CAPITAL INCOME TAXATION UNDER FULL LOSS OFFSET PROVISIONS OF A PROSPECT THEORY INVESTOR[1]

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ABSTRACT

In this paper we examine capital income taxation of a reference dependent sufficiently loss averse investor in a two period portfolio choice model under full loss offset provisions. Capital income taxation with loss offset provisions has been found to stimulate risk taking in expected utility models under certain assumptions about attitudes towards risk but would such effect be found under prospect theory type of preferences? We observe that the impact of capital income taxation depends on investors' reference levels relative to their endowment income and thus we explore capital income taxation for different types of loss averse investors in terms of their ambition. We consider the less ambitious investors to be the ones with relatively low reference levels (they avoid relative losses in both periods) and more ambitious investors to be those with relatively high reference levels. We analyze two types of more ambitious investors: investors with higher time preference (who experience relative losses only in the second period under the bad state of nature) and investors with lower time preference (who experience relative losses only in the first period). We observe that capital income taxation stimulates current consumption in most cases which encourages risk taking, although the final outcome would depend on the investors' degree of risk aversion, the rate of time preference and the tax rate in relation to certain thresholds. Current consumption could be discouraged for some ambitious type of investors that have relatively high second period reference levels but not necessary first period reference levels. In summary, to determine the impact of capital income taxation on the decision variables the reference levels in relation to endowment income play the most significant role. Ignoring reference depended preferences can lead to different conclusions for investors reaction to capital income taxation. We also find certain type of investors whose happiness level increases with capital income taxation under full loss offset provisions.

Keywords: prospect theory, loss aversion, consumption-savings decision, capital income tax

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INTRODUCTION

One of the most robust discovery in theoretical public finance has been that a proportional tax on risky returns with full loss offset will stimulate risk taking contrary to the popular view that taxes hurt investment activity. This important public policy finding for encouraging the undertaking of additional risky projects by rational risk averse investors originated from the seminal work by Domar and Musgrave (1944). With full loss offset, the tax reduces the return of the risky project but also its riskiness. In the absence of income effects a risk averse investor reacts to the tax increase by increasing investment in the risky asset in order to make the distribution of the after-tax return of the asset the same as that prior to taxation.[2] This reaction to the tax results in the expected utility remaining unchanged (Mossin, 1968). In the presence of income effects, Stiglitz (1969) showed that under reasonable attitudes towards risk, such as nondecreasing relative risk aversion, the encouragement to undertake more risky projects is still observed from a theoretical perspective. The importance of full loss offset provision to stimulate risk taking activity has had continued support for more general formulations (see Ahsan, 1974, 1989; Heaton, 1987; Eeckhoudt, Gollier and Schlesinger, 1997). However, this stimulus depends also on how the risk is handled by the public sector. If the public sector is no more efficient in handling risk, then risk taking will be discouraged with the tax on risky assets (see Gordon and Wilson, 1989; Ahsan and Tsigaris, 2009). The above literature uses the von Neumann-Morgenstern expected utility framework which assumes that investors are rational; are risk averse (but not loss averse nor risk lovers); they don’t place more importance to the status quo relative to other outcomes; they don’t compare their consumption or wealth to a reference level and they have objective (true) information on the probabilities of outcomes in different states of nature when making decisions (not subjective). However, in a breakthrough research Kahneman and Tversky (1979) conducted an experiment which showed that these assumptions are not valid and that a better model to describe such decisions under risk is prospect theory.[3]

Henceforth, the objective and main contribution of this research is to consider the response of a prospect theory type of investor arising from changes in a proportional capital income tax rate under full loss offset provisions on risk taking activity and consumption. We address the following questions for insights into the new approach of the literature on behavioural public finance: How would a loss averse and risk concerned (or risk lover) reference dependent investor react to a change in capital income taxation on risk taking in a two period consumption

2. This reaction is also observed when the return to the risk-free asset is zero or when the risk-free asset yields a positive return and the tax is imposed only on the excess return of the risky asset (i.e., on the risk premium).
3. See also Tversky and Kahneman (1992).

portfolio choice model? How does the prospect theory type investor's behaviour differ from the above predictions of the traditional public finance approach to these old questions on risk taking and taxation? Are there any new insights from the new approach which behavioral public finance theory can provide? For example, what additional assumptions will be needed to be imposed to generate an increased demand for risky assets from an increase in capital income taxation with full loss offset provisions? Does the investor shift preference towards current consumption and away from consumption in the future as in the traditional public finance approach to the old question of inter-temporal decisions from capital income taxation? Thus, in this paper we examine how in a two period prospect theory type of model an increase in capital income taxation, under full loss offset, affects a loss averse investor's behaviour in terms of: current and future consumption, savings, portfolio choice between a risk-free and risky assets as well as the happiness as measured by investor's expected indirect utility function. To arrive to some new insights a theoretical model developed by Hlouskova, Fortin and Tsigaris (2017) is used with an introduction of a proportional capital income tax on the returns to the risk-free and risky asset. Due to the behavioural economics type of preferences, we explore the reaction of prospect theory type investors whose behaviour depends on where their consumption reference levels are in relation to their endowment income levels. Hlouskova et al. (2017) show that the optimal solution for consumption and risk taking depends on the loss averse investor's degree of ambition determined by how the present value of endowment income differs from the present value of reference levels.

There have been experimental studies which attempt to analyze the effects of taxation on risk taking activity (see among others, Swenson, 1989; King and Wallin, 1990; Ackermann, Fochmann and Mihm, 2013; Blaufus et al., 2013; Fochmann, Kiesewetter and Sadrieh, 2012; Fochmann, Hemmerich and Kiesewetter, 2016; Fochmann et al., 2017; Fochmann and Hemmerich, 2017). Most of these experimental studies reach different conclusions from the theoretical literature and some support the popular view that taxes hurt risky investment even when full loss offset provisions are in place. Swenson (1989) and King and Wallin (1990) find, without accounting for behavioural biases, that a proportional tax with full loss offset will not have a significant effect on risk taking. On the other hand, Fochmann and Hemmerich (2017) and Fochmann et al. (2017) find in their experimental studies that a proportional income tax, even with full loss offset, results in a significant reduction in risk taking. They attribute this behavior to perceptual tax biases (not to rational tax effects) and explain such a negative reaction to the tax due to the investor's high cognitive load for solving complex problems. According to Fochmann and Hemmerich (2017) this reaction is consistent with Ackermann, Fochmann and Mihm (2013) who show that a reduction in the complexity of the problem posed on the subjects reduces the tax biases. It is also consistent with Kahneman and Tversky (1979) if the gross return is part of the reference level and with Thaler (1985) who provides explanation due to mental accounting.[4] There has also been some theoretical work on the effects of capital income taxation under prospect theory type of preferences (see Hlouskova and Tsigaris, 2012; Hlouskova et al., 2014; Mehrmann and Sureth-Sloane, 2017). Mehrmann and Sureth-Sloane (2017) find that tax loss offsets restrictions affect negatively investment in risky assets. Results from a one period model of Hlouskova and Tsigaris (2012) indicate that it is possible for a capital income tax increase not to stimulate risk taking (i.e., investing into a risky asset) even if the tax code provides attractive full loss offset provisions. This depends on how that tax affects the reference level and hence how it is perceived by investors. Risk taking can increase if the investor interprets part of the tax as a loss instead as a reduced gain. In this case the investor becomes risk seeking and responds by increasing total and private risk taking.

This new behavioural economics approach to the old questions of the field of public finance finds that investors that are driven by a self-enhancement motive (i.e., have relatively low reference levels) will have their present value of endowment income higher than the present value of their consumption reference levels. These investors are considered as less ambitious and in the limiting case represent investors who have expected utility preferences. Their optimal decisions are such that they manage to avoid relative losses from occurring in both periods.[5] Risk taking, future consumption and investors' expected indirect utility (happiness) are positively related to the first period optimal consumption relative to its reference level. Within the less ambitious investors we explore the effect of the tax for three subcases depending on where the reference level is in relation to the endowment income of the same period and/or in relation to the endowment income and reference level of the other period. These investors react to a capital income taxation in such a way that they continue to avoid relative losses in all subcases. To the extent that the capital income tax stimulates current consumption which most likely will occur with such a tax in most cases examined but not all, this contributes towards encouraging risk taking although the final outcome depends on various thresholds for the risk aversion parameter, the tax rate and the rate of time preference. A sufficiently high risk aversion will stimulate risk taking for the less ambitious investors, while those that are relatively less risk averse, combined with a lower tax rate and relatively impatient, increased taxation will discourage risk

4. The effects of tax on risk taking due to behavioral biases were studied in earlier work of Fochmann, Kiesewetter and Sadrieh (2012) who had reached the opposite conclusion.
5. Relative losses occur when consumption is under the corresponding reference level.

taking even with full loss offset provisions.[6] It is worth noting that when the first period reference level exceeds the endowment income in that period and the second period reference level is below the second period reference level (i.e., the investor is relatively more ambitious in the first than the second period) the tax increase results in a happier investor as measured by the indirect utility function. The same feature is observed also for investors whose both reference levels are below their corresponding endowment income under certain conditions such as being relatively impatient to consume in the future.

