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Hunting for superstars

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Abstract
The "superstar economy" is characterized by payoff functions that depend in a discontinuous way on the quality level of the corresponding products and services. Firm A might generate much higher returns than firm B, although A's product is only marginally superior to B's product. We look at an investor who considers to invest into start-ups that want to become active in one particular technological segment. Consequently only the very best few projects generate high returns. The investor is faced with a sequence of investment opportunities, observes the objective relative rankings of the corresponding projects seen so far, and must decide whether and how much to invest into the currently observed opportunity. Returns are realized at the end of the investment horizon. We derive the value functions and optimal investment rules for risk-neutral and risk averse investors. Under weak assumptions, the expected infinite horizon utility exceeds that of initial wealth. We show that for a risk-neutral investor "invest all or nothing", depending on the project's ranking and time of occurrence, is an optimal strategy. For a risk-averse investor the optimal rule is non-linear and path dependent. A simulation study is performed for risk-neutral and log-utility investors.

Keywords Economics of superstars · Optimal stopping · High-risk investments

JEL Classification: C60 · C81 · G24

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1 Introduction and motivation

The rapid growth of products and services marketed via the Internet increases the importance of considering revenues and profits generated by so called “superstars” or “superstar firms”. Examples are music streaming, search engines, Internet market places, news websites, where on the supply side goods can be marketed worldwide at almost zero marginal cost. At the same time consumers are often able to pick the—according to their perception—best product at no higher cost than the second best, third best etc. For example, since prices for streaming a film or listening to music on a particular platform are often fixed, there is no good reason to stream the second best film instead of the best (according to the consumer’s preferences), listen to the second best orchestra performing a particular piece of music instead of the best, etc. This is so, regardless of how small the perceived difference in quality between the films, orchestras, etc. is. Hence the ranking between competing artists, films, services, etc. becomes more important than the absolute quality level per se. Another important example are internet search engines. A keyword-search is “cost-less" (respectively the cost is always the same, users give away their data) and there is no good reason to use the second best search-engine instead of the best—at least this was the perception of most consumers during the early up to quite recent days of the internet economy.

By contrast, in the physical goods economy prices differ largely depending on quality levels or characteristics—that is why most of us are more likely to drive a medium-sized vehicle than a luxury car. Goods of similar quality can be sold at similar prices achieving at the same time similar market shares. Hence, for physical goods the payoffs depend continuously on the perceived quality level of these goods (see also heterogenous versions of the Cournot and the Bertrand model, e.g., [12] Chapter 16). Therefore, investing into firms that are active in the physical goods economy is much more predictable and less risky than investing into new economy firms.

Typically financing of projects related to Internet firms is performed by business angels and venture capitalists. In contrast to investment problems considered in the standard corporate finance literature, the investor is confronted with payoffs which depend on the rank of the corresponding project as already described in the above paragraphs. She faces cash flows arsing from investment projects where the main market share goes to the “winner”. Already the second-best makes a much smaller profit compared to the “winner”, let alone the lower-ranked competitors.

Gompers et al. [11] performed a survey among 885 venture capitalists. They [in their Table 5] compare the relevance of several factors, such as the team, the business model, the product, the market, etc. and find out that the founding team is the most important factor in the evaluation process. In addition, regarding valuation methods a bulk of venture capital firm use the net present value method or the internal rate of return, however “... 9% do not use any financial metrics”. By these two issues, a plausible way to approximate this behavior in a model is that investors are only able to rank the corresponding projects. Especially if the management team is very important, measuring the ability of a team on a cardinal scale is not very realistic, a ranking seems to be more plausible and provides an additional argument to consider rank dependent investment strategies.

As motivated by the above paragraphs, we study investment problems for superstar projects, that is, investment projects with rank dependent payoffs. To simplify the analysis, the financial resources the investor is able to spend is considered to be fixed, for example already provided by an investment fund. Entrepreneurs submit their business proposals. The venture capital investor is able to compare the corresponding projects with respect to their relative quality. That is, the relative rank of a project is observed. The profitability of a particular investment depends on its realized ranking among all projects. That is, the rank of the chosen project at the end of the investment horizon. Hence, the investment problem considered is different from standard portfolio optimization problems investigated in Finance (for a brief overview, see, e.g., [2] diversification effects can be expected by investing into multiplie investment opportunities in standard problems, while in our model investing into the best assests only is important). Due to competition among investors for superstar projects, the investor is forced to be quick with her decision. Taking this to the extreme, she has to decide on whether and how much to invest before she receives the next proposal. Based on the superstar property only those n alternatives with the highest quality characteristic generate positive returns in the future. Given these assumptions, this article obtains optimal investment strategies.

This article is related to the economics of superstars literature, finance literature, and to the mathematical literature on so called "secretary problems": Older literature on the economics of superstars, such as Frank and Cook [9], considers optimal remuneration schemes and when/where they are rank dependent (see, e.g., Pancs [21]). So called winner-take-all markets (see, e.g., [9]) are characterized by returns depending chiefly on the relative ranking of competitors and much less on the absolute quality of the products. Instead of looking at optimal payoffs schemes as discussed in the economics of superstar literature, the focus of this article is on optimal investment schemes when the expected payoffs strongly depend on the ranking of the corresponding investment project.

Recently Opp [20] developed an interesting dynamic equilibrium model describing the venture capital market. The framework considered in Opp [20] allows to obtain asset prices and returns in closed form. In particular, Opp [20] also derives equilibrium prices and abnormal expected returns. In Opp [20] at each point of time the most productive entrepreneur is selected (see Section 1.3.3 and 1.3.4 in Opp [20]). By contrast, in our model the n best projects generate returns at the end of the investment horizon T. The main task of the investor in our model is to optimally select the investment opportunities, where an investment project can turn out to be completely disastrous, if too many better projects appear in the unforeseen future.

Korteweg and Nagel [14] claim that linear factor pricing methods—as applied in standard asset pricing—are quite difficult to be applied to venture capital investing. As an alternative they propose to use stochastic discount factors and applied this methodology to empirical data from venture capital funds and individual investment projects. First, Korteweg and Nagel [14] inferred strong positive abnormal returns arsing from venture capital investments, while for venture capital funds these abnormal returns are close to zero. Probably a lot of the abnormal return generated by the corresponding investment projects is soaked up by costs arsing for the venture capital firm to evaluate business plans, select projects, and support the enterprises, etc. Hence, finding good investment projects is complicated, especially if the decision has to be quite fast due to competition with other venture capital firms. Second, the authors used data from *Sand Hill Econometrics* to apply their evaluation method to empirical data. The data base considers cash flows from already realized investment projects, while this papers models the time period before and obtains optimal investment rules. In this paper we also perform a simulation study to get some further insights into the investment behavior implied by our optimal rules. To do this, estimates of the expected gross returns become necessary. To get estimates for gross returns of superstar projects we use estimates from Cochrane [7] in our simulation study.

In the Mathematics literature investment decisions where only the winner ($n = 1$) obtains a positive return are solved in Bruss and Ferguson [3]. The case where the two best alternatives

($n = 2$) generate a positive return is considered in Łebek and Szajowski [15]. To the best of our knowledge these two articles are the only papers that consider investment models with rank dependent payoffs. The optimal investment strategy for the linear case with general n is related to Mucci [18]'s solution of the secretary problem with generalized payoffs. That paper is a generalization of the classical secretary problem with rank dependent payoffs (see the seminal paper of [5]). In the latter two articles exactly one secretary has to be chosen, while in former papers and in the current article a budget exists which can be invested into multiple alternatives and need not be invested at all. In all the previously cited articles of this paragraph the returns are non-stochastic. In this article we allow for stochastic returns and extend the results obtained in Bruss and Ferguson [3] and Łebek and Szajowski [15] to the case where the n best alternatives result in a positive payoff.

This article is organized as follows: Sect. 2 describes the investment problem in more detail and provides an almost closed form solution for an investor with a monotone increasing and continuous Bernoulli utility function. Then Sect. 3 investigates the risk-neutral case for $n \geq 1$ projects and deterministic returns. Section 4 considers stochastic returns. Finally, Sect. 5 presents results from a simulation study to investigate how optimal investment behavior changes with the attitude towards risk and the number of time periods considered.

2 The investment problem

In this section we follow Bruss and Ferguson [3] and consider $t = 1, \ldots, T$ periods of time. An investor is endowed with initial wealth $w_0 > 0$. At each period of time an investment opportunity, described by a uniformly iid random variable X_t on the unit interval [0, 1] occurs. The random variable X_t describes the unobserved quality level of the project occurring at time t. Let $\rho_t(X_s) := \sum_{v=1}^{t} \mathbf{1}_{X_v \geq X_s}$, $s \leq t$, and $\rho_T(X_s) := \sum_{v=1}^{T} \mathbf{1}_{X_v \geq X_s}$ denote the relative (at period t) and the absolute rank of X_s. Correspondingly, for a realization x_s, $\rho_t(x_s)$ and $\rho_T(x_s)$ denote the realized relative rank of x_s at time t, respectively the realized final rank. Hence the largest draw arising in $1, \ldots, t$ has relative rank 1 and the second best relative rank 2, etc. At time t, the period where the kth best observation (of the draws that occurred so far) was realized is denoted by τ_t^k. Note that τ_t^k depends on $x_1, \ldots, x_t$ and is therefore a function of time t.

In any period t the investor can invest any amount b_t, with $0 \leq b_t \leq w_{t-1}$ into the risky project showing up at this moment. But at time t the investor cannot invest in projects that occurred in one of the previous periods (recall is impossible). We assume that at time t the investor remembers past investments b_s and the current relative ranks of the past realizations x_s, $s \leq t-1$. However, she never observes the values x_s, but only the ranks among the past realizations.[1] The investment history at time t is $\mathbf{b}_{t-1} := (b_1, \ldots, b_{t-1}) \in \mathbb{R}_{\geq 0}^{t-1}$. By investing b_t the remaining budget is $w_t = w_{t-1} - b_t$.

Returns can be deterministic or stochastic. In the deterministic case the gross-returns per unit invested are described by $\left(\beta_t^i\right)_{t=1,\ldots,T;\, i=1,\ldots,n}$. In more detail, β_t^j abbreviates the deterministic return per unit invested in period t, if the draw of period t becomes the overall j-th best draw at time T. This is of course a simplifying assumption reflecting the idea that the selection period is relatively short compared to the future point in time when the returns finally realize as often observed with high-risk investment projects. If the lifetimes of the investment projects are different, this can be accounted for by discounting the payoffs to a

[1] By contrast in the so called full information case—briefly considered in Sect. 3 and the Appendix F—the investor is able to observe x_t and therefore to compare the absolute quality levels x_s, $s \leq t$.

common reference period. To keep the notation simple these discount factors are already contained in the β_t^i, that is:

Remark 1 A deterministic discount factor δ, $\delta \in (0, 1]$, can be included in a straightforward way by considering $\beta_t^j = \tilde{\beta}^j \delta^t$ or $B_t^j = \tilde{B}^j \delta^t$, etc. Stochastic discount factors δ_t, and time dependent $\tilde{\beta}^j$, $\tilde{B}^j$ can incorporated accordingly.

Our model has the following limitations: First, in contrast to what is often observed on the venture capital market, we abstract from investment stages. Second, we need to assume that there are no returns obtained before the last investment opportunity occurs. As long as all returns realize after the last investment opportunity shows up, differences in the time points of the realization of the returns can accounted for by choosing the β_j^t (or B_t^j) accordingly. In this case, β_j^t (or B_t^j) can also describe an accumulated return.

Suppose that the best n, $n \leq T$, projects yield a strictly positive return (almost surely non-negative and strictly positive with positive probability, in the case of stochastic returns). In the stochastic case considered in Sect. 4, the returns are denoted by $\left(B_t^i\right)_{t=1,\dots,T;\ i=1,\dots,n}$. We assume that B_t^i only depends on t, the rank of the corresponding X_t, and the (unobserved) realized value of X_t. For any realization x_s with absolute rank $\rho_T(x_s)$ larger than n the return is equal to zero. Note that by these assumptions on the returns the payoffs of the investment projects become more discontinuous. While small changes in technological shocks usually have small impacts on asset prices and returns in production based asset pricing models (see, e.g., [4] Chapter 7), in our model small changes in shocks X_t can have large effects if the ranking changes.

Next, we describe the $n(n+1)/2$-dimensional vector of potential payoffs $\mathbf{y}_{t-1}$ coming from past investments. Let $y_{i|\ell}^t \geq 0$ denote the payoff of the currently i-th best alternative—after X_t realized—if it becomes ℓ-th best at T. For example, $y_{1|1}^t = \beta_{\tau_t^1}^1 b_{\tau_t^1}$ is the payoff resulting from the corresponding investment $b^{\tau_t^1}$ if the current best alternative (i.e. $x_{\tau_t^1}$, where $\tau_t^1 \leq t$) remains the best at time T, $y_{1|2}^t = \beta_{\tau_t^1}^2 b_{\tau_t^1}$ is the payoff if the current best alternative becomes the second best at T, $y_{2|2}^t = \beta_{\tau_t^2}^2 b_{\tau_t^2}$ is the payoff if the current second-best alternative remains the second-best at T, etc.

Note that $y_{i|\ell}^t$ where $\ell < i$ is infeasible (since "better draws stay better"), while $y_{i|\ell}^t = 0$ for all $\ell > n$ by the assumption that only the best n projects can result in a positive return and the returns of all other projects are zero. If $t < n$ the corresponding $y_{j|\cdot}^t$ with $t < j \leq n$ are set to zero. Let ℓ_i denote the final rank (at time T) of the i-th best observation at time t. Hence, $\ell_i \geq i$ and $\ell_i \leq i + T - t$ given that there are $T - t$ remaining draws. For any pair $y_{i|\ell_i}^t$, $y_{j|\ell_j}^t$, where $i < j$, we get $\ell_i < \ell_j$. By considering the current n best draws after X_t realized, we obtain $\frac{n(n+1)}{2}$ terms $y_{j|\ell_j}^t := \beta_{\tau_t^j}^{\ell_j} b_{\tau_t^j} \geq 0$ and the lower triangular $n \times n$ matrix

$$\begin{pmatrix} \beta_{\tau_t^1}^1 b_{\tau_t^1} & & & & \\ \beta_{\tau_t^1}^2 b_{\tau_t^1} & \beta_{\tau_t^2}^2 b_{\tau_t^2} & & & \\ \beta_{\tau_t^1}^3 b_{\tau_t^1} & \beta_{\tau_t^2}^3 b_{\tau_t^2} & \beta_{\tau_t^3}^3 b_{\tau_t^3} & & \\ \vdots & \vdots & \vdots & \ddots & \\ \beta_{\tau_t^1}^n b_{\tau_t^1} & \beta_{\tau_t^2}^n b_{\tau_t^2} & \beta_{\tau_t^3}^n b_{\tau_t^3} & \dots & \beta_{\tau_t^n}^n b_{\tau_t^n} \end{pmatrix}$$

$$= \begin{pmatrix} y^t_{1|1} & & & & \\ y^t_{1|2} & y^t_{2|2} & & & \\ y^t_{1|3} & y^t_{2|3} & y^t_{3|3} & & \\ \vdots & \vdots & \vdots & \ddots & \\ y^t_{1|n} & y^t_{2|n} & y^t_{3|n} & \cdots & y^t_{n|n} \end{pmatrix} =: \mathcal{Y}_t. \tag{1}$$

The remaining upper triangular part of this matrix is zero since $y^t_{j|\ell_j} = 0$ for all $\ell_j < j$. To obtain the row vector $\mathbf{y}_t \in \mathbb{R}^{n(n+1)/2}$ the lower triangular part of $\mathcal{Y}_t$ is vectorized, in formal terms

$$\mathbf{y}_t = \left(y^t_{1|1}, \ldots, y^t_{1|n}, y^t_{2|2}, \ldots, y^t_{2|n}, \ldots, y^t_{n-1|n-1}, y^t_{n-1|n}, y^t_{n|n} \right).$$

The following paragraph describes the transition of $\mathcal{Y}_{t-1}$ to $\mathcal{Y}_t$: Depending on the realization of X_t, $\mathcal{Y}_{t-1}$ changes to $\mathcal{Y}_t$ in period t. Consider a realization x_t with realized rank $\rho_t(x_t) = j$ and let $\boldsymbol{\beta}^{j:n}_t := \left(\beta^j_t, \ldots, \beta^n_t\right)$. Recall that $\boldsymbol{\tau}_t = \left(\tau^1_t, \ldots, \tau^n_t\right)^\top$ denotes the points of time where the x_s, $s \le t$, with the ranks $1, \ldots, n$ had realized. If the rank of x_t is larger than n, then $\boldsymbol{\tau}_{t-1}$ remains the same: $\boldsymbol{\tau}_t = \boldsymbol{\tau}_{t-1}$. If x_t has rank j, $1 \le j \le n$, then we obtain the new $\boldsymbol{\tau}_t = \left(\tau^1_{t-1}, \ldots, \tau^{j-1}_{t-1}, t, \tau^j_{t-1}, \ldots, \boldsymbol{\tau}^{n-1}_{t-1}\right)^\top$. This update of $\boldsymbol{\tau}_{t-1}$ is abbreviated by $\boldsymbol{\tau}_{t-1}(x_t)$ resulting in the new $\boldsymbol{\tau}_t$. If the agent invests b_t, then the new matrix of payoffs is provided by

