CONSTRAINTS ON GOVERNMENT INVESTMENT OPPORTUNITIES AND THE CHOICE OF DISCOUNT RATE

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CONTENTS

Errata Sheet for

"Constraints on Government Investment Opportunities and the Choice of Discount Rate"

The author apologizes for failing to correct a number of typographical errors in the text. Most of these should cause no important difficulty. However, the following errors are potentially troublesome and should be noted by the reader:

Page 3. On line 5, "inter-time period" should read "intra-time period".

Page 12. $\frac{1 + \alpha_t}{1 + i_t} \geqq \frac{1}{v_t}$ should read $\frac{1 + p_t}{1 + i_t} \geqq \frac{1}{v_t}$

Page 13. $p_t \geqq (1 - a_t)\, i_t + \alpha_t$ should read $p_t \geqq (1 - a_t)i_t + a_t r_t$

Page 19. In the expression just before (11) the index of summation should be τ.

In the sentence immediately following expression (11), for "we find $\gamma > 1$", read "we find $v > 1$".

Page 24. In expression (19) an equality sign is missing after the column vector of x's, and the right hand side of the equality should be

$$\begin{bmatrix} \gamma^1 - 1 \\ \gamma^2 - 1 \end{bmatrix}.$$

The denominators in the right hand side of expression (22) should be the same, namely, $1 - \gamma^1 s^1 - \gamma^2 s^2$.

Introduction

In his well known paper on the choice of discount rate for public investment decisions, William J. Baumol (1968) noted the apparent impossibility of reconciling the arguments in favor of using a pure time preference rate with those in favor of using the marginal rate of return to private investment, the "opportunity cost" of funds. On the one hand it would seem there is a clear opportunity for welfare gain in undertaking an investment with rate of return in excess of a social time preference rate, however determined. On the other hand, it would clearly be possible to do even better if the rate or return on private investment exeeded the return on the government project.

The way out of this dilemna, however, already expressed by Kenneth Arrow (1966), is to articulate explicitly the optimization problem, including its constraints, confronting the government. Assuming there is a solution to this optimization problem, there is, correspondingly, a "correct" discount rate (or, more generally, term structure of discount rates) applicable to the project. In their recent book, Arrow and Mordecai Kurz employ this method in formulating a series of optimal control problems, corresponding to various possible combinations of "instruments" available to the government. For those cases in which the set of policy tools is too limited to enable the attainment of full optimality, Arrow and Kurz are able to describe the properties of the second best investment policy, which amounts to resolving the dilemna raised by Baumol.

The Arrow-Kurz analysis is carried out at a high level of abstraction and mathematical sophistication. One price for this elegance is a rather bare-bones model of the economy, so that it is not immediately clear what their approach implies for day-to-day project selection. Furthermore their advanced methods confine an appreciation of their work to a relatively small number of readers. This paper had its origins in an attempt to probe the practical implications of one of the more remarkable conclusions of Arrow's earlier paper, and to explain it in simple terms. That conclusion can be roughly stated as follows: if capital market imperfection takes the form of a fixed marginal propensity to save (independent of the rate of return), the optimal government investment policy in the long run is to invest to the point where the marginal rate of return on government capital equals the marginal rate of social time preference, <u>regardless of the rate of return on private capital</u>.

As will be seen below, it is possible to obtain this result, which Arrow derived with the tools of the calculus of variations, using a very simple line of argument. Of course, in simplifying the analysis, such niceties as the demonstrable existence of optimal policies must be sacrificed. On the other hand, it is possible at the same time, without difficulty, to enrich the model considerably as far as its correspondence with applied policy situations is concerned. As a consequence, it can be clearly seen just what sort of a special case the conclusion just summarized represents. Remarkable at least to me, and an important reason for preparing this paper, is the apparent robustness of the policy of discounting according to a pure time preference rate to plausible sorts of variations in the parameters describing the actual second best policy.

This paper concerns the problem of investment under certainty. It is implicity assumed, furthermore, that it makes sense for the government to base its investment decicions on preferences about aggregate consumption flows. That is, it is assumed that either the inter-time period income distributional effects of investment choices are unimportant, or mechanisms are at hand to adjust the income disbrutution to any desired extent within each time period.

I have not attempted here to provide an extensive bibliography. The works already cited perform this function admirably, and for references to some of the even more recent work on the topic see Harold M.Somers (1971). In an area as well developed as this one has become it becomes very difficult to know the sources of one's ideas, and I have despaired at making compre nensive acknowledgements. My debt to the work of Arrow and Kurz is, of course, great. I would like to acknowledge and recommend as well as paper by Martin S.Feldstein (1970), which I read after the basic ideas for this paper had been worked out. Feldstein's and my approaches are very similar, although we choose to stress different aspects of the problem.

Outline of the Paper

Section I contains the basic argument and the most important conclusions. It would be possible to take away the main message without reading further. The remaining sections are principally concerned with the amplifications and extensions required to enable the theoretical conclusions to be practically applied.

In Section II I discuss the procedure for calculating the shadow price of capital, which plays a central role in the theoretical analysis. In Section III and IV the model is extended to recognize many different private investment sectors and multi-period projects. In Section V I offer some concluding remarks about the prospects for detailed application of the model, coming back at that point to the thesis that discounting at a pure time preference rate a defensible rule of thumb for government choices.

I. The Basic Argument in a Simple Model

The essense of the analysis is adequately expressed in context of a simple model, in which it is assumed there is only one kind of private capital and all the yield from an investment takes place in the immediately following period. In such a model, in fact, one does not need to make a distinction between capital stock and investment flow. Each period the whole capital stock is up for grabs , and may be consumed or invested. I assume that the objective of the government is simply to obtain the best time stream of consumption:

$$. \; . \; c_t, \; c_{t+1}, \; . \; . \; .$$

Exactly how the government determines the relative desirability or "social value" of alternative consumption time streams need not concern us. From any given set of preferences over consumption paths emerges quite naturally a sequence of "social rates of time preference", i_t, conesponding to each sequence of consumption levels, c_t. To calculate i_t we ask the question, by how much must consumption be increased in period t+1 to compensate for a one unit (let us measure consumption in "dollars") reduction in period **t** consumption. The former amount is defined to be $1+i_t$. The more valuable, relatively, is the earlier consumption, the higher is the social rate of time preference. Of course, we would expect the value of i_t to depend upon the assumed levels of c_t and d_{t+1}, but strictly speaking, it may depend upon the entire stream of consumption.