On the other hand, investors that are driven by a self-improvement motive have higher reference levels than endowment income in present value. These investors find optimal solutions such that they cannot avoid relative losses in either the second period in the bad state of nature if they are relatively impatient or in the first period to avoid relative losses in the future if they are relatively more patient to consume in the future relative to the present. We label these investors as more ambitious investors and explore the effect of capital income taxation for three subcases depending on the relations among reference levels and endowment incomes. For ambitious but impatient investors we find that in most cases reaction to capital income taxation is to reduce losses in the bad state of nature. In all three subcases the tax stimulates risk taking. Surprisingly in two out of three cases the tax increases the investor's happiness level.

For more ambitious but also relatively patient investors the reaction of the capital income taxation, in cases where the second period reference level is higher than the respective endowment income, is to increase relative losses in the first period further. The impact on risk taking is similar to the case of a less ambitious household in that risk aversion and tax rate have to be higher then some thresholds to stimulate risk taking. It is again worth mentioning that an investor with relative high first period reference level but not a second period will be happier with the tax increase.

In section 2 the model set-up is presented. This is followed with section 3 where the impact of capital income tax on less ambitious investors is analyzed. In section 4 we examine the more ambitious investors with a higher rate of time preference and those with a lower rate of time preference. Section 5 concludes.

MODEL SET-UP

In a two period life cycle model an investor has a non-stochastic first period

6. Students as subjects used in experimental studies could be of the relatively impatient type which may explain why risk-taking falls with a tax under full loss offset provisions as in Fochmann and Hemmerich (2017).

exogenous labor income, $Y_1 > 0$, which it can allocate to current consumption, C_1, risk-free investment, m, and risky investment, $\alpha \geq 0$, where the sum of the risky and risk-free investment are savings S. Thus, in the first period

$$Y_1 = C_1 + m + \alpha = C_1 + S \tag{1}$$

Let $\tau \in (0,1)$ represents capital income tax. We consider two assets, a risk-free asset with a net after tax return $(1-\tau)r_f > 0$ and a risky asset with stochastic after tax return $(1-\tau)r_g$ in the good state of nature, which occurs with probability p, $0 < p < 1$, and with after tax return $(1-\tau)r_b < 0$, in the bad state of nature, which occurs with probability $1-p$. To ease the exposition, let us introduce the following notation

$$\bar{r}_f = (1-\tau)r_f$$
$$\bar{r}_s = \begin{cases} \bar{r}_g = (1-\tau)r_g & if \quad s = g \\ \bar{r}_b = (1-\tau)r_b & if \quad s = b \end{cases}$$

We assume that

$$-1 < r_b < 0 \leq r_f < r_g \tag{2}$$

which implies that also after tax returns follow the same inequality, namely: $-1 < \bar{r}_b < 0 \leq \bar{r}_f < \bar{r}_g$. Finally, we assume that $\mathbb{E}(r) = p\, r_g + (1-p)r_b > r_f$. This implies that also $\mathbb{E}(\bar{r}) = p\, \bar{r}_g + (1-p)\bar{r}_b > \bar{r}_f$.

In the second period the investor consumes

$$C_{2s} = \begin{cases} (1+\bar{r}_f)m + (1+\bar{r}_g)\alpha + Y_2 & if \quad s = g \\ (1+\bar{r}_f)m + (1+\bar{r}_b)\alpha + Y_2 & if \quad s = b \end{cases}$$

where $Y_2 \geq 0$ is the non-stochastic income in the second period and $s \in \{b, g\}$. Note that $C_{2g} \geq C_{2b}$ as $\alpha \geq 0$ and $\bar{r}_g > \bar{r}_b$, where C_{2g} is the second period investor's consumption in the good state of nature and C_{2b} in the bad state of nature. The investor is allowed to consume the non stochastic future income Y_2 in the first period, as long as consumption exceeds zero in either period and savings are negative. Hence, the investor can partially borrow from the risk-free asset m against its future income. The earnings from total investments are equal to $(1 + \bar{r}_f)m + (1+\bar{r}_s)\,\alpha,\ s \in \{b, g\}$. Based on this and (1) the consumption in the second period is

$$C_{2s} = \begin{cases} \left(1+\bar{r}_f\right)(Y_1 - C_1) + (\bar{r}_g - \bar{r}_f)\alpha + Y_2 & if \quad s = g \\ \left(1+\bar{r}_f\right)(Y_1 - C_1) + (\bar{r}_b - \bar{r}_f)\alpha + Y_2 & if \quad s = b \end{cases} \tag{3}$$

The investor's preferences are described by the following reference based utility function

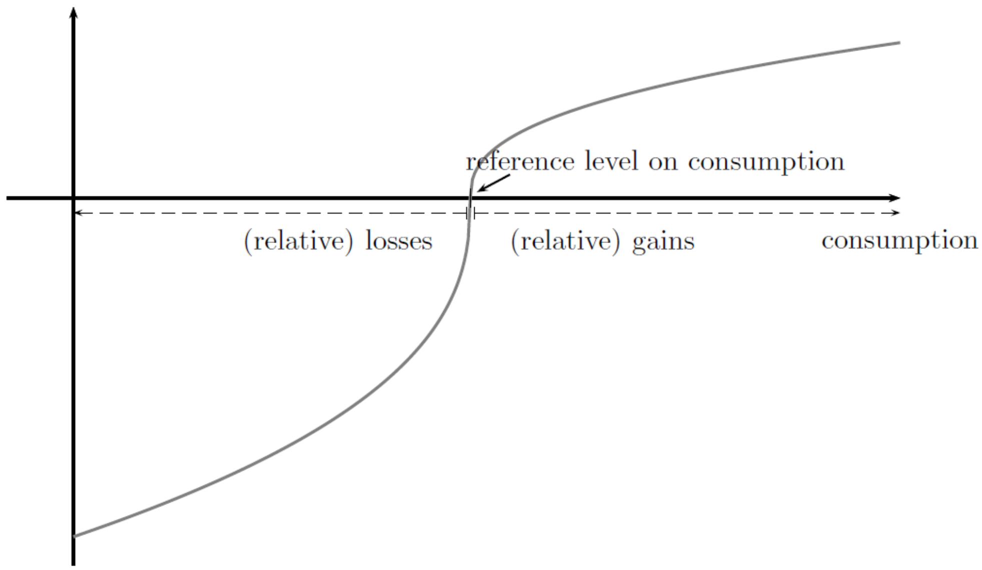
$$U(C_1, \alpha) = V(C_1 - \bar{C}_1) + \delta\, V(C_2 - \bar{C}_2) \tag{4}$$

where $\bar{C}_1$ and $\bar{C}_2$ are exogenous consumption reference (or comparison) levels, such that $\max\{\bar{C}_1, \bar{C}_2\} < Y_1 + \frac{Y_2}{1+r_f}$, δ is the discount factor, $0 < \delta < 1$, and $V(\cdot)$ is a prospect theory (S-shaped) value function defined as

$$V(C_i - \bar{C}_i) = \begin{Bmatrix} \frac{(C_i - \bar{C}_i)^{1-\gamma}}{1-\gamma}, & C_i \geq \bar{C}_i \\ \\ -\lambda \frac{(\bar{C}_i - C_i)^{1-\gamma}}{1-\gamma}, & C_i < \bar{C}_i \end{Bmatrix} \tag{5}$$

for $i = 1,2$. Parameter $\lambda > 1$ is the loss aversion parameter and $\gamma \in (0,1)$ is the parameter determining the curvature of the utility function. If consumption is above the reference level there are (relative) gains, if consumption is below the reference level there are (relative) losses. The utility has a kink at the consumption reference level and it is steeper for losses than for gains, i.e., a decrease in consumption is more severely penalized in the domain of losses than in the domain of gains. Finally, the utility function is concave above the reference point and convex below it. The investor is thus risk averse in the domain of gains (i.e., above the consumption reference level) and risk seeking in the domain of losses (i.e., below the consumption reference level), see Figure 1.