$\mathcal{Y}_t = \mathcal{Y}_{t-1}\left(b_t \boldsymbol{\beta}^{j:n}_t\right)$, where the function

$$\mathcal{Y}_{t-1}\left(b_t \boldsymbol{\beta}^{j:n}_t\right) := \begin{pmatrix} y^{t-1}_{1|1} & & & & & & & \\ y^{t-1}_{1|2} & y^{t-1}_{2|2} & & & & & & \\ \vdots & \vdots & \ddots & & & & & \\ y^{t-1}_{1|j-1} & y^{t-1}_{2|j-1} & \cdots & y^{t-1}_{j-1|j-1} & & & & \\ y^{t-1}_{1|j} & y^{t-1}_{2|j} & \cdots & y^{t-1}_{j-1|j} & \beta^j_t b_t & & & \\ y^{t-1}_{1|j+1} & y^{t-1}_{2|j+1} & \cdots & y^{t-1}_{j-1|j+1} & \beta^{j+1}_t b_t & y^{t-1}_{j|j+1} & & \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \ddots & \\ y^{t-1}_{1|n} & y^{t-1}_{2|n} & \cdots & y^{t-1}_{j-1|n} & \beta^n_t b_t & y^{t-1}_{j|n} & \cdots & y^{t-1}_{n-1|n} \end{pmatrix} = \begin{pmatrix} y^t_{1|1} & & & & \\ y^t_{1|2} & y^t_{2|2} & & & \\ y^t_{1|3} & y^t_{2|3} & y^t_{3|3} & & \\ \vdots & \vdots & \vdots & \ddots & \\ y^t_{1|n} & y^t_{2|n} & y^t_{3|n} & \cdots & y^t_{n|n} \end{pmatrix}. \tag{2}$$

The dimension of $\boldsymbol{\beta}^{j:n}_t$ determines the column j where $b_t \boldsymbol{\beta}^{j:n}_t$ is inserted in (2). If $\rho_t(x_t) > n$, then $\mathcal{Y}_t = \mathcal{Y}_{t-1}$. We obtain the new vector of payoffs $\mathbf{y}_t$ by vectorizing $\mathcal{Y}_{t-1}\left(b_t \boldsymbol{\beta}^{j:n}_t\right)$. An alternative approach to storing $\mathbf{y}_t$ is to work with the investment histories $\mathbf{b}_t = (b_1, \ldots, b_t)$. By means of $\mathbf{b}_t$ and $\boldsymbol{\tau}_t$, $\mathbf{y}_t$ can be obtained in a straightforward way.

The investor maximizes expected utility. The Bernoulli utility function $u(\cdot)$ is monotone increasing and continuous [that is, we allow for risk-averse, risk-neutral or risk-loving behavior].

Venture capitalists and business angels are typical investors in high-risk investment projects. Venture capital firms can be considered as institutionalized investors that have various opportunities to diversify, such that we consider them as risk-neutral investors. Business angels usually have less opportunities to diversity, hence we consider them as risk-averse.[2]

The remaining budget after period $T-1$ is w_{T-1} and the information on prior investments is provided by $\mathbf{y}_{T-1}$. Then, the term $\mathscr{V}_{T-1}(w_{T-1}, \mathbf{y}_{T-1})$ denotes the conditional expected utility when applying an optimal investment strategy in period T.[3]

[2] We thank one anonymous referee for this comment.

[3] For the existence of expected utility for the deterministic and the stochastic case see Appendix D.

Accordingly, $\mathscr{V}_{t-1}(w_{t-1}, \mathbf{y}_{t-1})$ denotes the conditional expected utility of the final wealth given that the investor applies an optimal investment strategy from t onward [subject to an investment history described by w_{t-1} and $\mathbf{y}_{t-1}$]. After investing b_t, the remaining budget becomes $w_t = w_{t-1} - b_t$ and $\mathbf{y}_{t-1}$ becomes $\mathbf{y}_t$, as already described above. In particular, the value functions $\mathscr{V}_t(\cdot)$ for $t = T$ and $t = T - 1$ are

$$\mathscr{V}_T(w_T, \mathbf{y}_T) = u(w_T + \sum_{j=1}^{n} y_{j|j}^T) \text{ and}$$

$$\mathscr{V}_{T-1}(w_{T-1}, \mathbf{y}_{T-1}) = \frac{T-n}{T} u\left(w_{T-1} + \sum_{j=1}^{n} y_{j|j}^{T-1}\right) + \frac{1}{T}\sum_{j=1}^{n} u\left(w_{T-1}\max\{1, \beta_T^j\} + \sum_{i=1}^{j-1} y_{i|i}^{T-1} + \sum_{i=j}^{n-1} y_{i|i+1}^{T-1}\right). \quad (3)$$

To see this, note that at $t = T$ there are no more steps to go and we obtain a final wealth of $w_T + y_{1|1}^T + \cdots + y_{n|n}^T$. w_T is the remaining non-invested budget, and $y_{j|j}^T$ the payoffs of the first to n-th best alternatives, when there are no further steps to go. If there is still one step to go, that is $t = T - 1$, we get $w_{T-1} + y_{1|1}^{T-1} + \cdots + y_{n|n}^{T-1}$ if the last draw X_T has a (final) rank larger than n. The probability of this event is $\frac{T-n}{T}$. With a probability of $1/T$ we get a draw of rank j, where $j \leq n$. In this case the current best $j - 1$ draws remain the $j - 1$ best, the realization x_T becomes the j-th best, while the current j-th to n-th best draws become the $j + 1$ to $n + 1$-th best. The payoff of the $n + 1$-th best draw is zero by the model assumptions. At $t = T$ the best strategy is to invest all the remaining budget w_{T-1} into this alternative, if and only if the relative (= absolute) rank of x_T is equal to $j \leq n$ and $\beta_T^j \geq 1$. For example, with a probability of $1/T$ the T-th draw becomes the best one. In this case the former best draw becomes the new second best, yielding the payoff $y_{1|2}^{T-1}$, while the former second best becomes third best, etc. For the value function we get the recursion

$$\begin{aligned}
&\mathscr{V}_{t-1}(w_{t-1}, \mathbf{y}_{t-1}) \\
&= \max\left\{\frac{t-n}{t}, 0\right\} \mathscr{V}_t(w_{t-1}, \mathbf{y}_{t-1}) \\
&\quad + \frac{1}{t}\max\nolimits_{0 \leq b_t \leq w_{t-1}} \mathscr{V}_t\left(w_{t-1} - b_t, \mathbf{y}_{t-1}\left(b_t \boldsymbol{\beta}_t^{1:n}\right)\right) + \\
&\quad + \frac{1}{t}\max\nolimits_{0 \leq b_t \leq w_{t-1}} \mathscr{V}_t\left(w_{t-1} - b_t, \mathbf{y}_{t-1}\left(b_t \boldsymbol{\beta}_t^{2:n}\right)\right) + \\
&\quad \vdots \\
&\quad + \frac{1}{t}\max\nolimits_{0 \leq b_t \leq w_{t-1}} \mathscr{V}_t\left(w_{t-1} - b_t, \mathbf{y}_{t-1}\left(b_t \boldsymbol{\beta}_t^{\min\{n,t\}):n}\right)\right) \\
&= \max\left\{\frac{t-n}{t}, 0\right\} \mathscr{V}_t(w_{t-1}, \mathbf{y}_{t-1}) \\
&\quad + \frac{1}{t}\sum_{j=1}^{\min\{n,t\}} \max\nolimits_{0 \leq b_t \leq w_{t-1}} \mathscr{V}_t\left(w_{t-1} - b_t, \mathbf{y}_{t-1}\left(b_t \boldsymbol{\beta}_t^{j:n}\right)\right).
\end{aligned} \quad (4)$$

The current value described in (4) is obtained as follows: The first term accounts for draws worse than the n-th best, in which case $w_t = w_{t-1}$ and $\mathbf{y}_t = \mathbf{y}_{t-1}$. The probability of this

event is $\frac{t-n}{n}$ [provided that $t > n$ and zero else]. The further n terms consider expected utility contributions arising if the relative rank of the draw x_t is $j = 1, \ldots, n$. If x_t is the j-th best, the payoffs resulting from investing b_t in this opportunity are described by $b_t \boldsymbol{\beta}_t^{j:n}$, in addition, in this case, we have $y^t_{i+1|k} = y^{t-1}_{i|k}$ for $i = j, \ldots, n-1$, $k = i+1, \ldots, n$, and $\mathbf{y}_t = \mathbf{y}_{t-1}\left(b_t \boldsymbol{\beta}_t^{j:n}\right)$.
For $t = 0$ we get

$$\mathscr{V}_0(w_0) = \max_{0 \le b_1 \le w_0} \mathscr{V}_1\left(w_0 - b_1, b_1 \boldsymbol{\beta}_1^{1:n}, 0, \ldots, 0\right). \tag{5}$$

For $t = 1$, we have $\tau_t^1 = 1$, $\tau_t^2, \ldots, \tau_t^n = 0$, $\mathbf{y}_t = \mathbf{y}_1 = \left(y^1_{1|1}, \ldots, y^1_{1|n}, 0, \ldots, 0\right)$, and for $n > 1$ we get[4]

$$\begin{aligned}
&\mathscr{V}_1\left(w_1, y^1_{1|1}, \ldots, y^1_{1|n}, 0, \ldots, 0\right)\\
&= \frac{1}{2}\mathbf{1}_{(\rho_2(x_2)=1)} \max_{0 \le b_2 \le w_1} \mathscr{V}_2\left(w_1 - b_2, b_2 \boldsymbol{\beta}_1^{1:n}, y^1_{1|2}, \ldots, y^1_{1|n}, 0, \ldots, 0\right)\\
&+ \frac{1}{2} \max_{0 \le b_2 \le w_1} \mathscr{V}_2\left(w_1 - b_2, y^1_{1|1}, \ldots, y^1_{1|n}, b_2 \boldsymbol{\beta}_1^{2:n}, 0, \ldots, 0\right)
\end{aligned}$$

while for $n = 1$ we obtain

$$\begin{aligned}
&\mathscr{V}_1\left(w_1, y^1_{1|1}, \ldots, y^1_{1|n}, 0, \ldots, 0\right)\\
&= \frac{1}{2}\mathbf{1}_{(\rho_2(x_2)=1)} \max_{0 \le b_2 \le w_1} \mathscr{V}_2\left(w_1 - b_2, b_2 \boldsymbol{\beta}_1^{1:n}, y^1_{1|2}, \ldots, y^1_{1|n}, 0, \ldots, 0\right)\\
&+ \frac{1}{2} \mathscr{V}_2\left(w_1, y^1_{1|1}\right).
\end{aligned} \tag{6}$$

The value functions $\mathscr{V}_t(w_t, \mathbf{y}_t)$, $t < n$, also follow from (4) in the same way, where the last $n - t$ columns of $\mathcal{Y}_t$ are zero.

Suppose that we follow the strategy "do nothing in the periods $> t$". Before X_{t+1} realizes the decision maker knows w_t and $\mathbf{y}_t = \left(y^t_{1|1}, \ldots, y^t_{n(t)|n}, 0, \ldots, 0\right)^\top$, where $n(t) := \min\{n, t\}$. If we do nothing in the remaining periods, we obtain $w_t + y^t_{1|\ell_1} + \cdots + y^t_{n(t)|\ell_{n(t)}}$, where ℓ_j, $j = 1, \ldots, n(t)$, are the final ranks for best $n(t)$ draws at the end of period t. The probabilities of these ranks ℓ_j, $j = 1, \ldots, n(t)$, are

$$p_t\left(\ell_1, \ldots, \ell_{n(t)}\right) = \frac{\binom{T - \ell_{n(t)}}{t - n(t)}}{\binom{T}{t}}. \tag{7}$$

These probabilities are derived in Appendix A. Then, the conditional expected utility of the strategy "do nothing in the remaining periods $> t$" is given by

$$\mathscr{W}_t(w_t, \mathbf{y}_t) := \sum_{\substack{\ell_1, \ldots, \ell_{n(t)}: \\ 0 < \ell_1 < \cdots < \ell_{n(t)} \le T - t + n(t)}} p_t\left(\ell_1, \ldots, \ell_{n(t)}\right) u\left(w_t + \sum_{j=1}^{n(t)} y^t_{j|\ell_j}\right). \tag{8}$$

Let $Z\left(\ell_1, \ldots, \ell_{n(t)+1}; w_t - b_{t+1}, \mathbf{y}_t(b_{t+1} \cdot \boldsymbol{\beta}_{t+1}^{j:n})\right) := (w_t - b_{t+1}) + y^t_{1|\ell_1} + \cdots + y^t_{j-1|\ell_{j-1}} + \beta_{t+1}^{\ell_j} b_{t+1} + y^t_{j|\ell_{j+1}} + \cdots + y^t_{n(t)|\ell_{n(t)+1}}$ denote a final payoff for the strategy where

[4] Eq. (6) follows from (4). For $n = 1$, the matrix $\mathcal{Y}_t = y^1_{1|1}$.

$b_{t+1} > 0$ is invested, if past and current investments have final ranks $\ell_1, \ldots, \ell_{n(t)+1}$ [and the time-$t+1$ draw has rank j at period $t+1$] while $Z\left(\ell_1, \ldots, \ell_{n(t)+1}; w_t, \mathbf{y}_t(0 \cdot \boldsymbol{\beta}_{t+1}^{j:n})\right) := w_t + y_{1|\ell_1}^t + \cdots + y_{j-1|\ell_{j-1}}^t + 0 + y_{j|\ell_{j+1}}^t + \cdots + y_{n(t)|\ell_{n(t)+1}}^t$ is the payoff for the strategy "doing nothing" after time t if past (up to and including time t) investments have final ranks $\ell_1, \ldots, \ell_{j-1}, \ell_{j+1}, \ldots, \ell_{n(t)+1}$. Endowed with this notation for our investment problem we get:

Theorem 1 *Consider a monotone increasing and continuous utility function $u(\cdot)$ and the conditional expected utility of the strategy "do nothing in the periods $> t$" defined in (8). For $t = 0, 1, \ldots, T$, the value function $\mathscr{V}_t(w_t, \mathbf{y}_t)$ satisfies*

$$\mathscr{V}_t(w_t, \mathbf{y}_t) = \mathscr{W}_t(w_t, \mathbf{y}_t) + r_t\,(w_t, \mathbf{y}_t)\,. \tag{9}$$

The residual terms $r_t\,(w_t, \mathbf{y}_t)$ are obtained as follows: $r_T(w_T, \mathbf{y}_T) = 0$ and the $r_t\,(w_t, \mathbf{y}_t)$, $t = T-1, \ldots, 1$, follow from the recursion

$$\begin{aligned} r_t(w_t, \mathbf{y}_t) =& \max\left\{\frac{t+1-n}{t+1}, 0\right\} r_{t+1}(w_t, \mathbf{y}_t) \\ &+ \frac{1}{t+1} \sum_{j=1}^{\min\{t+1,n\}} \max_{0 \le b_{t+1} \le w_t} \Bigg[r_{t+1}\left(w_t - b_{t+1}, \mathbf{y}_t\left(b_{t+1}\boldsymbol{\beta}_{t+1}^{j:n}\right)\right) \\ &+ \sum_{\substack{\ell_1,\ldots,\ell_{n(t)+1}: \\ 0<\ell_1<\cdots<\ell_{n(t)+1}\le T-t+n(t)}} p_{t+1}\left(\ell_1, \ldots, \ell_{n(t)+1}\right)\left(u\left(\tilde{Z}\right)\right. \\ &\left.- u\left(Z\left(\ell_1, \ldots, \ell_{n(t)+1}; w_t, \mathbf{y}_t(0 \cdot \boldsymbol{\beta}_{t+1}^{j:n})\right)\right)\right) \Bigg] \end{aligned} \tag{10}$$

where $\tilde{Z} = Z\left(\ell_1, \ldots, \ell_{n(t)+1}; w_t - b_{t+1}, \mathbf{y}_t(b_{t+1}\boldsymbol{\beta}_{t+1}^{j:n})\right)$. We have $r_t(w_t, \mathbf{y}_t) \ge 0$ for all $t = 1, \ldots, T$ and $\mathscr{W}_0(w_0) = w_0$. The optimal investments b_{t+1}, $t = 0, 1, \ldots, T-1$, are the maximizers of the expressions inside the squared brackets in (10). If $\beta_t^j < 1$ for all t and j, then $b_t = 0$ for all $t = 1, \ldots, T$. $\beta_t^j \ge 1$ for some t and j is necessary but not sufficient for $b_t > 0$.

For the proof see Appendix B.