Assume that investment in private sector capital has a marginal one-period rate of return r_t in period t. That is, one dollar invested in period t increases the total number of dollars available in period t+1 by $(1+r_t)$ dollars. An increase in the private capital stock at any time may be expected to have an impact on the entire future consumption stream, since some of the proceeds are likely to be consumed in the next period, while a further part will be reinvested. to affect consumption still farther in the future. Obviously, it may be a difficult task to identify the "perturbation" of the consumption stream resulting from a one dollar change in private capital at time t, but let us imagine that this has been done, and let v_t stand for the discounted value of the sequence of consumption changes, where the discount factors are derived in the usual manner from the sequence i_t, of social rates of time preference.

It is important to understand what v_t is. Imagine that a unit of time t capital drops like mana from heaven, affecting the whole stream of consumption starting at time t+1. Presumably this new consumption stream is superior to the original one from a social point of view (although we don't even really need to assume this). Now we ask in this new consumption stream by how much could we reduce the amount of consumption in period t, and still have a consumption stream as valuable as the original, pre-mana, consumption stream. A little reflection should convince that the maximum such amount is v_t. In this sense, v_t is the "social value" of a unit of private capital at time t. Note carefully that v_t is measured in units of time t consumption. To repeat, v_t is the amount of extra consumption at time t which is equivalent from a social point of view to the sequence of extra consumptions which would result from an additional unit of private capital formation. Because of their analytical function, I shall adopt Feldstein's (1970) practice of calling the v_t's simply the "shadow prices" of private capital. There is a prevalent presumption that, for a variety of reasons, most of them falling under the heading of "capital market imperfections", too little private investment is undertaken. If this is true at time t then v_t exceeds one dollar , and I shall generally assume this is the case. However, except to avoid dividing by zero, there is no formal necessity to restrict v_t at all. I postpone until the next section the question of how one might come to quantitative grips with the shadow price of private capital. Note in passing that to say that v is constant through time is roughly equivalent to saying that capital market imperfection is becoming neither more nor less severe.

Now let us suppose that the government is contemplating undertaking an investment in period t which has a rate of return p_t. That is, one dollar invested in the government opportunity in question will make available an increase of $(1+p_t)$ dollars in period t+1. The question now is, under what conditions would an incremental unit of the government investment lead to a better time stream of consumption (from the social point of view). The answer to this question is made difficult by the fact that, through its influence on private capital formation in periods t and t+1, the government investment decision affects the entire future consumption sequence. However, having equipped ourselves with the sequence of social values of capital, v_t, we are in a position to evaluate these indirect effects more easily.

Let a_t be the amount by which private capital formation is decreased as a result of the financing of an addition dollar of government investment. Under the usual full employment assumption $(1-a_t)$ is the corresponding loss in period t consumption.[1)]

For some forms of finance we would expect to find $a_t = s_t$, the marginal propensity to save out of disposable income, although clearly we shall wish to associate different a_t's with different techniques of financing government investment.[2)] Denote by α_t the amount by which private capital in period t is increased as a consequence of an increase of $ 1 in the output of the government sector. For the case in which revenue is raised by direct taxation of consumer citizens and in which the implicit income from the government project is treated exactly like ordinary, after-tax income, $a_t = \alpha_t = s_t$. (This is the Arrow-Kurz assumption). However, we must admit the possibility that these parameters differ.

It is now a simple matter to tote up the various effects of raising an additional dollar in period t to finance government investment. Table 1 shows the impact on consumption and capital formation in periods t and t + 1.

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1) It would not be difficult to recognize the possibility of underemployed resources, and hence to include explicitly separate parameters for the investment and consumption effects here, but it would rather clutter up the argument.

2) The use of the general approach taken here to analyse the choice of financing method is the principal subject in Feldstein (1970).

	PERIOD	
	t	$t + 1$
change in consumption	$-(1 - a_t)$	$(1 + p_t)(1 - \alpha_{t+1})$
change in private capital formation	$-a_t$	$(1 + p_t)\alpha_{t+1}$

Table 1: The Effect of Undertaking an Additional $ 1 of Government Investment at Time t

We know that the value in period t consumption units of $ 1 of private investment is v_t. Hence the effective loss in period t consumption as a result of the government investment is $1 - a_t + a_t v_t$ dollars. Similarly, the value of the increase in consumption and private investment in period t + 1, expressed in period t + 1 consumption units, is $(1 + p_t)(1 - \alpha_{t+1} + \alpha_{t+1} v_{t+1})$. To compare these two sums, we must introduce the social rate of time preference, i_t. Discounting the effective t + 1 consumption increase to period t, we obtain an expression for the net gain from the whole transaction:

$$(1) \qquad -(1 - a_t + a_t v_t) + \left(\frac{1 + p_t}{1 + i_t}\right)(1 - \alpha_{t+1} + \alpha_{t+1} v_{t+1}).$$

By definition of i_t, we shall attain a more valued consumption stream if we undertake government investment so long as expression (1) is positive, a condition expressed by inequality (2),

$$(2) \qquad \frac{1 + p_t}{1 + i_t} \geq \frac{1 + a_t(v_t - 1)}{1 + \alpha_{t+1}(v_{t+1} - 1)}$$

which determines by how much, if at all, the rate of return, p, on the government investment must exceed the social time preference rate, i, in order to make the investment worthwhile.

Some Important Special Cases

It is useful to consider the implications of condition (2) as a government investment criterion in special cases corresponding to various assumptions about the coefficients a_t, α_t and v_t.