Figure 1. Prospect theory (S-shaped) value function.



The investor maximizes the following expected utility as given by (4) and (5)

$$\text{Max}_{(C_1,\alpha)}: \quad \mathbb{E}(U(C_1,\alpha)) = V(C_1 - \bar{C}_1) + \delta\, \mathbb{E}(V(C_2 - \bar{C}_2))$$

$$\text{such that:} \qquad C_1 \geq 0, \qquad C_{2b} \geq 0 \ \text{ and } \ \alpha \geq 0$$

Based on this and (3) the investor's maximization problem can be formulated as follows

$$\left.\begin{array}{llll} \text{Max}_{(C_1,\alpha)}: & \mathbb{E}(U(C_1,\alpha)) & = & V(C_1 - \bar{C}_1) \\ & & + & \delta\, \mathbb{E}(V\left((1+\bar{r}_f)(Y_1 - C_1) + Y_2 + (\bar{r}_s - \bar{r}_f)\alpha - \bar{C}_2\right)) \\ \text{such that:} & 0 \leq & C_1 & \leq Y_1 + \frac{Y_2 - (\bar{r}_f - \bar{r}_b)\alpha}{1+\bar{r}_f}, \\ & 0 \leq & \alpha & \leq \frac{(1+\bar{r}_f)Y_1 + Y_2}{\bar{r}_f - \bar{r}_b} \end{array}\right\} \tag{6}$$

Note that the upper bound on C_1 follows from $C_{2b} \geq 0$ and the upper bound on α follows from the imposition of the upper bound on C_1 exceeding zero, i.e.

$Y_1 + \frac{Y_2-(\bar{r}_f-\bar{r}_b)\alpha}{1+\bar{r}_f} \geq 0$. The condition on α means that short sales are not allowed.

As a measure of the riskiness of future consumption (also known as private risk taking) we also introduce the standard deviation of the second period consumption of the investor

$$\sigma_{C_2} = (1-\tau)\alpha\,\sigma_r = (1-\tau)\alpha\sqrt{p(1-p)}(r_g - r_b) \tag{7}$$

where σ_r is the standard deviation of the return of the risky asset, namely, $\sigma_r = \sqrt{p(1-p)}(r_g - r_b)$.

LESS AMBITIOUS INVESTORS

In this section, we explore an investor with a present value of endowment income greater than or equal to the present value of reference consumption levels, $Y_1 + \frac{Y_2}{1+\bar{r}_f} \geq \bar{C}_1 + \frac{\bar{C}_2}{1+\bar{r}_f}$, where the discount rate is the after tax return to the risk-free asset. This will be expressed as

$$\overline{\Omega} = \left[Y_1 + \frac{Y_2}{1+\bar{r}_f} - \left(\bar{C}_1 + \frac{\bar{C}_2}{1+\bar{r}_f}\right)\right] \geq 0 \tag{8}$$

Hlouskova et al. (2017) consider such an investor to be driven by a self enhancement motive. This motive makes the investor feel good and thus could increase his/her self-esteem. For example, the investor, to feel good, could be comparing the present value of his/her wealth with others of lower economic status and also could be attempting to maintain this relative position in society. This is possible by equating the present value of the reference consumption levels to the present value of endowment income of investors with the lower economic status. Another example would be when the investor abolishes the reference levels and makes his/her choices based on the expected utility model. The less ambitious investor is interested in avoiding relative losses[7] in both periods and decides on consumption and risk taking accordingly.

Based on Hlouskova, Fortin and Tsigaris (2017), the solution for a sufficiently loss averse investor of preferences given by (6) becomes

7. Throughout the paper "relative losses" will refer to the consumption below the corresponding reference level while "relative gains" will refer to the consumption above the corresponding reference level.

$$C_1^* = \bar{C}_1 + \frac{\bar{\Omega}}{1+\frac{\bar{M}}{1+\bar{r}_f}} \tag{9}$$

$$\alpha^* = \frac{\left(1-K_0^{\frac{1}{\gamma}}\right)}{r_f-r_b+K_0^{\frac{1}{\gamma}}(r_g-r_f)} \times \frac{\bar{M}}{1-\tau} \times \frac{\bar{\Omega}}{1+\frac{\bar{M}}{1+\bar{r}_f}}$$

$$= \frac{\left(1-K_0^{\frac{1}{\gamma}}\right)}{r_f-r_b+K_0^{\frac{1}{\gamma}}(r_g-r_f)} \times \frac{\bar{M}}{1-\tau} \times (C_1^* - \bar{C}_1) \tag{10}$$

where K_0 and $\bar{M}$ are given by (30) and (35). The optimal solution of current consumption relative to its reference level depends positively on the present value of endowment income net of the present value of consumption reference levels, $\bar{\Omega}$. If $\bar{\Omega} > 0$, then current consumption will exceed its reference level resulting in relative gains in the first period, while if $\bar{\Omega} = 0$ then current consumption will equal to the first period reference level. The fraction of $\bar{\Omega}$ in (9), namely $\frac{1}{1+\frac{\bar{M}}{1+\bar{r}_f}}$, is investor's marginal propensity to consume (MPC) which is the increase in consumption in the first period from a unit increase in the present value of endowment income keeping the reference levels constant.[8] The sufficiently loss averse investor will invest a fraction of this relative gain in the risky asset. Furthermore, it can be shown that optimal second period consumption in the good and in bad state of nature will also exceed the second period reference level resulting in relative gains also in the second period, see (46) and (47) in the appendix. Second period consumption also increases when current relative gains increase. Thus, the investor is avoiding relative losses from occurring in either period and in any state of nature that materializes in the second period. Also of note is that the loss aversion parameter is not part of the optimal decisions because relative losses are avoided.[9]

CAPITAL INCOME TAXATION FOR THE LESS AMBITIOUS INVESTORS

Next we show the impact of an increase in capital income taxation on the decision variables under three different cases assuming that the present value of

8. Note that marginal propensity to consume of less ambitious investor is less than one.
9. However the solution can be found only for a sufficient loss averse investor. For more details in this matter, see Hlouskova et al. (2017).

endowment income exceeds the present value of reference consumption levels ($\bar{\Omega} > 0$).[10] First, when the first period consumption reference level is below the first period endowment income and the second period consumption reference level is below the second period endowment income, i.e., when $\bar{C}_1 < Y_1$ and $\bar{C}_2 < Y_2$. Second, when the first period consumption reference level exceeds the first period endowment income while the second period reference level is below the second period endowment income such that the present value of endowment income is in the excess to the present value of consumption reference levels, i.e., when $\bar{C}_1 > Y_1$, $\bar{C}_2 < Y_2$ and $Y_1 + \frac{Y_2}{1+r_f} - \left(\bar{C}_1 + \frac{\bar{C}_2}{1+r_f}\right) > 0$. In this case the investor is relatively more ambitious in the first period than in the second period. Third, when the first period consumption reference level is below the first period endowment income, $\bar{C}_1 < Y_1$, but the second period reference level exceeds the second period endowment income, $\bar{C}_2 > Y_2$ while $Y_1 + Y_2 > \bar{C}_1 + \bar{C}_2$ and thus investor is relatively more ambitious in the second period than in the first period..

CASE 1: $\bar{C}_1 < Y_1$, $\bar{C}_2 < Y_2$

In the case of low reference levels in both periods, we find that an increase in capital income taxation will increase current consumption, see (54), and reduce second period consumption in both states of nature, see (57) and (58). An increase in capital income taxation increases the price of future consumption, stimulating current consumption, and thus also current relative gains, as it is cheaper at the expense of future consumption. This effect can be seen as an increase in investor's marginal propensity to consume, i.e., the tax increases $\text{MPC} = \frac{1}{1+\frac{\bar{M}}{1+\bar{r}_f}}$, while keeping $\bar{\Omega}$ constant, see (9) and (53). Since the second period reference level is below the second period endowment income, the tax also increases $\bar{\Omega}$, keeping the MPC constant, see (49).[11] Thus, this income effect also increases first period consumption and thus reinforces the increase in consumption due to the substitution effect (i.e., the increase in MPC).

