & c_{1N} & c_{1,N+1} & \cdots \\ \vdots & & & \ddots & & & & \ddots \end{pmatrix},$$

where $c_{\beta 1}, \ldots, c_{\beta N-1}$ and c_{N-1+j}, $j \ge 1$ are zero for the stock case (see Eq. (12)). For the flow case $c_{11}, \ldots, c_{N-1,1} = I_n$ and $c_{1N}, \ldots, c_{2N-1,1} = -I_n$ (see Eq. (16)). For the case of stock and flow variables the corresponding coordinates of $c_{1N}, \ldots, c_{2N-1,1}$ are zero for stock variables.

Let

$$A = \begin{pmatrix} A_\beta \\ A_1 \\ \vdots \\ A_{p-1} \end{pmatrix}, \quad T = \begin{pmatrix} T_\beta \\ T_1 \\ \vdots \\ T_{p-1} \end{pmatrix}, \text{ and } R := c^{-1}T = \begin{pmatrix} R_\beta \\ R_1 \\ \vdots \\ R_{p-1} \end{pmatrix}. \tag{A.9}$$

Observe that for the stock case (the flowcase is treated analogously)

$$\begin{aligned} \bar{C}_{b,c}\,\bar{A}_{b,c}^{-1} &= \left(\begin{pmatrix} I_r & c_{\beta 1} & \dots & c_{\beta N-1} & 0 & \dots \\ 0 & I_n & I_n & \dots & I_n & 0 & \dots \\ & & & \sharp & & & \end{pmatrix} A^N \right) c^{-1} T \underbrace{T^{-1} c A^{-N} c^{-1} T}_{\bar{A}_{b,c}^{-1}} \\ &= \begin{pmatrix} I_r & c_{\beta 1} & \dots & c_{\beta N-1} & 0 & \dots \\ 0 & I_n & I_n & \dots & I_n & 0 & \dots \\ & & & \sharp & & & \end{pmatrix} c^{-1} T = \begin{pmatrix} [c]_{(1:n+r,1:m)} \\ \sharp \end{pmatrix} c^{-1} T = \begin{pmatrix} T_\beta \\ T_1 \\ \sharp \end{pmatrix} \end{aligned} \tag{A.10}$$

where "$\sharp$" denotes some matrix entries which are not important here. Note that $\bar{A}_{b,c} = T^{-1} c A_b c^{-1} T$. From Steps 1 and 2, we obtain $\bar{A}_c := T^{-1} c A c^{-1} T = T^{-1} c Q \Lambda Q^{-1} c^{-1} T$. $A c^{-1} T = c^{-1} T \bar{A}_c$ and

$$AR = A \begin{pmatrix} R_\beta \\ R_1 \\ \vdots \\ R_{p-1} \end{pmatrix} = \begin{pmatrix} (I_r + \beta'\alpha) R_\beta + \beta'\Phi_1 R_1 + \cdots + \beta'\Phi_{p-1} R_{p-1} \\ \alpha R_\beta + \Phi_1 R_1 + \cdots + \Phi_{p-1} R_{p-1} \\ R_1 \\ \vdots \\ R_{p-2} \end{pmatrix} = \begin{pmatrix} R_\beta \bar{A}_c \\ R_1 \bar{A}_c \\ R_2 \bar{A}_c \\ \vdots \\ R_{p-1} \bar{A}_c \end{pmatrix} = R\bar{A}_c \tag{A.11}$$

$$AR = R\bar{A}_c = c^{-1} \begin{pmatrix} T_\beta \bar{A}_c \\ T_1 \bar{A}_c \\ T_2 \bar{A}_c \\ \vdots \\ T_p \bar{A}_c \end{pmatrix} = c^{-1} \begin{pmatrix} T_\beta T^{-1} c A c^{-1} T \\ T_1 T^{-1} c A c^{-1} T \\ T_2 T^{-1} c A c^{-1} T \\ \vdots \\ T_p T^{-1} c A c^{-1} T \end{pmatrix}. \tag{A.12}$$

Now, $R_\beta = c_\beta^{-1} T$ and $R_1 = c_1^{-1} T$, where $c_\beta^{-1} := \left[c^{-1}\right]_{(1:r,1:m)}$ and $c_1^{-1} := \left[c^{-1}\right]_{(r+1:r+n,1:m)}$. Therefore, we receive R_i for $i = 2, \dots, p-1$, given $R_1 = c_1^{-1} T_1$ from the recursion $R_{i+1} = R_i \bar{A}_c^{-1}$, for $i = 1, \dots, p-2$.