The effect of a rising or falling shadow price of private capital works in the same direction under all meaningful assumptions about the other coefficients. A glance at conditions (2) will show that increasing v_{t+1}, holding v_t constant, tends to raise the denominator on the right hand side and hence to reduce the critical rate at which p_t becomes an acceptable rate of return on the government project. The effect of decreasing v_{t+1}, holding v_t constant, is in the opposite direction. We shall interpret this result below. For the moment we simply note that to consider the consequences of varying the other parameters it is sufficient to look at the cases for which $v_t = v$, a constant.

There are four extreme cases of special interest.

Case 1. $a_t = \alpha_{t+1}$. An additional dollar's worth of output from a government project causes the same increment in private capital formation as does a reduction of \$ 1 in the amount raised through the financing instrument corresponding to which a_t is defined. This case is the rather remarkable one dist-inguished in the Introduction. Substituting into condition (2) we see that the acceptance condition for the public investment becomes

$$\frac{1 + p_t}{1 + i_t} = \frac{1 + a_t(v - 1)}{1 + a_t(v - 1)} = 1$$

This says that <u>government investments should be accepted as long as the marginal rate of return exceeds the time preference rate</u>. The surprising aspect of this case is that the rate of return in the private sector, sometimes called <u>the</u> opportunity cost of funds, does not enter the calculation at all. The reason is not far to seek. By our assumption that $\alpha_{t+1} = a_t$, we have assured that for every dollar of reduced investment in period t, the government project puts back $(1 + p_t)$ dollars of increased investment in period t + 1. The issue then is, how much must we increase investment in period t + 1 to compensate for a loss of one unit in period t. Since $v_t = v_{t+1}$, the answer is, clearly, $(1 + i_t)$ dollars.

This case has not received much attention before, which is somewhat puzzling, since the assumptions involved seem, upon reflection, to be rather plausible. However, the assumptions of Case 2 are much more frequently encountered.

<u>Case 2.</u> $a_t = 1$, $\alpha_{t+1} = 0$. Under these assumptions <u>all</u> of the resources for the project come from investment and the output induces no increase in private investment. Condition (2) becomes

$$\frac{1 + p_t}{1 + i_t} \geq v_t.$$

This case is usually interpreted to require a rate of return on the government project at least equal to the rate of return in the private sector. Note, however, that this is actually not strong enough in general. Under the conditions in which v_t, i_t and s_t (which we have called the marginal propensity to save) are all constant, we shall see in section II that $v_t > \Upsilon = \frac{1 + r}{1 + i}$, indicating that the government project must have a return rate in excess of r to pass muster under the assumptions of Case 2. The reason for this is simply that the government project does not generate the favorable repercussions on future capital formation which the private investment does.

Case 3. $a_t = 0$, $\alpha_{t+1} = 1$. In this case all resources for the project come from consumption and all yield is converted into private capital. Condition (2) becomes

$$\frac{1 + \alpha_t}{1 + i_t} \geq \frac{1}{v_t}$$

For obvious reasons this case would allow as desirable projects for which rate of return is actually below the time preference rate, and *a fortiori* below the rate of return in the private sector.

Case 4. A Two-Period World. Many analyses of the public investment problem have employed a two-period model, with the second period taken to represent "the future". For some purposes this is a satisfactory procedure, but in the present context it amounts to a very special case of our general analysis, and one with no particular claim to policy relevance. Since in a two-period world it makes no sense to speak of investment in the second period, we have $\alpha_{t+1} = 0$. The other special aspect of the two-period world is the ease with which one can calculate the value of v_t. For with no period t + 1 capital formation going on, a unit increment to period t capital generates $(1 + r_t)$

extra units of period t + 1 consumption. The period t consumption value of this, v_t, is $(1 + r_t)/(1 + i_t)$. Substituting these special assumptions into condition (2) and doing a little algebraic manipulation, we see that the critical condition becomes

$$p_t \gtreqless (1 - a_t)\, i_t + \alpha_t$$

i.e., the condition that the yield on the government investment exceed a weighted average of the rate of time preference and the private investment yield, the weights being the proportions in which the resources are taken out of consumption and private investment, respectively. Note the very special assumptions under lying this conclusion: no future capital formation consequences of increments in <u>either</u> government or private investment.[3)]

Summing Up the Analysis of the Simple Model

The essential conclusions to be drawn from the simple model are obtained from an examination of condition (2) for accepting a government investment project with rate of return p_t. Assuming that all the parameters of (2) are constant through time, we can write it as

$$(3) \qquad \frac{1 + p_t}{1 + i} \gtreqless \frac{1 + a\,(v - 1)}{1 + \alpha\,(v - 1)} .$$

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(3) Peter Diamond (1968) derived this same condition, at the same time stressing the limitations of a two-period model. Agnar Sandmo and Jacques H. Drèze (1971) also obtain this formula in a two-period model.

From (3) we conclude that the required yield on the government project should exceed the rate of time preference if the dollar reduction in current private investment per dollar withdrawn to finance the government project exceeds the dollar increase in private investment per dollar of value of the project's output ($a > \alpha$). (We assume $v > 1$). The required yield on the government project falls below the time preference rate if the inequality is reversed. The extent of the divergence in each case depends upon v; the larger is v, the larger is the divergence.

As we have pointed out, a tendency for v to grow with time favors government investment, _ceteris paribus._ The reason for this is that the government investment serves in part to _shift private investment toward the future_. The more rapidly v is rising, the less future private investment do we need to offset the loss of any given amount of current investment. By the same line of reasoning, a tendency for v to fall with time raises the minimum acceptable rate of return on government projects.

Although there remains much to be said in the following sections about how one might estimate the values of the various parameters, especially v_t, and about how the simple model can be generalized, the most important theoretical points have already been established. For the analysis has shown how it is possible to reach a definite resolution of the difficulty described by Baumol and referred to in the introductory section. The solution generally lies, interestingly enough, not on either horn of the dilemna, time preference or private productivity discounting, and may not even lie between the two apparent extremes.

