ECONOMIC GROWTH
AND
EDUCATIONAL ACTIVITY

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Research Memorandum Nr. 66

März 1972

CONTENTS

I. INTRODUCTION

Although there exists a huge number of both, models of economic growth and models of educational planning, it has, to the author's knowledge, not been attempted to integrate this two kinds of models. Models of economic growth neglect the problems occuring because of the need for an educated labour force by aggregating labor into one single kind. Models of education on the other hand either abstract totally from non-educational economics (paediometric models,as M. Blaug calls them) or take the rest of the economy as exogenously given. For example the well known Correa-Tinbergen-Bos model uses in its early stage a very simple Harrod model to describe the economic development and then finds out the optimal (in some sense) adaptation of the educational system. In its later stage this model takes the economic growth rate as exogenously given. Similarly, most linear programming models, based on cost-benefit-analysis, e.g. that of S. Bowles for Northern Nigeria, take the economy and its development as exogenously given[1)] and so does the so-called manpower forecasting approach.

Certainly it makes life easier or more exactly; it makes the models simpler, if we leave out the feedback of the educational sector on the rest of the economy. But, on the other hand, it is a commonplace that there is a general interdependence in the economic world. If educational planning aims to influence the

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1) This is not the case e.g. for J. Benerd's "General Optimization Model for the Economy and Education", published in Mathematical Models in Educational Planning, OECD Paris 1967. But even this "model as it stands at present does not accordingly allow any choice between techniques; and the costs of educating the labour force have no impact on the techniques adopted." (p. 212). But at least "the structure of the programme, however, enables it to incorporate interchangable technical processes fairly easily." (p. 212).

(long term) economic growth rate, how can we take this same growth rate as exogenously given in our models?

To incorporate education in a growth model means to desaggregate labour. There are good reasons to disaggregate labour in such models, where we desaggregate capital. This argument leads us to linear growth models of the von Neumann type, first developed by von Neumann in 1932 and published 1937 in Vienna[1], and further developed e.g. by Kemeny, Morgenstern, and Thompson in 1956[2] and by Morishima[3] in the sixties. To the author's knowledge there is only one attempt, , to include education in linear growth models, given by Karl Förstner[4]. But, at least to the author's opinion, this generalization is seriously misleading (e.g. Förstner uses only one activity vector for both sectors, which implies some very unrealistic implicit assumptions) and Förstner did not prove the existence of an equilibrium for his model.

Since the models in this paper do catch the interdependence of both sectors, the goods producing one and the educational one, it can be looked at in two ways: as a generalization of the von Neumann model or as a generalization of educational models.

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1) J. von Neumann, "Über ein ökonomisches Gleichungssystem und eine Verallgemeinerung des Browerschen Fixpunktsatzes"; Ergebnisse eines mathematischen Kolloquiums, No. 8 (1937), translated as "A Model of General Economic Equilibrium", Review of Economic Studies, Vol. 13, No. 33 (1945-46).

2) J.G. Kemeny, O. Morgenstern, and G.L. Thompson, "A Generalization of the von Neumann Model of an Expanding Economy", Econometrica, Vol. XXIV (1956).

3) M. Morishima, "Theory of Economic Growth", Oxford 1969.

4) K. Förstner, "Wirtschaftliches Wachstum bei vollständiger Konkurrenz und linearer Technologie" in R. Henn (ed.) "Methods of Operations Research IV", Meisenheim am Glan, 1967.

But in both cases it should not be interpreted as a description of the real world and I want to point this out very clearly and stress it very firmly. The model cannot give a good picture of the real world because too many and too important features of social reality are not included, e.g. individual choice of education is in general not or not enough (in order to use the model for description of reality) based on things like discounted life time income or to give another simple reason: we just don't have perfect competition. Nevertheless, the von Neumann type models seem to be very powerful instruments for problems of planning and for pointing out some separated features of an economy. And it gives more evidence for their powerfulness that for simple cases it is very easy to include educational activities.

Although the reader is assumed to be familiar with the model of Kemeny-Morgenstern-Thompson and with Morishima's model, in section II those two models are outlined very shortly. Of course, I don't need to mention that these few pages cannot catch the ingenuity, deepness and power of those models.

Since all the models developed in this paper are very similar to those two models, they include on top of new restrictions and assumptions all those of their basis, the Kemeny-Morgenstern-Thompson model or the Morishima model, respectively.[1] But as well

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1) Some assumptions become more restrictive if we include education,e.g. neglecting exernal effects is very restrictive for education.

they include some advantages of those models, e.g. the possibility of incorporating some kinds of technical progress, although those kinds are rather limited to an equal over all technical progress, if one wants to go from comparative statics to dynamic models.

Our models can be characterized into two types according to their basis: those based on the Kemeny-Morgenstern-Thompson model, developed in section III, and those based on Morishimas model, developed in section IV. In an Appendix we give some simple ideas about optimal allocation of investment between "physical capital" and "educational capital". If "students" do get less income and savings are positively related to income, it turns out that there must be relative more investment in education because it is in some sense "cheaper" since it causes higher savings. This argument is already stated in the literature by several authors[1)] and can be proved formally by classical methods, which also give an idea of how much investment in education is "cheaper".

Finally I may mention that, of course, this paper can only be a start and that nearly all problems are still unsolved. But nevertheless this paper may give a flavour of an approach in a certain direction and it may give more evidence that linear growth models of the von Neumann type can be, at least if further developed, very strong instruments in economic theory.

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1) cf. E.F. Denison, "The sources of economic growth in the United States and the alternatives before us". New York 1962, pp. 77-78.

II. THE PURE COMMODITY ECONOMY

Linear growth models are based on von Neumann's famous model, developed as early as 1932. In 1956 John G. Kemeny, Oskar Morgenstern, and Gerald L. Thompson have given a generalization of von Neumann's model, referred to as KMT-model in the following, and in the sixties further generalizations have been developed by several authors, especially by Michio Morishima, whose model will be shown in its basic structure.

1. The Kemeny-Morgenstern-Thompson-Model

A linear technology is defined by an input matrix and an output matrix, which we will denote by A and B, respectively. Each row of A and B corresponds to a process, each column to a commodity. If we have m processes and n commodities, A and B are of order $(m \times n)$, and a_{ij} is the amount of commodity j used in process i and b_{ij} is the amount of commodity j produced in process i.

In the KMT-Model there are two further assumptions concerning the technology:

(i) every process uses some inputs, i.e., goods produced in the preceding time period, and

(ii) every good can be produced in the economy, i.e., given a good, there exists at least one process which can produce it.

Mathematically those two assumptions are equivalent to

(i) every row of A has at least one positive entry,

(ii) every column of B has at least one positive entry.

An activity vector x is a nonnegative row vector $(x_1, \ldots, x_m)$, where x_i gives the level of i-th process. If the i-th process is not used, x_i is obviously zero. Since we will demand that the economy produces something of positive value, at least one x_i must be positive and therefore we can normalize x such that

$\sum_i x_i = 1$. Because of the linearity of the technology and the exclusion of external effects, we can describe every state of production by a normalized vector x, giving the relative intensities of the processes used, and a scalar, giving the level of the production state.

A price vector y is a nonnegative column vector $(y_1, \ldots, y_n)'$, where y_j gives the price of the j-th commodity. Since we are interested in relative prices only and since we will assume that something of value must be produced, we can normalize y such that $\sum_j y_j = 1$.

An interest factor β is defined as $\beta = 1 + b/100$, where b is the rate of interest (per cent). Similar an expansion factor or growth factor α is defined as $\alpha = 1 + a/100$ with a being the growth rate (per cent).

The KMT-Model assumes unlimited supply of "land and labour", i.e., unlimited supply of the original means of production. As far as labour is concerned, Morishima disposes of this rather unrealistic assumption. The KMT-Model further assumes that there is no consumption (or that there is consumption of workers, but in fixed amounts, independent of prices, and that this consumption is included in the input matrix A). Morishima disposes of this assumption as well. Another assumption is that of the same constant length of production period for all processes. But, as Morishima has pointed out, this can always be reached by defining new (intermediate) goods and processes.

For a state of balanced growth we demand the fulfilment of five conditions:

(1) $xB \geqq \alpha xA$

i.e., the expansion rate α and the activity vector x must be such that the input in any period does not exceed the output of the preceding period.

(2) $xBy = \alpha xAy$

i.e., goods, available in amounts greater than needed, have zero price.

(3) $\beta Ay \geqq By$

i.e., absence of supernormal profits

(4) $\beta xAy = By$

i.e., a process, giving a profit less than the interest on the inputs, is not used.

(5) $xBy > 0$

i.e., something of positive value is produced in the economy.

Kemeny, Morgenstern, and Thompson have proved that such an equilibrium exists (but may not be unique) for every technology fulfilling (i) and (ii), i.e., there exist x, y, α and β such that (1) to (5) is fulfilled. In this state of balanced growth $\alpha = \beta > 0$, hence a = b, which is a special case of Sir Harrod's famous equation $g = s/\kappa$, the "warranted rate of growth" equals the savings ratio divided by the capital output ratio. In the case of no saving of workers this equation reduces to $g = s_c \rho$, the warranted rate of growth equals the capitalists' saving ratio times the profit rate. In the KMT-Model g corresponds to a and ρ to b and $s_c = 1$.
Another famous result of Kemeny, Morgenstern, and Thompson is the dual relationship of α and β. If there is a unique α, it is at the same time the highest technically possible expansion rate and the lowest interest factor, which makes a profitless system of prices possible. If the α is not unique in all solutions, the highest and the lowest α have the two properties respectively.

The proof proceeds by game theoretic terms. In exactly the same way we will proceed in proving the existence of a state of balanced growth in one of our generalized KMT-Models in the next section.

2. Morishima's Model

Morishima's model is a threefold generalization of the KMT-Model: Labour is no longer assumed to be available in unlimited supply, workers and capitalists do save and consume in positive amounts, and consumption depends on relative prices and real wages and profits. Morishima has proved the existence of a state of balanced growth, where the expansion factor of the economy equals the growth factor of the labour force, which is assumed to be a constant.

Let us first fix the notation and give the basic assumptions.
A, B, x, y, α, β have the same meaning like in the KMT-Model.
L ist a column vector, where l_i denotes the labor input in the i-th process.
Ω is the "real" wage, it is the money wage divided by the sum of the prices. In order to get low weights for the prices of non-consumption goods, the unity of a non-consumption good must be small relative to the unity of a consumption good.

s_w and c_w are the workers savings consumption ratios respectively. Similarly s_c and c_c are the savings and consumption ratios of the capitalists.

E is the sum of all profits and it is assumed that workers and capitalists share E according to their savings. Hence, if E_w denotes the workers share and E_c the capitalists share, we have the so-called "Pasinetti formula":

$$\frac{E_w}{s_w(E_w+W)} = \frac{E_c}{s_c E_c}$$

It is assumed that E exceeds $\frac{s_w}{s_c - s_w}$ W, where W denotes total wages, because otherwise the so-called silvery equilibrium would be of the anti-Pasinetti-type, where E_w equals E, i.e. workers could get all profits.

Workers, as well as capitalists are assumed to have similar and homothetic utility functions, hence for deriving demand for consumption purposes we can treat the labour force as one big worker and the set of capitalists as one big capitalist. Homothetic utility functions have the property that income elasticity is unity for all goods. Furthermore, we make the usual assumption of diminishing marginal rates of substitution between goods in all directions.

These assumptions and the normalization of prices give us two equations

$$e_j = g_j(y_1, \ldots, y_n) \quad (W + E_w),$$

where e_j is the workers demand for the j-th good and g_j is the Engel coefficient for the j-th good.

Similar, denoting by d_j capitalists demand for good j (for consumption) and by f_j the corresponding Engel coefficient, we have

$$d_j = f_j(y_1, \ldots, y_n)E_c$$

All Engel coefficients are assumed to be non-negative, finite and continuous for every non-negative set of normalized prices.

Further on the following equations must hold

$$\sum_j g_j(y_1, \ldots, y_n)y_j = c_w$$

$\sum_j f_j(y_i, \ldots, y_n) = c_c$, which follow from the budget constraints.

$W = \Omega \sum_i l_i x_i = \Omega xL$, Ω being the real wage rate

$$E_w = \frac{s_w}{s_c - s_w} W$$

$$E_c = E - \frac{s_w}{s_c - s_w} W$$

$$E = (B-1)[xAy + x\Omega L]$$

If g(y) and f(y) denote the corresponding (row) vectors to g_j and f_j and if we assume the labour force to grow at a constant geometrical rate ρ, we get six conditions for balanced growth (silvery equilibrium):

$$(1)\quad xB \geqq \alpha \left[xA + x\Omega L \left\{\frac{s_c}{s_c - s_w} g(y) - \frac{s_w}{s_c - s_w} f(y)\right\}\right] +$$

$$+ (B-1)(xAy + x\Omega L)f(y)$$

$$(2)\quad xBy = [\alpha + (B-1)c_c]\,(xAy + x\Omega L)$$

$$(3)\quad By \leqq B(Ay + \Omega L)$$

$$(4)\quad xBy = Bx(Ay + \Omega L)$$

$$(5)\quad xBy > 0$$

$$(6)\quad \alpha = \rho$$

(1) to (4) are the familiar conditions of non-negative excess supply, absence of supernormal profits and the rules of free goods and profitability prevailing. (5), also familiar, demands that something of positive value is produced and (6) assures the absence of unemployment, if - which is a necessary condition for all balanced growth models, the initial conditions are right. In the case of equation (6) this means, if there is no unemployment at the beginning, there will always be full employment.

Michio Morishima has proved the existence of a state of balanced growth, i.e. a state of silvery equilibrium, if two further conditions hold: labour is indespensable for production of commodities, each good can be produced and ϱ must not exceed a certain value.

In the KMT-Model we had $\alpha = \beta$. Since now there is capitalists' saving the new relationship is $(\alpha - 1) = (\beta - 1)s_c$ or from (6) $(\varrho - 1) = (\beta - 1)s_c$. Again it can be shown[1] that this is a special case of Sir Harrod's famous relationship $g = s/\kappa$.

1) Let Y denote national income, K capital and C total consumption. $\beta - 1$ is the return of capital, hence $\beta - 1 = \frac{E}{K}$. Since $E = E_c + E_w$, we have $s_c E = s_c E_c + s_c E_w$. From the Pasinetti formula

$$\frac{E_w}{s_w(W+E_w)} = \frac{E_c}{s_c E_c}$$ follows $s_c E_w = s_w(W+E_w)$, hence $s_c E = s_c E_c + s_w(W+E_w)$.

But this is total savings S, giving $s_c = \frac{S}{E}$. Hence $\varrho - 1 = \frac{E}{K}\frac{S}{E} = \frac{S}{K} = \frac{S}{Y} = s/\kappa$.

III. KMT-TYPE-MODELS INCLUDING EDUCATION

1. The Basic Model

In this section we will develop models including education, which can - by use of simple algebra - be reduced to the formalism of the KMT-Model, although, of course, it will differ in the interpretation. We are in the happy state, being able to use a method widely used in mathematics and other formal sciences: to solve a new problem simply by reduction to a solved one.

We can divide the economy of our model into two interdependent sectors: one sector being the realm of production of goods, corresponding to the models of a pure commodity economy; the second sector is the educational one. In both sectors we assume the technology to be linear and hence we will describe it by input matrices and output matrices. In this section we assume uneducated labour to be free available and that the wage for uneducated labour is included in the commodity input matrix, i.e., each uneducated worker gets for his wage commodities in fixed amounts. By wage we will mean throughout this chapter the money paid to an educated worker on top of his basket of goods.

Throughout this section we make use of the following assumptions: Linear technology in the production as well as in the educational sector. No consumption by the capitalists.. The wage of educated workers has either to be paid to capitalists for crediting the costs of education or is saved and used for giving credit, i.e., the worker is partly a capitalist in the second case. Workers getting educated do get the same wage they would get if they were working. This is rather an operational assumption, because we could dispose of it and subtract the wage (and the interest on it, if paid at the beginning of the production period) from the price of education. Uneducated labour is assumed to be free available and all people have the same abilities and faculties.[1)] It is assumed that it is possible to group educated labour such that every group containes a homogenous type of labour.

It is assumed that wages are paid at the beginning of each period, but it will be shown that this assumption has only a minor effect on the solution: only the relation between wages and prices of education changes by a discount factor.