The above theorem is an extension of Bruss and Ferguson [3] and Łebek and Szajowski [15], who consider the cases $n = 1$ and $n = 2$, respectively. We observe a split of the value function $\mathscr{V}_t\,(w_t, \mathbf{y}_t)$ into two parts. The term $\mathscr{W}_t\,(w_t, \mathbf{y}_t)$ is the expected utility, that after period-t uncertainty has resolved and period-t investments have been made, the investor expects to receive, if from that moment on she would passively lean back without investing any of the non-invested remaining budget w_t. This is the expected utility coming from past and present investments. As (8) shows, this term is easy to compute. The term $r_t\,(w_t, \mathbf{y}_t)$ is the additional expected utility that the investor expects to receive if she would instead invest optimally in the future, that is, in all the remaining periods. Since "leaning back" is always a feasible strategy, this additional expected utility must be non-negative.

For the optimal investment strategies we observe: (i) The maximization problem arsing from the term in squared brackets in (10) is non-linear in the investments b_{t+1}. Hence, in general, we get some optimal b_{t+1}^*, where $0 \le b_{t+1}^* \le w_t$. By contrast, for the risk-neutral case considered in Sect. 3, we show that a "bang-bang" strategy, that is b_{t+1}^* is either 0 or $w_t = w_0$, is an optimal strategy.

(ii) Consider the case of constant relative risk-aversion. In a purely static standard investment problem the optimal investment weights b/w, do not depend on the size of w. In the current investment model the investment weights b_t/w_t do not only depend on the relative rank of x_t and time t, but also on prior investments $\mathbf{b}_{t-1}$. To see this, note that prior investments enter via $\mathbf{y}_{t-1}$ into the utility maximization problem contained in the recursions (10).

(iii) Let us now consider the case when the number of periods considered becomes large. For linear utility, Bruss and Ferguson [3] [see Example 2.3.1] derived a closed form expression for the value function when the number of periods $T \to \infty$. In this case, given an initial wealth w_0 and a return $\beta_k^1 = \beta^1 > 1$ the value function is $w_0\beta^1 \exp(-(\beta^1 - 1)/\beta^1) > w_0$. Hence, even if the number of periods approaches infinity the investor expects to receive a strictly positive net return. As an example, for $\beta^1 = 2$ and an initial wealth of $w_0 = 1$, the value function becomes $\beta^1 \exp(-(\beta^1 - 1)/\beta^1) = 1.21306$. To consider the non-linear case we proceed as follows: In the *classical secretary problem* the goal is to maximize the probability to get the element with rank one when only the relative ranks are observed. Lindley [16] has shown that it is optimal to reject the first r^* options and then take the first alternative where the realized relative rank is equal to one. The number r^* follows from $r^* = \min\{1 \le r \le T | \sum_{k=r+1}^{T} \frac{1}{k-1} \le 1\}$. If we let the number of periods $T \to \infty$, r^* becomes $\frac{T}{e}$. With this strategy the probability of selecting the overall best approaches $\frac{1}{e}$ for $T \to \infty$. Using this, let us briefly remark on the case when T approaches infinity: Assume that $\beta_t^1 \ge 2$, for all t and the utility function $u(\cdot)$ is continuously differentiable. Consider the investment strategy: "Invest $\eta > 0$ in the first investment opportunity with relative rank 1 occurring at times larger than $\frac{T}{e}$, and otherwise do not invest at all". According to the solution of the classical secretary problem, the investor invests in the opportunity with absolute rank 1 with a probability of $\frac{1}{e}$. In this case the investor receives a payoff of $w_0 + (\beta_t^1 - 1)\eta$. With probability $\frac{1}{e}$ the opportunity with rank 1 occurred among the first $\frac{T}{e}$ periods and hence the investor will not invest and will end up with a final wealth of w_0. With the remaining probability of $\frac{e-2}{e}$ the investor invests and receives at least $w_0 - \eta$. This strategy yields the expected utility of $\frac{1}{e}u\left(w_0 + (\beta_t^1 - 1)\eta\right) + \frac{1}{e}u\left(w_0\right) + \frac{e-2}{e}u\left(w_0 - \eta\right)$. As $\frac{e-2}{e} < \frac{1}{e}$, for some small $\eta > 0$ and a monotone increasing and continuously differentiable utility function $u(\cdot)$, the inequality $\frac{1}{e}u\left(w_0 + (\beta_t^1 - 1)\eta\right) + \frac{1}{e}u\left(w_0\right) + \frac{e-2}{e}u\left(w_0 - \eta\right) > u(w_0)$ is satisfied. If there are further positive payoffs $\beta_t^j > 1$, $j > 1$, the expected utility of this strategy is larger and hence the condition on β^1 can be relaxed. Note that this inequality is independent of T. Obviously the expected utility of the above described strategy is a lower bound for the expected utility of the optimal strategy. Hence, we have:

Remark 2 Under the above assumptions on the β_t^1 and the utility function, the expected utility of the optimal strategy will be strictly larger than $u(w_0)$, even if T approaches infinity. In particular in the risk-neutral case, $\beta_t^1 > 1$, for all t, is sufficient for an expected payoff larger than w_0, also for $T \to \infty$.

One of the most unrealistic features of our model is that investment opportunities must be taken up immediately. One could modify this by allowing for each opportunity a fixed time window (e.g., 3 periods), in which the decision maker could invest in this opportunity. So, in period t one can invest into the opportunities that appeared in the periods $t-2$, $t-1$, and t. Then not much would change qualitatively. Of course, the investor would always wait till the last moment before investing in a given opportunity, to get as much information as possible.

3 Risk neutrality

This section considers the linear case, where $u(\cdot)$ is the identity function and the investor maximizes expected wealth. The risk-neutral case is mainly import for institutional investors such as venture capital firms. Suppose that the investor either invests 0 or the remaining budget w_t (which will turn out to be a feature of some optimal strategy for a risk-neutral investor). Then, by applying this strategy at t, the remaining budget w_t is either equal to w_0 [such that $b_1, \ldots, b_{t-1} = 0$ and $y_{j|\ell_j} = 0$] or the remaining budget is $w_t = 0$. In the second case the investor already invested $b_s = w_0$ in some period s, where $1 \leq s \leq t-1$. Hence, $\mathcal{Y}_{t-1}$ contains at most one non-zero column. The non-zero entries are $w_0 \boldsymbol{\beta}_s^{\rho_{t-1}(x_s):n} = \left(y_{\rho_{t-1}(x_s)|\rho_{t-1}(x_s)}, \ldots, y_{\rho_{t-1}(x_s)|n}\right)^\top$. For ranks larger than n the return of the investment is zero and knowledge of the rank of x_s is irrelevant. For the strategies "doing nothing in the remaining periods $> t$" and "invest w_0 or 0" we obtain

$$\mathscr{W}_t(w_t, \mathbf{y}_t) = \begin{cases} w_0 & \text{, if } b_s = 0 \text{ for all } s = 1, 2, \ldots, t \\ & \quad \text{and } w_t = w_0, \\ w_0 \sum_{\ell=\rho_t(x_s)}^{n(t)} p_t\left(\{\rho_T(x_s) = \ell\} | \rho_t(x_s)\right) \beta_s^\ell & \text{, if } b_s = w_0 \text{ at } s \in \{1, \ldots, t\} \\ & \quad \text{and } w_t = 0. \end{cases} \tag{11}$$

The probabilities $p_t\left(\{\rho_T(x_s) = \ell\} | \rho_t(x_s)\right)$ that x_s has final rank $\ell = \rho_t(x_s), \rho_t(x_s) + 1, \ldots$ given that its current rank [immediately after X_t realizes] is $\rho_t(x_s)$ follows from (7).[5] That is,

$$p_t\left(\{\rho_T(x_s) = \ell\} | \rho_t(x_s) = j\right) = \frac{\binom{\ell-1}{j-1}\binom{T-\ell}{t-j}}{\binom{T}{t}} \quad \text{for } \ell = j, \ldots, j+T-t\,. \tag{12}$$

We recursively define the additional value that one obtains when investing an amount of 1 optimally after period t. For $t = T$ we have $c_T = 0$. For $t < T$ we get

$$\begin{aligned} c_{t-1} =& \max\left\{\frac{t-n}{t}, 0\right\} c_t \\ &+ \frac{1}{t} \sum_{j=1}^{\min\{n,t\}} \max\left\{\sum_{\ell=j}^{n(t)} p_t\left(\{\rho_T(X_t) = \ell\} | \rho_t(X_t) = j\right)\left(\beta_t^\ell - 1\right), c_t\right\}. \end{aligned} \tag{13}$$

The residual term c_t is the expected contribution per unit of remaining capital to the value function by investing optimally instead of investing nothing in the last $T-t$ periods. In the proof of Theorem 2 we observe that $c_t = r_t(1, \mathbf{0})$. Then, for the linear case we obtain:

Theorem 2 *Consider a risk-neutral investor where $u(\cdot)$ is the identity $id(\cdot)$.*

(i) An optimal investment strategy is "invest $b_t = w_t = w_0$ at the first time t where

$$\sum_{\ell=\rho_t(x_t)}^{n(t)} p_t\left(\{\rho_T(x_t) = \ell | \rho_t(x_t)\}\right)\left(\beta_t^\ell - 1\right) \geq c_t$$

and 0 else".

[5] Note that these are negative hypergeometric distribution probabilities (see, e.g., Chow et al. [5], Ferguson [8][Chapter 2], and Chapter [13][page 47, equation 1.19]).

(ii) The value function $\mathscr{V}_t(w_t, \mathbf{y}_t)$ satisfies

$$\mathscr{V}_t(w_t, \mathbf{y}_t) = \mathscr{W}_t(w_t, \mathbf{y}_t) + w_t c_t$$
$$= \begin{cases} w_0(1 + c_t) & \text{, if } b_s = 0 \text{ for all } s = 1, \ldots, t \\ & \quad \text{and } w_t = w_0, \\ w_0 \sum_{\ell=\rho_t(x_s)}^{n} \; p_t\left(\{\rho_T\,(x_s) = \ell\}|\rho_t(x_s)\right) \beta_s^{\ell} & \text{, if } b_s = w_0 \text{ for one } s \in \{1, \ldots, t\} \\ & \quad \text{and } w_t = 0. \end{cases} \tag{14}$$

For proof see Appendix C.

The value function $\mathscr{V}_t(w_t, \mathbf{y}_t)$ is the sum of the conditional expected utility of the strategy "we do nothing in the remaining periods $> t$" $\mathscr{W}_t(w_t, \mathbf{y}_t)$ and the contribution of investing the remaining budget optimally in the remaining periods, i.e. $w_t c_t$. We observe that, if it is optimal to invest a positive amount, then it is also optimal to invest the whole remaining budget w_0. The secretary problem with generalized payoffs was already investigated in Mucci [18]. The chief difference to our model under risk-neutrality is that in the secretary problem a candidate has to be hired, while in our model the investor has the option of non-investing.

Remark 3 Note that the residual term in the risk-neutral case, $r_t(w_t) = w_t c_t$, does not depend on prior realizations of the random variable X_s, $s \leq t$. By contrast, in the general case the residual term is of the form $r_t(w_t, \mathbf{y}_t)$ [see Eq. (10)]. Hence, in the linear case the complexity of the optimization problem is reduced enormously, also from a computational point of view as observed in Sect. 5.

Let us now assume that $\beta_t^1 \geq \beta_t^2 \geq \cdots \geq \beta_t^n \geq 0$, for all $t = 1, \ldots, T$. It is intuitive that if the risk-neutral investor finds it optimal to invest at time t all her remaining budget in the current draw given that this draw has relative rank j, then she finds it also optimal to invest all her budget, if the draw at t has relative rank i with $i < j$. Her position is better in the latter scenario. This implies that for each point of time there exists a threshold d_t, such that the investor (upon having not invested before) invests all her budget at time t if the time-t draw has relative rank d_t or smaller.

Now let us assume in addition that the β_t^j do not depend on time. That is, there are $\beta^1 \geq \beta^2 \geq \cdots \geq \beta^n \geq 0$, such that $\beta_t^j = \beta^j$, for all $j = 1, \ldots, n$ and all $t = 1, \ldots, T$. Then it is also intuitive that if the risk-neutral investor finds it optimal to invest at time t all her remaining budget in the current draw of relative rank j, then she finds it also optimal to invest all her remaining budget at a later time, if that later draw has relative rank j. The reason for this is that in later periods there are less possibilities for the rank of a past investment to increase. This results in the following remark:

Remark 4 Consider a risk-neutral investor and assume that $\beta_t^1 \geq \beta_t^2 \geq \cdots \geq \beta_t^n \geq 0$, for all $t = 1, \ldots, T$.

1. Then there are rank-thresholds d_t, $t = 1, \ldots, T$ with $0 \leq d_t \leq n$ such that upon not having invested so far, the investor invests all the budget w_0 in period t if $\rho_t(x_t) \leq d_t$. If $d_s = 0$ for some period s, then this means that the investor never invests in that period.
2. If we assume in addition that the β_t^j do not depend on time, then these rank-thresholds are non-decreasing over time. That is $1 \leq t < s \leq T$ implies $d_t \leq d_s$. That means, these thresholds become less and less stringent over time.

For the cases $T = 3$ and $T = 5$ Appendix F obtains these thresholds for the case where $\beta_t^1 = 3$ and $\beta_t^2 = 2$, for all t. For $T = 3$ we get $d_1 = 0$, $d_2 = 1$, and $d_3 = 2$, while for $T = 5$ the thresholds are $d_1 = d_2 = 0$, $d_3 = d_4 = 1$, and $d_5 = 2$. That is, given the returns $\beta^1 = 3$

and $\beta^2 = 2$ and an investment horizon of $T = 3$ periods, the investor never invests at $t = 1$, she/he invests all the remaining wealth at $t = 2$ if the relative rank of the second draw is one, while she/he does not invest at $t = 2$ if the rank of the draw is two. At $t = 3$, the investor invests all the remaining wealth if the rank of the third draw is one or two, otherwise she/he does not invest.

Finally, let us compare the limited information to the full information case in a small example (all the calculations are provided in Appendix F, the case where only the best draw results in a positive return was already investigated in Bruss and Ferguson [3]). We once again consider $T = 3$ and $\beta^1 = 3$, $\beta^2 = 2$. For the limited information case, that is where the decision rule has to be rank dependent, we get the thresholds $d_1 = 0$, $d_1 = 1$ and $d_2 = 2$ and an expected wealth of $7/3 \approx 2.33$. For the full information case, that is if the investor is able to observe x_t at t, an agent invests into the first draw if and only if $x_1 \geq 0.679449$. At $t = 2$ she invests all the remaining wealth if and only if the relative rank is one and $x_2 \geq \frac{1-x_1}{2}$. At $t = 3$ all the remaining wealth is invested if the rank of the third draw is either 1 or 2, otherwise we do not invest. By applying this strategy under full information, the expected wealth is 2.59603. That is, already for a small number of periods the fully informed investor expects an additional wealth of approximately 0.2627. Hence, if an investor is only able to rank the projects compared to evaluating them quantitatively (observing x_t) the difference is quite costly.

4 Stochastic returns

Suppose that the returns per unit are random variables. These random variables are $B_1^j, \ldots, B_T^j$, $j = 1, \ldots, n$, the realizations are abbreviated β_t^j. The stochastic analogon of $\boldsymbol{\beta}_t^{j:n}$ is $\mathbf{B}_t^{j:n}$. All the returns are realized and therefore known by the investor after T. We assume $\mathbb{P}\left(B_t^j > 0\right) > 0$ and $\mathbb{P}\left(B_t^j \geq 0\right) = 1$ for all $j = 1, \ldots, n$ and $t = 1, \ldots, T$. We assume that the distribution of the return B_t^j is allowed to depend on t, the final rank $j = \rho_T(X_t)$, and the realization of X_t. The results of Sect. 2 can be extended to this stochastic case. Here, the deterministic returns are replaced by the stochastic returns B_t^j. The utility of the final wealth $u\left(w_t + \sum_{j=1}^{\min\{n,t\}} y_{j|\ell_1}^t + \cdots + y_{j|\ell_{n(t)}}^t\right)$ has to be replaced by the conditional expectation [6] of $u\left(w_t + \sum_{j=1}^{\min\{n,t\}} Y_{j|\ell_1}^t + \cdots + Y_{j|\ell_{n(t)}}^t\right)$, where $Y_{j|\ell}^t$ are the stochastic analogons of $y_{j|\ell}^t$ considered in (1). We assume that the corresponding expected values and conditional expectations exist in the following.[7] Let

$$\mathbb{E}_t\left(\mathscr{W}_t\left(w_t, \mathbf{Y}_t\right)\right) := \sum_{\substack{\ell_1,\ldots,\ell_{n(t)}:\\ 0<\ell_1<\cdots<\ell_{n(t)}\leq T-t+n(t)}} p_t\left(\ell_1, \ldots, \ell_{n(t)}\right) \cdot \mathbb{E}_t\left(u\left(w_t + \sum_{j=1}^{\min\{n,t\}} Y_{j|\ell_1}^t + \cdots + Y_{j|\ell_{n(t)}}^t\right)\right) \tag{15}$$

[6] The conditioning information consists of the realized relative ranks of the past investment projects and the investment histories at time t.