The second-best character of these conclusions, resulting from constraints on the government's investment opportunities, must be strongly stressed. How, it will be asked, can it possibly make sense for the government to invest in a project with a yield of 5% when there are projects available in the private sector yielding 10%? The answer is that this can make sense <u>only if the private sector investment is not also an investment opportunity for the government.</u> In our model the government participates in the private sector investment opportunity only indirectly, via its choice of public sector investments (and via its choice of financing instruments). By taking into account in advance the response of private sector investment to government actions it is possible to evaluate the entire stream of consequences of a particular choice. Because of the limited set of investment opportunities considered appropriate for the government, and because of the divergence between private and social value of private capital, it can follow that it is better to undertake the 5% project than not to undertake it, which is the real choice. The apparent 10% opportunity cost is no such thing, since the investment in question does not represent an "opportunity" for the government at all.

II. Estimating the Shadow Prices

It is a matter of tedious algebra to express the formal calculation of the shadow prices, v_t. To save on notation let us consider v_0, the value of an investmental dollar of capital at time 0. Such an increment implies that in period 1, income is larger than it would otherwise have been by an amount $(1 + r_0)$. This leads to an increment $(1 - s_1)(1 + r_0)$ in consumption in period 1, and an increase $s_1(1 + r_0)$ in period 1 investment. The period 2 income increase is thus $s_1(1 + r_0)(1 + r_1)$, of which $(1 - s_2)\ s_1\ (1 + r_0)\ (1 + r_1)$ is consumed and $s_2 s_1 (1 + r_0)(1 + r_1)$ invested. In general, the increase Δy_T in period t income resulting directly and indirectly from the \$ 1 investment increase at time 0 is given by

(4) $$\Delta y_1 = 1 + r_0$$

$$\Delta y_T = (1 + r_0) \prod_{j=1}^{T-1} s_j\ (1 + r_j),\ T \geq 2,$$

where the notation $\prod_{j=m}^{n} x_j$ represents the product of $x_m, x_{m+1}, \ldots x_n$.

The consumption increase Δc_T generated in period T by the \$ 1 time 0 investment increase is then simply $(1 - s_T)\ \Delta y_T$. To evaluate the throw-off at each date, taking period 0 consumption as numeraire, we need only multiply the consumption change at that date by the <u>discount factor</u> δ_r derived from the social time preference rates in the familiar manner:

$$(5) \quad \delta_\tau = \prod_{j=0}^{\tau-1} \frac{1}{(1 + i_j)}, \tau \geq 1;$$

$$\delta_0 = 1$$

So, for example, a $1 increase in period 1 consumption is worth $\delta_1 = \frac{1}{1 + i_0}$ dollars in period 0 consumption. Since $\Delta c_\tau = (1 - s_\tau) \Delta y_\tau$, the present value of the increment is

$$(6) \quad PV(\Delta c_\tau) = PV\left((1 - s_\tau) \Delta y_\tau\right)$$

$$= PV\left((1 - s_\tau)(1 + r_0) \prod_{k=1}^{\tau-1} s_k (1 + r_k)\right), \tau \geq 2$$

$$= \frac{(1 - s_\tau)(1 + r_0)}{1 + i_0} \prod_{k=1}^{\tau-1} s_k \frac{(1 + r_k)}{1 + i_k}$$

$$PV(\Delta c_1) = \frac{(1 - s_1)(1 + r_0)}{1 + i_0}$$

We are now in a position to write down the present value of the whole stream of consumption increments generated by a $ 1 increase in period 0 investment. Let us call this quantity v_0. Then

$$(7) \quad v_0 = \sum_{\tau=1}^{\infty} PV(\Delta c_\tau)$$

$$v_0 = (1 - s_1) \frac{(1 + r_0)}{1 + i_0} + \sum_{\tau=2}^{\infty} \left[(1 - s_\tau) \frac{(1 + r_0)}{(1 + i_0)} \prod_{k=1}^{\tau-1} s_k \frac{(1 + r_k)}{1 + i_k}\right].$$

The reader will not need to be told that to calculate v_0 in practice is likely to be a formidable task if the problem is treated in full generality. A number of reasonable simplifications will, however, bring the job within the realm of the possible.

The Case of Equal Time Preference and Rate of Return

Before we turn to this however, it should be useful to point out how things simplify when the marginal rate of return equals the social rate of time preference in every time period. In this case we have

(8) $$v_0 = (1 - s_1) + \sum_{\tau = 2}^{\infty} (1 - s_\tau) \prod_{k=1}^{\tau - 1} s_k$$

Writing out the first few terms,

$$v_0 = (1 - s_1) + (1 - s_2)\, s_1 + (1 - s_3)\, s_1 s_2 + (1 - s_4) s_1 s_2 s_3 + \dots$$

$$= 1 - s_1 + s_1 - s_1 s_2 + s_1 s_2 - s_1 s_2 s_3 + s_1 s_2 s_2 - s_1 s_2 s_3 s_4 + \dots$$

The n^{th} partial sum of this series is this given by

$$1 - s_1 s_2 s_3 \dots s_n$$

We are interested in the limit of this sequence of partial sums, which is clearly exactly 1, provided all but a finite number of the s's are strictly less than 1.

In this special case, then, the shadow price of a unit of investment at time zero is exactly 1, and this is just as it should be. For when the rate of time preference exactly equals the marginal rate of return in every period we <u>should</u> be indifferent between an extra dollar of consumption and an extra dollar of investment in every period, including period zero. Our calculations tell us that giving up $ 1 of consumption at time zero would generate a stream of future consumption increments just equal in value to that wich is given up.

The Case of Investment Coefficient, Rate of Time Preference and Rate of Return All Constant

In order to reduce the problem to manageable dimensions, assume $s_t = s$, $i_t = i$, $r_t = r$, where s, i, and r are constants. Then we have

$$(9) \quad v_0 = (1-s)\frac{(1+r)}{(1+i)} + \sum_{\tau=2}^{\infty} \frac{(1-s)(1+r)}{1+i}\left(\frac{s(1+r)}{1+i}\right)^{\tau-1}.$$

Defining γ,

$$(10) \quad \frac{1+r}{1+i} = \gamma,$$

(9) can be rewritten as

$$v_0 = (1-s)\gamma \sum_{\tau=1}^{\infty} (s\gamma)^{\tau-1},$$

or

$$(11) \quad v_0 = \frac{(1-s)\gamma}{1-s\gamma}, \text{ assuming } s\gamma < 1.$$

Since in most cases of interest, $r > i$, and hence $\gamma > 1$, we find $\gamma > 1$; an extra dollar invested at time zero generates a stream of consumption changes worth more than one dollar. Note, further, that generally $v_0 > \gamma = (1+r)/(1+i)$. For the special case $i = r$, we conclude $v_0 = 1$ directly from expression (11).