We further assume that there are perfect credit possibilities, so that every worker gets a credit for his education costs.

On top of this we assume for our first model: People live for ever. There is neither education by work nor is there output of commodities (as by-products, or, more important, depreciated machines and buildings) in the educational sector.

Let us start by explaining the symbols used:

The production sector:

input matrix for commodities	A_1 of order	$(m_1 \times n_1)$
input matrix for educated labour	L_1	$(m_1 \times n_2)$
output matrix of commodities	B_1	$(m_1 \times n_1)$
activity vector for the production sector	x^1	$(1 \times m_1)$
price vector for commodities	y^1	$(n_1 \times 1)$

The educational sector:

input matrix for commodities	A_2	$(m_2 \times n_1)$
input matrix for educated labour (used for work)	L_2	$(m_2 \times n_2)$

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1) Equivalently we can assume that there is no scarcity of abilities and that everyone's abilities are known.

input matrix for educated labour (getting educated)	L_3	of order $(m_2 \times n_2)$

(Note that non educated workers **not** getting educated are included in A_2 indirectly)

output matrix of educated labour	B_3	$(m_2 \times n_2)$[1]
activity vector for the educational sector	x^2	$(1 \times m_2)$
price vector for education	y^2	$(n_2 \times 1)$

The wage vector is denoted by w, order $(n_2 \times 1)$

x^1 and x^2 are normalized such that

$$\sum_{i=1}^{m_1} x_i^1 + \sum_{i=1}^{m_2} x_i^2 = 1$$

Similar y^1 and y^2 are normalized such that

$$\sum_{j=1}^{n_1} y_j^1 + \sum_{j=1}^{n_2} y_j^2 = 1$$

The expansion and the interest factor are α and β respectively.

For a state of balanced growth, or an equilibrium as we will call it in this context, we demand that neither goods nor educated labour can be used in amounts exceeding the corresponding available amounts, i.e., the amounts taken over from the preceding period. Further we demand the absence of supernormal

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1) Note that the concept of activity analysis allows to include drop outs very easily, they appear in B_3 as outputs of the degree of education they have reached. The unrealistic point in using this concept for educational activity is the assumption of linearity, i.e. that a process can be used at any level without altering the proportions of the inputs and outputs. This implies, to give a drastic example, that if we need one classroom and one teacher for, say 35 pupils in elementary school, we can use a process, which has as inputs one half of a classroom and a half of a teacher and as outputs 17,5 pupils.

profits and the rule of free goods and profitability being valid. This leads to the following equalities and inequalities:

(1.1) $\alpha x^1 A_1 + \alpha x^2 A_2 \leqq x^1 B_1$ availability of goods

(1.2) $\alpha x^1 L_1 + \alpha x^2 (L_1 + L_2) \leqq x^2 B_2 + x^1 L_1 + x^2 L_2$

Since people are assumed to live for ever, available educated labour consists of educated labour used in the last period $(x^1 L_1 + x^2 L_2)$ and educated labour "produced" in the last period $x^2 B_2$.

The rule of free goods demands equations (1.3) and (1.4):

(1.3) $\alpha x^1 A_1 y^1 + \alpha x^2 A_2 y^1 = x^1 B_1 y^1$

(1.4) $\alpha x^1 L_1 w + \alpha x^2 (L_2 + L_3) w = x^2 B_2 w + x^1 L_1 w + x^2 L_2 w$[1)]

(1.5) $\beta A_1 y^1 + \beta L_1 w \geqq B_1 y^1$

(1.6) $\beta A_2 y^1 + \beta (L_2 + L_3) w \geqq p$

where p is a column vector for the (gross-)earnings of each process in the educational sector. We will soon be able to express it in terms of B_2, L_3 and y^2.

(1.7) $\beta x^1 A_1 y^1 + \beta x^1 L_1 w = y^1 B_1 y^1$

(1.8) $\beta x^2 A_2 y^1 + \beta x^2 (L_2 + L_3) w = x^2 p$

(1.7) and (1.8) are the formal expression of the rule of profitability.

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1) It would also be logically correct to take y^2 instead of w and this would make things even a little simpler. But to my opinion the more fundamental relationaship is that in w.

Two highly connected problems must be solved: what is the relation of w and y^2 and what is p?

In this section we do not have a consumption aspect of education. Since the productive power of education is somehow expressed by the wage of educated labor, we expect, the price of education to be dependent on the wage of educated labor. Certainly, the price education of, say, type j y_j^2 will be zero, if there is an excess supply of educated labor of type j, which, in turn, implies $w_j = 0$. In a world of perfect competition, where people are not distinguished by different abilities, costs of education must equal discounted differential life time incomes. This is known to everyone informed about the field of economics of education. Of course, this can only be true as long as we deny the consumption effect of education. Hence, we will get different results in the next section, where we do assume that at least some kinds of education do increase people's utility, ie., we take care for consumption aspects of education.

But now we only care for the production aspect of education. Hence, in order to equate supply and demand of educated labor, education prices must be equal to differential lifetime incomes, discounted at the prevailing interest rate. If the price were higher, no one would want to get educated, since it would result in a financial loss and vice versa in the opposite case, if the price were lower. Hence there would be excess demand or excess supply of educated labor respectively.

Hence, we have $\frac{\beta}{\beta-1} w = y^2$ for $\beta > 1$. If $\beta = 1$, the discounted life time income $\sum_{\beta=1}^{\infty} \beta w$ does not converge and we are in trouble. But there is a way out.

If $\beta w = (\beta-1)y^2$ holds, hence if $\beta > 1$, we will show that $\alpha = \beta$ holds in a state of balanced growth. Hence, to each equilibrium $\alpha > 1$ there corresponds an equilibrium $\beta > 1$, which shows that if $\beta \leqq 1$ then $\alpha \leqq 1$. But if $\alpha \leqq 1$, we don't need any education, since people live for ever by assumption. x^2 becomes zero, since w_j and hence y_j would be zero, if x_j^2 is positive. Hence we have

$$(1.9) \quad \beta w = \max \left[(\beta-1) y^2 ,\ 0 \right]$$

Another interpretation of equation (1.9) is also possible. What are the costs of education per period? Obviously they are the interest charges on the price y^2 and since the credit for y^2 has to be taken at the end of the education period, which is the beginning of the first period with the higher wage for the worker the costs $(\beta-1)y^2$ accrue at the end of each period, starting with the first period after education. Since wages are paid at the beginning of each period, the financial advantage of education for the worker is his wage-value at the end of the period, hence it is βw. In equilibrium financial loss $(\beta-1)y^2$ must equal to financial win βw.

Now we easily recognize, what would happen if wages were paid at the end of each period. We would have w instead of βw in equation (1.9). But also we would have w instead of βw in (1.5) - (1.8). Since for every $\beta > 0$ we can multiply (1.4) by β, we will get the same equalities and inequalities, if we substitute $(\beta-1)y^2$ for βw or for w respectively. Only y^2 expressed in terms of wages changes.

So far we have only treated education of uneducated labor. If we denote discounted life time income of an eduacted worker of type j by $E(z_j)$, we know that $E(z_j) = y_j^2$. What is the price of educating an educated worker, say from type i to type j? Denote this price by y_{ij}^2. By exactly the same arguments used for equation (1.9) we must equate differential costs to differential income. Hence,

$$y_{ij}^2 = E(z_j) - E(z_i) = y_j^2 - y_i^2 \quad ^{1)}$$

We can interprete this result in the way that the educating institution gets y_j^2 for education up to type j, but has to pay y_i^2 , if the worker was educated up to type i already. It is the same, as if the educating institution paid the value of education, the worker has already received, and than gets paid for the whole educational level the worker has reached. Obviously this makes round about ways in education improfitable.

From $y_{ij}^2 = y_j^2 - y_i^2$ follows

(1.10) $p = B_2 y^2 - L_3 y^2$

The rest is easy, we have now done most of the work. All we need to do, is to substitute (1.10) in (1.6) and (1.8) and $(\beta-1)\bar{y}^2 = \beta w$. For this substitution in (1.4) we must assume $\beta > 1$, but since as we have shown (1.4) and the whole educational sector is not relevant if $\beta = 1$, this does not cause any troubles.

From a purely algebraic point of view we could take out y^2 of the inequalities and equalities as well. But in that case we would not arrive at the KMT-Model. The corresponding two person game, a zero-sum-game in the KMT-case, would be a two person non zero sum game, and Thompson's proof for the existence of an equilibrium would not be applicable.

The simple algebraic give:

(1.1*) $\alpha\left[x^1 A_1 + x^2 A_2\right] \leqq x^1 B_1$

(1.2*) $\alpha\left[x^1 L_1 + x^2 (L_2+L_3)\right] = x^1 L_1 + x^2 (B_2+L_2)$

(1.3*) $\alpha\left[x^1 A_1 y^1 + x^2 A_2 y^1\right] = x^1 B_1 y^1$

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1) Negative values do not matter, because we assume that there are no processes producing something by taking out education, i.e,, by destroying it, of workers.

$$(1.4^*)\quad \alpha\left[x^1L_1y^2 + x^2(L_2+L_3)y^2\right] = x^1L_1y^2 + x^2(B_2+L_2)y^2$$

$$(1.5^*)\quad \beta\left[A_1y^1 + L_1y^2\right] \geqq B_1y^1 + L_1y^2$$

$$(1.6^*)\quad \beta\left[A_2y^2 + (L_2+L_3)y^2\right] = B_2y^2 + L_2y^2$$

$$(1.7^*)\quad \beta\left[x^1A_1y^1 + x^1L_1y^2\right] = x^1B_1y^1 + x^1L_1y^2$$

$$(1.8^*)\quad \beta\left[x^2A_2y^2 + x^2(L_2+L_3)y^2\right] = x^2B_2y^2 + x^2L_2y^2$$

Define now the matrices A, B and vectors x,y:

$$A = \begin{bmatrix} A_1 & L_1 \\ A_2 & L_2+L_3 \end{bmatrix} \qquad B = \begin{bmatrix} B_1 & L_1 \\ 0 & B_2+L_2 \end{bmatrix}$$

$$x = (x^1 x^2) \qquad y = \begin{bmatrix} y^1 \\ y^2 \end{bmatrix}$$

We have $\sum_i x = 1$ and $\sum_j y = 1$

We assume (i) and (ii) similar section II for A and B. This means (i) every process uses some input (commodities or educated labor). Since each worker gets a commodity basket every row of A_1 has at least one positive entry, if we assume that every process uses some work. (ii) every good can be produced and any kind of educated labor is attainable. [1]) i.e.every row of B_2 has at least one positive entry.

Hence we demand, that every kind of education standard can be reached, i.e., every column of B_2 has at least one positive entry. For the KMT solution we would need this property for B only, but then, if L_1 contains unattainable educational standards, this may force the system to have an α as low as 1. This comes from the fact, that L_1 and L_2 occuring as outputs in B are inputs as well, hence if we want to increase this output we must increase first the inputs, which would be impossible, if we did not have other processes producing these outputs without using the same amount of inputs. This makes it clear, that, in fact, the v.Neumann path could never be reached, if there are unattainable educational types used on this path. Hence although there could be balanced "growth" at α at most 1, the initial state for such a growth could never be reached.

Finally let us demand that the economy contains something of positive value at the end of the period, i.e.,

(1.9*) $x^1B_1y^1 + x^1L_1y^2 + x^2B_2y^2 + x^2L_2y^2 > 0$

Making use of A, B, x and y we get:

(I.1) $\alpha xA \leqq xB$

(I.2) $\alpha xAy = xBy$

(I.3) $\beta Ay \geqq By$

(I.4) $\beta xAy = xBy$

(I.5) $xBy > 0$

Hence we have reached our aim: the reduction of our model to a model, which is formally a KMT-Model. Arrived at the island, we have been looking for, we do not need to bother about proofs, we can build upon earlier work. There exist α,β,x,y fulfilling (I.1)-(I.5). Also we have $\alpha = \beta > o$. By equation (1.9) we get the wage vector if $\beta > 1$. In the case of $\beta \leqq 1$ we get $x^2 = 0$, since the educational sector is not used. Hence, if we make $w = y^2 = o$, we get

(I.1') $\alpha x^1A_1 \leqq x^1B_1$

(I.2') $\alpha x^1A_1y^1 = x^1B_1y^1$

(I.3') $\beta A_1y^1 \geqq B_1y^1$

(I.4') $\beta x^1A_1y^1 = x^1B_1y^1$

(I.5') $x^1B_1y^1 > o$

The system (I.1') - (I.5') is a KMT-Model because it fulfills (i) and (ii). Hence, since (I.1') - (I.5') is a possible state of (I-1) - (I.5), there exists a state of balance growth in all cases.

Before developing a little more sophisticated models, let us have once more a look at our present one.

A and B are "general" input or output matrices respectively, they include both sectors. We notice, that educated labor, although used in both sectors, is not used up. People live for ever and work for ever, in fact, once we have a certain kind of worker, we have a certain amount of work of a certain kind, and since technology and hence economic demands on education do not change, this amount of certain work will always have a certain effect in the economy. Work is given, once we have the worker, and hence work does not appear as a production factor in our model. Instead of it educated labor appears as catalyst: it is an input, needed at the beginning of the period, and at the end of the period the same manpower turns up as an output. Another point of interest is that of the relations between the two sectors, the goods producing one and the educational one. Let us assume, that α and β are greater one, since we already know, that in the case of $\alpha \leqq 1$ we have the trivial case. Since for every worker the wage is the discounted interest on the price of his education, this must hold for the economy as a whole as well: the sum of all wages is the discounted interest on the stock of human capital (if we narrow this this term to education for simplicity). Or in another formulation: The wage sum plus the interest on it, i.e. β times the wage sum equals the interest on human capital. Obviously the factor β comes from the assumed time of wage payment. If wages were paid at the end of each period, the wage sum would equal the interest on human capital. But not only this stock - flow relation can be derived. The total wage sum, denote it by W, is $W = x^1 L_1 w + x^2(L_2 + L_3)\ w$. If we add equations (1.7) and (1.8) after substituting $B_2 y^2 - L_3 y^2$ for p (from (1.10), we get: $\beta W = x^2 B_2 y^2 - x^2 L_3 y^2 + x^1 B_1 y^1 - \beta(x^1 A_1 y^1 + x^2 A_2 y^1)$

But since $\alpha = \beta$, this reduces to $\beta W = x^2 B_2 y^2 - x^2 L_3 y^2$ because of (1.3) .

Hence the value of the wage sum at the end of each period (wage sum plus interest on it) equals the value produced in the educational sector. (For simplicity, we have no commodity output matrix in the educational sector. This means, there is no capital transferred, all capital is used up. But there is no difficulty at all to include such an output matrix, we only have to replace the zero matrix in the lower left corner of B by a commodity output matrix, say B_3. This will be done in the following models). Hence the wage sum, or more correctly its value at the end of a period, corresponds to stock and to flow values of education. The connection to the stock value, the interest on human capital, can be interpreted as an equality of investment in education and savings out of wages.[1)]

The equality of the flows enables the balanced growth, both sectors have one common expansion factor α. This is the reason, that, as follows from (1.3) substituting β for α, the value of the commodity inputs at the end of the period (i.e., value of inputs times β) equals the value of the output of the goods producing sector.

From (1.4*) follows the expected equation

(1.4') $$(\beta-1)\left[x^1 L_1 y^2 + x^2 (L_2 + L_3)\, y^2\right] = x^2 B_2 y^2 - x^2 L_3 y^2$$

where the second term on the left hand side of the equations is the stock of human capital and $(\beta-1)$ interest rate. Hence, as the two discussed equalities concerning the wage sum times β imply, the value of the output of the educational sector (net of value of transferred work $x^2 L_3 y^2$) equals the rate of interest on human capital.

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1) Since the differential wages cannot be spent by the worker but have to be paid back for received education, they are savings.

Finally a Marxian interpretation of equation (1.9) may be given. From (1.9) we have $w_j = \frac{B-1}{B} y_j^2 \, V_j$. $\frac{B-1}{B} y_j^2$ can be interpretated as "costs of reproduction" of work of kind j falling on one periods share. Hence wages equal the cost of reproduction of labor, a wellknown proposition of Marxian economics. This is not at all surprising, since the world of our model is a producing system, nothing more, and hence the value of education must be exactly its value for production, which is, in turn, the wage.