Step 4: (obtain $R = c^{-1}T$, T and $\beta, \Phi_1, \dots, \Phi_{p-1}$)

To retrieve T and R we proceed as follows: By means of (A.10) and (A.12), and the assumption $p \geq 2N$ we derive

$$T = cR = \begin{pmatrix} c_{\beta\beta} R_\beta + c_{\beta 1} R_1 + c_{\beta 2} R_2 + \cdots + c_{\beta N-1} R_{N-1} \\ 0 R_\beta + c_{11} R_1 + c_{12} R_2 + \cdots + c_{1N} R_N + c_{1,N+1} R_{N+1} + \cdots + c_{1,2N} R_{2N} \\ 0 R_\beta + 0 R_1 + R_2 + R_3 + \cdots + R_{N+1} \\ \vdots \\ R_N + R_{N+1} + \cdots + R_{2N-1} \\ \hline I_n R_{N+1} \\ \vdots \\ I_n R_{p-1} \end{pmatrix} . \tag{A.13}$$

Recall that for stock case $c_{1j} = I_n$, $j = 1, \ldots, N$, $c_{1j} = 0$, $j > N$, $c_{\beta j} = 0$, for $j \geq 1$, while for the flow case $c_{1j} = I_n$, $j = 1, \ldots, N$, $c_{1j} = -I_n$, $j = N+1, \ldots, 2N$, and $c_{\beta j} = -(N-j)\beta'$, for $j = 1, \ldots, N-1$.

From the above considerations T_1 can be obtained from (A.10). Since $R_{i+1} = R_i \bar{A}_c^{-1}$, Eq. (A.13) yields

$$T_1 = \begin{cases} R_1 + R_2 + \cdots + R_N & \text{, for the stock case,} \\ R_1 + R_2 + \cdots + R_N - R_{N+1} - \cdots - R_{2N} & \text{, for the flow case.} \end{cases} \tag{A.14}$$

In the above Step 3, we obtained $R_{i+1} = R_i \bar{A}_c^{-1}$, which results in

$$T_1 = \begin{cases} R_1 + R_2 + \cdots + R_N & \text{, for the stock case,} \\ R_1 + R_2 + \cdots + R_N - (R_1 + \cdots + R_N)\, \bar{A}_c^{-N} & \text{, for the flow case,} \end{cases} \tag{A.15}$$

such that $R_1 + \cdots + R_N = T_1$ for the stock and $R_1 + \cdots + R_N = T_1 \left(I_m - \bar{A}_c^{-N}\right)^{-1}$ for the flow case. As already obtained above, $R_{i+1} = R_i \bar{A}_c^{-1}$. This yields $R_1 + \cdots + R_N = R_1 \sum_{j=1}^{N} \bar{A}_c^{-j+1}$. Since $R_1 + \cdots + R_N$ follows from (A.15) we are also able to derive R_1 and therefore R_{i+1} by the recursion $R_{i+1} = R_i \bar{A}_c^{-1}$, $i = 2, \ldots, p-1$. Finally, we observe

$$\begin{aligned} T_2 &= R_2 + R_3 + \cdots + R_{N+1} = (R_1 + R_2 + \cdots + R_N)\, \bar{A}_c^{-1} \\ &\vdots \\ T_N &= R_N + R_{N+1} + \cdots + R_{N+N-1} = \ldots \\ T_{N+1} &= R_{N+1} \\ &\vdots \\ T_{p-1} &= R_{p-1}. \end{aligned} \tag{A.16}$$

Hence T_i, $i = 2, \ldots, p-1$, are provided by (A.15). Recall that T_β and T_1 follow from (A.10).