In terms of future consumption there are two opposite effects operating since relative gains in the second period are proportional to relative gains in the first period, see (46) and (47). The increase in the tax makes future consumption more expensive and thus the investor reduces the proportion allocated to the relative

10. The differentiation of results (consumption, investment in the risky asset, private risk taking and the indirect utility function) with respect to the tax rate is presented the appendix, see (54)-(60).

11. $\bar{\Omega}$ increases, as increasing tax rate reduces the discount rate increases the discount factor $\frac{1}{1+\bar{r}_f}$ and thus the present value of endowment income net of the reference levels (also in present value) as $\bar{C}_2 < Y_2$.

gains in the second period keeping the first period relative gains constant but this is partially offset since the tax increases relative gains in the first period and thus increasing second period relative gains. However, when both reference levels are below their respective income levels the former effect is stronger than the latter one reducing future consumption in both states of nature.[12] In spite of this, the investor will still make relative gains in the second period in both, good and bad, states of nature but not as much as it would have made without the capital income tax.

As seen in (10) the investment in the risky asset is proportional to the relative gain in the first period. The increase in the capital income tax affects the proportionality factor $\frac{\bar{M}}{1-\tau}$, keeping relative gains in the first period constant, and it affects the relative gains in the first period as described above, keeping constant the proportionality factor. There are two effects on the proportionality factor. The first effect is the Domar-Musgrave phenomenon where due to loss offset provisions the investor increases the risky asset in such a way that private risk taking remains unaffected. This effect is $\frac{\alpha^*}{1-\tau}$, see (55). The second effect is a negative one which reduces the Domar-Musgrave effect and can make it negative. This effect is $-\frac{\alpha^*}{1-\tau}\frac{\bar{r}_f}{\gamma(1+\bar{r}_f)}$, see again (55).[13] These two effects compose the impact of the tax on the proportionality factor making the overall impact of the tax ambiguous. On the other hand, the increase in the tax also increases the relative gains in the first period consumption, keeping constant the proportionality factor, and thus enhances the positive effect of the capital income tax on the investment in the risky asset.

Thus, a sufficient condition for risk taking to increase with increasing tax rate is that both effects (on the proportionality factor and on current relative gains) move in the same positive direction. This occurs when τ exceeds its threshold value $\bar{\tau} = 1 - \frac{\gamma}{(1-\gamma)r_f}$ (i.e., for $\tau > \bar{\tau}$), see (51) and (55). As for $\gamma \geq \frac{r_f}{1+r_f}$ is $\bar{\tau} \leq 0$ and thus for sufficiently risk averse investors investment in the risky asset increases when capital income tax increases for all τ. In summary, for less risk-averse investors, i.e., when $\gamma < \frac{r_f}{1+r_f}$, investment in the risky asset increases with increasing τ when $\tau > \bar{\tau}$, which is sufficient (not necessary) condition for $\frac{d\alpha^*}{d\tau} > 0$. For $\tau < \bar{\tau}$ (and

12. See (50), (57) and (58) in the appendix.

13. Note that if the tax was imposed only on the excess returns of the risky asset, i.e., $\bar{r}_f = r_f$ and $\bar{r}_s = r_f + (1-\tau)(r_s - r_f)$, $s \in \{g, b\}$, then there would be only the Domar-Musgrave effect. The negative impact on the proportion would be absent, see also (51). Thus, in this case for $C_{2s} = (1+r_f)(Y_1 - C_1) + (1-\tau)(r_s - r_f)\alpha + Y_2$ this would be a capital income tax which exempts the safe interest earned from total savings $S = Y_1 - C_1$.

thus for $\gamma < \frac{r_f}{1+r_f}$, to keep $\bar{\tau}$ positive) the impact of taxation on risk taking is ambiguous (i.e., risk taking can be both increasing or decreasing with increasing tax rate). However, sufficient condition for risk taking to decrease with increasing tax rate is that τ does not exceed certain threshold, namely when $\tau < \bar{\tau}_2 = 1 - \frac{\gamma(1+M)}{(1-2\gamma)r_f}$, if investor is not too risk averse, i.e., $\gamma < \frac{r_f}{1+2r_f}$ and has sufficiently high time preference (see Table 1 on the next page).

The indirect utility function (happiness) is positively affected by the increase in relative gains in the first period but this is offset by the reduction in relative consumption in the second period in both states of nature. The tax will increase $\bar{\Omega}$ which will increase happiness but the tax also increases the marginal propensity to consume distorting inter-temporal decisions away from future consumption which reduces happiness as measured by the indirect utility function (see (48)). Hence there is ambiguity with respect to the impact of the tax on the indirect utility. However, it can be shown that the sign of the impact of the tax on the indirect utility depends on the level of the tax rate relative to threshold level $\bar{\tau}_1$, see (59) and (61), where $\bar{\tau}_1$ is given by (40). The indirect utility function is decreasing (with increasing tax rate) when $\tau < \bar{\tau}_1$, is increasing (with increasing τ) when $\tau > \bar{\tau}_1$, and is neutral with changing τ when $\tau = \bar{\tau}_1$. It is intuitive that for a sufficiently small δ is $\bar{\tau}_1 < 0$ (i.e., in the case of very impatient investors who have a very high rate of time preference) is the indirect utility an increasing function of τ since importance is placed on the increase in the first period relative gains in consumption than in the reduction in second period relative gains. Hence, there is a stronger positive effect of current relative consumption from a higher tax on the indirect utility than a negative effect of the future relative consumption for investors with sufficiently large time preference (i.e., for sufficiently small δ). In this case, a higher capital income tax will make the investor happier.[14]

14. The following holds based on (59): happiness level is increasing with increasing tax rate if in the limit $\bar{C}_1 = Y_1$ but $\bar{C}_2 < Y_2$. On the other hand, if $\bar{C}_1 < Y_1$ and in the limit $\bar{C}_2 = Y_2$ then happiness level is decreasing with increasing tax rate.

Table 1. Sensitivity Results with Respect to τ for Less Ambitious Investors.

Case 1	$\bar{C}_1 < Y_1$ and $\bar{C}_2 < Y_2$						
	dC_1^*	dC_{2g}^*	dC_{2b}^*	$d\alpha^*$	$d\sigma_{C_2^*}$	dS	$d\mathbb{E}(U^*)$
$d\tau$	> 0	< 0	< 0	$\gtreqless 0$	< 0	< 0	$\gtreqless 0$
				> 0 when $\tau > \bar{\tau}$			> 0 when $\tau > \bar{\tau}_1$
				< 0 when $\tau < \bar{\tau}_2$ $\gamma < \frac{r_f}{1+2r_f}$ $\delta < \tilde{\tilde{\delta}}$			$= 0$ when $\tau = \bar{\tau}_1$ < 0 when $\tau < \bar{\tau}_1$
Case 2	$Y_1 < \bar{C}_1 < Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$ and $\bar{C}_2 < Y_2$						
	dC_1^*	dC_{2g}^*	dC_{2b}^*	$d\alpha^*$	$d\sigma_{C_2^*}$	dS	$d\mathbb{E}(U^*)$
$d\tau$	> 0	$\gtreqless 0$	$\gtreqless 0$	$\gtreqless 0$	$\gtreqless 0$	< 0	> 0
		> 0 when $\bar{C}_1 > \bar{C}_1^U$ $\gamma \geq \frac{r_f}{1+r_f}$	> 0 when $\bar{C}_1 > \bar{C}_1^U$ $\gamma \geq \frac{r_f}{1+r_f}$	> 0 when $\gamma \geq \frac{r_f}{1+2r_f}$	> 0 when $\bar{C}_1 > \bar{C}_1^U$ $\gamma \geq \frac{r_f}{1+r_f}$		
		< 0 when $\bar{C}_1 < \bar{C}_1^L$ $\delta < \tilde{\delta}$	< 0 when $\bar{C}_1 < \bar{C}_1^L$ $\delta < \tilde{\delta}$	< 0 when $\tau < \bar{\tau}_4$ $\gamma < \gamma_3$ $\delta < \tilde{\delta}$	< 0 when $\bar{C}_1 < \bar{C}_1^L$ $\delta < \tilde{\delta}$		
Case 3	$\bar{C}_1 < Y_1$ and $Y_2 < \bar{C}_2 < Y_1 - \bar{C}_1 + Y_2$						
	dC_1^*	dC_{2g}^*	dC_{2b}^*	$d\alpha^*$	$d\sigma_{C_2^*}$	dS	$d\mathbb{E}(U^*)$
$d\tau$	$\gtreqless 0$	< 0	< 0	$\gtreqless 0$	< 0	$\gtreqless 0$	< 0
	> 0 when $\bar{C}_2 < \bar{C}_2^L$			> 0 when $\gamma \geq \gamma_1$		< 0 when $\bar{C}_2 < \bar{C}_2^L$	
	< 0 when $\bar{C}_2 > \bar{C}_2^U$			< 0 when $\tau < \bar{\tau}_3$ $\gamma < \gamma_2$ $\delta < \tilde{\delta}$		> 0 when $\bar{C}_2 > \bar{C}_2^U$	