[7] Note that for the linear case finite expectation of B_t^j is sufficient for the existence of the value function. For the non-linear case, the existence of the corresponding conditional expectation depends on both the stochastic properties of B_t^j and the properties of the utility function $u(\cdot)$. See Appendix D.

denote the utility contribution of the strategy "do nothing in the remaining periods $> t$". In the stochastic setup the vector $\mathbf{y}_t$ becomes the random vector $\mathbf{Y}_t$, with coordinates $Y_{j|\ell}$. In addition, the stochastic payoffs when investing b_t are $Z_t\left(\ell_1,\ldots,\ell_{n(t)}; w_{t-1}-b_t, \mathbf{Y}_{t-1}(b_t\mathbf{B}_t^{j:n})\right) := (w_{t-1}-b_t) + Y_{1|\ell_1}^{t-1} + \cdots + Y_{j-1|\ell_{j-1}}^{t-1} + B_t^{\ell_j} b_t + Y_{j|\ell_{j+1}}^{t-1} + \cdots + Y_{n(t)-1|\ell_{n(t)}}^{t-1}$ denote a final payoff for the strategy where we invest $b_t > 0$, while $Z_t(\ell_1,\ldots,\ell_{n(t)}; w_{t-1}, \mathbf{Y}_{t-1}(0\cdot\mathbf{B}_t^{j:n})) := w_{t-1} + Y_{1|\ell_1}^{t-1} + \cdots + Y_{j-1|\ell_{j-1}}^{t-1} + 0 + Y_{j|\ell_{j+1}}^{t-1} + \cdots + Y_{n(t)-1|\ell_{n(t)}}^{t-1}$. Endowed with this notation for our investment problem we obtain:

Theorem 3 *Consider the investment problem investigated in Theorem 1, with stochastic returns B_t^j replacing the deterministic payoffs β_t^j. The value contribution of "doing nothing in the remaining periods $> t$" follows from (15). For $t = 0, 1, \ldots, T$, the value function $\mathbb{E}_t\left(\mathscr{V}_t(w_t, \mathbf{Y}_t)\right)$ satisfies*

$$\mathbb{E}_t\left(\mathscr{V}_t(w_t, \mathbf{Y}_t)\right) = \mathbb{E}_t\left(\mathscr{W}_t(w_t, \mathbf{Y}_t)\right) + \mathbb{E}_t\left(r_t\left(w_t, \mathbf{Y}_t\right)\right). \tag{16}$$

For the residual term we get $r_T\left(w_T, \mathbf{Y}_T\right) = 0$ and the recursion

$$\begin{aligned}\mathbb{E}_t\left(r_t(w_t, \mathbf{Y}_t)\right) =& \max\left\{\frac{t+1-n}{t+1}, 0\right\}\mathbb{E}_t\left(r_{t+1}(w_t, \mathbf{Y}_t)\right)\\ &+ \frac{1}{t+1}\sum_{j=1}^{\min\{t+1,n\}} \max_{0\le b_{t+1}\le w_t}\left[\mathbb{E}_t\left(r_{t+1}\left(w_t - b_{t+1}, \mathbf{Y}_t\left(b_{t+1}\mathbf{B}_{t+1}^{j:n}\right)\right)\right)\right.\\ &+ \sum_{\substack{\ell_1,\ldots,\ell_{n(t)+1}:\\ 0<\ell_1<\cdots<\ell_{n(t)+1}\le T-t+n(t)}} p_{t+1}\left(\ell_1,\ldots,\ell_{n(t)+1}\right)\mathbb{E}_t\left(\left[u\left(\tilde{Z}\right)\right.\right.\\ &\left.\left.\left.- u\left(Z\left(\ell_1,\ldots,\ell_{n(t)+1}; w_t, \mathbf{Y}_t(0\cdot\mathbf{B}_{t+1}^{j:n})\right)\right)\right]\right)\right]\end{aligned} \tag{17}$$

where $\tilde{Z} = Z\left(\ell_1,\ldots,\ell_{n(t)+1}; w_t - b_{t+1}, \mathbf{Y}_t(b_{t+1}\mathbf{B}_{t+1}^{j:n})\right)$. $\mathbb{E}_t\left(r_t(w_t, \mathbf{Y}_t)\right) \ge 0$ for all $t = 0, 1, \ldots, T$. The optimal investments b_{t+1}, $t = 0, \ldots, T-1$, are the b_{t+1} solving the maximization problems $\max_{0\le b_{t+1}\le w_t}\mathbb{E}_t[\cdot]$ in (17).

For a risk-neutral investors Theorem 3 implies

Corollary 1 *Consider a risk-neutral investor where $u(\cdot)$ is the identity. Suppose that B_t^j only depends on the realized rank of the random variable X_t. Then, we can follow the deterministic case investigated in Theorem 2. That is for any X_t with final rank j the expected return is $\beta_t^j = \mathbb{E}(B_t^j)$. This reduces the numerical burden in the simulation analysis where we need not approximate the conditional expectation terms by means of numerical integration.*

Finally, we show that for a time invariant distribution of the returns, the value function is non-increasing in T. Hence, in a market with more superstar candidates it is harder to successfully invest.

Theorem 4 *Consider a monotone increasing and continuous utility function $u(\cdot)$ and let $T \ge n \ge 1$ and $w_0 > 0$. Let $\mathbb{E}_0^T\left(\mathscr{V}_0(w_0)\right)$ denote the expected value function of the T period problem. Suppose that the distribution of $B_t^1, \ldots, B_t^n$ does not depend on t. Then $\mathbb{E}_0^T\left(\mathscr{V}_0(w_0)\right) \ge \mathbb{E}_0^{T+1}\left(\mathscr{V}_0(w_0)\right)$.*

Proof The following arguments are based on Chow et al. [5] and Assaf and Samuel-Cahn [1]. Consider the $T+1$ period problem, but where the investor is told at the beginning in which period the draw with final rank $T+1$ will occur. Obviously the value of this modified problem is at least as high as the original $T+1$ period problem. By ignoring the period of the worst draw, the investor faces the same problem as a T period investor. □

5 Simulation analysis and optimal investment

In this section we perform a simulation study for risk-neutral investors as well as risk-averse investors with Bernoulli utility function $u(w) = \ln(w)$. These cases are motivated by possibly different risk attitudes of venture capital firms and business angels. The initial wealth is $w_0 = 1$. For the non-linear case the computation of the residual term $r_t(w_t, \mathbf{y}_t)$ is numerically intensive. To see this, for each t, $t = 1, \ldots, T$, we have to obtain the optimal investment strategy b_t, given various w_{t-1} and $\mathbf{y}_{t-1}$. That is, we construct a grid for w_{t-1} and possible prior investments $\mathbf{b}_{t-1} = (b_1, \ldots, b_{t-1})$ resulting in $\mathbf{y}_{t-1}$. In particular, we work with a grid of 20 points including the end points 0 and $w_0 = 1$, such that w_{t-1} and b_t are multiples of $1/(20-1) = 0.0526$. More details are provided in Appendix E. By starting with $r_T = 0$, the residual terms $\mathbb{E}_t\left(r_t(w_t, \mathbf{y}_t)\right)$ can be obtained recursively, for $t = T, T-1, \ldots, 1$. For the linear case, the term c_t is computed within seconds. Since the computational burden increases rapidly in T and n, we work with $n = 2$ and $T = 5$, 10 and 20. For the linear case we additionally consider $T = 50$. The number of Monte-Carlo replications is $M = 5000$. [8]

Note that the random variables B_t^1 and B_t^2 represent the gross-returns. Performing the simulation study with realistic values, requires estimates of the distribution of the gross returns B_t^ℓ. Estimates of sample means based on empirical data are e.g. provided in Korteweg and Nagel [14][Table IV]. According to literature, the log-normal distribution provides a reasonable approximation of the return distribution. To approximate returns we use estimates of log-returns provided in Cochrane [7][Table 4; 'Parameter estimates in the round-to-round sample'], where the sample mean of the log-returns is 0.2 and the estimated standard deviation is 0.84, such that the mean arithmetic net-return is close to 1.73 and the sample variance is 1.36.

By using these numbers we have a first estimate for gross-returns of superstar projects. Up to our knowledge more precise estimates on the expected gross-returns of the first best, the seconds best, etc. and the corresponding variances are not available. In the following simulation study we assume that the first best has a gross-return higher than the expected gross-return following from Cochrane [7][Table 4], while the second best has an expected return below this value. In particular, the expectation of the gross-returns B_t^1 should be larger than the estimate $1 + 1.73$ obtained in Cochrane [7], while the expectation of B_t^2 should be smaller that 2.73. In particular, we assume that B_t^1 is log-normally distributed with mean parameter 0.8 and variance parameter $\sigma^2 = 0.8^2$, while B_t^2 is log-normally distributed with mean parameter $\mu_2 = 0.2$ and variance parameter $\sigma^2 = 0.8^2$, for all $t = 1, \ldots, T$. The arithmetic gross-returns B_t^1 have an expectation of $\exp\left(\mu_1 + \frac{1}{2}\sigma^2\right) = 3.0649$ and variance $\left(\exp\left(\sigma^2\right) - 1\right)\exp\left(2\mu_1 + \sigma^2\right) = 8.24$, while B_t^2 has expectation $\exp\left(\mu_2 + \frac{1}{2}\sigma^2\right) = 1.6820$ and variance $\left(\exp\left(\sigma^2\right) - 1\right)\exp\left(2\mu_2 + \sigma^2\right) = 2.54$, respectively. The probability $\mathbb{P}\left(B_t^1 \leq 1\right) \approx 15.87\%$ while $\mathbb{P}\left(B_t^2 \leq 1\right) \approx 40.13\%$. By the properties of the log-normal

[8] The software package `Matlab` was used.

distribution $\mathbb{P}\left(B_t^j > 0\right) > 0$. Hence, in our simulation study we allow for realizations of B_t^j which are smaller than one [negative net-returns] also for investment projects with final best or second best rank with positive probability. By the properties of the log-normal distribution the returns B_t^j are positive (almost surely).

Table 1 presents the sample mean (mean), the sample standard deviations (sd), the minimum (min), the median, and the maximum (max) of realizations of the *final wealth*, $w_T + Y_{1|1}^T + Y_{2|2}^T = w_T + b_{\tau_1^T} B_{\tau_1^T}^1 + b_{\tau_2^T} B_{\tau_2^T}^2$, the realizations of the *final payoff*, $Y_{1|1}^T + Y_{2|2}^T$, and the *final non-invested budget* w_T. For a risk-averse investor with logarithmic Bernoulli utility function we additionally consider realizations of final utility $u\left(w_T + Y_{1|1}^T + Y_{2|2}^T\right) = \ln\left(w_T + Y_{1|1}^T + Y_{2|2}^T\right)$ [by "final" we mean the point of time, when all investment decisions haven taken place and the random variables B_t^1 and B_t^2 are realized].

Table 1 shows that for the linear and the risk-averse case the sample means of the final wealth are decreasing in T. The mean final payoff in the linear case and mean final utility in the non-linear case decrease as expected from Theorem 4. In addition, Table 1 presents the minimum, the maximum and the median of the realized final wealth, the final payoff and the non-invested budget. [9] Since in the linear case—conditional on investing—the investor invests all budget w_0, we observe a minimum final budget of 0. The maximum of the non-invested final budget is $w_T = w_0 = 1$. Hence, with $T = 5$, 10, 20 and 50, we observe paths of the random variable X_t, $t = 1, \ldots, T$, where the investor does not invest. The maxima of the realizations of $w_T + Y_{1|1}^T + Y_{2|2}^T$ and $Y_{1|1}^T + Y_{2|2}^T$ obtained for the linear case are almost the same, differences are due to sampling effects. Comparing the linear to the risk-averse case—as can be expected by economic intuition—the mean final wealth is smaller while the average non-invested budget w_T is larger for the risk-averse investor (for any fixed T). In addition, we observe investment paths where the risk-averse investor invests all the remaining budget available. This happens if the relative rank is one and the number of remaining steps to go is one or zero, or if the relative rank is equal to two and there is no further step to go. For a final rank of $x_T \leq 2$, the decision of a log-utility investor to spend all the remaining budget into this risky alternative depends on the distribution of the returns and the remaining budget w_{T-1} (recall that the returns B_T^1 and B_T^2 are random variables).

Note that for the risk-neutral case and our assumption that the expectation of B_t^1 is larger than two, the argument of Remark 2 applies here. That is, for $T \to \infty$ the expected payoff is larger than the initial wealth w. In line with this, our simulation analysis shows that also for $T = 50$, the average final wealth remains significantly above the initial wealth $w = 1$ and the drop from $T = 10$ to $T = 50$ is rather small [around 4%].

By considering the Tables 2, 3, 4 and 5, we are able to have a closer look at the structure of the optimal investment paths. The tables show the average amounts invested at t in an investment alternative where the realized rank at t is 1 (or 2) [invest rank 1/invest in rank 2 in the corresponding tables]. For the log-utility case we also report the average of the corresponding amounts invested at period t (amount best/amount second best; for the linear case these amounts are equal to the average investments since $w_0 = 1$ and conditional on investing to invest all is optimal). For the log-utility case, we report the sample average of b_t/w_t and the corresponding sample means if the rank is 1 or 2, respectively.

Consider for example the case where $T = 5$. A risk-neutral investor invests for the first time at $t = 3$ if the relative rank of the realization of X_3 is equal to one. In the case of a

[9] Note that the final wealth $w_T + Y_{1|1}^T + Y_{2|2}^T$ remains strictly positive in the risk-averse case. The value 0.0000 for $T = 20$ and log-utility in Table 1 is actually small but strictly larger than zero.

Table 1 Descriptive statistics: we consider $n = 2$, and stochastic returns B_1 and B_2

	Mean	sd	Min	Median	Max
T = 5, linear					
Final wealth	2.2091	1.5681	0.0000	2.1596	4.9518
Payoff	1.9023	2.5229	0.0000	2.1596	4.9518
Final budget w_T	0.3068	0.2127	0.0000	0.0000	1.0000
T = 10, linear					
Final wealth	1.9672	1.8974	0.0000	1.7612	4.9519
Payoff	1.7278	2.5427	0.0000	1.7612	4.9519
Final budget w_T	0.2394	0.1821	0.0000	0.0000	1.0000
T = 20, linear					
Final wealth	1.9458	1.8384	0.0000	1.6614	4.9528
Payoff	1.6628	2.5768	0.0000	1.6614	4.9528
Final budget w_T	0.2830	0.2030	0.0000	0.0000	1.0000
T = 50, linear					
Final wealth	1.8894	1.8824	0.0000	1.5822	4.9529
Payoff	1.6124	2.5755	0.0000	1.5822	4.9529
Final budget w_T	0.2770	0.2003	0.0000	0.0000	1.0000
T = 5, log-utility					
Final wealth	2.0425	0.6938	0.1953	1.9729	4.9519
Payoff	1.8084	1.0217	0.0000	1.7744	4.9519
Final budget w_T	0.2341	0.1055	0.0000	0.0000	1.0000
Final utility	0.6234	0.1998	−1.6333	0.6795	1.5998
T = 10, log-utility					
Final wealth	1.8570	0.6659	0.0643	1.7432	4.9345
Payoff	1.5291	1.0568	0.0000	1.5106	4.9345
Final budget w_T	0.3279	0.1277	0.0000	0.1579	1.0000
Final utility	0.5128	0.2451	−2.7444	0.5557	1.5963
T = 20, log-utility					
Final wealth	1.8075	0.6924	0.0000	1.6640	4.9258
Payoff	1.4324	1.1117	0.0000	1.3827	4.9258
Final budget w_T	0.3751	0.1342	0.0000	0.2105	1.0000
Final utility	0.4733	0.3691	−36.7368	0.5092	1.5945

The random variables B_1 and B_2 follow log-normal distributions with mean parameters 0.8 and 0.2 and a variance parameter of 0.8^2. This figure presents the sample mean (mean), the sample standard deviations (sd), the minimum (min), the median and the maximum (max) of realizations of the Final Wealth $w_T + Y_{1|1}^T + Y_{2|2}^T$, the final Payoff $Y_{1|1}^T + Y_{2|2}^T$, the remaining budget w_T, and for a log-utility investor the Final Utility $u\left(w_T + Y_{1|1}^T + Y_{2|2}^T\right) = \ln\left(w_T + Y_{1|1}^T + Y_{2|2}^T\right)$. Results are obtained from 5, 000 simulation runs

risk-neutral investor all budget, i.e. $w_0 = 1$, is invested at $t = 3$. The probability that the relative rank of X_3 is equal to one is $\frac{1}{3}$. The number observed, 0.3264, is close to $\frac{1}{3}$. At $t = 4$ the investor invests in 17.14% of the random draws. Note that the probability that X_4 has relative rank of one is $1/4$. Since with a probability of $\frac{1}{3}$ the risk-neutral investor invested all the budget before $t = 4$, the probability that the relative rank of X_4 is one and the investor did

Table 2 Investment paths: We consider $n = 2$, $T = 5$ and stochastic returns B_1 and B_2

t	1	2	3	4	5
Risk-neutral					
Invest rank 1	0.0000	0.0000	0.3264	0.1714	0.1006
Invest rank 2	0.0000	0.0000	0.0000	0.0000	0.0948
Log-utility					
Invest rank 1	0.0000	0.4852	0.3378	0.2358	0.1510
Amount best	0.0000	0.3158	0.5823	0.6279	0.5402
Invest rank 2	0.0000	0.0000	0.3326	0.2532	0.1528
Amount second best	0.0000	0.0000	0.0798	0.3486	0.4682
b_t/w_t	0.0000	0.3158	0.3961	0.8099	0.9785
b_t/w_t rank 1	0.0000	0.3158	0.6960	1.0000	1.0000
b_t/w_t rank 2	0.0000	0.0000	0.0916	0.6329	0.9573

The random variables B_1 and B_2 follow log-normal distributions with mean parameters 0.8 and 0.2 and a variance parameter of 0.8^2. This table presents the average amounts invested at t in an investment alternative where the realized rank is 1 (2) (invest rank 1/invest in rank 2). For the log-utility case we also report the average of the corresponding amounts invested at period t (amount best/amount second best; for the linear case these amounts are equal to the average investments since $w_1 = 1$ and "all or nothing" is optimal). In addition, we report the sample average of b_t/w_t and the corresponding sample means if the rank is 1 or 2, respectively. Results are obtained from 5, 000 simulation runs

not invest before is $\frac{1}{4}\frac{2}{3} = \frac{1}{6} \approx 0.1667$. The value observed, 0.1714, is close to $\frac{1}{6}$. For $t = 5$ the investors starts to invest into the second best opportunity. The probabilities for the best and second best in this period are $\frac{1}{5}$ each time. With probability $\frac{1}{2}$ no investment happened in the first four periods. Thus, the corresponding probabilities to invest into the best or second best alternative in $t = 5$ are $\frac{1}{2}\frac{1}{5} = \frac{1}{10}$ each time. The corresponding numbers in the first and the second row of Table 2 are close to these values. With $T = 10$, 20 and 50, for the linear case investments in rank 1 follow from the probabilities that the relative rank is equal to one and the investor did not invest before. For $T = 10$, the probability that X_4 has rank one is $\frac{1}{4}$, etc. In addition, we observe that the risk-neutral investor starts to invest in periods larger than $T/3$ [of course this empirical observations depends on the distributions chosen for B_t^1 and B_t^2].