Step 5: (Obtain Σ_ν)

Let

$$\gamma_{\Delta_N y}(\kappa - \ell) := \mathbb{E}\, \Delta_N y_{t-\ell} \Delta_N y'_{t-\kappa} ,$$

$$
\begin{aligned}
\gamma_\beta(\kappa-\ell) &:= \mathbb{E}\,\beta' y_{t-\ell}(\beta' y_{t-\kappa})' , \qquad\qquad (A.17)\\
\gamma_{\beta,\Delta_N y}(\kappa-\ell) &:= \mathbb{E}\,\beta' y_{t-\ell}\Delta_N y_{t-\kappa}' = \left(\mathbb{E}\,\Delta_N y_{t-\kappa}(\beta' y_{t-\ell})'\right)' = \gamma_{\Delta_N y,\beta}(\ell-\kappa)' , \quad \text{and}\\
\Gamma_{rp} &:= \mathbb{E}\,\underline{x}_{t+1}\underline{x}_{t+1}' \in \mathbb{R}^{m\times m}\\
&= \begin{pmatrix}
\gamma_\beta(0) & \gamma_{\beta,\Delta y}(0) & \gamma_{\beta,\Delta y}(1) & \dots & \gamma_{\beta,\Delta y}(p-2)\\
\gamma_{\Delta y,\beta}(0) & \gamma_{\Delta y}(0) & \gamma_{\Delta y}(1) & \dots & \gamma_{\Delta y}(p-2)\\
\gamma_{\Delta y,\beta}(-1) & \gamma_{\Delta y}(-1) & \gamma_{\Delta y}(0) & \dots & \gamma_{\Delta y}(p-3)\\
 & & & \ddots & \\
\gamma_{\Delta y,\beta}(-p+2) & \gamma_{\Delta y}(-p+2) & \gamma_{\Delta y}(-p+3) & \dots & \gamma_{\Delta y}(0)
\end{pmatrix}\\
&= \begin{pmatrix}
\gamma_\beta(0) & \gamma_{\beta,\Delta y}(0) & \gamma_{\Delta y,\beta}(-1)' & \dots & \gamma_{\Delta y,\beta}(p-2)'\\
\gamma_{\Delta y,\beta}(0) & \gamma_{\Delta y}(0) & \gamma_{\Delta y}(-1)' & \dots & \gamma_{\Delta y}(-p+2)'\\
\gamma_{\Delta y,\beta}(-1) & \gamma_{\Delta y}(-1) & \gamma_{\Delta y}(0) & \dots & \gamma_{\Delta y}(-p+3)'\\
 & & & \ddots & \\
\gamma_{\Delta y,\beta}(-p+2) & \gamma_{\Delta y}(-p+2) & \gamma_{\Delta y}(-p+3) & \dots & \gamma_{\Delta y}(0)
\end{pmatrix} , \qquad (A.18)
\end{aligned}
$$

where $\underline{x}_t$ was defined in (10), (11). The last step follows from the fact that $\left(\underline{x}_t\right)_{t\in\mathbb{Z}}$ is stationary, such that Γ_{rp} has to be symmetric.

Let $S_\beta := (I_{r\times r}, , 0_{r\times n}, \dots, 0) \in \mathbb{R}^{r\times m}$, and $S_{\Delta_N y} := (0_{n\times r}, I_n, 0, \dots, 0) \in \mathbb{R}^{n\times m}$. Then (10) and (11) result in

$$
\begin{aligned}
&\gamma_{u^{\cdot}}(-hN) := \mathbb{E}\,u^{\cdot}_{t-hN}u^{\cdot\prime}_t = S_\beta c A^{hN} c^{-1} c\Gamma_{rp}c' S_\beta' = S_\beta A_{b,c}^h c\Gamma_{rp}c' S_\beta' ,\\
&\gamma_{u^{\cdot},\Delta_N y}(-hN) := \mathbb{E}\,u^{\cdot}_{t-hN}\Delta_N y_t' = S_\beta c A^{hN} c^{-1} c\Gamma_{rp}c' S_{\Delta_N y}' = S_\beta A_{b,c}^h c\Gamma_{rp}c' S_{\Delta_N y}' ,\\
&\gamma_{\Delta_N y}(-hN) = S_{\Delta_N y} c A^{hN} c^{-1} c\Gamma_{rp}c' S_{\Delta_N y}' = S_{\Delta_N y} A_{b,c}^h c\Gamma_{rp}c' S_{\Delta_N y}' , \quad \text{and}
\end{aligned}
$$

$$
\underbrace{\begin{pmatrix}
\gamma_{u^{\cdot}}(0) & \gamma_{u^{\cdot},\Delta_N y}(0)\\
\gamma_{u^{\cdot},\Delta_N y}(0) & \gamma_{\Delta_N y}(0)\\
\gamma_{u^{\cdot},\Delta_N y}(N) & \gamma_{\Delta_N y}(N)\\
\vdots & \\
\gamma_{u^{\cdot},\Delta_N y}((np-2)N) & \gamma_{\Delta_N y}((np-2)N)
\end{pmatrix}}_{\Gamma_{\beta\Delta_N y}}
=
\underbrace{\begin{pmatrix}
S_\beta\\
S_{\Delta_N y}\\
S_{\Delta_N y}A_{b,c}^N\\
S_{\Delta_N y}A_{b,c}^{2N}\\
\vdots\\
S_{\Delta_N y}A_{b,c}^{N(np-2)}
\end{pmatrix}}_{\mathcal{O}_N}
c\Gamma_{rp}c' \begin{pmatrix} S_\beta' \\ S_{\Delta_N y}' \end{pmatrix} .
$$