Notation: $\bar{\tau}$, $\bar{\tau}_1$, $\bar{\tau}_2$, $\bar{\tau}_3$, $\bar{\tau}_4$, γ_1, γ_2, γ_3, $\bar{C}_1^L$, $\bar{C}_1^U$, $\bar{C}_2^L$, $\bar{C}_2^U$, $\tilde{\delta}$ *and* $\tilde{\tilde{\delta}}$ *are given by (39), (40), (41), (42), (43), (22), (23), (24), (16), (17), (18), (19), (20) and (21) in the glossary. Note that under assumptions of stated cases the following holds:* $\bar{\tau} > \bar{\tau}_2$, $\frac{r_f}{1+2r_f} > \gamma_3$, $\gamma_1 > \gamma_2$, $\bar{C}_1^U > \bar{C}_1^L$ *and* $\bar{C}_2^U > \bar{C}_2^L$.

Even though risk taking might increase with increasing capital income taxation the standard deviation of future consumption (private risk taking) is a decreasing function in τ. This occurs because the direct impact of the capital income taxation on reducing volatility of future consumption is stronger than the indirect impact on volatility from a potential increase in risk taking activity. The summary can be found at the top part of Table 1.[15]

CASE 2: $Y_1 < \bar{C}_1 < Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$ AND $\bar{C}_2 < Y_2$

The second case of a less ambitious investor we consider is when the first period reference level exceeds the first period income and the second period reference level is below the second period exogenous income. This is a investor who is relatively more ambitious in the first than in the second period.

An increase in capital income taxation will stimulate current consumption and discourage total savings as in case 1. However, second period consumption in the bad and good state could be stimulated in this case. This happens if the first period reference level is above threshold $\bar{C}_1^U$, i.e., when investor is sufficiently ambitious in the first period and he/she is also relatively averse to risk, i.e., when $\gamma \geq \frac{r_f}{1+r_f}$. Also risk taking will increase with a relatively high aversion to risk, i.e., when $\gamma \geq \frac{r_f}{1+2r_f}$. The standard deviation of future consumption (i.e., private risk taking) can increase also when the first period reference level is above threshold $\bar{C}_1^U$ and investor is also relatively risk averse. However, future consumption in the bad and good states could be discouraged by capital income taxation if the first period reference level is below threshold $\bar{C}_1^L$ and the investor is relatively impatient (to consume in the future). In this case risk taking could be also discouraged if the investor is relatively impatient, the tax rate is below a certain threshold level, and diminishing sensitivity aversion to risk is relatively low.

A surprising result is that an increase in capital income taxation increases the indirect utility in this case. As in the previous case, the capital income tax increases $\bar{\Omega}$ which increases happiness which is partially offset by the increase in the marginal propensity to consume $\frac{1}{1+\frac{\bar{M}}{1+r_f}}$, see (48). However, the offset is not strong enough due to the first period reference level exceeding the first period income, leading to an increase in happiness, (see (59) in the appendix).[16]

CASE 3: $\bar{C}_1 < Y_1$ AND $Y_2 < \bar{C}_2 < Y_1 - \bar{C}_1 + Y_2$

15. Note that the solution for $\bar{C}_1 = Y_1$ and $\bar{C}_2 = Y_2$ is the following: $C_1^* = Y_1$, $C_{2b}^* = C_{2g}^* = Y_2$, $\alpha^* = 0$, $\sigma_{C_2^*} = 0$, $S^* = 0$, $\mathbb{E}(U^*) = 0$ and thus does not depend on the tax.

16. For more detailed explanation, see the middle part of Table 1 as well as its caption where the threshold levels of the tax rate, degree of risk aversion and time preference are presented.

In the final case, the first period reference level is lower than the investor's first period income but the second period reference level is higher than the second period endowment income. An example would be if the investor did not have second period endowment income. We still consider this investor as less ambitious in that it avoids relative losses but has a relatively high second period reference level to the second period endowment income. We say that this investor is relatively more ambitious in the second period than the first period.

Capital income taxation has the same directional effects as in the first case for all decision variables except for current consumption which now is ambiguous while in the first and second cases current consumption is stimulated by capital income tax. The impact on current consumption depends on the second period reference level. If the second period reference level is below threshold $\bar{C}_2^L$ then the current consumption is stimulated but if the second period reference exceeds a higher threshold level $\bar{C}_2^U$ then the current consumption is discouraged by capital income tax increase.[17] In terms of risk taking similar pattern is observed as in the first case except that the thresholds for the risk aversion parameter γ to stimulate risk taking has changed as well as the thresholds of the capital income tax and risk aversion in the case when an increase in capital income taxation discourages risk taking. Finally, the increase in capital income taxation reduces happiness as measured by the indirect utility, see (59). For summary of the results, see the bottom part of Table 1.

1. MORE AMBITIOUS INVESTORS

We now consider a sufficiently loss averse investor who has relatively high consumption reference levels.[18] By high reference levels we mean that the present value of endowment income is lower than the present value of consumption reference levels, where the discount rate is the after tax return to the risk-free asset, i.e., $Y_1 + \frac{Y_2}{1+\bar{r}_f} < \bar{C}_1 + \frac{\bar{C}_2}{1+\bar{r}_f}$ and thus $\bar{\Omega} < 0$, see (8). This investor is thus more ambitious and has higher aspirations which could be due to the self-improvement motive when trying to reach an ambitious target. For example, the investor compares its present value of endowment income (discounted by the return of after tax risk-free asset) with income of investors of higher economic status by taking

17. It is easy to see that $\bar{C}_2^L < \bar{C}_2^U$.

18. For more detailed conditions on sufficiently large loss aversion parameter λ, see Hlouskova, Fortin and Tsigaris (2017), Proposition 3, equations (20)-(23). Note that conditions for lower bounds on λ could be easily derived such that they do not depend on tax rate τ. As the presentation of these conditions is cumbersome, we decided to avoid their explicit statement in this paper. Note that Proposition 3 assumes also an upper bound on $\bar{C}_2$. This can be handled in a similar way. Supplementary material upon request is available.

his/her present value of consumption reference level to be equal to the present value of endowment income of a richer investor. Thus, to self-improve, the investor makes an upward comparison instead of downward comparison as in the less ambitious investor case previously examined. The ambitious investor is interested to catch up to the richer investor's wealth and as a result will not avoid relative losses from occurring.

There are two cases to consider: an investor with relatively higher time preference when $\delta \leq \bar{\delta}$ (less patient investor) and investor with relatively lower time preference when $\delta > \bar{\delta}$ (more patient investor) where the threshold value for δ is defined as follows

$$\bar{\delta} = \bar{\delta}(\tau) = \frac{1}{1-p}\left[\frac{r_g - r_f}{(1+(1-\tau)r_f)(r_g - r_b)}\right]^{1-\gamma} \qquad (11)$$

Note that the threshold of the time preference, $\bar{\delta}$, depends on the tax rate and is an increasing function in τ. For sufficiently small δ, namely when $\delta \leq \bar{\delta}(0)$, the behavior of the investor is of the one with relatively higher time preference *for any tax rate*. And if the probability of the good state to occur is sufficiently large then only this case is considered.[19] Let, on the other hand, p be sufficiently small (see (71) in the appendix) such that $\bar{\delta}(0) < 1$ and let, in addition, δ be sufficiently large, such that $\delta > \bar{\delta}(0)$. Then for smaller tax rate is investor's time preference above its threshold (i.e., $\delta > \bar{\delta}(\tau)$) and thus behaves as the investor with relatively lower time preference. When the tax rates exceeds a certain threshold value $\tilde{\tau}$,[20] then investor's discount factor δ becomes smaller than its threshold for all sufficiently large τ, i.e., $\delta \leq \bar{\delta}(\tau)$ for $\tau > \tilde{\tau}$, and thus the investor behaves then as the one with relatively higher time preference. For more details see the appendix.