Next, we consider the log-utility investor: For $T = 5$ the investor already starts to invest at $t = 2$ if the relative rank is one. This happens with probability $\frac{1}{2}$ and the amount $b_2 = 0.3158$ is invested. The relative frequencies where the risk-averse investor invests into an alternative with rank equal to one is close to $\frac{1}{2}$ at $t = 2$. For $t \geq 3$ the numbers in the 'invest rank 1' row are higher than in the risk-neutral case. This is caused by the fact that the risk-averse investor did not invest all the budget available at $t = 2$. The investor invests on average 0.5823 if the relative rank of x_3 is one and 0.0798 if the relative rank of x_3 is two. The final rows show the proportions of the remaining budget invested in the current best or current second best alternative. Note that for $t \geq 3$ the optimal investments b_3 also depend on the distribution of $Y^2_{1|1}$, $Y^2_{1|2}$ and $Y^2_{2|2}$. The ratio b_t/w_t is increasing in t for both, investing into the first best and investing into the second best draw, respectively.

Compared to the risk-neutral case, the risk-averse agent starts to invest in earlier periods. For example for $T = 10$, if the relative rank of x_3 is one the amount invested is $b_3 = 0.0526$ at $t = 3$. Hence, the risk-averse investor already invests smaller amounts into alternatives with rank one in earlier stages of the investment process. Since the expected returns are quite

Table 3 Investment paths: We consider $n = 2$, $T = 10$ and stochastic returns B_1 and B_2

t	1	2	3	4	5	6	7	8	9	10
Risk-neutral										
Invest rank 1	0.0000	0.0000	0.0000	0.2464	0.1476	0.1008	0.0708	0.0536	0.0404	0.0272
Invest rank 2	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0446	0.0292
Log-utility										
Invest rank 1	0.0000	0.0000	0.3362	0.2464	0.2076	0.2076	0.1548	0.1176	0.1080	0.0864
Amount best	0.0000	0.0000	0.0526	0.2256	0.3557	0.3557	0.5612	0.5699	0.5096	0.4836
Invest rank 2	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.1338	0.1234	0.1106	0.0826
Amount second best	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.2387	0.3402	0.4274	0.4996
b_t/w_t	0.0000	0.0000	0.0526	0.2288	0.3860	0.3860	0.5492	0.7052	0.9288	0.9952
b_t/w_t rank 1	0.0000	0.0000	0.0526	0.2288	0.3860	0.3860	0.7575	0.8670	1.0000	1.0000
b_t/w_t rank 2	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.3081	0.5510	0.8593	0.9902

The random variables B_1 and B_2 follow log-normal distributions with mean parameters 0.8 and 0.2 and a variance parameter of 0.8^2. This table presents the average amounts invested at t in an investment alternative where the realized rank is 1 (2) (invest rank 1/invest in rank 2). For the log-utility case we also report the average of the corresponding amounts invested at period t (amount best/amount second best; for the linear case these amounts are equal to the average investments since $w_1 = 1$ and "all or nothing" is optimal). In addition, we report the sample average of b_t/w_t and the corresponding sample means if the rank is 1 or 2, respectively. Results are obtained from 5, 000 simulation runs

Table 4 Investment paths: We consider $n = 2$, $T = 20$ and stochastic returns B_1 and B_2

t	≤ 5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Risk-neutral																
Invest rank 1	0.000	0.000	0.000	0.126	0.099	0.076	0.062	0.050	0.045	0.036	0.032	0.026	0.026	0.022	0.020	0.017
Invest rank 2	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.027	0.022	0.017	0.014
Log-utility																
Invest rank 1	0.000	0.168	0.142	0.131	0.117	0.099	0.090	0.090	0.074	0.074	0.067	0.061	0.062	0.052	0.054	0.051
Amount best	0.000	0.053	0.105	0.204	0.314	0.379	0.455	0.511	0.558	0.570	0.561	0.547	0.488	0.474	0.504	0.443
Invest rank 2	0.000	0.000	0.000	0.000	0.000	0.000	0.059	0.053	0.072	0.062	0.063	0.061	0.055	0.048	0.050	0.045
Amount second best	0.000	0.000	0.000	0.000	0.000	0.000	0.053	0.137	0.166	0.193	0.267	0.370	0.366	0.368	0.424	0.477
b_t/w_t	0.000	0.053	0.106	0.209	0.330	0.416	0.334	0.443	0.451	0.538	0.603	0.685	0.735	0.828	0.961	0.986
b_t/w_t rank 1	0.000	0.053	0.106	0.209	0.330	0.416	0.520	0.620	0.696	0.772	0.805	0.834	0.841	0.941	0.998	1.000
b_t/w_t rank 2	0.000	0.000	0.000	0.000	0.000	0.000	0.054	0.146	0.198	0.256	0.387	0.536	0.615	0.707	0.922	0.971

The random variables B_1 and B_2 follow log-normal distributions with mean parameters 0.8 and 0.2 and a variance parameter of 0.8^2. This table presents the average amounts invested at t in an investment alternative where the realized rank is 1 (2) (invest rank 1/invest in rank 2). For the log-utility case we also report the average of the corresponding amounts invested at period t (amount best/amount second best; for the linear case these amounts are equal to the average investments since $w_1 = 1$ and "all or nothing" is optimal). In addition, we report the sample average of b_t/w_t and the corresponding sample means if the rank is 1 or 2, respectively. Results are obtained from 5, 000 simulation runs

Table 5 Investment paths: We consider $n = 2$, $T = 50$ and stochastic returns B_1 and B_2

t	≤ 17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33
Risk-neutral																	
Invest rank 1	0.000	0.054	0.049	0.045	0.040	0.029	0.035	0.028	0.029	0.026	0.024	0.021	0.023	0.017	0.018	0.022	0.016
Invest rank 2	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
Invest rank 1	0.016	0.013	0.015	0.011	0.012	0.012	0.010	0.010	0.009	0.009	0.007	0.007	0.010	0.006	0.008	0.004	0.005
Invest rank 2	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.012	0.010	0.008	0.009	0.008	0.008	0.008	0.007	0.007	0.006

The random variables B_1 and B_2 follow log-normal distributions with mean parameters 0.8 and 0.2 and a variance parameter of 0.8^2. This table presents the average amounts invested at t in an investment alternative where the realized rank is 1 (2) (invest rank 1/invest in rank 2) high, all the remaining budget is invested at $t = T$ if the relative rank (= final rank) of x_T is one [to see this $b_t/w_t = 1$ for $t = T$ with a rank of one]. For x_T with a rank of two we observe that b_T/w_T is close but not equal to one. This effect is caused by paths where a larger w_T is available. For the given payoff distribution of B_T^2 the investor need not invest all the remaining budget. In our case, the probability that $\mathbb{P}\left(B_T^2 \leq 1\right) \approx 40.13\%$. Hence, for a sufficiently large remaining budget w_{T-1}, the risk-averse investor need not invest all the remaining w_{T-1}.

6 Conclusion

In this article we consider superstar investment problems, where the payoff function depends on the final ranking of the project. Our model investigates high-risk investment opportunities, where an investor is only able to rank the projects seen so far. Investment decisions have to take place immediately after a project shows up. Returns realize at the end when all investment decisions have been taken. Only for a small number of superstars positive returns can be expected.

We obtain the value function and the optimal investment rules for expected utility investors for deterministic as well as stochastic returns. In any case (e.g. risk-neutrality and risk-aversion) the value can be split up into the expected utility given that the investor does nothing in the remaining periods and the additional gain arising from investing the remaining budget optimally. We prove that—as might be expected—the value function is non-increasing in the investment horizon. What is perhaps more surprising is that the expected utility of our optimal strategy is strictly larger than the utility of the initial endowment, even if the number of projects approaches infinity (where we keep the returns of the n best investment opportunities constant).

We show that for the risk-neutral case a simple rank-threshold strategy is optimal for time invariant returns. These rank thresholds are decreasing over time. That is, the conditions to invest are less and less demanding. To invest "all or nothing" – conditional on whether such a threshold is satisfied—is an optimal strategy.

For a risk-averse investor the optimal strategy becomes non-linear. The optimal investment in period t depends on the rank of the current project and the prior investments. For the special case of log-utility, except for very last few periods (at most n periods), it is never optimal to invest all the remaining wealth.

We numerically implement our optimal strategies for a risk-neutral and a log-utility investor. The number of superstars in our simulation analysis is two. The assumed distribution of the returns is based on estimates in Cochrane [7]. In line with our theoretical findings, we observe that the value function is decreasing in the number of investment periods. On average the risk-neutral investor invests a larger share of the initial budget available than a log-utility investor. The risk-averse investor starts to invest earlier, but the amounts invested are smaller than in the risk-neutral case, where—upon investing—without loss of generality all the total budget available is invested.

In the next steps the model considered in this article could be modified to become more realistic. For example, we assumed that due to competition the investor is forced to decide immediately whether to invest into the current investment project. This assumption could be relaxed by allowing the investor to decide to invest into the current project today and in the next $m > 0$ periods. A further possible modification is to consider multiple investment rounds as observed on the venture capital market. To do this, a second signal could be included which is correlated with the ranking of the project in question, where the investor—after some time span—has to decide to either quit investing into the corresponding project or to impute further capital.

Throughout our analysis the number of time periods (i.e. the number of possible investment projects) is fixed at T. Hence, instead of working with a deterministic number of projects, an extension would be to describe the number of projects by a random variable. A starting point to investigate problems with a stochastic number of projects could be Bruss and Ferguson [3][Section 6].

A further interesting way to extend our approach is to consider ambiguity-averse investors. For the classical secretary problem Chudjakow and Riedel [6] obtained an optimal threshold strategy also for ambiguity-averse agents. The optimal strategy was obtained by applying the optimal stopping results derived in Riedel [22]. An extension to the full information best choice problem—originally investigated in Gilbert and Mosteller [10] – was obtained in Obradović [19].

A natural way to extend our approach is to augment the approach of Chudjakow and Riedel [6] to the investment problem considered in this article. Thereby the question arises whether under risk-neutrality and ambiguity-aversion a strategy of the form “invest all or nothing (conditional on time and rank)” is still optimal and under which assumptions the investment thresholds are shifted to earlier or later periods. While we are optimistic that the risk-neutral case can be extended to ambiguity-aversion (at least for deterministic returns), we expect that the non-linear case becomes even more involved.

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Data availability We do not work with any empirical data. Hence, no data can be provided. The Matlab code used in the simulation analysis is available on request.

Declarations

Conflict of interest There are no conflict of interest.

Ethical approval We do not work with any empirical data. For this reason, we are not aware of any ethical issues that could arise within this article.

A Probabilities

We use the convention that $\binom{p}{q} = 0$ for $p < q$ and $\ell_0 = k_0 = 0$.

Definition 1 Let $1 \leq k_1 < k_2 < \cdots < k_m \leq t$ denote the m realized ranks in ascending order and $1 \leq \ell_1 < \ell_2 < \cdots < \ell_m \leq T - t + k_m$. Define

$$p_t(\ell_1, \ldots, \ell_m | k_1, \ldots, k_m) = \begin{cases} \dfrac{\binom{T-\ell_m}{t-k_m} \prod_{i=1}^m \binom{\ell_i - \ell_{i-1} - 1}{k_i - k_{i-1} - 1}}{\binom{T}{t}}, & \text{if } k_i \leq \ell_i, \text{ for all } i = 1, \ldots, m, \\ 0 & \text{else}. \end{cases}$$

If $k_i = i$, for all $i = 1, \ldots, m$, we simply write $p_t(\ell_1, \ldots, \ell_m)$ for $p_t(\ell_1, \ldots, \ell_m | k_1, \ldots, k_m)$. Note that since $\binom{s}{0} = 1$, for any $s = 0, 1, \ldots,$ we have

$$p_t(\ell_1, \ldots, \ell_m) = \frac{\binom{T-\ell_m}{t-m}}{\binom{T}{t}}. \tag{18}$$

Consider m realized relative ranks $k_1, \ldots, k_m$ of some of the X_s, $s \leq t$ at time t. In the following we obtain the joint conditional probability that these particles have final ranks $1 \leq \ell_1 < \cdots < \ell_m \leq T - t + k_m$ after period T.

Proposition 1 *Let $m \leq t$, then $p_t(\ell_1, \ldots, \ell_m | k_1, \ldots, k_m)$ is the probability that draws that have relative ranks $1 \leq k_1 < k_2 \cdots < k_m \leq t$ at time t, have final ranks $1 \leq \ell_1 < \ell_2 \cdots < \ell_m \leq T - t + k_m$ at time T.*

Proof To establish Proposition 1 we apply backward induction: Consider the final period T. If $1 \leq k_1 = \ell_1 < k_2 = \ell_2 \cdots < k_m = \ell_m$, this probability has to be 1. Indeed,

$$p_T(\ell_1, \ldots, \ell_m | \ell_1, \ldots, \ell_m) = \frac{\binom{T-\ell_m}{T-\ell_m} \prod_{i=1}^m \binom{\ell_i - \ell_{i-1} - 1}{\ell_i - \ell_{i-1} - 1}}{\binom{T}{T}} = 1. \tag{19}$$

Since we did not prove yet that the $p_t(\ldots | k_1, \ldots, k_m)$ are probability distributions, we also need to consider the case that $k_i \neq \ell_i$ for at least one $i = 1, \ldots, m$ and show that then $p_T(\ell_1, \ldots, \ell_m | k_1, \ldots, k_m) = 0$. If there is an $i = 1, \ldots, m$ such that $k_i > \ell_i$, then let j be the smallest such index such that $k_j > \ell_j$. Hence, $\ell_j - \ell_{j-1} - 1 < k_j - k_{j-1} - 1$, in which case

$$\binom{\ell_j - \ell_{j-1} - 1}{k_j - k_{j-1} - 1} = 0$$

If there is an index i such that $\ell_i > k_i$, then let j be the largest such index. If $j < m$ then $\ell_{j+1} - \ell_j - 1 < k_{j+1} - k_j - 1$ and hence, $\binom{\ell_{j+1} - \ell_j - 1}{k_{j+1} - k_j - 1} = 0$. If $j = m$, then $T - \ell_m < T - k_m$ such that $\binom{T-\ell_m}{T-k_m} = 0$. This shows that in the case $t = T$, we have

$$p_T(\ell_1, \ldots, \ell_m | k_1, \ldots, k_m) = \begin{cases} 1 & \text{if } k_j = \ell_j \text{ for } j = 1, \ldots, m \text{ and} \\ 0 & \text{else}. \end{cases} \tag{20}$$