Note that $\mathcal{O}_N A_{b,c}^{-N} = \mathcal{O}$, where $\mathcal{O}$ is defined in (A.22). The matrix $\mathcal{O}$ has full column rank, as will be shown in Lemma 7, such that also $\mathcal{O}_N$ has full rank. Thus we obtain the first two column blocks of $\Gamma_{rp,c}$. Now looking at the specific structure of

$$
\Gamma_{rp,c} = \left(\begin{array}{cc|cccc}
\gamma_{u^{\cdot}}(0) & \gamma_{u^{\cdot},\Delta_N y}(0) & \gamma_{u^{\cdot},\Delta_N y}(1) & \gamma_{u^{\cdot},\Delta_N y}(2) & \cdots & \gamma_{u^{\cdot},\Delta_N y}(N-1)\\
\gamma_{\Delta_N y,u^{\cdot}}(0) & \gamma_{\Delta_N y}(0) & \gamma_{\Delta_N y}(1) & \gamma_{\Delta_N y}(2) & \cdots & \gamma_{\Delta_N y}(N-1)\\
\gamma_{\Delta_N y,u^{\cdot}}(-1) & \gamma_{\Delta_N y}(-1) & \gamma_{\Delta_N y}(0) & \gamma_{\Delta_N y}(1) & \cdots & \gamma_{\Delta_N y}(N-2)\\
\gamma_{\Delta_N y,u^{\cdot}}(-2) & \gamma_{\Delta_N y}(-2) & \gamma_{\Delta_N y}(-1) & \gamma_{\Delta_N y}(1) & \cdots & \gamma_{\Delta_N y}(N-3)\\
\vdots & \vdots & \vdots & \vdots & \vdots\ \vdots & \vdots\\
\gamma_{\Delta_N y,u^{\cdot}}(-(N-1)) & \gamma_{\Delta_N y}(-(N-1)) & \gamma_{\Delta_N y}(-(N-2)) & \gamma_{\Delta_N y}(-(N-3)) & \cdots & \gamma_{\Delta_N y}(0)\\
\gamma_{\Delta y,u^{\cdot}}(-N) & \gamma_{\Delta y,\Delta_N y}(-N) & \gamma_{\Delta y,\Delta_N y}(-(N-1)) & \gamma_{\Delta y,\Delta_N y}(-(N-2)) & \cdots & \gamma_{\Delta y,\Delta_N y}(-1)
\end{array}\right|
$$

$$\left|\begin{array}{cccc}
\gamma_{u^{\cdot},\Delta y}(N) & \gamma_{u^{\cdot},\Delta y}(N+1) & \cdots & \gamma_{u^{\cdot},\Delta_N y}(p-2) \\
\gamma_{\Delta_N y,\Delta y}(N) & \gamma_{\Delta_N y,\Delta y}(N+1) & \cdots & \gamma_{\Delta_N y,\Delta_N y}(p-2) \\
\gamma_{\Delta_N y,\Delta y}(N-1) & \gamma_{\Delta_N y,\Delta y}(N) & \cdots & \gamma_{\Delta_N y,\Delta_N y}(p-3) \\
\gamma_{\Delta_N y,\Delta y}(N-2) & \gamma_{\Delta_N y,\Delta y}(N-1) & \cdots & \gamma_{\Delta_N y,\Delta_N y}(p-4) \\
\vdots & \vdots & \vdots & \vdots \\
\gamma_{\Delta_N y,\Delta y}(1) & \gamma_{\Delta_N y,\Delta y}(2) & \cdots & \gamma_{\Delta_N y,\Delta_N y}(p+2-(N-1)) \\
\gamma_{\Delta y}(0) & \gamma_{\Delta y}(1) & \cdots & \gamma_{\Delta y}(p+2-N)
\end{array}\right),\tag{A.19}$$