We first examine an investor who has a relatively lower discount factor for future utility of relative consumption. It discounts utility from future consumption at a relatively higher rate of time preference. For this investor relative losses cannot be avoided in the second period in the bad state of nature given investment in the risky asset. In the subsequent case, the ambitious investor has a relatively lower rate of time preference and thus places more importance in the utility of future consumption (i.e., δ should be sufficiently large). In this case, the investor will make decisions to avoid relative losses in the second period but at the sacrifice of making relative losses in the first period. We explore these two cases next starting from the investor who values current consumption relatively more than future

19. As in this case is $\bar{\delta}(0) \geq 1$.
20. The threshold value of the tax rate, $\tilde{\tau}$, is such that $\delta = \bar{\delta}(\tilde{\tau})$.

consumption given the assumed low discount factor for future utility.

MORE AMBITIOUS INVESTORS WITH A HIGHER TIME PREFERENCE

We assume in this subsection that $\delta \leq \bar{\delta}$. For sufficiently loss averse investor the solution is given by (see Hlouskova, Fortin and Tsigaris, 2017)

$$C_1^* = \bar{C}_1 + \frac{(-\bar{\Omega})}{\frac{\bar{M}(\lambda)}{1+\bar{r}_f} - 1} \qquad (12)$$

$$\alpha^* = \frac{\left[\left(\frac{1}{K_0}\right)^{1/\gamma} + \lambda^{1/\gamma}\right]}{r_g - r_f} \frac{\bar{k}}{1-\tau} (C_1^* - \bar{C}_1) \qquad (13)$$

where $\bar{M}(\lambda)$ is given by (36) in the glossary. The optimal solution is such that current consumption is above its reference level, i.e., the relatively impatient and sufficiently loss averse investor avoids relative losses in the first period. Risky investment is undertaken by this investor by investing a fraction of the relative gains of the first period. In addition, the optimal consumption in the second period in the good state of nature will exceed the second period reference level but optimal consumption in the bad state of nature will be below the corresponding reference level, see (74) and (75) in the appendix. Hence, the ambitious investor cannot avoid losses and as a result the loss averse parameter appears in the solutions. Risk taking and first period consumption will decrease with increasing loss averse parameter (see Hlouskova et al., 2017). It is important to note that first period consumption is an inferior good for this ambitious investor with a higher time preference who is experiencing losses in the bad state of nature. A marginal increase in the present value of endowment income will reduce the first period consumption by $\frac{1}{\frac{\bar{M}(\lambda)}{1+\bar{r}_f} - 1}$ and increase savings in the safe asset. Note that the risky asset is inferior as well and thus an increase in the present value of endowment income will reduce risk taking as well. A behavioral explanation is that the investor being loss averse is trying to increase the second period consumption under the bad state to reduce relative losses at the expense of reducing relative gains (in the first period and in the second period under the good state of nature). This is materialized by allocating the increased income into investment in the risk-free asset.

CAPITAL INCOME TAX: AMBITIOUS AND RELATIVELY IMPATIENT INVESTORS

We again consider three cases. First, we assume that the first period consumption reference level exceeds the first period endowment income of the investor and similarly, the second period consumption reference level exceeds the second period endowment income. This is an investor who is very ambitious in that not only the present value of endowment income is lower than the present value of consumption reference levels but in every period the endowment income is below its reference level.[21] Second, consumption reference in the first period exceeds income in the first period but consumption reference in the second period is below second period endowment income while $Y_1 + Y_2 < \bar{C}_1 + \bar{C}_2$. We consider this investor to be more ambitious in the first than second period. Third, consumption reference in the first period is below the first period income but consumption reference in the second period exceeds the income of that period while $Y_1 + \frac{Y_2}{1+r_f} < \bar{C}_1 + \frac{\bar{C}_2}{1+r_f}$. We consider this investor to be relatively more ambitious in the second than in the first period. Without these conditions the results are ambiguous in general.

CASE 1: $\bar{C}_1 > Y_1$ AND $\bar{C}_2 > Y_2$

Thus, if $\bar{C}_1 > Y_1$ and $\bar{C}_2 > Y_2$, an increase in capital income taxation will encourage current consumption. There are two effects operating in the same direction. First, the effect of the tax on $(-\bar{\Omega})$, keeping the marginal propensity to consume, MPC= $\frac{1}{\frac{\bar{M}(\lambda)}{1+\bar{r}_f}-1}$, constant. Second, the effect of the tax on the MPC, while keeping $\bar{\Omega}$ constant. The increase in the tax increases $(-\bar{\Omega})$ since $\bar{C}_2 > Y_2$ (see (49)), and the investor reacts by increasing consumption in the first period. Based on (73), the tax also stimulates MPC, keeping $\bar{\Omega}$ constant, which reinforces the former effect. Risk taking also increases, under these reference levels, due to full loss offset provision and the stimulus in first period relative gains from the tax. Contrary to the first case of a less ambitious investor, in this case there are no additional conditions on the tax rate or on the risk aversion because the stimulus in first period gains offsets the potential reduction arising from the impact of tax increase on the $\frac{\bar{k}}{1-\tau}$ term (see (13) and (52)). Furthermore, as $\bar{C}_1 > Y_1$ and investor is sufficiently loss averse then the increase in capital income tax reduces second period consumption in the good state of nature, see (79), and increases consumption in the bad state of nature.[22] We thus observe substitution of second period

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21. These conditions are sufficient for $\bar{\Omega} < 0$ but not necessary.

22. In the appendix it is shown, see (80), that relative gains in the second period are proportional to each other: $C_{2b}^* - \bar{C}_2 = -(\lambda K_0)^{1/\gamma}(C_{2g}^* - \bar{C}_2)$. Thus, a reduction in C_{2g}^* increases C_{2b}^* and thus reduces losses in the bad state of nature.

consumption away from the good state of nature towards the bad state of nature with an increase in the tax. This is done to reduce the relative losses in the bad state of nature by sacrificing some relative gains in the good state of nature given that loss aversion is present. The happiness level (the value of the indirect utility) will increase with increasing capital income taxation as the positive effect of current relative gains and reduction of relative losses in the bad state of nature are stronger than the negative effect caused by decrease in relative gains in the good state of nature. Finally the riskiness of future consumption decreases with increased taxation. The summary is presented in the top part of Table 2 and differentiations of these results can be found in the appendix, see (77)-(82).

CASE 2: $\bar{C}_1 > Y_1$ AND $Y_1 + Y_2 - \bar{C}_1 < \bar{C}_2 < Y_2$

When the relatively impatient, sufficiently loss averse and ambitious investor has its first period reference consumption above its first period income and its second period reference level below its endowment income in that period, then the impact of capital income taxation is as follows. Consumption in the first period will increase for a relatively less risk averse investor and decrease for more risk averse investor. As in the first case, consumption in the good state will fall and increase in the bad state with the tax increase. Recall that this investor is making relative losses in the bad state and relative gains in the good state. Hence capital income taxation causes the investor to reduce gains in the good state in order to reduce losses in the bad state of nature. Risk taking increases but the riskiness of future consumption falls due to the increase in capital income taxation. In this case as in the previous case the happiness level increases with capital income taxation. The summary can be found in the middle part of Table 2.

CASE 3: $\bar{C}_1 < Y_1$ AND $\bar{C}_2 > Y_2 + (1 + r_f)(Y_1 - \bar{C}_1)$

In this final case the impact of an increase in capital income taxation is as follows. Current consumption increases based on the same lines of arguments as in case 1, future consumption in the good state increases as $\bar{C}_1 < Y_1$, see (79), but decreases in the bad state which is the opposite of the previous two cases. The investor thus increases gains in the second period in the good state of nature by increasing losses in the bad state of nature. This is due to the relatively high second period reference level and a relatively low first period reference level. Both risk taking and the riskiness of future consumption (private risk taking) increase with an increase in capital income taxation. As $\bar{C}_1 < Y_1$, capital income taxation reduces happiness level for sufficiently loss averse investor, see (81), which is in contrast to the previous two cases discussed above. For the summary see the bottom part of Table 2.