We know from section II. that the solution of our model is not necessarily unique. Kemeny, Morgenstern and Thompson have investigated some properties of the model in the case of non-unique solutions. Their results, modified according to our interpretation of the A and B matrices, are fully applicable to our generalized KMT model. These relations are e.g.those proved in section 6 of the KMT paper .The existence of sub-economies [1] and the properties of the maximum respectively the minimum α, being the technically maximum possible expansion factor and the minimum interest factor for a profitless system respectively.

One point more may be of interest. What happens, if the "learning workers", represented by x^2L_3 do not get any wages or wages different from the wage of "working workers"? Obviously this is no contradiction to our theoretical model of competition (like e.g. different wages for working workers of the same kind or different prices for the same good

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1) If there are two equilibria with different αs, then commodities produced in the equilibrium with the larger α are free in the one with the smaller α. Processes used in positive amounts at the smaller α are not used in the equilibrium with the larger α.

would be) as long as discounted life time incomes do not differ. (Since consumption is fixed the time preference rate must equal the interest rate). The answer to our question is that it does not change anything. Losts of education includes price and incomes foregone. If the educating institution pays no or smaller (or larger) wages for the students, we can add the wage or the wage difference respectively on both sides of the profit inequality and profit equation. This will give the " augmented price", which equals costs for the individual. For this " augmented price" all equalities and inequalities of our model hold.

2. Further Generalizations

People do not live for ever and they do not work all their life time. Workers learn by working and the educational sector has a positive commodity output, at least the output of real input capital, depreciated for one period. These facts we will now put into the model developed. In order to avoid confusion, we will do this in two steps, in the first taking account of death and retiring as well as of the commodity output of the educational sector. The second step will be to include learning by doing into the model.

We assume, that during a period there is no change in the labor force for each type j of educated labor, there is a number $\gamma_j \leqq 1$, which gives the percentage staying in work.[1] Hence $1 - \gamma_j$ is percentage of labor of kind j, which is not in the labor force at the beginning of the next period.

Define the (diagonal) matrix G by

$$G = \begin{bmatrix} \gamma_1 & 0 & \dots & 0 & 0 \\ 0 & \gamma_2 & \dots & 0 & 0 \\ \hline 0 & 0 & \dots & 0 & \gamma_{n_2} \end{bmatrix}$$

(γ_j can be interpreted as a transition probability in a Markoff process.)

By the matrix B_3 we denote the commodity output matrix of the educational sector.

Notation and normalization concerning the symbols already used, have not changed. We still assume (i) and (ii) and demand, that something of positive value is produced.

The main problem, caused by the fact, that G is not equal to the identity matrix, is that of establishing the relationship between w and y. In deriving this relationship we will find,

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1) Note that this implies, that there is no variance although we will use probabilities.

that G depends on α, which makes it impossible to reach firm ground in form of the KMT model, since our matrix corresponding to B of the KMT model will depend on α. But fortunately we will recognize, that we can use Thompsons proof, if B is a continuous function of α.

Let us concentrate on labor of type j. Each worker has an age of a certain amount of periods. Denote the amount of the youngest ones by N_1, the amount of those one period older by N_2 and so on up to an upper limit, say N_r. The amount of all workers of type j denote by N. Obviously $\sum_{v=1}^{r} N_v = N$.

By p_i we denote the probability, that a worker out of N_v still works in the next period. We have $p_r = 0$. $1-p_r$ is the probability of dropping out of the labor force for a worker of group N_v. From those definitions follows $\gamma_j N = \sum_{v=1}^{r} p_v N_v$, hence

$$(2.1)\quad \gamma_j = \frac{1}{N} \sum_{v=1}^{r} p_v N_v$$

For the p_v's we make the assumption, that (iii) for all kinds of labor the p_v-s are non increasing,

$$\text{i.e.,} \quad p_{v+1} \leq p_v \quad \text{for all } v.$$

Further we have

$$E\left[z_j\right] = w_j + \frac{p_1}{\beta} w_j + \frac{p_1 p_2}{\beta} w_j + \dots\dots\dots + \frac{p_1 p_2 \cdots p_{r-1}}{\beta} w_j ,$$

hence if we define $p_o = 1$

$$(2.2)\quad E\left[z_j\right] = w_j \sum_{v=1}^{r} \frac{1}{\beta^{v-1}} \prod_{i=o}^{v-1} p_i$$

If the economy grows at a common rate of $(\alpha - 1)$

$$(2.3)\quad N_{v+1} = \frac{p_v}{\alpha} N_v \quad \text{for all } v = 1, \dots , r-1$$

$$\text{Hence} \quad \sum_{v=1}^{r} p_v N_v = \alpha \sum_{v=1}^{r} N_{v+1}$$

Since $N_{r+1} = 0$ $\sum_{v=1}^{r} N_{v+1} = N - N_1$, which gives together with (2.1) $\gamma_j = \alpha \frac{N-N_1}{N}$ or

(2.4) $(\alpha - \gamma_j) \; N = \alpha N_1$ [1)]

From (2.3) and $N = \sum_{v=1}^{r} N_v$ follows

(2.5) $N = N_1 \sum_{v=1}^{r} \frac{1}{\alpha^{v-1}} \prod_{i=0}^{v-1} p_i$

We demand like in our previous model

(2.6) $y_j^2 = E\left[z_j \right]$

We now proceed in the following way:
we demand that $\alpha = \beta$ and derive the relationship between w and y^2. This is of course, an unjustified condition for an equilibrium, but as we will find out, the condition is superfluous, since $\alpha = \beta$ will be an implication of the other equilibrium conditions. Since we do not aim to prove uniqueness of the equilibrium, but only its existence this is a feasible way of proceeding. Hence we demand

(2.7) $\alpha = \beta$

From (2.2), (2.5), (2.6) and (2.7) we get

$N = \frac{N_1}{w_j} \; y_j^{\,2}$, multiplied by $(\beta - \gamma_j)$ this gives $(\beta - \gamma_j) \; N =$
$= (\beta - \gamma_j) \; N_1 \frac{y_j^2}{w_j}$, which, in turn, because of (2.4) and

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1) An alternative proof would be:
$(\alpha - 1) = \frac{\alpha N_1 - (1 - \gamma_j)N}{N}$ the growth rate of the economy equals the growth rate of the labor of type j available. That implies (2.4). The proof is simpler bur less illustrative.

(2.7) implies $\beta N_1 = (\beta - \gamma_j) N_1 \frac{y_j^2}{w_j}$, hence (2.8) $\beta W_j =$

$= (\beta - \gamma_j) y_j^2$ for $\beta > \gamma_j$ and $w_j = y_j^2 = 0$ for $\beta \leq \gamma_j$

This must be true for all $j = 1, \ldots, m_1$, hence we can write

(2.8') $\beta w = (\beta I - G) y_j^2$ if $\beta > \gamma_j$ for all j

If $G = I$ we have (1.9)

In order to keep w non negative we demand

(2.9) $w_j = 0$ if $(\beta - \gamma_j) \leq 0$

Our problem is not solved completely yet, because we allow for educating educated labor. Hence we must find a relationship for y_{ij}^2, the price of education from step i to step j. We demand y_{ij}^2 to be equal to the discounted expected differential income, hence $y_{ij}^2 = E\left[z_j\right] - \beta\, (E\left[z_i\right] - w_i)$, where the second term is the expected income of labor of type i after one period. The rest is easy. Because of (2.6) and (2.8) we have $y_{ij}^2 = y_i^2 - \beta y_i^2 + \beta w_i = y_j^2 - \beta y_i^2 + (\beta - \gamma_i)\, y_i^2 =$

$= y_j^2 - \gamma_i y_i^2$, hence (2.10) $y_{ij}^2 = y_j^2 - \gamma_i y_i^2$

We are now in a position to formulate the model. The equalities and inequalities are the formal expressions of our familiar equilibrium conditions. As we have already discussed, we need to add the pseudo-condition $\alpha = \beta$.

(2.11) $\alpha x^1 A_1 + \alpha x^2 A_2 \leq x^1 B_1 + x^2 B_3$

(2.12) $\alpha x^1 L_1 + \alpha x^2 (L_2 + L_3) \leq x^2 B_2 + x^1 L_1 G + x^2 L_2 G$

(2.13) $\alpha x^1 A_1 y^1 + \alpha x^2 A_2 y^1 = x^1 B_1 y^1 + x^2 B_3 y^1$

(2.14) $\alpha x^1 L_1 w + \alpha x^2 (L_2 + L_3)\, w = x^2 B_2 w + x^1 L_1 G w + x^2 L_2 G w$

(2.15) $\beta A_1 y^1 + \beta L_1 w \geq B_1 y^1$

[1)]

$$(2.16)\quad \beta A_2 y^1 + \beta (L_2+L_3)w \geqq p + B_3 y^1$$

$$(2.17)\quad \beta x^1 A_1 y^1 + \beta x^1 L_1 w = x^1 B_1 y^1$$

$$(2.18)\quad \beta x^2 A_2 y^1 + \beta x^2 (L_2+L_3) w = x^2 p + x^2 B_3 y^1$$

$$(2.19)\quad \beta w = (\beta I - G)\, y^2 \text{ and } w_j = o \text{ if } (\beta - \gamma_j) \leqq o$$

$$(2.20)\quad \alpha = \beta$$

On top of this we demand for an equilibrium, that something of positive value is produced: $x^1 B_1 y^1 + x^2 B_3 y^2 + x^1 L_1 G y^2 + x^2 (B_2 + L_2 G) y^2 > o$.

The next problem to be solved is the evaluation of p in (2.16). If all workers getting educated stay in the labor force because of (2.10) p must equal $B_2 y^2 - L_3 G_y{}^2$ [2). If people die after

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1) B_3 is the commodity output matrix for the educational sector. If no goods are produced in that sector, i.e. no "by-products", then B_3 contents just the depreciated input goods. Hence in the case of no by-products $x^2 A_2 y^1 - x^2 B_3 y^1$ is the total depreciation of capital in the educational sector.

2) Let n_{ij}^h be the amount of workers getting educated from typ i to typ j in the h^{th} process and let p_h be the earnings for education of that process. Then $p_h = \sum_{ij} n_{ij}^h y_{ij}^2$ $i = 1 \dots n_2$ and $j = 1, \dots n_2$, where n_2 is the number of education types.
$p_h = \sum_{ij} n_{ij}^h y_{ij}^2 = \sum_j n_{ij}^h (y_j^2 - \gamma_i y_i^2) = \sum_j (y_j^2 \sum_i n_{ij}^h) - \sum_i (\gamma_i y_i^2 \sum_j n_{ij}^h)$ But $\sum_j n_{ij}^h = l_{hj}^3$ of L_3 and $\sum_i n_{ij}^h = b_{hj}^2$ of B_2, since $\sum n_{ij}^h$ is the amount of workers of degree i, which get educated to any other level in proecess h and similar $\sum_i n_{ij}^h$ is the amount of workers educated up to level j from any former degree. Hence $p_h = \sum_j y_j^2 b_{hj}^2 - \sum_i y_i^2 \gamma_i l_{hi}^3$

for all h, which gives (2.21), $p = (B_2 - L_3 G) y^2$.

their education, we assume they die always in fixed proportion (represented by ΔB_2) and hence we can include their death in the technology: they do not occur in B_2 although they appear in L_3. In this case we must assume, that the educating institution has costs $\Delta B_2 . y^2$, which means that the dead workers are an input of the process of education. On the other hand our assumption is based in the other assumption, that the dead workers also had to pay their education price. Hence effective "costs" are $\Delta B_2 y^2 - (\Delta B_2 y^2 - \Delta L_3 G y^2) = \Delta L_3 G y^2$ It is possible to developed a model, where these rather strange "costs" are excluded by paying wages to workers getting educated different to the correspondings ones of working workers, but I doubt. that it is worth while to spend room on it in this paper. It seems to be rather a minor point ,especially since we have already discussed, that different wages for workers getting educated relative to "working workers" does not imply a completely different model.

$L_3 G y^2$, call it "taking over costs" , is the amount the educational institutions have to pay (to the worker or his creditors respectively) if they claim $B_2 y^2$ as payments for education. Hence if we assume, people in education do not leave the labor force at the end of that period (or that the educational institution has to pay the "taking over costs" for such people as well), we can state:

(2.21) $p = B_2 y^2 - L_3 G y^2$, which gives

(2.16') $\beta A_2 y^1 + \beta (L_2 + L_3) w \geq B_2 y^2 - L_3 G y^2 + B_3 y^1$ and

(2.18) $\beta x^2 A_2 y^1 + \beta x^2 (L_2 + L_3) w = x^2 B_2 y^2 - x^2 L_3 G y^2 + x^2 B_3 y^1$

Before doing the algebraic, we show, that G is a continouos fuction of α. This is the case if and only if all γ_j are continuous functions of α.

From (2.1) and (2.3) we had $\gamma_j = \alpha \frac{N - N_1}{N} = \alpha - \alpha \frac{N_1}{N}$

which together with (2.5) gives

$$\gamma_j = \alpha \left[1 - \frac{N_1}{N_1 \sum_{v=1}^{r} \frac{1}{\alpha^{v-1}} \prod_{i=o}^{v-1} p_i} \right], \text{ hence}$$

$$(2.22) \quad \gamma_j = \left[1 - \frac{1}{\sum_{v=1}^{r} \frac{1}{\alpha^{v-1}} \prod_{i=o}^{v-1} p_i} \right] \text{ for } j = 1, \ldots, n_2$$

Hence G is a continous function of α, since for $v=1 \frac{1}{\alpha^{v-1}} p_o = 1$

In the case $p_i = 0$, completely new education is necessary in every period. In that case - as can be shown - the KMT model can very easily be derived, but there is no relevance in it.

We are now ready, to convert our model into one similar to a KMT model. The only difference is continous dependence of B, the input matrix on α.

From (2.8) to (2.21) follows:

(2.11*) $\alpha x^1 A_1 + \alpha x^2 A_2 \leqq x^1 B_1 + x^2 B_3$

(2.12*) $\alpha x^1 L_1 + \alpha x^2 (L_2+L_3) \leqq x^2 B_2 + x^1 L_1 G + x^2 L_2 G$

(2.13*) $\alpha x^1 A_1 y^1 + \alpha x^2 A_2 y^1 = x^1 B_1 y^1 + x^2 B_3 y^1$

(2.14*) $\alpha x^1 L_1 y^2 + \alpha x^2 (L_2+L_3) y^2 = x^2 B_2 y^2 + x^1 L_1 G y^2 + x^2 L_2 G y^2$

(2.15*) $\beta A_1 y^1 + \beta L_1 y^2 \geqq B_1 y^1 + L_1 G y^2$

(2.16*) $\beta A_2 y^1 + \beta (L_2 + L_3) y^2 \geqq B_2 y^2 + L_2 G y^2 + B_3 y^1$

(2.17*) $\beta x^1 A_1 y^1 + \beta x^1 L_1 y^2 = x^1 B_1 y^1 + x^1 L_1 G y^2$

(2.18*) $\beta x^2 A_2 y^1 + \beta x^2 (L_2+L_3) y^2 = x^2 B_2 y^2 + x^2 L_2 G y^2 + x^2 B_3 y^1$

(2.19*) $\left[x^1 B_1 + x^2 B_3 \right] y^1 + \left[x^1 L_1 G + x^2 (B_2 + L_2 G) \right] y^2 > o$

(2.20) $\alpha = \beta$

Inequality (2.19*) demands, that something of positive value is produced.

We now define the matrices A and B (α) - the α denoting the dependence on α - and the vectors x and y :

$$A = \begin{bmatrix} A_1 & L_1 \\ A_2 & L_2 + L_3 \end{bmatrix}$$

$$B(\alpha) = \begin{bmatrix} B_1 & L_1 G \\ B_3 & B_2 + L_2 G \end{bmatrix}$$

$$x = (x^1 \; x^2)$$

$$y = \begin{pmatrix} y^1 \\ y^2 \end{pmatrix}$$

B depends on α, since G depends on α. We have already proved, that this is a continuos dependence, hence B depends continuosly on α. G does not occur in A, hence A is independent of α. The equilibrium conditions can now be written in the form:

(II.1) $\quad \alpha xA \leqq xB(\alpha)$

(II.2) $\quad \alpha xAy = xB(\alpha)y$

(II.3) $\quad \beta Ay \geqq B(\alpha)y$

(II.4) $\quad \beta xAy = xB(\alpha)y$

(II.5) $\quad xB(\alpha)y > o$

(2.20) $\quad \alpha = \beta$

From (II.5) follows $xAy > o$, $\alpha > o$ and $\beta > o$. (II.2) and (II.4) imply $\alpha xAy = \beta xAy$, hence $\alpha = \beta$. Condition (2.20) is superfluos. A state of balanced growth is fully described by (II.1) to (II.5)

Next we want to show, that there exists a state fulfilling (II.1) to (II.5). Our proof follows exactly the route of the proof for the KMT-model.