We can, if we wish, come at the same problem from another angle. Let v_t be the value of an increment to investment at time t, expressed in time t consumption units. Now an increment of investment at time t generates extra consumption at t + 1 equal to $(1 + r_t)(1 - s_{t+1})$, and extra investment equal to $(1 + r_t)\, s_{t+1}$. The latter is equivalent to $(1 + r_t)\, s_{t+1} v_{t+1}$ units of period t + 1 consumption, by definition of v_t. Finally, since c_{t+1} units of t + 1 consumption is equivalent in social value to $c_{t+1}/(1+i_t)$ units of time t consumption, by definition of i_t, we conclude that $ 1 extra invested at time t has a value

$$(12)\quad v_t = \frac{(1 + r_t)\,(1 - s_{t+1}) + (1 + r_t)\, s_{t+1}\, v_{t+1}}{1 + i_t}$$

$$= (1 - s_{t+1} + s_{t+1}\, v_{t+1})\,\Upsilon_t,$$

where Υ_t is defined to be $(1 + r_t)/(1 + i_t)$. If we assume $\Upsilon_t = \Upsilon$ and $s_t = s$, constants, (12) describes a simple difference equation in v_t. The symmetry of the situation would seem to require v_t to be constant, and if we make this assumption explicity, $v_t = v$, a constant, then

$$(13)\quad v = (1 - s + sv)\,\Upsilon,$$

or

$$(14)\quad v = \frac{(1 - s)\,\Upsilon}{1 - s\,\Upsilon}\ .$$

Illustrative Numerical Value of the Shadow Price of Capital

The simple formulation which results from assuming the various coefficients constant makes it easy to get more feel for the magnitudes which might be involved in application. A sensible number for the before tax marginal rate of return in the corporate sector is 10%, or $r = .10$; a not wild level for the social rate of time preference might be 5%, $i = .05$. Assuming all investment to be in the corporate sector, a very large value of s would be .2. In this case $v = 1.06$, a dollar of private capital is worth about $ 1.06.

III. Generalizing to Many Private Investment Sectors

There are a number of ways in which the simple model can be generalized, and it would probably be of little use at this point to produce a catalogue of possibilities. However, in view of the previous treatment of this topic, by Baumol and others, we should consider explicitly the possibility of more than one private sector.

It is customary by now to think of private investment as being of two types, corporate and non-corporate, where decisions in the former sector are influenced by the corporation income tax. Hence, we start by considering a two-private-sector model.

Let a^1_t, then, represent the fraction of an incremental dollar raised to finance government activities which comes out of non-corporate capital formation. Let a^2_t be the corresponding value for corporate capital. Define α^1_t and α^2_t in the analogous manner, indicating the amounts of capital formation in sectors 1 and 2 induced by a \$ 1 increase in government output in period t. Let v^1_t and v^2_t be the appropriate shadow price of capital in the two sectors, given their respective rates of return, r^1_t and r^2_t. Table 2 shows the effect of undertaking an additional \$ 1 of government investment at time t in this model world.

	PERIOD	
	t	$t + 1$
Change in Consumption	$-(1 - a^1_t - a^2_t)$	$(1 + p_t)(1 - \alpha^1_{t+1} - \alpha^2_{t+1})$
Change in Non-Corporate Capital Formation	$-a^1_t$	$(1 + p_t)\alpha^1_{t+1}$
Change in Corporate Capital Formation	$-a^2_t$	$(1 + p_t)\alpha^2_{t+1}$

Table 2: The Effect of an Additional \$ 1 of Government Investment When There Are Two Private Investment Sectors

Expressed in period t consumption units, the resources taken out of the private sector in period t are valued at $(1 - a^1_t - a^2_t) + a^1_t v^1_t + a^2_t v^2_t$, while the value yield from the government project, expressed in period t + 1 consumption units, is

$$(1 + p_t)\left[(1 - \alpha^1_{t+1} - \alpha^2_{t+1}) + \alpha^1_{t+1} v^1_{t+1} + \alpha^2_{t+1} v^2_{t+1}\right.$$

This can be discounted to period t by social rate of time preference, so that we have all of the consequences of the decision expressed in period t consumption units. The criterion for acceptance of the government investment is, then

$$(15) \quad -(1 - a^1_t - a^2_t + a^1_t v^1_t + a^2_t v^2_t) + \frac{1 + p_t}{1 + i_t}\left[1 - \alpha^1_{t+1} - \alpha^2_{t+1} + \alpha^1_{t+1} v^1_{t+1} + \alpha^2_{t+1} v^2_{t+1}\right] \geq 0.$$

A little algebraic manipulation allows us to express this condition in the equivalent form,

$$(16) \quad \frac{1 + p_t}{1 + i_t} \geq \frac{1 + a^1_t(v^1_t - 1) + a^2_t(v^2_t - 1)}{1 + \alpha^1_{t+1}(v_{t+1} - 1) + \alpha^2_{t+1}(v^2_{t+1} - 1)} .$$

The interpretation of condition (16) runs along the same lines as our previous interpretation of condition (3), and presents no special difficulties. Of particular interest is the case $a^1_t = \alpha^1_{t+1}$, $a^2_t = \alpha^2_{t+1}$, in which dollars taken out of the private sector by the financing methods and implicit dollars put into the private sector in the form of yield on the government investment, are divided among consumption and investment in the two sectors in the same proportions. In this case condition (16) becomes simply

$$\frac{1 + p_t}{1 + i_t} \geq 1;$$

the government project's yield rate need be only as high as the social rate of time preference.