In the following induction step we show if the formula (7) is correct for some $t+1$ with $t+1 \le T$ and $m \le t$, it is also correct for t. This, together with the case $t = T$, then also shows that the $p_t(\ldots|k_1,\ldots,k_m)$ are probability distributions. The draw at time $t+1$ can either have relative rank strictly larger than k_m. This happens with probability $\frac{t+1-k_m}{t+1}$. Conditional on this event, the probability that the draws with relative ranks $k_1,\ldots,k_m$ at time t [and hence also at time $t+1$], have final ranks $\ell_1,\ldots,\ell_m$ at time T is $p_{t+1}(\ell_1,\ldots,\ell_m|k_1,\ldots,k_m)$. Or the draw x_{t+1} at time $t+1$ has relative rank $\rho_{t+1}(x_{t+1})$ such that $k_{j-1} < \rho_{t+1}(x_{t+1}) \le k_j$ for some $1 \le j \le m$, then the draws that had relative ranks $k_1,\ldots,k_m$ at time t now have relative ranks $k_1,\ldots,k_{j-1},k_j+1,k_{j+1}+1,\ldots,k_m+1$ at time $t+1$. This happens with probability $\frac{k_j-k_{j-1}}{t+1}$. Conditional on that event the probability that the draws with relative ranks $k_1,\ldots,k_m$ at time t have final ranks $\ell_1,\ldots,\ell_m$ is $p_{t+1}\left(\ell_1,\ldots,\ell_{j-1},\ell_j,\ldots,\ell_m|k_1,\ldots,k_{j-1},k_j+1,\ldots,k_m+1\right)$. Hence the probability that the draws with relative rank $k_1,\ldots,k_m$ at time t have final rank $\ell_1,\ldots,\ell_m$ is according to the induction hypothesis equal to

$$
\begin{aligned}
&\frac{t+1-k_m}{t+1} p_{t+1}(\ell_1,\ldots,\ell_m|k_1,\ldots,k_m) \\
&+\sum_{j=1}^{m} \frac{k_j-k_{j-1}}{t+1} p_{t+1}\left(\ell_1,\ldots,\ell_{j-1},\ell_j,\ldots,\ell_m|k_1,\ldots,k_{j-1},k_j+1,\ldots,k_m+1\right) \\
=\ &\frac{t+1-k_m}{t+1} \frac{\binom{T-\ell_m}{t+1-k_m}\prod_{i=1}^m \binom{\ell_i-\ell_{i-1}-1}{k_i-k_{i-1}-1}}{\binom{T}{t+1}} \\
&+\sum_{j=1}^{m} \frac{k_j-k_{j-1}}{t+1} \frac{\left[\prod_{i=1,\, i\neq j}^m \binom{\ell_i-\ell_{i-1}-1}{k_i-k_{i-1}-1}\right]\binom{\ell_j-\ell_{j-1}-1}{k_j+1-k_{j-1}-1}\binom{T-\ell_m}{t+1-(k_m+1)}}{\binom{T}{t+1}} \\
=\ &\frac{t+1-k_m}{t+1} \frac{\frac{T-\ell_m-t+k_m}{t+1-k_m}\binom{T-\ell_m}{t-k_m}\prod_{i=1}^m \binom{\ell_i-\ell_{i-1}-1}{k_i-k_{i-1}-1}}{\frac{T-t}{t+1}\binom{T}{t}} \\
&+\sum_{j=1}^{m} \frac{k_j-k_{j-1}}{t+1} \frac{\frac{\ell_j-k_j-(\ell_{j-1}-k_{j-1})}{k_j-k_{j-1}}\binom{T-\ell_m}{t-k_m}\prod_{i=1}^m \binom{\ell_i-\ell_{i-1}-1}{k_i-k_{i-1}-1}}{\frac{T-t}{t+1}\binom{T}{t}} \\
=\ &\frac{\binom{T-\ell_m}{t-k_m}\prod_{i=1}^m \binom{\ell_i-\ell_{i-1}-1}{k_i-k_{i-1}-1}}{\binom{T}{t}} \frac{T-\ell_m-(t-k_m)+\sum_{j=1}^m \ell_j-k_j-(\ell_{j-1}-k_{j-1})}{T-t} \\
=\ &\frac{\binom{T-\ell_m}{t-k_m}\prod_{i=1}^m \binom{\ell_i-\ell_{i-1}-1}{k_i-k_{i-1}-1}}{\binom{T}{t}} \frac{T-\ell_m-(t-k_m)+\ell_m-k_m}{T-t} \\
&\text{[Recall that } \ell_0=k_0=0] \\
=\ &\frac{\binom{T-\ell_m}{t-k_m}\prod_{i=1}^m \binom{\ell_i-\ell_{i-1}-1}{k_i-k_{i-1}-1}}{\binom{T}{t}} \frac{T-t}{T-t}
\end{aligned}
$$

$$= p_t(\ell_1, \ldots, \ell_m | k_1, \ldots, k_m). \tag{21}$$

□

The induction step in the proof above has shown in particular the following

Lemma 1 *Let* $1 \leq \ell_1 < \cdots < \ell_{n(t)} \leq T - t + n(t)$.

$$p_t\left(\ell_1, \ldots, \ell_{n(t)} | 1, \ldots, n(t)\right) = \frac{t+1-n(t)}{t+1} p_{t+1}\left(\ell_1, \ldots, \ell_{n(t)} | 1, \ldots, n(t)\right) + \frac{1}{t+1} \sum_{j=1}^{n(t)} p_{t+1}\left(\ell_1, \ldots, \ell_{j-1}, \ell_j, \ldots, \ell_{n(t)} | 1, \ldots, j-1, j+1, \ldots, n(t)+1\right).$$

B Proof of Theorem 1

In this section we prove Theorem 1. As a first step we show:

Lemma 2 *Let* $T \geq n$ *and* $n(t+1) = \min\{n, t+1\}$. *Then,*

$$\mathscr{W}_t(w_t, \mathbf{y}_t) = \frac{t+1-n(t+1)}{t+1} \mathscr{W}_{t+1}(w_t, \mathbf{y}_t) + \frac{1}{t+1} \sum_{j=1}^{n(t+1)} \mathscr{W}_{t+1}\left(w_t, \mathbf{y}_t(0 \cdot \boldsymbol{\beta}_{t+1}^{j:n})\right). \tag{22}$$

Proof By the definition of $\mathscr{W}_t(w_t, \mathbf{y}_t)$ provided in (8), we get

$$\begin{aligned}
\mathscr{W}_t(w_t, \mathbf{y}_t) &= \sum_{\substack{\ell_1, \ldots, \ell_{n(t)}: \\ 1 \leq \ell_1 < \cdots < \ell_{n(t)} \leq T-t+n(t)}} p_t\left(\ell_1, \ldots, \ell_{n(t)} | 1, \ldots, n(t)\right) u\left(w_t + \sum_{i=1}^{n(t)} y_{i|\ell_i}^t\right) \\
&\text{[apply Lemma 1]} \\
&= \sum_{\substack{\ell_1, \ldots, \ell_{n(t)}: \\ 1 \leq \ell_1 < \cdots < \ell_{n(t)} \leq T-t+n(t)}} \left[\frac{t+1-n(t)}{t+1} p_{t+1}\left(\ell_1, \ldots, \ell_{n(t)} | 1, \ldots, n(t)\right) u\left(w_t + \sum_{i=1}^{n(t)} y_{i|\ell_i}^t\right) + \right. \\
&\left. \frac{1}{t+1} \sum_{j=1}^{n(t)} p_{t+1}\left(\ell_1, \ldots, \ell_{n(t)} | 1, \ldots, j-1, j+1, \ldots, n(t)+1\right) u\left(w_t + \sum_{i=1}^{j-1} y_{i|\ell_i}^t + \sum_{i=j}^{n(t)} y_{i|\ell_i}^t\right)\right]
\end{aligned} \tag{23}$$

Note that

$$\begin{aligned}
&p_{t+1}\left(\ell_1, \ldots, \ell_{n(t)} | 1, \ldots, j-1, j+1, \ldots, n(t)+1\right) \\
&= \frac{\binom{T - \ell_{n(t)}}{t+1-n(t)-1}\binom{\ell_j - \ell_{j-1} - 1}{j+1-(j-1)-1}}{\binom{T}{t+1}} \\
&= \frac{\binom{T - \ell_{n(t)}}{t+1-n(t)-1}}{\binom{T}{t+1}} (\ell_j - \ell_{j-1} - 1)
\end{aligned}$$

$$= \sum_{\ell:\, \ell_{j-1}<\ell<\ell_j} p_{t+1}\left(\ell_1,\ldots,\ell_{j-1},\ell,\ell_j,\ldots,\ell_{n(t)}|1,\ldots,j-1,j,j+1,\ldots,n(t)+1\right) \tag{24}$$

By setting $\hat{\ell}_i = \ell_i$, for $1 \le i \le j-1$ $\hat{\ell}_j = \ell$ and $\hat{\ell}_i = \ell_{i-1}$, for $j+1 \le i \le n(t)+1$, we get

$$\sum_{\ell:\, \ell_{j-1}<\ell<\ell_j} p_{t+1}\left(\ell_1,\ldots,\ell_{j-1},\ell,\ell_j,\ldots,\ell_{n(t)}|1,\ldots,j-1,j,j+1,\ldots,n(t)+1\right)$$
$$= \sum_{\hat{\ell}_j:\, \hat{\ell}_{j-1}<\hat{\ell}_j<\hat{\ell}_{j+1}} p_{t+1}\left(\hat{\ell}_1,\ldots,\hat{\ell}_{n(t)+1}|1,\ldots,n(t)+1\right). \tag{25}$$

Hence,

$$p_{t+1}\left(\ell_1,\ldots,\ell_{n(t)}|1,\ldots,j-1,j+1,,\ldots,n(t)+1\right) u\left(w_t + \sum_{i=1}^{j-1} y^t_{i|\ell_i} + 0 + \sum_{i=j}^{n(t)} y^t_{i|\ell_i}\right)$$
$$= \sum_{\hat{\ell}_j:\, \hat{\ell}_{j-1}<\hat{\ell}_j<\ell_{j+1}} p_{t+1}\left(\hat{\ell}_1,\ldots,\hat{\ell}_{n(t)+1}|1,\ldots,n(t)+1\right) u\left(w_t + \sum_{i=1}^{j-1} y^t_{i|\hat{\ell}_i} + 0 + \sum_{i=j+1}^{n(t)+1} y^t_{i-1|\hat{\ell}_i}\right). \tag{26}$$

Note that if $n(t)=n$, then $\hat{\ell}_{n(t)+1} = \ell_{n(t)} > n$ and hence $y^{t+1}_{n(t)+1|\hat{\ell}_{n(t)+1}} = 0$. Next,

$$\sum_{\ell_1,\ldots,\ell_{n(t)} \le T-t+n(t)} \sum_{\ell:\, \ell_{j-1}<\ell<\ell_j} p_{t+1}\left(\ell_1,\ldots,\ell_{n(t)}|1,\ldots,j-1,\right.$$
$$\left. j+1,,\ldots,n(t)+1\right) u\left(w_{t-1} + \sum_{i=1}^{j-1} y^t_{i|\ell_i} + 0 + \sum_{i=j}^{n(t)} y^t_{i|\ell_i}\right)$$
$$= \sum_{\hat{\ell}_1,\ldots,\hat{\ell}_{n(t)} \le T-t+n(t)} p_t\left(\hat{\ell}_1,\ldots,\hat{\ell}_{n(t)+1}|1,\ldots,j,\ldots,n(t)+1\right)$$
$$\times u\left(w_{t-1} + \sum_{i=1}^{j-1} y^t_{i|\hat{\ell}_i} + 0 + \sum_{i=j+1}^{n(t)+1} y^t_{i-1|\hat{\ell}_i}\right)$$
$$= \mathscr{W}_t\left(w_{t-1}, \mathbf{y}_{t-1}\left(0\cdot\boldsymbol{\beta}^{j:n}_t\right)\right). \tag{27}$$

Also, note that

$$\mathscr{W}_{t+1}(w_t,\mathbf{y}_t) = \sum_{\substack{\ell_1,\ldots,\ell_{n(t)}:\\ 1\le\ell_1<\cdots<\ell_{n(t)}\le T-t-1+n(t)}} p_{t+1}\left(\ell_1,\ldots,\ell_{n(t)}|1,\ldots,n(t)\right) u\left(w_t + \sum_{i=1}^{n(t)} y^t_{i|\ell_i}\right).$$

Finally,

$$\mathscr{W}_t(w_t,\mathbf{y}_t) = \sum_{\substack{\ell_1,\ldots,\ell_{n(t)}:\\ 1\le\ell_1<\cdots<\ell_{n(t)}\le T-t+n(t)}} p_t\left(\ell_1,\ldots,\ell_{n(t)}|1,\ldots,n(t)\right) u\left(w_t + \sum_{i=1}^{n(t)} y^t_{i|\ell_i}\right)$$
$$= \frac{t+1-n(t)}{t+1}\mathscr{W}_{t+1}(w_t,\mathbf{y}_t) + \frac{1}{t+1}\sum_{j=1}^{n(t)} \mathscr{W}_{t+1}\left(w_t, \mathbf{y}_t\left(0\cdot\boldsymbol{\beta}^{j:n}_{t+1}\right)\right). \tag{28}$$

Now, either $n \le t$ and we have $n(t+1) = n(t) = n$. Or $n > t$ and then $n(t)+1 = n(t+1) = t+1$. In this case, we have

$$\mathscr{W}_{t+1}\left(w_t, \mathbf{y}_t\left(0 \cdot \boldsymbol{\beta}_{t+1}^{n(t+1):n}\right)\right) = \mathscr{W}_{t+1}\left(w_t, \mathbf{y}_t\right).$$

In both cases it follows that

$$\mathscr{W}_t(w_t, \mathbf{y}_t) = \frac{t+1-n(t+1)}{t+1}\mathscr{W}_{t+1}(w_t, \mathbf{y}_t) + \frac{1}{t+1}\sum_{j=1}^{n(t+1)} \mathscr{W}_{t+1}\left(w_t, \mathbf{y}_t(0 \cdot \boldsymbol{\beta}_{t+1}^{j:n})\right).$$

□

Having shown the recursion for $\mathscr{W}_t\,(w_t, \mathbf{y}_t)$, we can now turn to:

Proof of Theorem 1 Equation (9) defines the residual term $r_t(w_t, \mathbf{y}_t) = \mathscr{V}_t(w_t, \mathbf{y}_t) - \mathscr{W}_t(w_t, \mathbf{y}_t)$. Hence, it remains to show that the recursion Eq. (4) and the function defined in (8) result in this residual term for all time periods considered. To do this, we use the left hand side of the recursion equation, i.e. $\mathscr{V}_t(w_t, \mathbf{y}_t)$, and subtract the left side in (22), i.e. $\mathscr{W}_t(w_t, \mathbf{y}_t)$. In addition we subtract the right side of the Eq. (22) from the right hand side in (4). This results in the following equation

$$\begin{aligned}
r_t(w_t, \mathbf{y}_t) &= \max\left\{\frac{t+1-n}{t+1}, 0\right\}\left[\mathscr{W}_{t+1}(w_t, \mathbf{y}_t) + r_{t+1}(w_t, \mathbf{y}_t) - \mathscr{W}_{t+1}(w_t, \mathbf{y}_t)\right] \\
&\quad + \frac{1}{t+1}\sum_{j=1}^{\min\{t+1,n\}} \max_{0 \le b_{t+1} \le w_t}\Big[\mathscr{W}_{t+1}\left(w_t, \mathbf{y}_t(b_{t+1}\boldsymbol{\beta}_{t+1}^{j:n})\right) + r_{t+1}\left(w_t, \mathbf{y}_t(b_{t+1}\boldsymbol{\beta}_{t+1}^{j:n})\right) \\
&\quad - \mathscr{W}_{t+1}\left(w_t, \mathbf{y}_t(0 \cdot \boldsymbol{\beta}_{t+1}^{j:n})\right)\Big] \\
&= \max\left\{\frac{t+1-n}{t+1}, 0\right\} r_{t+1}(w_t, \mathbf{y}_t) + \frac{1}{t+1}\sum_{j=1}^{\min\{t+1,n\}} \max_{0 \le b_{t+1} \le w_t} \\
&\quad \left[\mathscr{W}_{t+1}\left(w_t - b_{t+1}, \mathbf{y}_t(b_{t+1}\boldsymbol{\beta}_{t+1}^{j:n})\right) - \mathscr{W}_{t+1}\left(w_t, \mathbf{y}_t(0 \cdot \boldsymbol{\beta}_{t+1}^{j:n})\right) + r_{t+1}\left(w_t, \mathbf{y}_t(b_{t+1}\boldsymbol{\beta}_{t+1}^{j:n})\right)\right].
\end{aligned}$$

It remains to show that $\mathscr{W}_{t+1}\left(w_t - b_{t+1}, \mathbf{y}_t(b_{t+1}\boldsymbol{\beta}_{t+1}^{j:n})\right) - \mathscr{W}_{t+1}\left(w_t, \mathbf{y}_t(0 \cdot \boldsymbol{\beta}_{t+1}^{j:n})\right)$ is equal to

$$\begin{aligned}
&\sum_{\substack{\ell_1,\ldots,\ell_{n(t)+1}: \\ 0<\ell_1<\cdots<\ell_{n(t)+1}\le T-t+n(t)}} p_{t+1}\left(\ell_1, \ldots, \ell_{n(t)+1}\right)\Big[u\left(Z\left(\ell_1, \ldots, \ell_{n(t)+1}; w_t - b_{t+1}, \mathbf{y}_t(b_{t+1}\boldsymbol{\beta}_{t+1}^{j:n})\right)\right) \\
&\quad - u\left(Z\left(\ell_1, \ldots, \ell_{n(t)+1}; w_t, \mathbf{y}_t(0 \cdot \boldsymbol{\beta}_{t+1}^{j:n})\right)\right)\Big]
\end{aligned}$$