We will proceed as follows: First we define $M_\alpha = B(\alpha) - \alpha A$ and show the equivalence of (II.1) to (II.4) to a zero-sum-game with the payoff matrix M_α and α such that the value of the game $v(M_\alpha)$ equals zero. Next we show that $v(M_\alpha)$ depends continuosly on α and that for some $\bar{\alpha}$ $(M_\alpha) > o$ and for some $\underline{\alpha}$ $v(M_\alpha) < o$ for $\bar{\alpha}$ and $\underline{\alpha}$ both positive. Hence there must be an α^*, such that $v(M_\alpha) = 0$ and the existence of a solution for (II.1) to (II.5) is assured, if we show, that for at least one α^* $xB(\alpha)y > o$ holds.

Define $M_\alpha = B(\alpha) - \alpha A$. We rewrite (II.1) to(II.4)

(II.1')	$0 \leqq x M_\alpha y$
(II.2')	$0 = x M_\alpha y$
(II.3')	$0 \geqq M_\alpha y$
(II.4')	$0 = x M_\alpha y$

Obviously this corresponds to a zero-sum-game, where M_α is the payoff matrix and x and y are the probability vectors for the strategies of the maximizing (controling the rows) and the minimizing (controling the columns) player respectively. (Remember that x and y are normalized in such a way that $\sum_i x_i = 1$ and $\sum_j y_i = 0$, and, of course, activities and prices must be non negative, hence $x_i \geqq o \; \forall i$ and $y_j \geqq o \; \forall j$)

From game theory we know, that for every M_α, there exist x^* and y^*, such that $xM_\alpha y^* \leqq x^* M_\alpha y^* \leqq x^* M_\alpha y$ holds and $x^* M_\alpha y^* = v(M_\alpha)$ is called the value of the game M_α and x^*, y^* are optimal strategies. The optimal strategies may not be unique, but in the case of a two person zero-sum-game, the value of the game $v(M_\alpha)$ is unique.

It has been proved, that the value of a matrix game depends contionuosly on the elements of the payoff matrix. Since the elements of M_α depend continuosly on α, we know that:

The value $v(M_\alpha)$ of the matrix game depends continuosly an α.

Because of (i) and (ii) we have:
for every i there exists a j such that $a_{ij} > o$ and for every j there exists an i such that $b_{ij} > o$, where $A = (a_{ij})$ and $B(\alpha) = (b_{ij})$. It can be shown that this implies $v(-A) < 0$ and $v(B(\alpha)) > o$.

For simplicity we denote $v(M_\alpha)$ by $v(\alpha)$. If $\alpha = 0$, $M_\alpha = B(\alpha)$, hence $v(o) > o$. We have now to prove, that for some $\underline{\alpha}$, large enough, $v(\underline{\alpha}) < 0$ holds. In a matrix game, the value v cannot decrease, if we increase some of the elements of the payoff matrix and do not decrease any of the elements. Hence if $\bar{M} \geqq M$, we have $v(\bar{M}) \geqq v(M)$. We know that $G(\alpha) \leqq I$, I being the identity matrix, hence if we define

$$\bar{B} = \begin{bmatrix} B_1 & L_1 \\ B_3 & B_2+L_2 \end{bmatrix}$$

we have $\bar{B} \geqq B(\alpha)$ and $\bar{B}$ is independent of α and fulfills (ii). Define $\bar{M}_\alpha = \bar{B} - \alpha A$. It has been proved, that for any A and $\bar{B}$ fulfilling (i) and (ii) and being independent of α, there exists an $\underline{\alpha}$, large enough such that $v(\bar{M}_{\underline{\alpha}}) < 0$. Since $\bar{B} \geqq B(\alpha)$, $\bar{M}_\alpha \geqq M_\alpha$, hence $v(\bar{M}_{\underline{\alpha}}) \geqq v(M_\alpha)$. This gives $0 > v(\bar{M}_{\underline{\alpha}}) \geqq v(M_{\underline{\alpha}})$, hence we have proved, that $v(\underline{\alpha}) < 0$ for a large enough positive α.

Let us summarize: $v(\bar{\alpha}) > 0$ for $\bar{\alpha} = 0$ and $v(\underline{\alpha}) < 0$ for a certain $\underline{\alpha} > 0$. Since $v(\alpha)$ is a continuos function of α, there must exist at least one α^*, such that $v(\alpha^*) = 0$.

We still have to prove, that there exists an α^*, giving $v(\alpha^*) = 0$ <u>and</u> $x\,B(\alpha)y > o$, x and y being optimal strategies (they may not be unique) for the matrix game M_{α^*}.

First, we have to prove, that $v(M_\alpha) = v(\alpha)$ is a monotonically decreasing function of α. Let $\hat{\alpha} > \alpha$. Then $M_{\hat{\alpha}} - M_\alpha = B(\hat{\alpha}) - B(\alpha) - (\hat{\alpha} - \alpha)A$

$$B(\hat{\alpha}) - B(\alpha) = \begin{bmatrix} 0 & L_1(G(\hat{\alpha}) - G(\alpha)) \\ 0 & L_2(G(\hat{\alpha}) - G(\alpha)) \end{bmatrix}$$

$$M_{\hat{\alpha}} - M_{\alpha} = \begin{bmatrix} -(\hat{\alpha} - \alpha) A_1 & L_1\left[G(\hat{\alpha}) - G(\alpha) - (\hat{\alpha} - \alpha) I\right] \\ -(\hat{\alpha} - \alpha) A_2 & -(\hat{\alpha} - \alpha) L_3 + L_2\left[G(\hat{\alpha}) - G(\alpha) - (\hat{\alpha} - \alpha) I\right] \end{bmatrix}$$

Obviously $v(\alpha)$ is monotonically decreasing, if

$G(\hat{\alpha}) - G(\alpha) < (\hat{\alpha} - \alpha) I$, which is equivalent to

$\gamma_j(\hat{\alpha}) - \gamma_j(\alpha) < \hat{\alpha} - \alpha$ for all j.

From (2.22) we know $\gamma_j = \alpha - \dfrac{\alpha}{\sum_{v=1}^{r} \frac{1}{\alpha^{v-1}} \prod_{i=o}^{v-1} p_i}$

Hence $\gamma_j(\hat{\alpha}) - \gamma_j(\alpha) = \hat{\alpha} - \alpha - \left[\dfrac{\hat{\alpha}}{\sum_{v=1}^{r} \frac{1}{\hat{\alpha}^{v-1}} \prod_{i=o}^{v-1} p_i} - \dfrac{\alpha}{\sum_{v=1}^{r} \frac{1}{\alpha^{v-1}} \prod_{i=o}^{v-1} p_i} \right]$

From $\hat{\alpha} > \alpha$ follows $\frac{1}{\hat{\alpha}^{v-1}} \leqq \frac{1}{\alpha^{v-1}}$ for $v \geqq 1$ with the strict inequality for $v > 1$. Hence the first term inside the brackets has a greater numerator than the second term and a denominator at most as high as the second term in the brackets. Hence the expression inside the brackets must be strictly positive, hence $\gamma_j(\hat{\alpha}) - \gamma_j(\alpha) < \hat{\alpha} - \alpha \ \forall j$, hence $G(\hat{\alpha}) - G(\alpha) - (\hat{\alpha} - \alpha) I < o$, hence $M_{\hat{\alpha}} \leqq M_{\alpha}$ hence $v(\hat{\alpha}) \leqq v(\alpha)$.

Again we will use the proof of the KMT model and we will only pick out the point, where the dependence of $B(\alpha)$ on α causes differences. From the monotony and $v(\underline{\alpha}) < 0$ and $v(\bar{\alpha}) > 0$ follows, that the α^*s with $v(\alpha^*) = 0$ form a closed intervall, having positive finite and real boundaries.

Hence $\underline{\alpha}^* \leqq \alpha^* = \bar{\alpha}^*$, where $\underline{\alpha}^*$ and $\bar{\alpha}^*$ are the lower and upper boundary respectively. For a fixed $\bar{\alpha}^*$ $B(\bar{\alpha}^*)$ is fixed. Hence we can use the KMT result, that if $v(\bar{\alpha}^*) = 0$ and $xB(\bar{\alpha}^*)y = o$ for all saddle points (x,y) of $M_{\bar{\alpha}^*}$, then for all j

$$\sum_i \hat{x}_i \left(b_{ij}(\bar{\alpha}^*) - (\bar{\alpha}^* + \delta)\, a_{ij}\right) \geqq 0$$ would hold for a certain $\hat{x}$,

where δ is a positive number: $\delta > o$.

Let us now have a look on $B(\bar{\alpha}^* + \delta)$. From (2.1) we have

$\gamma_i = \sum_{v=1}^{r} p_v \frac{Nv}{N}$, where $N = \sum_v N_v$. We know from (2.3), that

$\frac{Nv+1}{N_v} = \frac{pv}{\alpha}$, hence $\frac{Nv+1}{N_v}$ is decreasing if α increases. γ_j is

a weighted sum of the p_v-s, if α increases, the weights of the p_vs having a low v increase relative to those having higher v-s, because $\frac{N_v}{N_{v+1}}$ increases with α. Since by assumption (iii) the

p_v-s are non increasing with the v-s, i.e., $p_{v+1} \leqq p_v$, we have the result, that γ_j is a non decreasing function of α, because the higher weights are connected with at least as high numbers. Since all γ_j are nondecreasing functions of α, G and hence B is nondecreasing, if α increases.

Hence for $\delta > o$ we have:

$$b_{ij}(\bar{\alpha}^* + \delta) \geqq b_{ij}(\bar{\alpha}^*),$$

$$\sum_i \hat{x}_i \left[b_{ij}(\bar{\alpha}^* + \delta) - (\bar{\alpha}^* + \delta)\, a_{ij} \right] \geqq o$$ would hold for all j

or $\qquad \hat{x} M_{\bar{\alpha}^* + \delta} \geqq o.$

Let $(\tilde{x},\tilde{y})$ be a saddle point of $M_{\bar{\alpha}^* + \delta}$, then

$$\hat{x} M_{\bar{\alpha}^* + \delta}\, \tilde{y} \leqq \tilde{x} M_{\bar{\alpha}^* + \delta}\, \tilde{y} = v(\bar{\alpha}^* + \delta)$$

Since $v(\bar{\alpha}^*) = 0$ and $v(\alpha)$ is monotonically decreasing $v(\bar{\alpha}^* + \delta)$ must equal zero, $v(\bar{\alpha}^* + \delta) = o$. But this a contradiction to $\bar{\alpha}^*$ being the maximal α^*. Hence to $M_{\bar{\alpha}^*}$ there must exist a saddle point $(\bar{x},\bar{y})$, such that $\bar{x}\, B(\bar{\alpha}^*)\, \bar{y} > o$. This completes our proof of the existence of a state of balanced growth.

Again we can interprete our model straight forward: A and B are the generalized input and output matrices respectively. They both contain commodities as well as labor, but now labor is not just a catalyst any more, since (if G is not the identity matrix) some labor is used up in production and this "wear and tear of labor" formally comes in by our matrix G. So, e.g., we have L_1 labor input in the commodity production sector, but the output of that sector is only L_1G for labor. The difference is subject to wear and tear. Having this in mind a look at our matrices A and $B(\alpha)$ shows, that the interpretation as generalized input and output matrices is very intuitive.

Next let us analyze the relation of the wage sum to the stock of human capital (we have already pointed out, that by human capital we mean only education in this paper) and on the other hand, to the flow of earnings (gross of costs) to the educational sector. We expect to find similar relations to those of the basic model.

The value of the wage sum at the end of each period is B times the wage sum. If we denote by l the (row) vector of educated labor available, i.e. l_j is the stock of labor of typ j in the economy, we have

$$BW = Blw = \left[BlI - lG \right] y^2 = Bly^2 - lGy^2 = Bly^2 - ly^2 + ly^2 - lGy^2 = (B-1)ly^2 + l(I-G)y^2$$

$$BW = (B-1)\, ly^2 + l\,(I-G)\, y^2$$

The first term $(\beta-1)ly^2$ is the interest on human capital. In the basic model, where people worked for ever, this was equal to βW. But now we have a second term in the equation and since $(I-G)$ corresponds to the rate of workers leaving the labor force $l\,(I-G)y^2$ is the value of human capital leaving the labor force. Again this shows that "wage savings" (confer note 1 on page 22) equal investment in human capital, giving "total savings" equal "total investment".

Hence we arrived at a very intuitive result:

The wage sum, discounted forward to the end of the period, equals the sum of two values, the interest on human capital and the value of human capital leaving the labor force.

The relation wage sum - earnings of the educational sector for education is the same as in the basic model:

$$\beta W = \beta x^1 L_1 w + \beta x^2 (L_2+L_3)\ w$$

From (2.17) and (2.18) we get:

$$\beta x^1 L_1 w + \beta x^2 (L_2+L_3) w = x^1 B_1 y^1 + x^2 p + x^2 B_3 y^1 - \left[\beta x^1 A_1 y^1 + \beta x^2 A_2 y^1 \right]$$

Since $\alpha = \beta$ and because of (2.13) this reduces to

$\beta\ W = x^2 p$, the earnings of the educational sector, i.e., investment in educations equal the wage sum, if both are evaluated at the same point of time.

Finally let us have a look on the problem of drop outs, workers, who do not reach their educational aim. Their ratio is assumed to be fixed and it is included in the mapping $L_3 \rightarrow B_2$. In fact, technologically, there is no difference, if a worker of level i is educated to level j by intention or if the intention was to educate him up to level h, but he could not get so far. The risk of failing bears, in our model, the educational sector. The worker pays only for the degree of education he has actually reached, not for the one he was intended to get educated.

We are now ready to build in learning by doing. Formally this is done by substituting L_1G, the labor output of the goods producing sector by L_4 and L_2G by L_5. About L_4 and L_5 we make the following assumptions, which seem plausible because of the forgoing:

> L_4 and L_5 depend continuosly on α, they do not decrease if α increases and thirdly,
> $L_4(\hat{\alpha})-L_4(\alpha) \leqq (\hat{\alpha}-\alpha)\, I$ and
> $L_5(\hat{\alpha}) - L_5(\alpha) \leqq (\hat{\alpha}-\alpha)\, I$, if $\hat{\alpha} \geqq \alpha$

The equations of our model are:

$$(3.1)\quad \alpha x^1 A_1 + \alpha x^2 A_2 \leqq x^1 B_1 + x^2 B_3$$
$$(3.2)\quad \alpha x^1 L_1 + \alpha x^2 (L_2+L_3) \leqq x^2 B_2 + x^1 L_4 + x^2 L_5$$
$$(3.3)\quad \alpha x^1 A_1 y^1 + \alpha x^2 A_2 y^1 = x^1 B_1 y^1 + x^2 B_3 y^1$$
$$(3.4)\quad \alpha x^1 L_1 w + \alpha x^2 (L_2+L_3) w = x^2 B_2 w + x^1 L_4 w + x^2 L_5 w$$
$$(3.5)\quad \beta A_1 y^1 + \beta L_1 w \geqq B_1 y^1 + L_4 y^2 - L_1 G y^2$$
$$(3.6)\quad \beta A_2 y^1 + \beta (L_2 + L_3)\, w \geqq B_2 y^2 - L_3 G y^2 + L_5 y^2 - L_2 G y^2 + B_3 y^1$$
$$(3.7)\quad \beta x^1 A_1 y^1 + \beta x^1 L_1 w = x^1 B_1 y^1 + x^1 L_4 y^2 - x^1 L_1 G y^2$$
$$(3.8)\quad \beta x^2 A_2 y^1 + \beta x^2 (L_2+L_3)\, w = x^2 B_2 y^2 - x^2 L_3 G y^2 + x^2 L_5 y^2 - x^2 L_2 G y^2 + x^2 B_3 y^1$$
$$(3.9)\quad \beta w = (\beta I - G) y^2 \text{ and } y^2_j = w_j = 0 \text{ if } (\beta - \gamma_j) \leqq 0$$
$$(3.10)\quad \alpha = \beta$$
$$(3.11)\quad x^1 B_1 y^1 + x^2 B_3 y^1 + x^2 B_2 y^2 + x^1 L_4 y^2 + x^2 L_5 y^2 > 0$$

In setting up this equations for describing a state of equilibrium we draw on the results of the forgoing model, giving us the prices of education and the pseudo condition (3.10). G is implicit in L_4 and L_5.