we see the following relations

$$\begin{aligned}
\Gamma_{rp,c}^{(2+h)} &= \Gamma_{rp,c}^{(2)}(h) \qquad \text{for} \quad h = 1, \ldots, m-2 \\
\Gamma_{rp,c}(h) &= A_c^h \Gamma_{rp,c} \qquad \text{for} \quad h = 1, 2, \ldots \\
\Gamma_{rp,c}(2+h) &= A_c^h \Gamma_{rp,c}^{(2)},
\end{aligned}\tag{A.20}$$

where by $\Gamma_{rp,c}^{(j)}(h)$, we denote the j-th column block of $\Gamma_{rp,c}(h)$. The first equation follows from the structure of the autocovariances of the states, i.e. $\Gamma_{rp,c}(h) = \mathbb{E}\, x_{t+h} x_t'$ for $h \in \mathbb{N}_0$, the second equation follows from the Lyapunov equations. Hence, we receive all columns of $\Gamma_{rp,c}$ by using the recursions in (A.20) and therefore of $\Gamma_{rp} = c^{-1} \Gamma_{rp,c} c^{-1'}$. Finally, again by using the Lyapunov equations we have all second moments of $(\Delta y_t)_{t\in\mathbb{Z}}$ and $(u_t^{\mathcal{S}})_{t\in\mathbb{Z}}$.

Now Σ_ν retained by using the "high frequency Yule-Walker type equations", that is,

$$\begin{aligned}
&\Delta y_t - \alpha\beta' y_{t-1} - \Phi_1 \Delta y_{t-1} - \cdots - \Phi_{p-1}\Delta y_{t-p+1} = \nu_t \\
&\Delta y_t \Delta y_t' - \alpha\beta' y_{t-1}\Delta y_t' - \Phi_1 \Delta y_{t-1}\Delta y_t' - \cdots - \Phi_{p-1}\Delta y_{t-p+1}\Delta y_t' = \nu_t \Delta y_t' \\
&\underbrace{\mathbb{E}\Delta y_t \Delta y_t'}_{\gamma_{\Delta y}(0)} - \alpha \underbrace{\mathbb{E}\beta' y_{t-1}\Delta y_t'}_{\gamma_{\beta y \Delta y}(1)} - \Phi_1 \underbrace{\mathbb{E}\Delta y_{t-1}\Delta y_t'}_{\gamma_{\Delta y}(1)} - \cdots - \Phi_{p-1} \underbrace{\mathbb{E}\Delta y_{t-p+1}\Delta y_t'}_{\gamma_{\Delta y}(p-1)} = \underbrace{\mathbb{E}\nu_t \Delta y_t'}_{\Sigma_\nu}.
\end{aligned}\tag{A.21}$$

Hence, also generic identifiability of Σ_ν is established.
Finally we prove continuity of π^{-1}. This involves two steps: 1. The continuity of the mapping from the observed second moments to the parameters of a canonical minimal realisation $(\bar{A}_{b,c}, \bar{B}_{b,c}, \bar{C}_{b,c})$ (say the echelon form):
Recall that the set of transfer functions with McMillan-degree m, call it $\tilde{M}(m)$, can be decomposed in disjoint pieces corresponding to different Kronecker indices summing up to m. The set of transfer functions where the first m rows of the Hankel matrix are a basis of the row space of the Hankel matrix is generic in $\tilde{M}(m)$ [(w.r.t. the pointwise topology for $\tilde{M}(m)$ (see Hannan and Deistler 2012, p. 65)]. This set is also called the "generic neighbourhood". As has been shown in Step 5 above, $\Gamma_{r,pc}$ from equation (A.19) has full rank m. We know that the linear dependencies in the Hankel matrix of the transfer function, say $\tilde{\mathcal{H}}$, and the Hankel matrix of the second moments, say $\tilde{\mathcal{H}}_\gamma$, are the same (for the definitions see Anderson et al. 2016a,). Now since $\Gamma_{rp,c}$ is the upper left $m \times m$ block of $\tilde{\mathcal{H}}_\gamma$, we know that the first m rows of $\tilde{\mathcal{H}}$ are a basis of the

row space of $\tilde{\mathcal{H}}$. Therefore Θ_I is a subset of the generic neighbourhood.