By using the definition of $\mathscr{W}_{t+1}\left(w_t - b_{t+1}, \mathbf{y}_t(b_{t+1}\boldsymbol{\beta}_{t+1}^{j:n})\right)$ provided in (8), we obtain

$$\begin{aligned}
&\mathscr{W}_{t+1}\left(w_t - b_{t+1}, \mathbf{y}_t(b_{t+1}\boldsymbol{\beta}_{t+1}^{j:n})\right) - \mathscr{W}_{t+1}\left(w_t, \mathbf{y}_t(0 \cdot \boldsymbol{\beta}_{t+1}^{j:n})\right) \\
&= \sum_{\substack{\ell_1,\ldots,\ell_{n(t)+1}: \\ 0<\ell_1<\cdots<\ell_{n(t)+1}\le T-t+n(t)}} p_{t+1}\left(\ell_1, \ldots, \ell_{n(t)+1}\right) u\left(w_t - b_{t+1} + \sum_{i=1}^{j-1} y_{i|\ell_i}^t + \beta_{t+1}^j b_{t+1} + \sum_{i=j}^{n(t)} y_{i|\ell_{i+1}}^t\right) \\
&\quad - \sum_{\substack{\ell_1,\ldots,\ell_{n(t)+1}: \\ 0<\ell_1<\cdots<\ell_{n(t)+1}\le T-t+n(t)}} p_{t+1}\left(\ell_1, \ldots, \ell_{n(t)+1}\right) u\left(w_t + \sum_{i=1}^{j-1} y_{i|\ell_i}^t + 0 + \sum_{i=j}^{n(t)} y_{i|\ell_{i+1}}^t\right)
\end{aligned}$$

[Note that $Z\left(\ell_1, \ldots, \ell_{n(t)+1}; w_t - b_{t+1}, \mathbf{y}_t(b_{t+1}\boldsymbol{\beta}_{t+1}^{j:n})\right)$

$$
\begin{aligned}
&:= (w_t - b_{t+1}) + y^t_{1|\ell_1} + \cdots + y^t_{j-1|\ell_{j-1}} + \\
&\beta^{\ell_j}_{t+1} b_{t+1} + y^t_{j|\ell_{j+1}} + \cdots + y^t_{n(t)|\ell_{n(t)+1}} \text{and} \\
&Z(\ell_1, \ldots, \ell_{n(t)+1}; w_t, \mathbf{y}_t(0 \cdot \boldsymbol{\beta}^{j:n}_{t+1})) := w_t + y^t_{1|\ell_1} + \ldots \\
&+ y^t_{j-1|\ell_{j-1}} + \beta^{\ell_j}_{t+1} 0 + y^t_{j|\ell_{j+1}} + \cdots + y^t_{n(t)|\ell_{n(t)+1}}] \\
= & \sum_{\substack{\ell_1,\ldots,\ell_{n(t)+1}: \\ 0<\ell_1<\cdots<\ell_{n(t)+1}) \le T-t+n(t)}} p_{t+1}\left(\ell_1, \ldots, \ell_{n(t)+1}\right) \Big[u\left(Z\left(\ell_1, \ldots, \ell_{n(t)+1}; w_t - b_{t+1}, \mathbf{y}_t(b_{t+1} \boldsymbol{\beta}^{j:n}_{t+1})\right)\right) \\
& - u\left(Z\left(\ell_1, \ldots, \ell_{n(t)+1}; w_t, \mathbf{y}_t(0 \cdot \boldsymbol{\beta}^{j:n}_{t+1})\right)\right) \Big].
\end{aligned} \tag{29}
$$

□

C Proof of Theorem 2

Proof of Theorem 2 Consider $r_t(w_t, \mathbf{y}_t)$ obtained in Theorem 1. For the risk-neutral case $u(\cdot)$ is equal to the identity map.

This yields $u\left(Z\left(\ell_1, \ldots, \ell_{n(t)+1}; w_t - b_{t+1}, \mathbf{y}_t(b_{t+1}\boldsymbol{\beta}^{j:n}_{t+1})\right)\right) - u\Big(Z\Big(\ell_1, \ldots, \ell_{n(t)+1}; w_t, \mathbf{y}_t(0 \cdot \boldsymbol{\beta}^{j:n}_{t+1})\Big)\Big) = Z\left(\ell_1, \ldots, \ell_{n(t)+1}; w_t - b_{t+1}, \mathbf{y}_t(b_{t+1}\boldsymbol{\beta}^{j:n}_{t+1})\right) - Z\Big(\ell_1, \ldots, \ell_{n(t)+1}; w_t - 0, \mathbf{y}_t(0 \cdot \boldsymbol{\beta}^{j:n}_{t+1})\Big) = b_{t+1}\left(\beta^{\ell_j}_{t+1} - 1\right)$.

For $t = T$ we get $r_T = 0$, which is of course proportional to w_T. Next we consider $t = T - 1$, where

$$
r_{T-1}(w_{T-1}) = \frac{1}{T} \sum_{j=1}^{n} \max_{0 \le b_T \le w_{T-1}} \Bigg[\sum_{\substack{\ell_1,\ldots,\ell_n: \\ 0<\ell_1<\cdots<\ell_n \le n+1}} p_T\left(\ell_1, \ldots, \ell_n\right) b_T \left(\beta^{\ell_j}_T - 1\right) \Bigg] \tag{30}
$$

From (30) we observe that $b_T = w_{T-1}$ is optimal if $\left(\beta^{\ell_j}_T - 1\right) > 0$. For $\left(\beta^{\ell_j}_T - 1\right) < 0$ $b_T = 0$ if optimal, while for $\left(\beta^{\ell_j}_T - 1\right) = 0$ any $b_T \in [0, w_{T-1}]$ is optimal [that is we have a correspondence]. In the following we assume that $b_T = w_{T-1}$ if $\left(\beta^{\ell_j}_T - 1\right) \ge 0$. Based in this result, we observe that $r_{T-1}(w_{T-1}) = w_{T-1} r_{T-1}(1) = w_{T-1}\left(\beta^{\ell_j}_T - 1\right)$. Note that for the linear case $r_{T-1}(w_{T-1}, \mathbf{y}_{T-1})$ obtained in the non-linear case, becomes independent of $\mathbf{y}_{T-1}$, hence we write $r_{T-1}(w_{T-1})$. By a standard induction step we observe that also $r_t(w_t, \mathbf{y}_t)$ does not depend on $\mathbf{y}_t$. Therefore we write $r_t(w_t)$. Next, for any t, $t = 1, \ldots, T$, we have to show $r_t(w_t) = w_t r_t(1)$. We assume that this relationships holds for $t + 1$ and show that it also hold for t. We consider

$$
\begin{aligned}
& r_t(w_t) \\
= & \max\left\{\frac{t+1-n}{t+1}, 0\right\} \underbrace{w_t}_{=w_{t+1} \text{ for } b_{t+1}=0} r_{t+1}(1) \\
& + \frac{1}{t+1} \sum_{j=1}^{\min\{t+1,n\}} \max_{0 \le b_{t+1} \le w_t} \Bigg[\sum_{\substack{\ell_1,\ldots,\ell_{n(t+1)}: \\ 0<\ell_1<\cdots<\ell_{n(t+1)} \le T-t-1+n(t+1)}} p_{t+1}\left(\ell_1, \ldots, \ell_{n(t+1)}\right) b_{t+1}\left(\beta^{\ell_j}_{t+1} - 1\right) + \underbrace{(w_t - b_{t+1})}_{w_{t+1}} r_{t+1}(1) \Bigg] \\
= & \max\left\{\frac{t+1-n}{t+1}, 0\right\} w_t r_{t+1}(1)
\end{aligned}
$$

$$+\frac{1}{t+1}\sum_{j=1}^{\min\{t+1,n\}}\max_{0\le b_{t+1}\le w_t}\underbrace{\left[(w_t-b_{t+1})r_{t+1}(1)+b_{t+1}\sum_{\substack{\ell_1,\dots,\ell_{n(t+1)}:\\ 0<\ell_1<\dots<\ell_{n(t+1)}\le T-t-1+n(t+1)}}p_{t+1}\left(\ell_1,\dots,\ell_{n(t+1)}\right)\left(\beta_{t+1}^{\ell_j}-1\right)\right]}_{=:S(w_t,b_{t+1})}. \tag{31}$$

The term $S(w_t, b_{t+1})$ is affine linear in b_{t+1}. We observe that investing $b_{t+1} = w_t$ if $\sum_{\substack{\ell_1,\dots,\ell_{n(t+1)}:\\ 0<\ell_1<\dots<\ell_{n(t+1)}\le T-t-1+n(t+1)}} p_{t+1}\left(\ell_1,\dots,\ell_{n(t+1)}\right)\left(\beta_{t+1}^{\ell_j}-1\right) \ge r_{t+1}(1)$ and zero else is an optimal strategy. Consider (31). The maximum of the term $S(w_t, b_{t+1})$ is $w_t\sum_{\substack{\ell_1,\dots,\ell_{n(t+1)}:\\ 0<\ell_1<\dots<\ell_{n(t+1)}\le T-t-1+n(t+1)}} p_{t+1}\left(\ell_1,\dots,\ell_{n(t+1)}\right)\left(\beta_{t+1}^{\ell_j}-1\right)$ if we invest $b_{t+1} = w_t$ and $w_t r_{t+1}(1)$ if $b_{t+1} = 0$. Hence,

$$\max_{0\le b_{t+1}\le w_t} S(w_t,b_{t+1}) = w_t\max\left\{r_{t+1}(1),\sum_{\substack{\ell_1,\dots,\ell_{n(t+1)}:\\ 0<\ell_1<\dots<\ell_{n(t+1)}\le T-t-1+n(t+1)}}p_{t+1}\left(\ell_1,\dots,\ell_{n(t+1)}\right)\left(\beta_{t+1}^{\ell_j}-1\right)\right\}.$$

$$\begin{aligned} r_t(w_t) &= \max\left\{\frac{t+1-n}{t+1},0\right\}w_t r_{t+1}(1)\\ &\quad+\frac{1}{t+1}\sum_{j=1}^{\min\{t+1,n\}}w_t\max\left\{r_{t+1}(1),\sum_{\substack{\ell_1,\dots,\ell_{n(t+1)}:\\ 0<\ell_1<\dots<\ell_{n(t+1)}\le T-t-1+n(t+1)}}p_t\left(\ell_1,\dots,\ell_{n(t+1)}\right)\left(\beta_{t+1}^{\ell_j}-1\right)\right\}\\ &= w_t r_t(1). \end{aligned} \tag{32}$$

We had already shown that the residual term fulfills $r_t(w_t, \mathbf{0}) = w_t r_t(1, \mathbf{0})$. In Eq. (13) we defined c_t. Note that $c_t = r_t(1, \mathbf{0})$. For any period $> t$ we observe that given some w_t, the investment strategy invest all the remaining budget w_t is optimal if $S(w_t, b_{t+1})$ is maximized with some $b_{t+1} > 0$, otherwise it is optimal to invest zero. This results holds for all $t = 1, \dots, T$. Hence, investing all the remaining budget is optimal for all $t \ge 1$, which of course implies that investing all or nothing is optimal.

Next, consider $\mathscr{W}_t(w_t, y_t)$ obtained in Theorem 9. From the above paragraphs we already know investing all or nothing is an optimal strategy. Therefore, we get

$$\begin{aligned} \mathscr{W}_t(w_t,\mathbf{y}_t) &= \sum_{\substack{\ell_1,\dots,\ell_{n(t+1)}:\\ 0<\ell_1<\dots<\ell_{n(t+1)}\le T-t-1+n(t+1)}}p_t\left(\ell_1,\dots,\ell_{n(t+1)}\right)w_t+\sum_{j=1}^{n(t+1)}y_{j|\ell_j}^t\\ &= \begin{cases} w_0 & \text{, if we did not invest before or at} t,\\ w_0\sum_{j=\rho_t(x_s)}^{n(t+1)}p_t\left(\{\rho_T(x_s)=j\}\right)\beta_s^j & \text{, if we invested before} w_0 \text{at} s\le t, \end{cases} \end{aligned} \tag{33}$$

where $p_t(\{\rho_T(x_s) = \ell\})$ is provided by (12). Note that Proposition (1) applied to the case $m = 1$, yields $p_t(\{\rho_T(x_s) = \ell\})$. Hence, also $\mathscr{W}_t(w_t, \mathbf{y}_t)$ and $\mathscr{V}_t(w_t, \mathbf{y}_t)$ are proportional to $\mathscr{W}_t(1, \mathbf{y}_t)$ and $\mathscr{V}_t(1, \mathbf{y}_t)$, respectively. □

D On the existence of expected utility

This section discusses the existence of expected utility for deterministic and stochastic returns. In the main text this existence is assumed.

Deterministic Returns: We consider a utility function $u : \mathbb{R}_{\geq 0} \to \mathbb{R} \cup \{-\infty\}$. This function is continuous, monotone increasing and satisfies $u(w_0) > -\infty$. The final wealth is provided by $w_{T+} = w_T + y^T_{1|1} + \cdots + y^T_{n|n} = w_T + \sum_{t=1}^T b_t \beta_t^{\rho_T(x_t)}$. Since $0 \leq \beta_t^\ell \leq \max\{1, \max_{t,\ell} \beta_t^\ell\} =: \beta^{\max}$, we observe that $w_{T+} \in \left[0, \beta^{\max} w_0\right]$.

Hence, $u\left(\beta^{\max} w_0\right) > -\infty$. By contrast $u\left(w_{T+}\right)$ can be unbounded from below, for example, in the case of log-utility. Since $\mathbb{E}\left(u(w_0)\right) = u(w_0) > -\infty$, a strategy where $\mathbb{E}\left(u\left(w_{T+}\right)\right) = -\infty$ cannot be optimal. In other words, the strategy $b_1 = 0, \ldots, b_T = 0$ is better than a strategy $b_1, \ldots, b_T$ where $u\left(w_{T+}\right) = -\infty$ with strictly positive probability. An optimal strategy, say $b_1^*, \ldots, b_T^*$, is at least as good as the strategy over never investing, that results in $u(w_0)$. Hence, for log-utility the investor does not choose a strategy where the probability $\mathbb{P}\left(w_{T+} = 0\right) > 0$. Consequently, for deterministic returns the expected utility exists. There is a non-empty subset of strategies containing the optimal strategy.

Stochastic Returns: In this case $w_{T+} = w_T + Y^T_{1|1} + \cdots + Y^T_{n|n} = w_T + \sum_{t=1}^T b_t B_t^{\rho_T(x_t)}$, while $w_{T+} \geq 0$ an upper bound as in the deterministic case need not exist.

Risk-neutrality: In the case of a risk-neutral agent, we work with the identity function for $u(x)$. Then, $0 \leq \mathbb{E}\left(w_{T+}\right) = \mathbb{E}\left(w_T + \sum_{t=1}^T b_t B_t^{\rho_T(x_t)}\right) \leq w_0 + w_0 \mathbb{E}\left(\sum_{t=1}^T B_t^{\rho_T(x_t)}\right)$. Hence, $\mathbb{E}\left(B_t^\ell\right) < \infty$, for all t and ℓ, is sufficient for the existence of $\mathbb{E}\left(w_{T+}\right)$.

Risk-aversion: In this case the utility function $u(x)$ is concave. Suppose that $\mathbb{E}\left(B_t^\ell\right) < \infty$, for all t and ℓ, as in the risk-neutral case. Then, by Jensen's inequality $\mathbb{E}\left(u\left(w_{T+}\right)\right) \leq u\left(\mathbb{E}\left(w_{T+}\right)\right)$. Since $\mathbb{E}\left(w_{T+}\right)$ exists under the above assumptions and only strategies where $\mathbb{E}\left(w_{T+}\right) \geq w_0$ can be optimal, we observe that $-\infty < u(w_0) \leq u\left(\mathbb{E}\left(w_{T+}\right)\right) < \infty$. As in the non-stochastic case there can be strategies where $\mathbb{E}\left(u\left(w_{T+}\right)\right) = -\infty$. Since $-\infty < u(w_0)$, these strategies can be excluded a-priori. Hence, by the assumption $\mathbb{E}\left(B_t^\ell\right) < \infty$, for all t and ℓ, the existence of $\mathbb{E}\left(u\left(w_{T+}\right)\right)$ follows for all optimal strategies.