We assume, that both sectors get paid for the education of "working workers". Of course, this must not be explicitely, but can take the form of, e.g.lower wages. Conceptually we can divide the actual wage in the full wage and the charges for education. It does not seem unrealistic, that a job giving education, i.e., giving good prospects for high earnings in the future gets less paid at the present. We can write (3.5) in the form

$\beta A_1 y^1 + \beta L_1 w - (L_4 - L_1 G) y^2 \geqq B_1 y^1$ and interprete $\beta L_1 w - (L_4 - L_1 G) y^2$ as "actual" wage costs.

Note, that the goods production sector may use a process only because it brings a return because of the education connected with the production. Similar an education process - like already in the previous model - may be used only because it is so productive in producing commodities. Hence the distinction between the two sectors is not so sharp any more.

Substituting (3.9) in (3.4) to (3.8) we can rewrite the equilibrium conditions:

$$(3.1^*)\quad \alpha x^1 A_1 + \alpha x^2 A_2 \leqq x^1 B_1 + x^2 B_3$$

$$(3.2^*)\quad \alpha x^1 L_1 + \alpha x^2 (L_2 + L_3) \leqq x^2 B_2 + x^1 L_4 + x^2 L_5$$

$$(3.3^*)\quad \alpha x^1 A_1 y^1 + \alpha x^2 A_2 y^1 = x^1 B_1 y^1 + x^2 B_3 y^1$$

$$(3.4^*)\quad \alpha x^1 L_1 y^2 + \alpha x^2 (L_2 + L_3) y^2 = x^2 B_2 y^2 + x^1 L_4 y^2 + x^2 L_5 y^2$$

$$(3.5^*)\quad \beta A_1 y^1 + \beta L_1 y^2 \geqq B_1 y^1 + L_4 y^2$$

$$(3.6^*)\quad \beta A_2 y^1 + \beta (L_2 + L_3) y^2 \geqq B_2 y^2 + L_5 y^2 + B_3 y^1$$

$$(3.7^*)\quad \beta x^1 A_1 y^1 + \beta x^1 L_1 y^2 = x^1 B_1 y^1 + x^1 L_4 y^2$$

$$(3.8^*)\quad \beta x^2 A_2 y^1 + \beta x^2 (L_2 + L_3) y^2 = x^2 B_2 y^2 + x^2 L_5 y^2 + x^2 B_3 y^1$$

$$(3.10^*)\quad \alpha = \beta$$

$$(3.11^*)\quad x^1 B_1 y^1 + x^2 B_3 y^1 + x^1 L_4 y^2 + x^2 (B_2 + L_5) y^2 > 0$$

We define:

$$A = \begin{bmatrix} A_1 & L_1 \\ A_2 & L_2+L_3 \end{bmatrix} \qquad B(\alpha) = \begin{bmatrix} B_1 & L_4 \\ B_3 & B_2+L_5 \end{bmatrix}$$

$x = (x^1 \; x^2)$ and $y = \begin{pmatrix} y^1 \\ y^2 \end{pmatrix}$

and x and y are normalized such that

$$\sum_i x_i^{\,1} + \sum_i x_i^{\,2} = 1 \text{ and } \sum_j y_j^{\,1} + \sum_j y_j^{\,2} = 1.$$

Hence (3.1^*) to (3.8^*) and (3.11^*) can be rewritten:

(III.1) $\alpha xA \leqq xB$

(III.2) $\alpha xAy = xBy$

(III.3) $\beta A_y \geqq By$

(III.4) $\beta xAy = xBy$

(III.5) $xBy > 0$

From (III.1) to (III.5) follows $\alpha = \beta$ and (3.10^*) is superfluos. From our assumptions about L_4 and L_5 we have the same properties for $B(\alpha)$ and M_α like in the previons model: $B(\alpha)$ is a continuos, non decreasing function of α and M_α is a non increasing function of α. Hence again $v(\alpha)$, the value of the matrix game M_α is monotonically decreasing (in a weak sense, i.e. non increasing) in α. Hence all the premises of our existence proof are fulfilled and this establishes the existence of an $\alpha \doteq \beta$, an x vector, giving the activities, and a y vector giving the (relative) prices, such that (III.1) to (III.5) is fulfilled.

Since (3.9) equals (2.19) $\beta W = (\beta-1)\, ly^2 + l\,(I-G)\, y^2$ still holds and as well is its interpretation still true (savings equal to investment). The second relation concerning the wage sum needs a slight modification: From (3.3), (3.7) and (3.8) we get $\beta W =$
$= x^1 (L_4 - L_1G)y^2 + x^2 (B_2 - L_3G + L_5 - L_2G)\, y^2$, the wage sum equals the discounted earnings for education in both sectors, hence again βW equals investment in education.

IV. MORISHIMA TYPE MODELS INCLUDING EDUCATION

1. The simple model

After we have given some definitions and assumptions we will develop the model and give some interpretations. Finally, we will prove the existence of a balanced growth path.

Let Ω denote the real wage rate and the vector w the differential wage. By l, a column vector of the dimesnion $m_1 + m_2$ we denote the total demand of labour used in the processes. Hence l_i is the amount of labour (total, i.e. not distinguished by type) needed for the i'th process at unity level, where the i'th process is a production process if $i \leq m_1$ and an education process if $i > m_1$. In L_1, L_2 and L_3 we include now uneducated labour and in w we have an element 0 for uneducated labour correspondingly.

We now come to consumption and we will deal with consumption of commodities and consumption aspects of education. We assume that there is no workers' saving and no capitalists' consumption. We further assume that if the differential wage is positive, there is no education purely for consumptional purposes, i.e., every educated person wants to join the labour force as an educated worker, if the wages for a worker with his kind of education are higher than those of workers with a lower education.

Before handling consumption of commodities we must get over the problem of consumption aspects of education. It seems that it is impossible to make a clean cut, since investment and consumption aspects of education are just two sides of one indivisible thing. Hence, in order to solve the problem, we must evaluate the two aspects (without trying to "divide" education itself).[1, 2] Consequently and making the model more difficult, consumption of education will appear in the "monetary", i.e., the cost inequalities and equations, respectively.

The way consumption aspects of education are evaluated in this model, is the following. We assume that with each type of education j there is associated an h_j with $0 \leq h_j < 1$ giving the fraction of Ω the individual is willing to pay in each period for the consumption value of education and we assume that

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1) Cf. W.G. BOWEN, "Assessing the Economic Contribution of Education: An Appraisal of Alternative Approaches". (Higher Education Report of the Committee under the Chairmanship of Lord Robbins 1961-1963, London, H.M.S.O. 1963, Appendix IV, pp. 73-76).

2) Note that consumption aspects of education include also such things like more pleasure or less displeasure at the work one can get with better education. Reasons for such pleasure may be e.g. intellectual satisfaction, being the boss, working in clean and heated rooms etc.

all individuals have the same utility functions, hence the same h_j. Since w_j, the excess of the wage over Ω for a worker of type j. has to be paid back for the price of education (which we will show) we have to base individual consumption decision on Ω for all (educated and not educated) workers. The total consumption value of education is the discounted sum of the amount paid per period.

We assume that the utility function, common to all individuals, is such that all income elasticities are unity.[1)] For education this is implied by assuming h_j to be independent of Ω . As far as commodities are concerned, this enables us to trace out the demand vector, if we define by $\bar{g}_j$ the amount of commodity j a worker demands if his wage net of payments for received education is unity.

Let the h_j's form a column vector h and the $\bar{g}_j$'s a row vector $\bar{g}$. Since a worker with education j pays Ω h_j every period his wage net of those payments is Ω $(1-h_j)$, hence his demand vector is Ω $(1-h_j)\bar{g}$. Let d be a row vector where d_j gives the amount of labor of type j available in the society. Note that this even in equilibrium is not necessarily equal to $x^1 L_1 + x^2(L_2+L_3)$, the vector having as its j'th element the amount of labor of type j actually used, because if a type of education has very high utility, its differential wage may be zero because there are more workers of this kind available than needed and those workers will partly work in other jobs, although this is rather exceptional. Denoting $1-h_j$ by $\hat{h}_j$ and $\hat{h}$ = e-h, where e is a column vector with all elements being 1, the total demand vector is (assuming full employment)

$$\sum_j \Omega \hat{h}_j d_j \bar{g} = \Omega\, d\hat{h}\bar{g}.$$

1) Hence there are no effects of changes in the distribution of the wage sum among the workers.

We assume h and $\bar{g}$ to be continuos functions of y^1, the price vector of the commodities. Hence, we allow for both, substitution between commodities and substitution of education for commodities and vice versa.

Let us assume all human beings have equal abilities.

What are now the equilibrium conditions which equate demand and supply of education? In this very abstract model which neglects the great influence of social variables we can state that in equilibrium costs of education for an individual, i.e., the price of education must equal the benefits of education, because otherwise either everybody or nobody would demand education. Hence we must have (since wages accrue at the beginning of each period but prices belong to the end):

$$(4.1)\quad \frac{\beta}{\beta-1}\,(w_j + \Omega h_j) = y_j^2 \qquad \text{for all } j = 1, \ldots, n_2 \quad \text{if } \beta > 1$$

where w_j is the differential wage and y_j^2 ist the price of education j. In vector notation (4.1) can be expressed as

$$(4.2)\quad \beta w = (\beta-1)\,(y^2 - \frac{\beta}{\beta-1}\,\Omega h),\ \beta > 1$$

or defining

$$(4.3)\quad \bar{y}^2 = y^2 - \frac{\beta}{\beta-1}\,\Omega h$$

which can be interpreted as the investment price of education, we get

$$(4.4)\quad \beta w = (\beta-1)\bar{y}^2 \text{ for } \beta > 1.$$

Once we know y^1, $\bar{y}^2$, Ω and β we can calculate w as well as y^2, since h does depend on y^1 only.

By h_{ij} denote the fraction of the real wage an individual is willing to pay each period for the consumption value of education from type i up to type j. Assuming that the value of education is independent of the path to it, we get

$$(4.5)\quad h_{ij} = h_j - h_i$$

The equilibrium condition for additional education are that in each case the price equals the benefits, i.e.,

$$(4.6)\quad \frac{\beta}{\beta-1}\left[(w_j - w_i) + \Omega h_{ij}\right] = y^2_{ij} \quad ^{1)} \quad \text{for } \beta > 1$$

giving $y^2_{ij} = \frac{\beta}{\beta-1}(w_j + \Omega h_j) - \frac{\beta}{\beta-1}(w_i - \Omega h_i)$ because of (4.5) hence we get

$$(4.7)\quad y^2_{ij} = y^2_j - y^2_i \quad ^{1)} \quad \text{for } \beta > 1,$$

which decreases the number of unknowns by a large amount.

(4.7) can be interpreted in the following way: An educating institution gets y^2_j for educating from level i up to level j but has to pay y^2_i, the value of the previous education.

What is the revenue of an educational process at unity level? It is for the i'th process the value of the commodity output

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1) More exactly we should express (4.6) and (4.7) in the form

$$y^2_{ij} = \max\left\{\frac{\beta}{\beta-1}\left[(w_j - w_i) + \Omega h_{ij}\right],\ 0\right\} \quad \text{and}$$

$$y^2_{ij} = \max\left\{y^2_j - y^2_i,\ 0\right\}$$

in order to avoid negative prices. But this is not necessary if we assume that there are no processes producing something by destroying education.

$\sum_j b^3_{ij} y^1_j$, b^3_{ij} being the i, j'th element of the output matrix B_3, plus the value of increased education $\sum_j b_{ij} y_j - \sum l^3_{ij} y^2_j$, b^2_{ij} and l^3_{ij} being elements of B_2 and L_3 respectively. Hence the vector having as its i'th element the revenue of the i'th educational process, if used at unity level, is $(B_2 - L_3)y^2 + B_3 y^1$.

Before we can state the model a few more assumptions must be specified. Assume that the labor force grows at a constant rate giving an expansion factor ϱ (= 1 + growth rate). Assume further that each process uses some positive amount of some kind of labor, i.e., $l > 0$, and that for each good and each kind of education there exists a process producing it.

One final assumption is necessary, which assures that the growth factor ϱ is not too high with respect to the technology. The assumption we use for that purpose will become clear in the next pages and ist power will be seen in the proof of the existence theorem. Now we just state it.: Let A and B be the following two matrices:

$$A = \begin{bmatrix} A_1 & L_1 \\ A_2 & L_1 + L_2 \end{bmatrix} \qquad B = \begin{bmatrix} B_1 & L_1 \\ B_3 & B_2 + L_2 \end{bmatrix}$$

let α_0 be the smallest possible growth factor of the KMT model associated with A and B; we assume that ϱ is smaller than α_0: $\varrho < \alpha_0$.

For a state of balanced growth we demand that the demand for goods and the demand for educated labor do not exceed their supplies, that there are no profits exceeding the one associated with the common profit rate β-1 (i.e., no excess profits) and that the rule of free goods (goods available in excess of demand have zero prize) and the rule of profitability (processes with a profit less than the one associated with the profit rate β-1 are used at zero level) prevail. In order to have full employment for ever the system must grow with the expansion factor ϱ

(with full employment being one of the initial conditions). In order to keep our relations concerning w and y^2 valid we demand that there is a positive rate of interest if $\varrho > 1$ [1], i.e., $\beta > 1$, if $\varrho > 1$. Finally, we want the real wage to be positive because this assures that something of value is produced in an equilibrium.

Formally these equilibrium conditions are: [2]

$$(4.8)\quad \alpha x^1 A_1 + \alpha x^2 A_2 + \alpha \Omega\, d\hat{h}\bar{g} \leqq x^1 B_1 + x^2 B_3$$

i.e., no excess demand for goods.

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1) If $\varrho \leqq 1$ there will be no positive activity in the educational sector (since workers work for ever), hence, the system reduces to the production sector and we don't need to care about w and y^2.

2) In order to make things shorter I did not start with the nominal wage and unnormalized prices, although this case the hj's would be the same and in all of the following equations and inequalities except in (4.8) and (4.10) we would have just prices p^1, p^2 (corresponding to commodities and education) and the nominal wage ν. In (4.8) and (4.10) we would still have Ω but prices p in (4.10). Similar to the following procedure we had to define the "investment price-vector of education":

$$\bar{p}^2 = p^2 - \frac{\beta}{\beta-1}\nu h.$$

After definition of normalized price vectors y^1 and y^2 by

$$y_j^1 = \frac{p_o^1}{\Sigma_k p_k^1 + \Sigma_k \bar{p}_k^2}, \quad \bar{y}_j^2 = \frac{\bar{p}_j^2}{\Sigma_k p_k^1 + \Sigma_k \bar{p}_k^2} \quad \text{and of the real}$$

wage by $\Omega = \dfrac{\nu}{\Sigma_k p_k^1 + \Sigma_k \bar{p}_k^2}$ the problems occuring in this context are discussed in M. Morishima, Theory of Crowth, Oxford 1969. Rearranging properly we would get the system (IV.1) to (IV.6).

(4.9) $\alpha x^1L_1 + \alpha x^2(L_2 + L_3) \leqq x^2B_2 + x^1L_1 + x^2L_2$

i.e., no excess demand for educated labor. Since workers are assumed to work for ever the supply is old supply plus new supply.

(4.10) $\alpha x^1A_1y^1 + \alpha x^2A_2y^1 + \alpha \Omega d\hat{h}\bar{g}y^1 = x^1B_1y^1 + x^2B_3y^1$

i.e., rule of free goods for commodities.