Table 2. Sensitivity Results with Respect to τ for Sufficiently Loss Averse and More Ambitious Investors with High Time Preference.

Case 1	$\bar{C}_1 \geq Y_1$, $\bar{C}_2 \geq Y_2$ and $\bar{C}_1 - Y_1 + \bar{C}_2 - Y_2 > 0$						
	dC_1^*	dC_{2g}^*	dC_{2b}^*	$d\alpha^*$	$d\sigma_{C_2^*}$	dS	$d\,\mathbb{E}(U^*)$
$d\tau$	> 0	< 0	> 0	> 0	< 0	< 0	> 0
Case 2	$\bar{C}_1 > Y_1$ and $Y_1 + Y_2 - \bar{C}_1 < \bar{C}_2 < Y_2$						
	dC_1^*	dC_{2g}^*	dC_{2b}^*	$d\alpha^*$	$d\sigma_{C_2^*}$	dS	$d\,\mathbb{E}(U^*)$
$d\tau$	$\gtrless 0$	< 0	> 0	> 0	< 0	$\gtrless 0$	> 0
	> 0 when $\gamma < \gamma_4$					> 0 when $\gamma > \gamma_5$	
	< 0 when $\gamma > \gamma_5$					< 0 when $\gamma < \gamma_4$	
Case 3	$\bar{C}_1 < Y_1$ and $\bar{C}_2 > (1 + r_f)(Y_1 - \bar{C}_1) + Y_2$						
	dC_1^*	dC_{2g}^*	dC_{2b}^*	$d\alpha^*$	$d\sigma_{C_2^*}$	dS	$d\,\mathbb{E}(U^*)$
$d\tau$	> 0	> 0	< 0	> 0	> 0	< 0	< 0

Notation: γ_4 and γ_5 are given by (25) and (26). Note that $\gamma_4 < \gamma_5$.

MORE AMBITIOUS INVESTORS WITH A LOWER TIME PREFERENCE

Here we explore a sufficiently loss averse investor who is not that impatient and thus has a low rate of time preference as it relates to future utility of consumption. Namely, we assume that $\delta > \bar{\delta}$.[23] Based on Hlouskova, Fortin and Tsigaris (2017) the solution is as follows

$$C_1^* = \bar{C}_1 - \frac{\lambda^{1/\gamma}}{\lambda^{1/\gamma} - \bar{\lambda}^{1/\gamma}}(-\bar{\Omega}) \tag{14}$$

$$\begin{aligned} \alpha^* &= \frac{1 - K_0^{1/\gamma}}{\bar{r}_f - \bar{r}_b + K_0^{1/\gamma}(\bar{r}_g - \bar{r}_f)} \times \frac{\bar{\lambda}^{1/\gamma}}{\lambda^{1/\gamma} - \bar{\lambda}^{1/\gamma}} \times \left[(1 + \bar{r}_f)(-\bar{\Omega})\right] \\ &= \frac{1 - K_0^{1/\gamma}}{\bar{r}_f - \bar{r}_b + K_0^{1/\gamma}(\bar{r}_g - \bar{r}_f)} \frac{\bar{M}}{\lambda^{1/\gamma}}(\bar{C}_1 - C_1^*) \end{aligned} \tag{15}$$

where $\bar{\lambda} = \left(\frac{\bar{M}}{1+\bar{r}_f}\right)^{\gamma}$.[24] In this case we find that the sufficiently loss averse investor will lower the consumption in the first period below its reference level in order to have consumption in the second period above the reference level in both states of nature (good and bad) as it values future relatively more than the present and thus wants to avoid relative losses in the second period. The investor will also invest in the risky asset. This type of investor sees first period consumption as normal and

23. Note that this assumption is feasible, i.e., $\bar{\delta} < 1$, if $\tau < \tau^U$, see (72) in the appendix.
24. Note that an increase in the capital income tax will reduce $\bar{\lambda}$, see (53) in the appendix.

not inferior. Normality is re-established just like in the case of the less ambitious investor but an increase in the present value of endowment income by one unit will increase first period consumption by more than one unit since the MPC $= \frac{\lambda^{1/\gamma}}{\lambda^{1/\gamma}-\bar{\lambda}^{1/\gamma}} > 1$. This implies that savings will fall when the present value of endowment income increases, i.e., savings become inferior.

Thus, the increase in investor's endowment income is allocated such that relative losses in the first period are reduced as much as possible (given investor's aversion to losses) as well as relative gains in the second period, see (14), (84) and (85). The increase in consumption depends amongst other factors on the loss aversion parameter as well as on the capital income tax rate. An increase in the capital income tax will reduce the increase in consumption from a unit increase in the present value of endowment income (i.e., the MPC). Risk taking activity will decrease with an increase in the present value of endowment income.

CAPITAL INCOME TAX: AMBITIOUS INVESTORS WITH A LOW TIME PREFERENCE RATE

Here again the same three cases are examined as with the ambitious investor with a higher time preference rate.

CASE 1: $\bar{C}_1 > Y_1$ AND $\bar{C}_2 > Y_2$

In this case an increase in capital income taxation reduces current consumption and thus current relative losses become larger. There are two opposing effects operating but one is more powerful than the other, given the assumptions about the reference levels. First, the increase in capital income tax reduces the marginal propensity to consume, keeping $\bar{\Omega}$ constant, which increases current consumption because $\bar{\Omega} < 0$. Second, the tax reduces $\bar{\Omega}$, keeping MPC constant, which reduces current consumption by a much larger amount resulting in an overall reduction provided both reference levels are above their respective endowment income. The impact of capital income taxation on risk taking is ambiguous. Risk taking is stimulated by increased tax for sufficiently risk averse investor, namely when $\gamma \geq \frac{r_f}{1+r_f}$. On the other hand, risk taking is discouraged by capital income tax when the tax rate does not exceed certain threshold, namely $\tau < 1 - \frac{\gamma}{(1-2\gamma)r_f}$. Note that this threshold is positive when the investor is less risk averse, namely when $\gamma < \frac{r_f}{1+2r_f}$. Second period consumption is reduced in both states of nature when tax increases. Given that the investor is making relative gains in the second period, they are reduced with increasing tax rate (while given relative patience of the investor the tax increases relative losses in the first period). Finally, the happiness level decreases with increasing tax rate τ, as $\bar{C}_2 > Y_2$ and investor is sufficiently loss averse, see (92) and the riskiness of future consumption (private risk taking $\sigma_{C_2^*}$) decreases with the tax rate as well. The summary can be found in the top part of Table 3.

CASE 2: $\bar{C}_1 > Y_1$ AND $Y_1 + Y_2 - \bar{C}_1 < \bar{C}_2 < Y_2$

This relatively patient sufficiently loss averse investor with first period consumption reference level exceeding the first period income and with a second period reference level being below the second period income reacts to capital income taxation by increasing current consumption (and thus decreasing savings) in contrast to the previous case where the investor reduced current consumption. Thus, in this case the positive effect of the tax via the MPC is stronger than the negative income effect. Risk taking impact is ambiguous. It is encouraged by the tax when tax exceeds a certain threshold, while it is discouraged by the tax if the tax does not exceed threshold $1 - \frac{\gamma}{(1-\gamma)r_f}$ and if investor is sufficiently risk averse. Expected indirect utility increases which is driven by decreasing current relative losses and sufficient degree of loss aversion.[25] See the middle part of Table 3 for summary of the results.

CASE 3: $\bar{C}_1 < Y_1$ AND $\bar{C}_2 > Y_2 + (1 + r_f)(Y_1 - \bar{C}_1)$

This relatively patient sufficiently loss averse investor has its current reference level below its first period income while its second period consumption target exceeds its corresponding endowment income. Current consumption is reduced with the capital income tax increase as in case 1 above making relative losses larger. Future consumption in the bad and good state will increase (with increasing tax rate) when investor is relatively risk averse and this increases the riskiness of future consumption as measured by the concept of private risk taking. The tax increase is not sufficient to reduce riskiness of future consumption due to the increase in consumption in both states of nature, good and bad. Risk taking also increases for relatively risk averse investors. On the other hand, future consumption as well as the private risk decrease with increasing tax rate for less risk averse investors. Risk taking decreases as well for relatively small tax rate and less risk averse investors. Finally, happiness falls with an increase in the tax rate in this last case.