Risk-loving or general continuous and monotone utility function: In the case where $u(x)$ is convex or can be S-shaped, inverse S-shaped, etc. In this case neither $\mathbb{E}\left(u\left(w_{T+}\right)\right) = +\infty$ nor $\mathbb{E}\left(u\left(w_{T+}\right)\right) = -\infty$ can be excluded a-priori. Even worse, strategies resulting in $\mathbb{E}\left(u\left(w_{T+}\right)\right) = +\infty$ if this is possible, are optimal is this case. Hence, in general, for a monotone increasing and continuous $u(x)$ the existence $\mathbb{E}\left(u(w_{T+})\right)$ has to be checked on a case by case basis. A sufficient for the existence of $\mathbb{E}\left(u(w_{T+})\right)$ would be bounded support of B_t^ℓ for all t, ℓ.

E Simulation study

In the following we describe how the residual term residual terms $r_t(w_t, \mathbf{y}_t)$ is obtained. As a byproduct we obtain optimal investments b_t as function of the history $\mathbf{y}_{t-1}$, the current budget w_{t-1} (before investment takes place at t) and the realized rank $\rho_t(x_t)$. The main numerical burden in our simulation study is to obtain the residual terms $r_t(w_t, \mathbf{y}_t)$ for various w_t and prior investments resulting in $\mathbf{y}_t$. To do this we consider a grid on the interval $[0, w_0]$, with a step width of $\kappa := 1/(\gamma - 1)w_0$, such that 0 and w_0 are end points of this grid. In our simulation runs we work with $w_0 = 1$ and $\gamma = 20$, resulting in $\kappa = 0.0526$. w_t and b_t are only allowed to be multiples of κ, where $b_t = \lambda_b \kappa$, $w_t = \lambda_w \kappa$ and $\lambda_b,\ \lambda_w$ are integers ≥ 0 and $\leq \gamma = 20$.

From Theorem 1 it follows that $r_T = 0$. In the final investment step the optimal b_T has to be obtained for various ranks of X_T. The ranks $1, \ldots, n$ are relevant for the investment decision, $b_T = 0$ for a rank $> n$. In particular, optimal investments have to be obtained for all $\gamma = 20$ possible values of w_{T-1} and n relevant ranks of X_T. Then, for example with $t = T$ and $n = 2$, for the n opportunities with the lowest n ranks we observe three relevant cases: (i) the ranks remain the same, (ii) the current best becomes second best and (iii) the current second best third best or the current best remains the best and the current second best becomes third best. For a general t there can be at most $T(T-1)/2$ cases. For general n we get $\binom{T}{n}$ such opportunities. To construct an object of the same dimension for each t in our numerical tool, we consider $T(T-1)/2$ opportunities for $n = 2$, where we know that some of them are infeasible for t close to T. In addition, by working with γ grid points γ^n investment constellations are possible (some of them are infeasible, e.g. is the sum of the amounts invested in t are larger than the budget available in t). Hence, $\gamma \times \binom{T}{n} \times \gamma^n$ scenarios have to be considered. Some of them are of course infeasible (e.g., if $w_{t-1} - b_t < 0$ or the rank cannot become possible since the number of steps to go is too low). By solving for the optimal b_t (approximately optimal since b_t is forced to be a number of the grid considered), we obtain $r_{T-1}(w_{T-1}, \mathbf{y}_{T-1})$. In a next step we use the recursion (10) to obtain $r_{T-2}(w_{T-2}, \mathbf{y}_{T-2})$, given $r_{T-1}(w_{T-1}, \mathbf{y}_{T-1})$ and the $\gamma \times \binom{T}{n} \times \gamma^n$ scenarios which have to be considered. For each possible scenario the optimal b_t on the grid have to be obtained. We do this by evaluating at $b_t = \lambda_b \kappa$, where $\lambda_b \kappa \leq w_{t-1}$ and take that b_t where a maximum is obtained. Note that we assume that the returns depend on the rank only and not on t (With payoffs depending on time the numerical complexity would increase additionally). Then this recursive procedure (10) is used to derive $r_{t-2}(w_{t-2}, \mathbf{y}_{t-2})$. Hence, the numerical complexity is given by $(\gamma \times \binom{T}{n} \times \gamma^n)T$, where for larger T and n a lot of possible outcomes at T and their corresponding probabilities have to be calculated. Therefore, we work with a relatively small γ, $T = 5, 10, 20$ and $n = 2$.

After we have obtained $r_t(w_t, \mathbf{y}_t)$, we already know the optimal investment strategy b_t given a history $\mathbf{y}_{t-1}$, a budget w_{t-1} and a realized rank $\rho_t(x_t)$. Hence, to derive samples of investment paths we only have to draw for each path T iid. uniformly distributed random variables on the unit interval. Then for each t the realized relative rank of $x_1, \ldots, x_t$ has to be obtained. The optimal b_t follows from the recursive procedure used to obtain $r_t(w_t, \mathbf{y}_t)$. After considering these optimal investment decisions for $t = 1, \ldots, T$, the investor obtains the corresponding final payoff $w_T + \sum_{j=1}^{n} y_{j|j}^T$. For the stochastic case $\mathbb{E}_t\,(r_t(w_t, \mathbf{y}_t))$ obtained by choosing the optimal b_t in each investment step. For the stochastic case the random variables $B^1_{\tau_1^T}, \ldots, B^n_{\tau_n^T}$ realize at T.

The number of Monte-Carlo replications is $M = 5000$. [10]

[10] The software package `Matlab` was used.

F Optimal strategies for small T

The following subsection obtains the optimal investment rules for the limited information case for a risk-neutral investor for $T = 3$ and $T = 5$ respectively. We set $\beta_t^1 = 3$ and $\beta_t^2 = 2$, for all t. We know from Sect. 3 that investing all the remaining wealth is an optimal strategy for risk-neutral investors. By time invariant β^j, the investment thresholds, denoted d_t in Remark 3, can be obtained by comparing the expected payoffs if we invest in the corresponding period to the expected payoffs if we do not invest. In addition, we obtain the optimal investment strategy for the full information case under risk-neutrality for $T = 3$. In this case the investor observes x_t, at $t = 1, \ldots, T$. Also in the full information case investing all or nothing is an optimal strategy (for a proof see [17]).

F.1 Full versus limited information for $T = 3$

Limited Information: At $t = T = 3$ an investor invests all the remaining wealth if the relative $=$ absolute rank of the draw x_3 is equal to 1 or 2, for a rank of 3 the agent does not invest. Hence, $d_3 = 2$ (invest if the relative rank is ≥ 2).

At $T = 2$: (2.i) Suppose that the realized relative rank of x_2 is $\rho_2(x_2) = 1$. If the agent invests, then she/he expects

$$2/3 * 3 + 1/3 * 2 = 1/3 * (6 + 2) = 8/3$$

while if she/he does not invest the expected payoff is

$$1/3 * 3 + 1/3 * 2 + 1/3 * 1 = 1/3 * (3 + 2 + 1) = 2 < 8/3.$$

Hence, $\rho_2(x_2) = 1$ results in investing all the remaining wealth. (2.ii) Suppose that $\rho_2(x_2) = 2$. Then, investing at $t = 2$ results in an expected wealth of

$$1/3 * 2 + 2/3 * 0 = 1/3 * (2) = 2/3$$

while we still expect a payoff of 2 if we do not invest. Hence, $d_2 = 1$ (invest if the relative rank at $t = 2$ is 1).

Finally at $t = 1$ the relative rank of the first draw has to be equal to 1. The rank statistics are uniformly on $\{1, \ldots, T\}$, hence $\mathbb{P}(\rho_3(X_1)) = j) = 1/3$, $j = 1, \ldots, 3$. The expected payoff is

$$1/3 * 3 + 1/3 * 2 + 1/3 * 0 = 1/3 * (5) = 5/3.$$

If we proceed we get

$$1/2 * 8/3 + 1/2 * 2 = 1/6 * (8 + 6) = 14/6 = 7/3 > 5/3$$

where 8/3 is the expected payoff we get if $\rho_2(x_2) = 1$ (see above) and 2 the expected payoff if $\rho_2(x_2) = 2$, both events has a probability of 1/2. This results in $d_0 = 0$ (do not invest at $t = 1$). That is, we get $d_1 = 0$, $d_2 = 1$, $d_3 = 2$, the expected wealth is $\mathscr{V}_0(1) = 7/3$.

Full Information: In this case the investor observes the realization of X_t in period t. At $t = T = 3$ an investor invests all the remaining wealth if the relative $=$ absolute rank of the draw X_3 is equal to 1 or 2, for a rank of 3 the agent does not invest. Hence, $d_3 = 2$ (invest if the relative rank if ≥ 2). Hence, the result remains the same as in the limited information case for $t = T$.

At $T = 2$: (2.i) Suppose that the realized relative rank of x_2 is $\rho_2(x_2) = 1$. Hence, $x_2 \geq x_1$ and x_1 has relative rank 2. If the agent invests, then she/he expects

$$3x_2 + 2(1 - x_2) = 2 + x_2$$

while if she/he does not invest the expected payoff is

$$3(1 - x_2) + 2(x_2 - x_1) + 1x_1 = 3 - x_2 - x_1.$$

Comparing these terms shows that we invest if the rank of x_2 is one and $x_2 \geq \frac{1-x_1}{2}$.

(2.ii) Now suppose that the relative rank of x_2 is 2 and therefore $x_1 > x_2$. If the agent invests, then she/he expects

$$2x_2 + 0(1 - x_2) = 2x_2$$

while if she/he does not invest the expected payoff is

$$3(1 - x_1) + 2(x_1 - x_2) + 1x_2 = 3 - x_2 - x_1.$$

Now the agent also invests if the rank of x_2 is two and $x_2 \geq \frac{3-x_1}{3} = 1 - \frac{x_1}{3}$.

Finally, we consider $T = 1$ where x_1 has rank 1. If we invest we expect

$$3x_1^2 + 4(1 - x_1)x_1.$$

(1.i) Conditional on the case $x_1 \geq x_2$, x_2 is uniformly distributed in the interval $[x_1, 1]$. If we do not invest at $t = 1$ but invest at $t = 2$, we expect

$$\frac{1}{1 - x_1} \underbrace{\int_{\max\left(\frac{1-x_1}{2}, x_1\right)}^{1} (2 + x_2) dx_2}_{=: I_1\left(x_1, \underbrace{\max\left(\frac{1-x_1}{2}, x_1\right)}_{a_1}, 1\right)}$$

while if we do not invest in $t = 1$ and $t = 2$ we expect

$$\frac{1}{1 - x_1} \underbrace{\int_{x_1}^{\max\left(\frac{1-x_1}{2}, x_1\right)} (3 - x_1 - x_2) dx_2}_{=: I_2(x_1, x_1, a_1)}.$$

(1.ii) Next conditional on the case $x_1 < x_2$, x_2 is uniformly distributed in the interval $[0, x_1)$. If we do not invest at $t = 1$ and invest at $t = 2$, we now expect

$$\frac{1}{x_1} \underbrace{\int_{\min\left(1-\frac{x_1}{3}, x_1\right)}^{x_1} 2x_2 dx_2}_{=: I_3\left(x_1, \underbrace{\min\left(1 - \frac{x_1}{3}, x_1\right)}_{a_3}, x_1\right)}$$

while if we neither invest in $t = 1$ nor in $t = 2$ we expect

$$\underbrace{\frac{1}{x_1}\int_0^{\min\left(1-\frac{x_1}{3},x_1\right)} (3 - x_1 - x_2)dx_2}_{=:I_4(x_1,0,a_3)} .$$

Then the expected payoff if we do not invest at $t = 1$ is given by $(1 - x_1)\left[\frac{1}{1-x_1} I_1\left(x_1, a_1, 1\right) + \frac{1}{1-x_1} I_2\left(x_1, x_1, a_1\right)\right] + x_1\left[\frac{1}{x_1} I_3\left(x_1, a_3, x_1\right) + \frac{1}{x_1} I_4\left(x_1, 0, a_3, \right)\right] = I_1\left(x_1, a_1, 1\right) + I_2\left(x_1, x_1, a_1\right) + I_3\left(x_1, a_3, x_1\right) + I_4\left(x_1, 0, a_3, \right)$. The above integrals result in the three cases:

1. $x_1 \geq 3/4$, in which case $\max\left(\frac{1-x_1}{2}, x_1\right) = x_1$ and $\min\left(1 - \frac{x_1}{3}, x_1\right) = 1 - \frac{x_1}{3}$. In this case the second integral I_2 is zero. By comparing $3x_1^2 + 4(1 - x_1)x_1$ to $I_1\left(x_1, a_1, 1\right) + I_2\left(x_1, x_1, a_1\right) + I_3\left(x_1, a_3, x_1\right) + I_4\left(x_1, 0, a_3, \right)$, we break even at $x_1 \approx 0.6822 < 3/4$. Hence, we always invest into the first draw if $x_1 \geq 3/4$.
2. $1/3 \leq x_1 < 3/4$, in which case $\max\left(\frac{1-x_1}{2}, x_1\right) = x_1$ and $\min\left(1 - \frac{x_1}{3}, x_1\right) = x_1$. In this case the second and the third integral are zero, that is $I_2 = I_3 = 0$. We break even at $x_1 \approx 0.679449$. Hence, we always invest into the first draw if $x_1 \geq 0.679449$ (and we do not invest for $0.679449 > x_1 \geq 1/3$).
3. $x_1 < 1/3$, where $\max\left(\frac{1-x_1}{2}, x_1\right) = \frac{1-x_1}{2}$ and $\min\left(1 - \frac{x_1}{3}, x_1\right) = x_1$. In this case the third integral $I_3 = 0$ is zero. We break even $x_1 \approx 0.7802$. Hence, we do not invest into x_1.

By the above cases we get the value function for $t = 1$ and an initial budget of 1 in the full information case:

$$\mathcal{V}_1(1, x_1)^{full\,information} = \begin{cases} 3x_1^2 + 4(1 - x_1)x_1 & \text{if } x_1 \geq 0.6798449 =: a^I \ , \\ I_2(x_1, 1/3, a^I) & \text{if } 1/3 \leq x_1 < a^I \ , \\ I_3(x_1, 0, 1/3) & \text{if } x_1 < 1/3 \ . \end{cases} \tag{34}$$

Hence, for the full information case the optimal investment strategy is given by

- Invest at $t = 1$ if and only if $x_1 \geq 0.679449$.
- Invest at $t = 2$ if $x_2 \geq x_1$ and $x_2 \geq \frac{1-x_1}{2}$ or if $x_2 < x_1$ and $x_2 \geq 1 - \frac{x_1}{3}$.
- Invest if the rank of x_3 is ≤ 2.

Are we investing into x_2 of rank 2 at $t = 2$? If x_1 is 2/3 and therefore close to a^I, $x_2 \geq 1 - 1/3(2/3) = 7/9 > a^I$ (also if $x_1 = a^I$ we get $x_2 \geq 1 - a^I/3 > a^I$). Therefore, for the given returns $\beta^1 = 3$ and $\beta^2 = 2$ the investor does not invest into x_2 if the realized rank is equal to two.

Finally we obtain the value function for $t = 0$ by means of

$$\mathcal{V}_0(1)^{full\,information} = \int_0^1 \mathcal{V}_1(1, x_1)^{full\,information} dx_1 = 2.59603 \ . \tag{35}$$

F.2 Thresholds d_t for $T = 5$

Limited Information: This works in a similar way to the case where $T = 3$. At $t = T = 5$, all the remaining wealth is invested if the relative = absolute rank of the draw x_5 is equal to 1 or 2, for a ranks of ≥ 3 the agent does not invest. Hence, $d_3 = 2$ (invest if the relative rank if ≥ 2).

At $T = 4$: (4.i) Suppose that the realized relative rank of x_4 is $\rho_2(x_2) = 1$. If the agent invests, then she/he expects

$$4/5 * 3 + 1/5 * 2 = 1/5 * (12 + 2) = 14/5 = 2.8$$

while if she/he does not invest the expected payoff is

$$1/5 * 3 + 1/5 * 2 + 3/5 * 1 = 1/5 * (3 + 2 + 3) = 8/5 = 1.6 < 2.8.$$

Hence, $\rho_4(x_4) = 1$ results in investing all the remaining wealth. (4.ii) Suppose that $\rho_4(x_4) = 2$. Then, investing at $t = 2$ results in an expected wealth of

$$3/5 * 2 + 2/5 * 0 = 6/5 = 1.2 < 1.6$$

while we still expect a payoff of 2 if we do not invest. Hence, $d_4 = 1$ (invest if the relative rank at $t = 4$ is 1).

For $T = 3$ and a relative rank of x_3 of 1, we get

$$3/5 * 3 + 3/10 * 2 = 1/10 * (3 * 2 * 3 + 6) = 24/10 = 2.6.$$

If we do not invest the expected payoff is

$$1/4 * 2.8 + 3/4 * 1.6 = 1.9 < 2.6.$$

Since d_t are non-decreasing in t, we get $d_3 = 1$ (see Remark 3)).

For $T = 2$ and a relative rank of x_2 of one we get

$$2/5 * 3 + 2/10 * 2 = 2.$$

If we do not invest the expected payoff is

$$1/3 * 2.4 + 2/3 * 1/4 * 2.8 + (2/3 * 3/4 * 1/5)1.6 > 2.$$

Hence $d_1 = d_2 = 0$.

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