(4.11) $\alpha x^1L_1w + \alpha x^2(L_2 + L_3)w = x^2B_2w + x^1L_1 w + x^2L_2w$

i.e., rule of free goods for educated labor. The differential wage is zero if its supply exceeds its demand.

(4.12) $\beta A_1y^1 + \beta L_1(w + \Omega e) \geqq B_1y^1$

i.e., no excess profits in the production sector.

(4.13) $\beta A_2y^1 + \beta(L_2 + L_3)(w + \Omega e) \geqq (B_2 - L_3)y^2 + B_3y^1$

i.e., no excess profits in the education sector.

(4.14) $\beta x^1A_1y^1 + \beta x^1L_1(w + \Omega e) = x^1B_1y^1$

i.e., rule of profitability

(4.15) $\beta x^2A_2y^1 + \beta x^2(L_2 + L_3)(w + \Omega e) = x^2(B_2 + L_3)y^2 + x^2B_3y^1$

(4.16) $\alpha = \rho$

(4.17) $\beta > 1$ if $\rho > 1$

(4.18) $\Omega > 0$

(4.19) $\beta w = (\beta - 1)\bar{y}^2$

i.e., the equilibrium condition stated in (1.4).

Using the definition of $\bar{y}^2$ given in (4.3) and eliminating α and w by use of (4.19) and (4.16) we get the following conditions for (4.11) to (4.16) and (4.19):

(4.20) $\rho x^1A_1 + \rho x^2A_2 + \rho \Omega d\hat{h}\bar{g} \leqq x^1B_1 + x^2B_3$

$$(4.21)\quad \rho\, x^1 L_1 + \rho\, x^2(L_2 + L_3) \leqq x^2 B_2 + x^1 L_1 + x^2 L_2$$

$$(4.22)\quad \rho\, x^1 A_1 y^1 + \rho\, x^2 A_2 y^1 + \rho\, \Omega\, d\hat{h}\bar{g}y^1 = x^1 B_1 y^1 + x^2 B_3 y^1$$

$$(4.23)\quad \rho\, x^1 L_1 \bar{y}^2 + \rho\, x^2(L_2 + L_3)\bar{y}^2 = x^2 B_2 \bar{y}^2 + x^1 L_1 \bar{y}^2 + x^2 L_2 \bar{y}^2$$

i.e., the investment price of education is zero if the corresponding labor is not scarce.

$$(4.24)\quad \beta A_1 y^1 + \beta L_1 \bar{y}^2 + \beta\, \Omega\, L_1 e \geqq B_1 y^1 + L_1 \bar{y}^2$$

$$(4.25)\quad \beta A_2 y^1 + \beta(L_2 + L_3)\bar{y}^2 + \beta\, \Omega\, (L_2 + L_3)e \geqq B_2 \bar{y}^2 + L_2 \bar{y}^2 + B_3 y^1 + \frac{\beta}{\beta-1}\Omega\,(B_2 - L_3)h$$

$$(4.26)\quad \beta x^1 A_1 y^1 + \beta x^1 L_1 \bar{y}^2 + \beta\, \Omega\, x^1 L_1 e = x^1 B_1 y^1 + x^1 L_1 \bar{y}^2$$

$$(4.27)\quad \beta x^2 A_2 y^1 + \beta x^2 (L_2 + L_3)\bar{y}^2 + \beta\, \Omega\, x^2 (L_2 + L_3)e = x^2 B_2 \bar{y}^2 + x^2 L_2 \bar{y}^2 + x^2 B_3 y^1 + \frac{\beta}{\beta-1}\Omega\, x^2 (B_2 - L_3)h$$

The total amount paid for education out of Ω is Ωdh, hence, since workers are assumed to consume all their income

$$(4.28)\quad \Omega\, x\mathbb{1} = \Omega\, d\hat{h}gy^1 + \Omega\, dh$$

hence $gy^1 = 1$. (since $xl = de$ = total amount of labor)

We state (or recall) the following definitions:

$$A = \begin{bmatrix} A_1 & L_1 \\ A_2 & L_2+L_3 \end{bmatrix} \qquad B = \begin{bmatrix} B_1 & L_1 \\ B_3 & B_2 + L_2 \end{bmatrix}$$

$x = (x^1 x^2)$ and $\sum_i x_i^1 + \sum_i x_i^2 = \sum_i x_i = 1$

$y = \begin{pmatrix} y^1 \\ \bar{y}^2 \end{pmatrix}$ normalized such that $\sum_j y_j^1 + \sum_j \bar{y}_j^2 = \sum_j y_j = 1$

The normalization of the prices is closely related to the definition of Ω as the real wage. Problems occuring in this context are discussed in M. Morishima, "Theory of Economic Growth".

$$l = \begin{bmatrix} L_1 \\ \\ L_2 + L_3 \end{bmatrix} e \qquad E = \begin{bmatrix} 0 \\ \\ B_2 - L_3 \end{bmatrix}$$

$g = (\bar{g}\ 0)$, hence $gy = \bar{g}y^1 = 1$

Hence, an equilibrium is defined by

(IV.1) $\quad \varsigma xA + \varsigma \Omega d\hat{h}g \leqq xB$

(IV.2) $\quad \varsigma xAy + \varsigma \Omega xl - \varsigma \Omega dh = xBy$

(IV.3) $\quad \beta Ay + \beta \Omega l - \frac{\beta}{\beta - 1} \Omega Eh \geqq By$

(IV.4) $\quad \beta xAy + \beta \Omega xl - \frac{\beta}{\beta - 1} \Omega xEh = xBy$

(IV.5) $\quad \Omega > 0$

(IV.6) $\quad \beta > 1$ if $\varsigma > 1$

In the case $\varsigma \leqq 1$ the system reduces to the production sector and since there is no educational sector there are no payments (on the equilibrium path) hence we can take $h = 0$. But in general the system would need a certain stock of educated labor as initial condition in order to be able to reach the equilibrium path.

Before trying to give some interpretations we can look at two more special cases: $\Omega = 0$ and $h = 0$.

If $\Omega = 0$ we get a model of the KMT type if we replace (IV.5) by $xBy > 0$ and ρ by α and leave out (IV.6). A possible equilibrium would be $\Omega = 0$, x, y and $\alpha = \beta$ the equilibrium values for this KMT model, if ρ happens to be of the equilibrium α's. But we have assumed the contrary by our assumption $\alpha_0 > \rho$. So in fact we could generalize this assumption to : either $\alpha_0 > \rho$ or $\rho = \alpha^*$ with α^* being any solution of the KMT model, but this is a minor point.

If $h = 0$, i.e., there is no education with positive consumption aspects, (IV.1) to (IV.5) reduce to the Morishima model with $c_c = s_w = 0$, but like in the KMT case A and B are "generalized" input and output matrices. Because in this case $\beta = \rho$ we don't need the condition $\beta > 1$. Hence in the case $h = 0$ we get a model formally identical to the Morishima model and we dont't need to prove anything.

In the following we assume $\rho > 1$ (having treated already the case $\rho \leq 1$). In this case the amounts of educated labor available are the amounts of the preceding period plus new educational outputs minus the students (i.e., plus net educational output). Hence, $\rho d = d + x^2 (B_2 - L_3)$, i.e., $\rho d = d + xE$, hence

(4.29) $d = \frac{1}{\rho - 1} xE$ and equation (ii) becomes

$$\rho xAy + \rho \Omega xl - \frac{\rho}{\rho - 1} \Omega xEh = xBy \text{ implying}$$

$$(\rho - \beta)(xAy + \Omega xl) + \left(\frac{\beta}{\beta - 1} - \frac{\rho}{\rho - 1}\right) \Omega xEh = 0 \text{ because of (IV.4)}$$

Obviously, $\varrho = \beta$ is a solution of this equation. It is easy to show that it is the only one: assume $\varrho > \beta$; then $-1/\varrho > -1/\beta$ hence $1-1/\varrho > 1-1/\beta$ hence $\frac{\varrho}{\varrho-1} < \frac{\beta}{\beta-1}$. Hence $\varrho - \beta > 0$ and $(\frac{\beta}{\beta-1} - \frac{\varrho}{\varrho-1}) > 0$. Since the other terms are nonnegative and $\Omega x1$ is positive, since $1 > 0$ and $\Omega > 0$, there is no solution with $\varrho > \beta$. Similar, there is no solution with $\beta > \varrho$, hence $\beta = \varrho$ is the only solution. But this makes (IV.6) redundant and we finally write the model in the following form (for $\varrho > 1$):

(IV.1') $\varrho xA + \frac{\varrho}{\varrho-1} \Omega xE\hat{h}g \leq xB$

(IV.2') $\varrho xAy + \varrho \Omega x1 - \frac{\varrho}{\varrho-1} \Omega xEh = xBy$

(IV.3') $\beta Ay + \beta \Omega 1 - \frac{\beta}{\beta-1} \Omega Eh \geq By$

(IV.4') $\beta xAy + \beta \Omega x1 - \frac{\beta}{\beta-1} \Omega xEh = xBy$

(IV.5') $\Omega > 0$.

Now we can try to interprete the model.

The generalized matrices A and B are easy to interprete. In our model the workers produce labor but they don 't get used up themselves. This can be regarded in that way that any process using a certain amount of labor has the corresponding workers as inputs as well as outputs. The worker is a labor producing machine without suffering wear and tear. Hence, L_1 and L_2 appear in the output matrix B. This shows that A and B can easily be regarded as an input matrix and an output matrix.

The term $\frac{\beta}{\beta-1} \Omega xEh$ obviously is the (capitalized) consumption value of the net production of education, since xE is the vector giving the net production of education for every type and Ω h is the vector giving the consumption values of education for one period, $\beta-1$ in the denominator is for the capitalization and the β in the numerator comes from the fact that payments for consumption of education are paid at the beginning of each period and we are evaluating the consumption value of additional net output of education at the end of the period.

Total payments for consumption of education at the beginning of the next period, i.e., at the end of the present one, are $\rho \Omega dh$ equal to $\frac{\rho}{\rho-1}\Omega xEh$. This gives an interesting result: in equilibrium, where $\beta = \rho$, in each period the addition to the consumption value of education equals the total payments for consumption aspects of education (received in earlier periods).

Total capital consists of three components: commodities used for production, human capital and the wage goods (since wages are paid praenumerando). The value of the wage goods must equal Ωxl, the amount spent by the workers (on commodities and education). It is easy to see that capital equals $xAy + \Omega xl$. Denoting savings by S and Investment by I we get (since savings equal profits)

$$S = (\beta-1)(xAy + \Omega xl) \quad ^{1)} \quad \text{and}$$

$$I = (\rho-1)(xAy + \Omega xl) .$$

Hence in equilibrium, where $\beta = \rho$, savings must equal investment $S = I$, the famous Keynesian equality.

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1) Note that the sum of the differential wages $x^1 L_1 w + x^2(L_2 + L_3) w$ are part of the savings because it cannot be spent by the workers since they have to use it for paying for received education.

Denoting capital $xAy + \Omega xl$ by K we get $\varrho - 1 = \frac{I}{K}$. Denote national income by Y and defining $s = \frac{S'}{Y}$ and $K = \frac{K}{Y} \cdot Y$ being equal to the sum of S and Ωxl grows at the same rate ϱ as S and K do, hence s and K are constant. Writing $\varrho - 1 = \frac{I}{K}\frac{I}{K} = \frac{s}{k}$ we got Sir Roy Harrod's famous equation for the growth rate: $\varrho - 1 = \frac{s}{k}$.

Denote by W the sum of differential wages $W = x^1 L_1 w\, x^2 + (L_2 + L_3)\, w$. Since they are paid at the beginning of the period their value at the end is BW. From equations (4.14) and (4.15) we get

$$BW = x^1 B_1 y^1 + x^2 B_3 y^1 - Bx^1 A_1 y^1 - Bx^2 A_2 y^1 - B\,\Omega xl + x^2(B_2 - L_3)y^2.$$

Since in equilibrium $\alpha = B = \varrho$ (4.10) and (4.28) give

$$x^1 B_1 y^1 + x^2 B_3 y^1 - Bx^1 A_1 y^1 - Bx^2 A_2 y^1 - B\,\Omega xl + B\,\Omega dh = 0.$$

Hence, $BW = x^2(B_2 - L_3)y^2 - B\,\Omega dh = x^2(B_2 - L_3)y^2 - \frac{B}{B-1}\,\Omega xEh$

because of (4.29). Since $xEh = x^2(B_2 - L_3)h$ by substituing y^2 by $\bar{y}^2$ (confer (4.3)) we get $BW = x^2(B_2 - L_3)\bar{y}^2$. The right hand part of the equation is the increase in the value [1] of "human capital" (narrowly defined in this context as educational capital), because it is the production of education evaluated at the end of the period. Hence, in equilibrium the sum of differential wages must equal the investment in human capital, both evaluated at the end of the period. On the other hand (4.10) shows that the value of inputs of goods, including wage commodities and evaluated at the end of the period, equals the value of the output of commodities. These two flow equalities are necessary for balanced growth, because otherwise one

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1) Evaluating only the investment aspect of education.

sector would expand faster than the other.

Finally we can also derive a stock equality. Human capital, denoted by K_h, is the value of the investment aspect of education, i.e.,

$$K_h = x^1 L_1 \bar{y}^2 + x^2(L_2 + L_3)\bar{y}^2$$

Because of $B = \varrho$ and (4.23)

$$BK_h = x^2 B_2 \bar{y}^2 + x^1 L_1 \bar{y}^2 + x^2 L_2 \bar{y}^2 = x^2(B_2 - L_3)\bar{y}^2 + K_h$$

hence, $K_h = \frac{1}{B-1} x^2(B_2 - L_3)\bar{y}^2$ (or, more intuitively, equals

$\frac{1}{\varrho - 1} x^2 (B_2 - L_3)\bar{y}^2$).

But since $x^2(B_2 - L_3)\bar{y}^2 = BW$ we get $BW = (B-1)K_h$, an intuitive result: The sum of differential wages equals the interest on human capital, both evaluated at the end of the period. This is the effect in the aggregate of our equilibrium condition that costs of education must equal its benefits.

Finally we will prove the following theorem:

<u>Theorem:</u> There exists at least one n-tuple $(x; y^T; \Omega)$, y^T denoting the transposed of y, such that (IV.1') to (IV.5') and hence (IV.1) to (IV.6) hold.

<u>Proof:</u>

We will proceed by making use of Brouwer's fixed point theorem: If a continuos point to point mapping assignes to every point of a non-empty, compact and convex subset of the Euclidean space a point of this subset, then there exists at least one point which is mapped into itself.

Let B be equal to ϱ ; $B = \varrho$.

Define excess demand F and excess profits D:

$F = \rho xA + \frac{\rho}{\rho-1} \Omega xE\hat{h}g - xB$ with the j'th component denoted Fj

$D = By + \frac{\rho}{\rho-1} \Omega Eh - \rho Ay - \rho \Omega 1$ with the i'th component D_i.

We can prove the Walras law for $\beta = \rho$:

(4.30) $Fy + xD = 0$ since $\hat{h} = e - h$ and $de = x1$.

Hence it is enough to prove that an n-tuple $(x;y^T;\Omega)$ fulfils $F \leqq 0$, $D \leqq 0$ and $\Omega > 0$.

Define $F_j^* = \max (F_j, 0)$ and $\min (F_j, M)$ for a very large M for all j.

$D_i^* = \max (D_i, 0)$ for all i.

Since g and h are continuos functions of y^1, F_j^*'s and D_i^*'s are continuos functions fo x, y and Ω.

Now we need our assumption $\rho < \alpha_e$. A theorem of the KMT model states that the smallest expansion factor is the smallest interest factor such that there exists a price vector y such that

$(B - \alpha_o A)\, y \leqq 0$, i.e., giving a system without excess profits.

If $\Omega = 0$ $D = (B - \rho A)y$ and since we have assumed $\rho < \alpha_0$ there is no price vector y such that $D \leqq 0$. Hence $\Omega = 0$ implies $\sum_i D_i^* > 0$.

From $\hat{h} > o$ (because $hj < 1$ for all j) follows $d\hat{h} > o$, which together with (4.29) implies $\frac{1}{\rho-1} \times E\hat{h} > o$. Hence there exists a large enough finite $\bar{v}$, such that $\Omega = \bar{v}$ implies that there is positive excess demand for at least one good no matter what activity vector x is taken,

i.e., $\Omega = \bar{v} \Longrightarrow \sum_j F_j^* > 0$.