Calculating the Two Shadow Prices

If we assume constant propensities to save/invest, constant rates of return , constant social rate of time preference and constant values of the two shadow prices, v^1 and v^2, we c calculate the/ /latter without particular difficulty. By a line of reasoning exactly analogous to that leading to equations (12) to (14), we conclude that v^1 and v^2 must satisfy simultaneous equations (17):

$$(17) \quad v^1 = \frac{1 + r^1}{1 + i}\left[1 + s^1 (v^1 - 1) + s^2 (v^2 - 1)\right]$$

$$v^2 = \frac{1 + r^2}{1 + i}\left[1 + s^1 (v^1 - 1) + s^2 (v^2 - 1)\right];$$

Define γ^1 and γ^2, analogous to γ in the one-sector analysis, and the new variables x^1 and x^2:

$$(18) \quad \gamma^1 = \frac{1 + r^1}{1 + i}, \gamma^2 = \frac{1 + r^2}{1 + i}, \; x^1 = v^1 - 1, \; x^2 = v^2 - 1.$$

Then (17) implies

$$x^1 = \gamma^1 \left[1 + s^1 x^1 + s^2 x^2 \right] - 1$$

$$x^2 = \gamma^2 \left[1 + s^1 x^1 + s^2 x^2 \right] - 1$$

or

$$(19) \quad \begin{bmatrix} 1 - \gamma^1 s^1 & -\gamma^1 s^2 \\ -\gamma^2 s^1 & 1 - \gamma^2 s^2 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} \quad \begin{bmatrix} {}^1 \quad - 1 \\ {}^2 \quad - 1 \end{bmatrix}.$$

Denoting the matrix of coefficients in (19) by A, we have

$$(20) \quad \det(\mathbf{A}) = (1 - \gamma^1 s^1)(1 - \gamma^2 s^2) - \gamma^1 \gamma^2 s^1 s^2$$
$$= 1 - \gamma^1 s^1 - \gamma^2 s^2.$$

Assuming (20) is not zero,

$$(21) \quad A^{-1} = \frac{1}{1 - \gamma^1 s^1 - \gamma^2 s^2} \begin{bmatrix} 1 - \gamma^2 s^2 & \gamma^1 s^2 \\ \gamma^2 s^1 & 1 - \gamma^1 s^1 \end{bmatrix}$$

and

$$(22) \quad \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} = A^{-1} \begin{bmatrix} \gamma^1 - 1 \\ \gamma^2 - 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\left(\dfrac{1 - \gamma^1 - s^2 \gamma^2 + s^2 \gamma^1}{1 - \gamma^1 s^1 - \gamma^2 s^2}\right) \\ -\left(\dfrac{1 - \gamma^2 - s^1 \gamma^1 + s^1 \gamma^2}{s - \gamma^1 s^1 - \gamma^2 s^2}\right) \end{bmatrix}.$$

After a little algebraic manipulation, we can write

$$(23)\quad \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \left(\frac{1 - s^1 - s^2}{1 - \gamma^1 s^1 - \gamma^2 s^2} \right) \begin{bmatrix} \gamma^1 \\ \gamma^2 \end{bmatrix}$$

Under the assumption sometimes made that the social rate of time preference is equal to the rate of time preference common to all individuals in a competitive capital market, and that this is in turn equal to the rate of return in the non-corporate sector, $r^1_t = i_t$, i.e. $\gamma^1 = 1$. Notice that, because some of the throw-off from each sector is invested in the other, even in this case generally $v^1 > 1$.

Increasing the number of private investment sectors beyond 2 is evidently a simple matter formally. The government investment test changes from (16) to

$$(24)\quad \frac{1 + p_t}{1 + i_t} \geq \frac{1 + \sum_{j=1}^{m} a^j_t \, (v^j_t - 1)}{1 + \sum_{j=1}^{m} \alpha^j_{t+1} \, (v^j_{t+1} - 1)} ,$$

where m is the number of sectors. For the case in which the various coefficients, and thus the shadow prices, are constant over time we can calculate them in the manner of expression (23) from

$$\begin{bmatrix} v^1 \\ v^2 \\ \vdots \\ v^m \end{bmatrix} = \left(\frac{1 - \sum_{j=1}^{m} s^j}{1 - \sum_{j=1}^{m} \gamma^j s^j} \right) \begin{bmatrix} \gamma^1 \\ \gamma^2 \\ \vdots \\ \gamma^m \end{bmatrix}$$

where $\gamma^j = (1 + r^j)/(1 + i)$

IV Treatment of Multiperiod Returns

Thus far, the analysis has been carried on as though the government were always in a position of choosing an investment requiring inputs of \$ 1 in period t and producing its entire output \$ 1 $(1 + p_t)$ in period t + 1. This permitted us to express the investment criterion in terms of the relationship between a well-defined rate of return (essentially, the one-period internal rate of return) and corresponding rate of time preference. However, it is a relatively simple matter to deal with more complex patterns of returns.

We capture most of the interesting aspects of this generalization in supposing that our typical government project can be described by two time sequences of numbers: b_t denoting the current dollar (consumption equivalent) value of the government service provided in period t and e_t denoting the additional resources which must be raised in period t to carry out the project. Note that in any given period either b_t or e_t or both <u>may be negative.</u> A negative value of b_t corresponds to costs such as environmental damage caused by the project. A negative value of e_t may result when the project yields a cash return -- for example, its output is sold on the market -- which may be applied toward a reduction in taxes in that period. Negative values for these variables may occur in another way as well. Our analysis applies equally to investment and disinvestment. <u>Not</u> undertaking a project is itself a project; if project A is described by $\{b_t, e_t\}$, the project "not undertaking A" is described by $\{-b_t, -e_t\}$.

Note that we ar here assuming only one form of benefits and only one form of finance. More generally, we could describe a project by a sequence of vectors (b^1_t, b^2_t, . . . , b^m_t, e^1_t, . . . , e^r_t), where b^j_t = dollars worth of benefits of the j'th type (of m types in all) in period t, and e^k_t = dollars required from finance source k (of r sources, e.g., sales tax, debt issue, etc.) in period t.