2. Note that from a given minimal realisation $(\bar{A}_{b,c}, \bar{B}_{b,c}, \bar{C}_{b,c})$ of $\theta \in \Theta_I$ all transformations involved in the retrieval algorithm described above are continuous.

Lemma 7 *Suppose that Assumptions 1 and 2 hold. The matrix*

$$\mathcal{O} = \begin{pmatrix} S_\beta A_{b,c} \\ S_{\Delta_N y} A_{b,c} \\ S_{\Delta_N y} A_{b,c}^2 \\ \vdots \\ S_{\Delta_N y} A_{b,c}^{n(p-1)} \end{pmatrix} \tag{A.22}$$

is of full column rank $m = r + n(p-1)$.

Proof The proof is very similar to the proof that the observability matrix is of full rank in Anderson et al. (2016a)[Proof of Theorem 7, p. 823].

Since the matrix c is of full rank m we are allowed to consider A^N and A. To see this, let $\tilde{q}_i$ now denote an eigenvector of cAc^{-1} with eigenvalue λ_i, then $\left(cA^N c^{-1}\right)\tilde{q}_i = cA^{N-1}c^{-1}cAc^{-1}\tilde{q}_i = \lambda_i cA^{N-1}c^{-1}\tilde{q}_i = \lambda_i^N \tilde{q}_i$.

In addition, if q_i is an eigenvector of A, then $\tilde{q}_i = cq_i$ is an eigenvector of $A_{b,c}$.

Moreover, $A_{b,c}^j \tilde{q}_i = cA^{jN}c^{-1}cq_i = \lambda_i^{Nj} cq_i$. The eigenvalues of A are such that $\lambda_i \neq \lambda_j$ implies $\lambda_i^2 \neq \lambda_j^2$, the eigenvectors of A and A^2 coincide. To see this, let $q_i \in \mathbb{R}^m$ and $\lambda_i \in \mathbb{R}$ denote an eigenvector and an eigenvalue of the matrix A. Then, $Aq_i = \lambda_i q_i$ and $A^2 q_i = AAq_i = \lambda_i Aq_i = \lambda_i^2 q_i$; for $N > 2$ this works in the same way. Therefore it is sufficient to look at the eigenvectors and eigenvalues of the matrix A.

Similar to Anderson et al. (2016a)[Lemma 2] we have shown in the proof of the above Theorem 3 that the first $r + n$ components of an eigenvector of A or cAc^{-1} are not equal to a vector of zeros. Therefore, by the Popov-Belevitch-Hautus (PBH)-eigenvector test (see, e.g., Kailath 1980, p. 135), the matrix $\mathcal{O}$ has full *column* rank $r + n(p-1)$. That is,

$$\begin{pmatrix} \left(A^N - \lambda_i^N I_m\right) \\ \begin{pmatrix} S_\beta \\ S_{\Delta y} \end{pmatrix} A^N \end{pmatrix} q_i = \begin{pmatrix} 0_{m\times 1} \\ \lambda_i^N \, [q_i]_{1:n+r} \neq 0_{n+r\times 1} \end{pmatrix}. \tag{A.23}$$

□

E Proof of Theorem 5

The proof is constructed as follows: For each of the identifiability conditions in Assumption 2, we suppose that **(Ij)** is violated for $j = 1, \ldots, 6$ and show that there

exists no “observationally equivalent” $\theta \in \Theta_I$.
Suppose **(I1)** or **(I2)** are violated for $\theta_{\neg I}$, then it follows that the McMillan degree of $\tilde{k}(\tilde{z})$ is less than m. Hence there exists no $\theta \in \Theta_I$ with the same auto-covariance function $\tilde{\gamma}$, which is granted by $\tilde{K}(0) = I_{\tilde{n}}$.

Suppose **(I3)** or **(I4)** are violated, then the minimal realisation of $\bar{A}_{b,c}$, which is directly obtained from $\tilde{\gamma}$ has eigenvalues $\lambda_i^N = \lambda_j^N$ for some $i \neq j$, and thus **(I4)** is violated.

Suppose that neither of the conditions in **(I5)** hold, then by Eq. (A.5), we have $C_b q = 0$ and the system is not observable (and therefore of McMillan degree smaller than m).

Suppose that condition **(I6)** is not satisfied, then after going through steps Steps 1 and 2 of the retrieval algorithm in the proof of Theorem 4, we obtain in Eq. (A.8) that $S_{n_f}^{(1)} q_i = 0$ for some i and therefore we are outside of Θ_I already.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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