Define the following mappings:

$$(4.31) \quad \xi_i = \frac{1}{1 + \sum_i D_i^*} (x_i + D_i^*)$$

$$(4.32) \quad \eta_i = \frac{1}{1 + \sum_j F_j^*} (y_j + F_j^*)$$

$$(4.33) \quad v = \frac{\bar{v}}{\bar{v} + \sum_j F_j^* + \sum_i D_i^*} \left(\omega + \sum_i D_i^*\right)$$

Denote by the set Z the set of all vectors $(x;y^T;\omega)$ such that

$$x_i \geq 0 \quad \& \quad \sum_i x_i = 1$$

$$y_i \geq 0 \quad \& \quad \sum_i y_i = 1$$

$$0 \leq \omega = \tilde{v}.$$

Obviously, Z is a non-empty convex compact subset of the Euclidian space.

The mappings (4.31) to (4.33) map every point of Z into Z, because if $(x, y^T, \omega) \in Z$, then

$$\xi_i \geq 0 \qquad \sum_i \xi_i = \frac{\sum x_i + \sum D_i^*}{1 + \sum D_i^*} = 1$$

$$\eta_i = 0 \qquad \sum_j \eta_j = \frac{\sum y_i + \sum F_j^*}{1 + \sum F_j^*} = 1 \text{ and}$$

$$0 \leq v \leq \bar{v} \text{ because } 0 \leq \frac{\omega + \sum D_i^*}{\bar{v} + \sum F_j^* + \sum D_i^*} \leq 1.$$

Since the mappings are continuos (because the denominators cannot vanish) Brouwer's fixed point theorem can be applied. Hence, there must exist at least one point $(\tilde{x}, \tilde{y}^T, \tilde{\omega})$ with:

$$(4.34)\quad \tilde{x}_i = \frac{1}{1 + \sum_i \tilde{D}_i} (\tilde{x}_i + \tilde{D}^*_i)$$

$$(4.35)\quad \tilde{y}_j = \frac{1}{1 + \sum_j \tilde{F}^*_j} (\tilde{y}_j + \tilde{F}^*_j)$$

$$(4.36)\quad \tilde{\Omega} = \frac{\bar{v}}{\bar{v} + \sum_j \tilde{F}^*_j + \sum_i \tilde{D}^*_i} (\tilde{\Omega} + \sum_i \tilde{D}^*_i)$$

where the $\sim$ denote the fixed point values.

We have now to prove that every fixed point is an equilibrium. From (4.36) we get

$$(4.37)\quad \tilde{\Omega} (\sum_j \tilde{F}^*_j + \sum_i \tilde{D}^*_i) = \bar{v} \sum_i \tilde{D}^*_i$$ since $\bar{v} > 0$ and $F^*_j \leq M$

$\tilde{\Omega} = 0$ would imply $\sum_i \tilde{D}^*_i = 0$. But we know that $\tilde{\Omega} = 0$ implies $\sum_i D^*_i > 0$, a contradiction to (4.37). Hence, $\tilde{\Omega} > 0$.

Assume $\sum \tilde{D}^*_i > 0$. (4.34) gives

(4.38) $\tilde{x}_i \sum \tilde{D}^*_i = \tilde{D}^*_i$. Hence (for $\sum_i \tilde{D}^*_i > 0$) $\tilde{x}_i$ is zero if and only if $\tilde{D}^*_i$ is zero, hence, $\tilde{x}_i \tilde{D}^*_i = \tilde{x}_i \tilde{D}_i$ and $\sum \tilde{x}_i D^*_i = \tilde{x}D$. Multiplying (4.38) by $\tilde{x}_i$ and summing up we get

$\sum \tilde{x}^2_i \sum \tilde{D}^*_i = \sum \tilde{x}_i \tilde{D}^*_i$ and the left expression is positive if $\sum \tilde{D}^*_i$ is positive.

Hence, $\tilde{x}\tilde{D} > 0$ if $\sum_i \tilde{D}^*_i > 0$.

By the same reasons $\tilde{F}\tilde{y} \geq 0$ if $\sum_j F^*_j > 0$. But since $xD + Fy \equiv 0$ either $\sum_i \tilde{D}^*_i = 0$ or $\sum_j F^*_j = 0$ or both.

Let $\sum_i \tilde{D}^*_i = 0$. (4.37) becomes $\tilde{\Omega} \sum_j \tilde{F}^*_j = 0$. Since $\tilde{\Omega} > 0$

$\sum_i \tilde{D}^*_i = 0$ implies $\sum_j \tilde{F}^*_j = 0$.

Let $\sum_j F_j^* = 0$, then (4.37) becomes $\tilde{\Omega} \sum_i \tilde{D}_i^* = \bar{v} \sum_i \tilde{D}_i^*$. If $\tilde{\Omega}$ were equal to $\bar{v}$ $\sum_j \tilde{F}_j^*$ would be positive, because we have chosen $\bar{v}$ large enough, hence $\tilde{\Omega} < \bar{v}$, hence $\sum_i \tilde{D}_i^* = 0$, hence $\sum \tilde{F}_j^* = 0$ implies $\sum_i \tilde{D}_i^* = 0$.

Since at least one, $\sum_i \tilde{D}_i^* =$ or $\sum_j \tilde{F}_j^* = 0$, both must equal zero. Hence, we have $\tilde{F} \leq 0$, $\tilde{D} \leq 0$, $\tilde{F}\tilde{y} = 0$, $\tilde{x}\tilde{D} = 0$ (because $Fy + xD = 0$ and x and y being non-negative) and $\tilde{\Omega} > 0$. This proves that the fixed point is an equilibrium and completes the proof of the thecrem.

2. One further generalization

In this last model we use all assumptions of the previous one except that we allow now for workers leaving the labor force and for learning by experience. Hence again we assume that workers do not save that capitalists do not consume and that $g > \alpha_c > 1$. We further assume that for any worker there is a given probability γ, which is the probability of continuing to work in the following period for any worker. We assume that the percentage of workers of each type continuing to work in the following period is exactly γ (i.e. we assume that there is no variance). A worker who has left the labor force does not pay any longer for his education (and has no income).

Since we demand that the price of education equals the value of the benefits it gives we get equation

(5.1) $\beta\,(w_j + \Omega h_j) = (\beta - \gamma)\; y_j^2$ for $\beta > \gamma$ and

since by assumption $h_{ij} = h_j - h_i$

(5.2) $y_{ij}^2 = y_j^2 - y_i^2$ for $\beta > \gamma$.

By the investment price vector of education we define

(5.3) $\bar{y}^2 = \frac{\beta}{\beta-\gamma} w$, (which is equal to $y^2 - \frac{\beta}{\beta-\gamma}\Omega h$) for $\beta > \gamma$

We define two matrices L_4 and L_5 corresponding to L_1 and L_2 respectively. L_4 is the labor output matrix for the production sector and L_5 the one for the educational sector as far as non-students are concerned. Hence in L_4 are the workers of L_1 but improved by learning by doing and diminished by those leaving the labor force (i.e. multiplied by γ). L_5 corresponds to L_2 in exactly the same way.

Because of (5.2) profits out of education are described by the vectors $(L_4 - \gamma L_1)\, y^2$ and $\left[(B_2 - \gamma_{\gamma} L_3) + (L_5 - \gamma L_2)\right] y^2$ for both sectors if each process is used at unity level.

The equilibrium conditions are:

(5.4) $\alpha x^1 A_1 + \alpha x^2 A_2 + \alpha \Omega x l \bar{g} - \alpha \Omega dh\bar{g} \leqq x^1 B_1 + x^2 B_3$

(5.5) $\alpha x^1 L_1 + \alpha x^2 (L_2 + L_3) \leqq x^2 B_2 + x^1 L_4 + x^2 L_5$

(5.5) $\alpha x^1 A_1 y^1 + \alpha x^2 A_2 y^1 + \alpha \Omega x l - \alpha \Omega dh = x^1 B_1 y^1 + x^2 B_3 y^1$

(5.6) $\alpha x^1 L_1 w + \alpha x^2 (L_2 + L_3) w = x^2 B_2 w + x^1 L_4 w + x^2 L_5 w$

(5.7) $\beta A_1 y^1 + \beta L_1 (w + \Omega e) \geqq B_1 y^1 + (L_4 - \gamma L_1) y^2$

(5.8) $\beta A_2 y^1 + \beta (L_2 + L_3)(w + \Omega e) \geqq \left[(B_2 + L_5) - \gamma (L_2 + L_3)\right] y^2 + B_3 y^1$

(5.9) $\beta x^1 A_1 y^1 + \beta x^1 L_1 (w + \Omega e) = x^1 B_1 y^1 + x^1 (L_4 - \gamma L_1) y^2$

(5.10) $\beta x^2 A_2 y^1 + \beta x^2 (L_2 + L_3)(w + \Omega e) = x^2 \left[(B_2 + L_5) - \gamma (L_2 + L_3)\right] y^2 + x^2 B_3 y^1$

(5.11) $\Omega > 0$

(5.12) $\alpha = \varrho$

(5.13) $B > \gamma$

(5.14) $y^2 = \frac{B}{B-\gamma}(w + \Omega h)$

If there is balanced growther at a rate $\alpha - 1$, then $\alpha d = \gamma d + x^1(L_4 - \gamma L_1) + x^2\left[(B_2 + L_5) - \gamma(L_2 + L_3)\right]$ or defining $x = (x^1\ x^2)$ and $E = \begin{bmatrix} -L_4 - \gamma L_1 \\ B_2 + L_5 - \gamma(L_2 + L_3) \end{bmatrix}$

(5.15) $d = \frac{1}{\alpha - \gamma} xE$

Hence we can a define an equilibrium by:

(5.16) $g x^1 A_1 + g x^2 A_2 + g \Omega x 1\bar{g} - \frac{g}{g-\gamma}\Omega xEh\bar{g} \leqq x^1 B_1 + x^2 B_3$

(5.17) $g x^1 L_1 + g x^2 (L_2 + L_5) \leqq x^2(B_2 + L_5) + x^1 L_4$

(5.18) $g x^1 A_1 y^1 + g x^2 A_2 y^1 + g \Omega x 1 - \frac{g}{g-\gamma}\Omega xEh = x^1 B_1 y^1 + x^2 B_3 y^1$

(5.19) $g x^1 L_1 \bar{y}^2 + g x^2 (L_2 + L_3)\bar{y}^2 = x^2 (B_2 + L_5)\bar{y}^2 + x^1 L_4 \bar{y}^2$

(5.20) $B A_1 y^1 + B L_1 \bar{y}^2 + B \Omega L_1 e - \frac{B}{B-\gamma}\Omega(L_4 - \gamma L_1)h \geqq B_1 y^1 + L_4 \bar{y}^2$

(5.21) $B A_2 y^1 + B (L_2 + L_3)\bar{y}^2 + B \Omega (L_2 + L_3)e - \frac{B}{B-\gamma}\Omega\left[(B_2 + L_5) - \gamma(L_2 + L_3)\right]h \geqq (B_2 + L_5)\bar{y}^2 + B_3 y^1$

(5.22) $B x^1 A_1 y^1 + B x^1 L_1 \bar{y}^2 + B \Omega x^1 L_1 e - \frac{B}{B-\gamma}\Omega x^1 (L_4 - \gamma L_1)h = x^1 B_1 y^1 + x^1 L_4 \bar{y}^2$

(5.23) $\beta x^2 A_2 y^1 + \beta x^2 (L_2+L_3)\bar{y}^2 + \beta\,\Omega\, x^2 (L_2+L_3)e - \frac{\beta}{\beta-\gamma}\Omega\, x^2 \cdot$

$$\cdot \left[(B_2+L_5) - \gamma (L_2+L_3) \right] h = x^2 (B_2+L_5)\bar{y}^2 + x^2 B_3 y^1$$

(5.11) $\Omega \geq 0$

(5.12) $\beta > \gamma$

Defining:

$$A = \begin{bmatrix} A_1 & L_1 \\ A_2 & L_2+L_3 \end{bmatrix} \qquad B = \begin{bmatrix} B_1 & L_4 \\ B_3 & B_2+L_5 \end{bmatrix}$$

$x = (x^1 \; x^2)$ normalized such that $\sum x_i = 1$

$y = \begin{pmatrix} y^1 \\ \bar{y}^2 \end{pmatrix}$ normalized such that $\sum y_j = 1$

$g = (\bar{g} \; 0)$ with dimension $(n_1 + n_2)$

Remembering that l is by definition $l = \begin{bmatrix} L_1 e \\ (L_2+L_3)e \end{bmatrix}$

the model finally can be written:

(V.1) $\rho x A + \rho\,\Omega\, x l g - \frac{\rho}{\rho-\gamma}\,\Omega\, x E h g \leq x B$

(V.2) $\rho x A y + \rho\,\Omega\, x l - \frac{\rho}{\rho-\gamma}\Omega\, x E h = x B y$

(V.3) $\beta A y + \beta\,\Omega\, l - \frac{\beta}{\beta-\gamma}\Omega\, E h \geq B y$

(V.4) $\beta x A y + \beta\,\Omega\, x l - \frac{\beta}{\beta-\gamma}\Omega\, x E h = x B y$

(V.5) $\Omega \geq 0$

The condition (5.12) $B > \gamma$ becomes redundand because from (V.2) and (V.4) follows $B = \varrho$ which in turn unplies $B > \gamma$ because $\varrho > 1$ and $\gamma \leq 1$.

The model again reduces to Morishima's model if h = o and to the KMT model if $\Omega = o$. Again we have $B = \varrho$ and the proof for the existence of such an balanced growth path is exactly the same like in the foregoing model, since mathematically the only difference is, that now we have $\varrho - \gamma$ and $B - \gamma$ instead of $\varrho - 1$ and $B-1$ in the denominator. Hence we don't need to go through the proof again.

The model has a similar interpretation like the foregoing one. A is a kind of generalized input, B a generalized output matrix. The term $\frac{B}{B-\gamma} \Omega xEh$ stands for the (capitalized) consumption value of the net production of education. Since $\varrho \Omega dh$, equal to $\frac{\varrho}{\varrho-\gamma} \Omega xEh$ are the total payments for consumption of education for one period and since in equilibrium $B = \varrho$, we have again the result that in equilibrium in each period payments for educational consumption equal the capitalized consumption value of the net increase in education. Further, like in the foregoing model, savings must equal investment, since savings equal $(B-1)(xAy + \Omega xl)$ and investment $(\varrho-1)(xAy + \Omega xl)$ and $B = \varrho$. It followe by the same arguments used in the last model, that also Sir Harrod's equation holds, the growth rate equals the savings-income ratio divided by the capital-output ratio.

Denoting the sum of differential wages by W we have $W = x^1 L_1 w + x^2 (L_2 + L_3)\, w$ and from (5.5), (5.9), (5.10) and $\alpha = B = \varrho$ we get $BW = x^1 (L_4 - \gamma L_1) y^2 + x^2 \left[(B_2 + L_5) - \gamma (L_2 + L_3) \right] y^2 - B \Omega dh$. Since $B \Omega dh = \frac{B}{B-\gamma} \Omega xEh$ and since $\bar{y}^2 = y^2 - \frac{B}{B-\gamma} \Omega h$ we finally get $BW = xE\bar{y}^2$ or in more detail $BW = x^1 (L_4 - \gamma L_1) \bar{y}^2 + x^2 \left[(B_2 + L_5) - \gamma (L_2 + L_3) \right] \bar{y}^2$. The sum of differential wages equals the increase (net of those leaving the labor force) of the investment value of educational capital. (5.5) and $\alpha = \beta = \rho$ show that the input of commodities, including wage goods, discounted foreward to the end of the period equals the value of outputs of commodities. These equalities make balanced growth possible.