Having constructed similar lines of reasoning in the previous two cases we can move fairly quickly to the appropriate criterion in this case. Things will be simplified somewhat if we define shadow prices for benefits and revenues as follows:

$$(26) \qquad \beta_t = 1 + \sum_{j=1}^{m} \alpha^j_t \, (v^j_t - 1)$$

$$\varepsilon_t = 1 + \sum_{j=1}^{m} a^j_t \, (v^j_t - 1);$$

where the sums in (26) are taken over all of the private sector investment possibilities distinguished by the model. β_t is the social value of a dollar's worth of benefits when account is taken of the induced effects on private capital formation. Similarly, ε_t is the social value of the lost consumption and capital formation resulting from raising an additional dollar of finance.

In period t the government project produces b_t dollars in benefits, worth $b_t\beta_t$ dollars in period t consumption when the influence on private capital formation is taken into account. The project requires that e_t dollars in revenue be raised, at a cost of $e_t \, \varepsilon_t$ dollars in period t consumption when the influence on private capital formation is taken into account. The net effect is $b_t \beta_t - e_t \ \varepsilon_t$, which is equivalent to $\delta_t (b_t \beta_t - e_t \ \varepsilon_t)$ units of period 0 consumption. The project is worthwhile if

$$(27) \qquad \sum_{t=0}^{T} \delta_t \ (b_t \beta_t - e_t \ \varepsilon_t) \geq 0,$$

where the project has its last direct payoff or resource requirement in period T.

We may check that condition (27) gives us back the criterion for one-period investments which we have already derived. For such an investment, we have $b_{t+1} = 1 + p_t$, $b_\tau = 0$ otherwise; $c_t = 1$, $c_\tau = 0$ otherwise. Then (27) becomes

$$(27a) \qquad \delta_{t+1} \ (1 + p_t) \ \beta_{t+1} = \delta_t \ \ \varepsilon_t \gtreqless 0.$$

Recall from the definition of δ_τ that $\delta_{t+1} / \delta_t = \frac{1}{1 + i_t}$.

Dividing (27a) by δ_t, it becomes

$$(27b) \qquad \frac{1 + p_t}{1 + i_t} = \frac{\varepsilon_t}{\beta_{t+1}}$$

Substituting definition (26) for ε_t and β_{t+1}, (27b) becomes

$$(27c) \qquad \frac{1 + p_t}{1 + i_t} \gtreqless \frac{1 + \sum_{j=1}^{m} a^j_t (v^j_t - 1)}{1 + \sum_{j=1}^{m} \alpha^j_{t+1} \ (v^j_{t+1} - 1)}$$

which is the previously derived condition (24) for one-period investments.

As usual, special cases do much to reveal the character of the criterion. If an additional dollar of benefits is divided among consumption and the different sorts of private investment in the same way every year and if the shadow prices of the different sorts of private capital are constant, then $\beta_t = \beta$, a constant. If an additional dollar of financing comes out of consumption and the different sorts of investment in the same proportions every year and if the shadow prices of capital are constant, then $\varepsilon_t = \varepsilon$, a constant. Then (27) becomes

$$(28) \qquad \beta \sum_{t=0}^{T} \delta_t b_t - \varepsilon \sum_{t=0}^{T} \delta_t e_t \geq 0,$$

which, assuming the appropriate expressions are positive, may also be written

$$(29) \qquad \frac{\sum_{t=0}^{T} \delta_t b_t}{\sum_{t=0}^{T} \delta_t e_t} \geq \frac{\varepsilon}{\beta} \quad .$$

Condition (29) says that a project is worth undertaking if the discounted stream of benefits divided by the discounted stream of costs (which are defined very precisely to refer to changes in revenue raised by a particular financing mode) exceeds a calculable critical level, ε/β.

Condition (29) is a form of a benefit cost ratio. It is derived here simply to show how our criterion relates to those put forth by others. It is probably usually a mistake to use ratio criteria in practice. The net present value approach is always correct, whereas ratio criteria can lead to difficulties where there are mutually exclusive or otherwise interrelated projects in question, or where capital budget constraints are involved.

A special case of case (29) may be of interest, namely that in which all the financing for the government project displaces a single kind of private capital, so that $a_t = 1$, and the yield from the project induces no private investment (usually somewhat loosely described as "all yield consumed"), so that $\alpha_t = 0$. As far as the consumption-investment division of resources withdrawn by and output yielded by the project are concerned, these assumptions are those least favoring government investment. Referring to definitions (26) we see that in this case $\beta = 1$ and $\varepsilon = v$. Condition (29) becomes

$$(30) \qquad \frac{\sum_{t=0}^{T} \delta_t b_t}{\sum_{t=0}^{T} \delta_t e_t} \gtreqless v,$$

requiring the benefit cost ratio to exceed the shadow price of private capital. Interpretation of (30) is facilitated by writing it in the original form of (29),

$$(31) \qquad \sum_{t=0}^{T} \delta_t (b_t - v\, e_t) \gtreqless 0.$$

Here we see that to account for these rather extreme assumptions to the disadvantage of government investment what is required is to weigh dollar expenditures by a factor v in calculating net benefits to be discounted at the social rate of time preference. Since we have previously calculated a reasonable value of v to be 1.06 it seems likely that the choices made under rule (31) will in many cases be very close to those made under the simple rule, "maximize present value of the stream of net benefits ($b_t - e_t$), discounting at the social rate of time preference."

A case frequently encountered in practice is the choice among alternative expenditure streams, corresponding to different technical methods of producing a given service stream. In the language of the model above, we have a fixed stream of gross benefits, $\bar{b}_t$, and must choose among alternative expenditures streams e_t. Here the appropriate procedure is to discount at the time preference rate, provided it is assumed that the method of financing the alternative streams is such that the private investment loss per dollar of financing is the same in every period for every alternative considered ($\varepsilon_t = \varepsilon$) and the shadow prices of private capital stocks are constant. Then the problem becomes one of finding the expenditure stream to maximize

$$\sum_{t=0}^{T} \delta_t \left(\bar{b}_t \beta_t - e_t \varepsilon\right) ,$$

which is the one which minimizes

$$\varepsilon \sum_{t=0}^{T} \delta_t e_t .$$

This is obviously the same choice which minimizes

$$\sum_{t=0}^{T} \delta_t e_t ,$$

the present value of the expenditure stream, discounted at the social rate of time preference.

The same rule applies in the last special case, in which the investment-inducing effects of an additional dollar of benefits are exactly the same as those of a reduction of a dollar in financing and the shadow prices of the various kinds of private capital are constant through time. Then we have $\mathcal{E} = \beta$, and condition (27) becomes

$$(32) \qquad \sum_{t=0}^{T} \delta_t (b_t - e_t) \geqq 0.$$

Although (32) does represent a special case, its preconditions are not implausible, requiring simply that extra dollars taxed away and extra dollars received (usually implicitly) in benefits are treated as about the same thing by the private sector.

V. Concluding Remarks

In a sense, condition (27) with its associated definitions, is the general solution to the public investment problem and hence the general conclusion of this analysis. The next step required is to put empirical flesh on the theoretical structure, in the form of actual estimates of the various coefficients of condition (27). However, one should not be over-optimistic about obtaining a set of coefficients which can be applied to any government investment problem to lead to a correct choice. The principal reasons for this are:

A. The precise financing technique providing the source of funds for any government project is often unclear or not even well-defined, and these sources vary from project to project. Do the funds to pay for a subsidy to the merchant marine come from corporation income taxes? personal income taxes? changes in Federal debt? A strictly correct answer would not even be assured if expenditure laws specified the source of finance. For example, Federal highway programs are nominally financed by gasoline taxes, and this may be the end of the story. However, it is also possible that gasoline taxes would be about what they are anyway. An increase in highway expenditures in this case forces some other program to find its financing in another revenue device, say, the personal income tax. Then the source of finance for the highway expenditure for our purposes is, in fact, the personal income tax. Furthermore, the source of finance for a project may well be funds which would otherwise have financed another government activity. In this case, then, the dollars for the project in question "cost" the foregone benefits from the alternative activity. We need not labor further the extreme difficulty of establishing the financing source.

B. Knowing the financing source, we face great difficulty in establishing how much of an incremental dollar from that source derives from consumption, how much from various forms of private capital (and, we might add, how much from unemployed resources, in effect, from nowhere). It is, of course, not sufficient to know simply the nominal payers of a tax. For example, locating the incidence of the corporation income tax is the subject of controversy within the economics profession, with no settlement in sight.

It is possible that identifying the precise incidence of a financing mode can be sidestepped. Since we are here concerned not with matters of distributional equity, but of intertemporal efficiency, we need only worry about the quantitative reaction of consumption and investment to increments in financing from each mode. These coefficients might be stable over time and identifiable by econometric techniques.

C. On the output side, matters are not much better. Assuming, as we have been, that one can reasonably estimate dollar values of the services from the government project, determining the influence of these flows on private investment is likely to be extremely difficult. Some headway may be made by examining the character of the flow involved. Is it more like consumption or more like savings? If the government undertakes to provide medical care for the aged, presumably this reduces the incentive for citizens to accumulate a reserve against this possibility. We might expect, then, that providing an additional dollar's worth of this insurance protection will lead individuals to reduce their private savings by about a dollar and to increase consumption by a like amount. At the other extreme, benefits from public parks may be effectively pure substitutes for private consumption. A family receiving these benefits reduces its expenditures on film and baseball game attendance, and increases its savings accordingly.

Note, though, that what a service looks like may tell only part of the story. The park example illustrates this. While it is true that the recreational services are of a nature which we would usually label "consumption", there is no guarantee that these services replace other consumption. They may be additive, they may even induce a reduction in labor supply and a net decrease in private saving ($\alpha < 0$, a case we have implicitly ruled out for most of our analysis).

Most benefits probably are between these extremes in their influence, and many no doubt are treated exactly as any other form of income, or the benefits may actually occur as income. An irrigation project, for example, increases the incomes of landowners in the affected area, and possibly the income of cooperating factors as well.

In view of these very serious obstacles to a precise implementation of (27), we would do well to consider what rules of thumb are likely to make sense and the circumstances in which they are likely to lead us astray. The general thrust of this analysis has been supportive of the rule of thumb:

(33) <u>In the absence of reasonably clear evidence to the contrary, treat v as constant and α^j_t and a^j_t as constant and equal. In other words, attempt to maximize present value of net dollar flows (including dollar equivalents of nonmarketed effects), discounting at the social rate of time preference.</u>

Assuming v, α_t and a_t constants, reasonable extremes of condition (27) are given by the cases of (28) corresponding to $\beta = v$, $\varepsilon = 1$ and $\beta = 1$, $\varepsilon = v$. The latter case we have already discussed, as it leads to the equivalent criteria (30) and (31). In other words in the case in which all financing comes out of investment and all benefits increase consumption we should multiply financing changes by a factor v before calculating net benefits. If $\beta = 1$ and $\varepsilon = v$ and we nevertheless use our rule of thumb (33) we will, of course, undertake some socially unprofitable projects. In the case of a project just barely worthwhile, so that

(34) $$\sum \delta_t b_t = \sum \delta_t e_t$$

undertaking the project will lead to a net loss equal to $(v-1)\sum \delta_t e_t$. By our "reasonable" value of v, this would amount to approximately 6% of the resources involved in the project.

At the other extreme is the case in which $\beta = v$, $\varepsilon = 1$, all yield from the project leads to an equivalent value increase in private investment and all finance for the project derives from private investment. Here rule of thumb (33) is too conservative. The barely worthwhile project for which (34) holds, will actually generate a net profit of $(v-1)\sum \delta_t e_t$.

Thus, while the simplification effected by rule of thumb (33) is extreme and of great practical value, degree of likely error associated with it appears modest in comparison, say, with the uncertainty surrounding cost estimates. If this is accepted, attention must next be debvoted to establishing a truly acceptable value for the social rate of time preference, since the particular number chosen is of greatest consequence for decisions among projects of any considerable duration.

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