The investment value of educational capital K_h equals $x^1 L_1 \bar{y}^2 + x^2 (L_2+L_3) \bar{y}^2$, its decrease subject to workers leaving the labor force (i.e., wear and tear), denote it by D_h, equals $x^1 (1-\gamma) L_1 \bar{y}^2 + x^2 (1-\gamma)(L_2+L_3)\bar{y}^2$ or $D_h = (1-\gamma)K_h$. From $\alpha = \beta = \rho$ and (5.19) follows that $\beta K_h = x^2(B_2-L_5)\bar{y}^2 + x^1 L_4 \bar{y}^2$. The right hand side of the last equation in turn equals $x^1(L_4 - \gamma L_1)\bar{y}^2 + x^2\left[(B_2+L_5) - \gamma(L_2+L_3)\right]\bar{y}^2 + K_h - D_h$, which equals $\beta W + K_h - D_h$. Hence $\beta K_h = \beta W + K_h - D_h$ or $\beta W = (\beta-1) K_h + D_h$, the sum of differential wages (disconnted forward to the end of the period) equals the interest on human capital plus the depreciation of human capital, evaluated both at investment prices. This seems to be an intuitive result.

Since again $\alpha = \beta = \rho$ one could assume different probabilities for continuing working γ_j for each type of educated labor j. As in chapter III.2. we could procced by defining a diagonal matrix G with γ_j as the j's element in the diagonal. Neither the existence proof nor the interpretations would change in a substantial way.

V. A TURNPIKE THEOREM FOR THE BASIC MODEL

In general we do not observe belanced growth nor do we really want it in the sense that we have a strong preference for it. Turnpike theorems have shown that the balanced growth path of the von Neumann model has some normative properties. Under certain assumptions an efficient growth path will tend to the von Neumann growth path. Since the assumptions used are rather restrictive, one should be cautious to give those theorems too much weight for problems outside of the pure realm of theory. [1)]

The formal equivalence of the KMT model and our basic model suggests, that the turnpike theorems based on the KMT model hold as well for our basic model. In the following we show that this is true for Morishima's turnpike theorem (and in fact I feel very sure, that it holds as well for others, e.g. that of Radner).

The reader is referred to Morishima's "Theory of Economic Growth" Chapter X: Maximization of Bequests: The first turnpike theorem, pp 179 - 197. We will only give the headlines.

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1) Turnpike theorems are also used for investigations in problems of stability of the growth equilibrium. These problems are not analyzed here, but for the KMT type models given in section III similar results about stability probably hold for our generalized models because of the formal equivalence. But it would be very misleading to use those models, which exclude all social aspects of education and even the consumption component of education, for an analysis of behavior. It seems that those models are more apt to give a basis for the developement of models of planning than for describing economic or educational reality.

Let A and B be our generalized input and output matrix respectively. We assume that capitalists do not consume and that workers get their wage in form of goods in fixed proportion, which is included in A (augmented input matrix).

A state with the max $\alpha = \bar{\alpha}$ fulfilling (I.1) to (I.5) of section III we define as a turnpike with the equilibrium value $\bar{X}$ defined as turnpike intensity vector and $\bar{y}$ as turnpike price vector. We assume (in addition to the assumptions for the basic model):

(i) $\bar{x}$ is unique (i.e. the system is to some extend indecomposable)
(ii) $\bar{y}$ is strictly positive. (i.e., there are no free goods, hence all inequalities (I.1) are equations).

The task is to maximize a vector b^* giving the desired proportion of goods and educational standards at the end of the planning period of length T. It is assumed that there is no use of that amount of any good, which exceeds the desired proportion, hence any excess can be discarded so that we are left with the desired proportions.

An efficient path or a "DOSSO-path" [1)] is a path which maximizes the amount of goods and educational standards in their desired proportions b^* subject to the restrictions imposed by the technology. Hence it maximizes a scalar u , where u gives the level at which b^* is available at period T. Formally a Dosso path is defined by:

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1) R. Dorfman, R.A. Samuelson & R.M. Solow, "Linear Programming and Economic Analysis" (New York 1958)

$$\max\ u$$

$$\left.\begin{aligned} &\text{subject to} \quad \sum_i b_{ij}\, q_i(t-1) \geqq \sum_i a_{ij} q_i\ (t) \\ &\qquad\qquad\qquad\qquad\qquad \text{for all } t = o, \dots, T-1 \\ &\text{and} \quad \sum_i b_{ij} q_i\ (T-1)\ \geqq u\, b_j^* \end{aligned}\right\} \begin{aligned} &\text{for all} \\ &j = 1, \dots, n \end{aligned}$$

where a_{ij} and b_{ij} are the elements of A and B respectively and $q_i\ (\tau)$ is the intensity of process i at time period τ. The problems can be solved forming on Lagrange expression

$$L = u - \sum_{j=1}^{n} \sum_{t=1}^{T-1} p_j\ (t) \left[\sum_i a_{ij} q_i(t) - \sum_i b_{ij} q_i\ (t-1) \right] -$$

$$- \sum_j p_j(T) \left[u \sum_j b_j^* - \sum_i b_{ij} q_i\ (T-1) \right]$$

where the $p_j(t)$'s are the unknown Lagrange multipliers. This unplies the Kuhn - Tucker conditions:

$$\frac{\partial L}{\partial q_i(t)}\ q_i^*\ (t) = - q_i^*\ (t) \left[\sum_j a_{ij} p_j(t) - \sum_j b_{ij} p_j(t+1) \right] =$$

$$= o \text{ for } t = o, \dots, T-1$$

$$\frac{\partial L}{\partial u^*}\ u^* = u^* \left[1 - \sum b_j^*\ p_j(T) \right] = 0$$ where the asterisk denotes the maximizing values.

We can write this in the form

$$(6.1) \quad \sum_j b_{ij} p_j(t+1) \begin{cases} = \sum_j a_{ij} p_j(t) & \text{if} \quad q_i^*\ (t) > o \\ \leqq \sum_j a_{ij} p_j(t) & \quad q_i^*\ (t) = o \end{cases}$$

$$(6.2) \quad \sum_j b_j^* p_j\ (T) \begin{cases} = 1 & \text{if } u^* > o \\ \geqq 1 & \text{if } u^* = o \end{cases}$$

This shows that the Lagrange multipliers have a familiar interpretation: they are prices, all evaluated at the present (hence they are discounted prices).

One question arives: what are the wages ?

The $p_j(t)$ are education prices for $j > n_1$. We see that in production educated labor runs through: its input price is $p_j(t)$, its output price is $p_j(t+1)$ for all $j > n_1$.
Obviously the (differential) wage is the difference:

(6.3) $w_j(t) = p_{j+n_1}(t) - p_{j+n_1}(t+1)$.

We can check it for the Neumann case, where, writing y_j^2, for y_j^2 (o) and w_j for $w_j(o)$, $p_{j-n_1}(t) = \frac{1}{\beta^t} y_j^2$ hence

$$w_j(t) = \frac{1}{\beta^t} y_j^2 - \frac{1}{\beta^{t+1}} y_j \text{ or } \beta^{t+1} w(t) = \beta y_j^2 - y_j^2 \text{ or}$$
$$\beta^{t+1} w_j(t) = (\beta-1) y_j^2$$

Since $w_j(t)$ is the present value of wage at time t, hence disconnted, i.e., $w_j = \frac{1}{\beta^t} w_j(t)$ we get the familiar equation $\beta w_j = (\beta-1) y_j^2$.

Two more assumptions are necessary:

(iii) The initial stock vector b (o) is given such that the economy can move from b (o) to a point on the turnpike in an finite number of periods.

(iv) b^* is given such that the economy can move in a finite number of periods from a point on the turnpike to a state with the stock composition b^*.

In order that the turnpike theorem holds further conditions are necessary, which are derived by Morishima. In order to keep things simple we will only give a sufficient condition. If it is not fulfilled, then there are still some cases where the theorem holds:

We can define a top process as a process where (I.3) holds with equality, i.e., a process, which is profitable. Let A^* and B^* be the matrixes A and B respectively, where only top processes are included. Then the sufficient condition is that (v) the characteristic equation $|B^* - \lambda_i A^*| = o$ is such that there is no negative or complex root λ_i that has absolute value $\bar{\alpha}$. (λ is the vector of latent roots for systems of difference equations).

If (i) to (v) hold, then Morishima's turnpike theorem holds:

If the programming period T is sufficiently long, then any Dosso path starting from b (o) will remain most of that period within a very small neighborhood of the turnpike.

Since formally there is no difference between our model and the KMT model, which Morishima uses for his turnpike, Morishima's proof is valid for our basic model as well. The only difference is in the interpretation: A and B are generalized matrices, i.e. they include education and so do b(o) and b^* include educated labor besides commodities.

APPENDIX:

Optimal Allocation of Investment

In this appendix we deal with quite different things: we look at problem of maximizing the absolute increase in national income with respect to investment in capital, i.e. "physical capital", and investment in education subject to a restriction of the Keynesian type, that (in some sense) savings must equal investment both taken ex ante) in equilibrium. This problem can be solved quite easily by classical methods and the solution has a fairly nice interpretation.

Let us first specify the notation:

C Consumption, S Savings

Y national income, defined such, that it does not include increase in "human capital".

$\dot{Y} = \frac{dY}{dt}$ absolute change in national income per time unit.

I_p Investment in "physical" capital

I_e Investment in "educational" capital

I_e^f "Incomes foregone" because of education, which is a part of I_e

Hence, $I_e - I_e^f$ is the part of the educational investment, which is actually spent and creates new income.

$\pi = \frac{I_e - I_e^f}{I_e}$ and we assume π to be a constant. Obviously $\pi \leq 1$ and $\pi < 1$ if $I_e^f > 0$.

We assume

(A.1) $\dot{Y} = g\ (I_p, I_e)$ with $\frac{\partial g}{\partial I_j} > 0$ $\frac{\partial^2 g}{\partial I_j^2} < 0$ for j=p, e

Of course, in general $\dot{Y}$ will depend not only on investment but on capital, physical and educational, as well. This is in fact included in our assumption (A.1), but at any given moment the capital stock is fixed and hence becomes a parameter influencing our function g. Since the capital stock is fixed at any moment, we assume that the only variables which do influence $\dot{Y}$ and are at the same time subject to economic policy are I_p and I_e.

We have the "Keynesian" equalities:

(A.2) $Y = C + S$ and

(A.3) $Y = C + I_p + (I_e - I_e^f)$, hence

$S = I_p + I_e - I_e^f$ or

(A.4) $S = I_p + \pi I_e$ and we assume that S is a twice continuously differentiable function of Y.

(A.5) $S = S(Y)$ 1) and $\frac{dS}{dY} > 0$.

The optimization problem is that of maximizing $\dot{Y}$ subject to the constraint A(4):

(A.6) $\max \dot{Y}$

subj. to $S(Y) = I_p + \pi I_e$

We can solve the problem easily by classical methods. Forming the Langrange-expression

$$L = g(I_p, I_e) + \lambda (I_p + \pi I_e - S(Y))$$

We get the following necessary conditions:

(A.7) $\frac{\partial L}{\partial I_p} = \frac{\partial g}{\partial I_p} + \lambda - \lambda \frac{\partial S}{\partial I_p} = 0$

$\frac{\partial L}{\partial I_e} = \frac{\partial g}{\partial I_e} + \pi \lambda - \lambda \frac{\partial S}{\partial I_e} = 0$

which implies

$$\frac{\partial g / \partial I_p}{\partial g / \partial I_e} = \frac{\partial S / \partial I_p - 1}{\partial S / \partial I_e - \pi} \quad \text{or}$$

---

1) Note that this assumption implies, that savings, hence consumption is not influenced by increase in educational capital, because this is not included in Y. This means that incomes foregone reduce consumption, although expected life time income might not have changed or might even have increased because of the investment in education.

(A.8) $\frac{\partial g}{\partial I_p}\frac{\partial S}{\partial I_e} = \pi \frac{\partial g}{\partial I_p} + \frac{\partial g}{\partial I_e} \quad \frac{\partial S}{\partial I_p} - \frac{\partial g}{\partial I_e}$;

given unitial conditions, Y is a function of $\dot{Y}$ 1), hence

(A.9) $S(Y) = h(\dot{Y})$. It follows, that $\frac{\partial S}{\partial I_j} = \frac{\partial h}{\partial I_j}$, j = p,e and

(A.10) $\frac{\partial S}{\partial I_p} = \frac{dS}{dg} \quad \frac{\partial g}{\partial I_p}$ and

$\frac{\partial S}{\partial I_e} = \frac{dS}{dg} \quad \frac{\partial g}{\partial I_e}$ since $\dot{Y} = g(I_p, I_e)$.

This implies

$$\frac{\partial g}{\partial I_p}\ \frac{\partial S}{\partial I_e} = \frac{\partial g}{\partial I_p}\ \frac{dS}{dg}\ \frac{\partial g}{\partial I_e} = \frac{dS}{dg}\frac{\partial g}{\partial I_p}\ \frac{\partial g}{\partial I_e} = \frac{\partial S}{\partial I_p}\ \frac{\partial g}{\partial I_e} \quad \text{or}$$

(A.11) $\frac{\partial g}{\partial I_p}\ \frac{\partial S}{\partial I_e} = \frac{\partial g}{\partial I_e}\frac{\partial S}{\partial I_p}$. Hence (A.8) reduces to

(A.12) $\frac{\partial g / \partial I_e}{\partial g / \partial I_p} = \pi$

This proves: If the distribution of investment between physical and educational investment is optimal, the ratio of the marginal products of educational and physical investment i.e.their rate of substitution equals the proportion of the educational investment, which has actually to be spent.

Obviously this is an analogon to the well-known result of the theory of production, that the rate of substitution between any two inputs equals their prices-ratio. In fact (A.1) can be interpreted as a production function, where the "product" is increase in national income, and π can be interpreted as a price ratio, as the following shows. Sine investment is measured in

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1) $Y(t) = Y(t_o) + \int_{t_o}^{t+t_o} \dot{Y}(\tau)\, d\tau$

money terms, one dollar of physical investment costs one dollar. This is not the case for educational investment: one dollar of educational investment needs π dollars ($\pi \leq 1$) for expenditures, where as $(1-\pi)$ dollars are financed by increased savings because incomes foregone reduce sonsumption. This formally proves Dennis-ons [1] proposition, that if incomes foregone reduce consumption it pays to invest more in education even if its marginal return (hence its internal rate of interest) is lower than that of investment in physical capital.[2] [3]

It is easy to show, that neglecting the fact, that educational investment may cause higher savings leads to non optimal solutions. Suppose the available resources for total investment are a constant I. Neglecting the "Keynesian" equality (A.4) we would set up the optimization problem in the following form:

$$\max \dot{Y} \text{ subject to } I_p + I_e = I.$$

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1) Dennison, The Sources of Economic Growth, pp. 77-78.

2) Note that Y is relatively lower if I_e is higher, since one dollar more for I_e causes only π dollar more income.

3) Of course, one has to stress, that this is not so if higher investment in education does not give incentives for more students, who are willing to substitute education and its economic and non-economic advantages for present income, i.e., the argument does not hold, if higher income foregone, because in this case π would become 1 for the marginal investment in education.

The Langrange expression is

$$L = g\,(I_p, I_e) + \lambda\,(I_p + I_e - I)$$

giving the 1st order optimum conditions:

$$\frac{\partial L}{\partial I_p} = \frac{\partial g}{\partial I_p} + \lambda = o \text{ and}$$

$$\frac{\partial L}{\partial I_e} = \frac{\partial g}{\partial I_e} + \lambda = o \qquad \text{or}$$

$$\frac{\partial g}{\partial I_p} = \frac{\partial g}{\partial I_e}$$

, which means that the morginal products of both, investment in physical capital and investment in educational capital, must equal. This is the result a simple comparison of benefits of a one dollar investment in different sectors would give, the result of building the decision on an comparison of different internal rates of interest. But this is too naive and the Keynesian approach equality gives one more reason of a purely economic kind for high investment in education. Our general results can very easily be verified with special functions, e.g. a Cobb-Douglas function for $\dot{Y}$.

Finally let us have a look at the second order conditions for our constraint maximum problem (A.6). In order that the conditions (A.7) indeed specify a maximum problem, the principle minors of the bordered matrix of the second derivatives must alternate in sign. But we can show, that the extremum must be a maximum, in a simpler way. Assume (A.12) does not hold, let I_p be greater than the optimal I_p^*. Hence because of $\frac{\partial^2 \dot{Y}}{\partial I_p^2} < o$

$$\frac{\partial g / \partial I_e}{\partial g / \partial I_p} > \pi$$

, hence reducing I_p by an incremental amount and adding it to I_e would increase $\dot{Y}$. Hence (A.7) gives indeed a maximum, because similar arguments hold, if I_p is smaller than the optimal I_p^*.