25 Sufficient degree of loss aversion and decreasing current relative losses overpower the decrease in future relative gains. For more detail, see the appendix and (92).

The summary of the sensitivity results can be found in Table 3 and the differentiation of results is presented in the appendix, see (88)-(93).

Table 3. Sensitivity Results with Respect to τ for Sufficiently Loss Averse and More Ambitious Investors with Low Time Preference.

Case 1	$\bar{C}_1 > Y_1$ and $\bar{C}_2 > Y_2$						
	dC_1^*	dC_{2g}^*	dC_{2b}^*	$d\alpha^*$	$d\sigma_{C_2^*}$	dS	$d\,\mathbb{E}(U^*)$
$d\tau$	< 0	< 0	< 0	$\gtrless 0$	< 0	> 0	< 0
				> 0 when $\gamma \geq \frac{r_f}{1+r_f}$			
				< 0 when $\tau < \bar{\tau}_5$ $\gamma < \frac{r_f}{1+2r_f}$			
Case 2	$\bar{C}_1 > Y_1$ and $Y_1 + Y_2 - \bar{C}_1 < \bar{C}_2 < Y_2$						
	dC_1^*	dC_{2g}^*	dC_{2b}^*	$d\alpha^*$	$d\sigma_{C_2^*}$	dS	$d\,\mathbb{E}(U^*)$
$d\tau$	> 0	< 0	< 0	$\gtrless 0$	< 0	< 0	> 0
				> 0 when $\tau > \bar{\tau}_6$			
				< 0 when $\tau < \bar{\tau}$ $\gamma < \frac{r_f}{1+r_f}$			
Case 3	$\bar{C}_1 < Y_1$ and $\bar{C}_2 > (1+r_f)(Y_1 - \bar{C}_1) + Y_2$						
	dC_1^*	dC_{2g}^*	dC_{2b}^*	$d\alpha^*$	$d\sigma_{C_2^*}$	dS	$d\,\mathbb{E}(U^*)$
$d\tau$	< 0	$\gtrless 0$	$\gtrless 0$	$\gtrless 0$	$\gtrless 0$	> 0	< 0
		> 0 when $\gamma > \gamma_6$	> 0 when $\gamma > \gamma_6$	> 0 when $\gamma \geq \frac{r_f}{1+r_f}$	> 0 when $\gamma > \gamma_6$		
		< 0 when $\gamma < \gamma_7$	< 0 when $\gamma < \gamma_7$	< 0 when $\tau < \bar{\tau}_6$ $\gamma < \gamma_8$	< 0 when $\gamma < \gamma_7$		

Notation: $\bar{\tau}$, $\bar{\tau}_5$, $\bar{\tau}_6$, γ_6, γ_7, γ_8 are given by (39), (44), (45), (27), (28) and (29) in the glossary. Note that under assumptions of stated cases the following holds: $\gamma_6 > \gamma_7$, $\frac{r_f}{1+r_f} > \gamma_8$ and $\bar{\tau}_6 > \bar{\tau}$.

SUMMARY

If discount factor δ is such that $\delta \leq \bar{\delta}(0)$ then only results presented in Table

2 apply.[26] Note that this condition is unbinding for sufficiently large p, namely

$$p \geq 1 - \left[\frac{r_g - r_f}{(1 + r_f)(r_g - r_b)}\right]^{1-\gamma}$$

as then $\bar{\delta}(0) \geq 1$. However, if p is sufficiently small, see condition (71), and thus $\bar{\delta}(0) < 1$, and if in addition δ is such that $\delta > \bar{\delta}(0)$ then for sufficiently small τ is $\delta > \bar{\delta}(\tau)$ and thus results presented in Table 3 hold while for sufficiently large τ is $\delta \leq \bar{\delta}(\tau)$ and thus results presented in Table 2 apply. If this is true (i.e., $\delta > \bar{\delta}(0)$) then in case 1 ($\bar{C}_1 > Y_1$ and $\bar{C}_2 > Y_2$) are current consumption, future consumption in the bad state of nature and the happiness decreasing for smaller tax rate but increasing for larger tax rates. This could hold also for investment in risky asset (when investor's risk aversion is sufficiently small), i.e., risk taking decreases with increasing τ and after τ exceeds its threshold then risk taking increases with increasing tax rate. Future consumption in the good state of nature as well as the private risk taking, are decreasing with increasing τ for all tax rates.

Regarding case 2 ($\bar{C}_1 > Y_1$ and $\bar{C}_2 < Y_2$), future consumption in the bad state of nature again decreases with increasing τ for $\tau < \tilde{\tau}$, where $\tilde{\tau}$ is such that $\delta = \bar{\delta}(\tilde{\tau})$, and increases with increasing τ when $\tau \geq \tilde{\tau}$. Different dynamics can occur also for current consumption which at first increases with increasing tax rate and after the tax threshold is exceeded then the current consumption can decrease (with increasing τ) when investor is sufficiently risk averse (see the middle block of Table 2). In addition, if the investor is not too much risk averse then the investment in the risky asset decreases at first (with increasing τ that is sufficiently small) and then it increases when the tax rate exceeds its tax threshold. Future consumption in the good state of nature and private risk taking are decreasing for all τ and the happiness level increases for all τ.

In case 3 ($\bar{C}_1 < Y_1$ and $\bar{C}_2 > Y_2$) the results are the same as in case 1 for current consumption and investment in the risky asset. Future consumption in the good state of nature (as well as the private risk taking) is increasing with increasing tax rate for all τ if investor is sufficiently risk averse. However, if investor is not too risk averse (see the last block of Table 3) then the future consumption in the good state is discouraged by smaller tax rates and then encouraged by larger tax rates (see Tables 2 and 3). Future consumption in the bad state of nature is always discouraged by the tax if investor is not too risk averse. However, if he/she is too risk averse, then the future consumption in the bad state is encouraged by smaller

26. As $\bar{\delta}(\tau)$ is increasing in τ and thus $\delta < \bar{\delta}(\tau)$ for all range of the tax rate.

tax rates and discouraged by larger tax rates. Finally, happiness level is discouraged by tax rates for all levels of τ.

Note that while for smaller tax rates the future consumption under both states of nature responses in the same way (in terms of the direction) to the tax change, impact of the larger tax rates on future consumption is opposite for the good state as for the bad state.

CONCLUSION

In this paper we conduct a positive analysis of the impact of capital income taxation, with full loss offset provisions, on consumption, risk taking and the indirect utility function for different types of sufficiently loss averse investors in a two period two asset portfolio savings model. The impact of the tax is reference dependent and in relation to the investors endowment income. For less ambitious investors, capital income taxation encourages current consumption in most cases, and thus increases relative gains in the first period and reduces future consumption. The increased relative gain in the first period increases the demand for the risky asset. Furthermore, the risk taking adjustment under full loss offset (Domar-Musgrave phenomenon) is also present as in expected utility models but because the return of the risk-free asset is also taxed at the same rate as the risky asset, the adjustment is smaller which does not leave private risk taking and expected utility unchanged as in Mossin (1968). The final outcome on risk taking for less ambitious investors depends on the investor being relative risk averse, as well as where the tax rate and the rate of time preference is in relation to certain thresholds. Namely, capital income tax stimulates risk taking for a sufficiently high risk averse investor, while it discourages risk taking when tax rate is below a certain threshold and if investor’s degree of risk aversion is sufficiently low, and time preference sufficiently high.

More ambitious but sufficiently loss averse investors are affected differently by the tax. Investors that are more ambitious but also relatively impatient due to the high rate of time preference will increase risky investment with increased taxation. Current consumption is discouraged for some types of ambitious investors that are relatively patient with a low rate of time preference and have a relatively high second period reference levels but not necessary first period reference level. For the latter type of investors risk taking can increase for relatively risk averse investors similar to the less ambitious investors’ reaction to the tax. This research shows that reference levels in relation to endowment income play the most significant role in determining the outcome of capital income tax changes and should not be ignored. We also find certain type of investors whose indirect utility function increases with capital income taxation under full loss offset provisions.

Future research should examine the impact of capital income taxation on the investor's decisions under various assumptions regarding the way the risky tax revenue is handled by the government and by also making the second period reference level endogenous as in Hlouskova et al. (2019). One extreme way is to ignore the risky tax revenue to the public sector which is assumed in the analysis of this paper. The other end of the spectrum is, for the government, to return the tax revenue back to the investor in a lump sum stochastic form in which case the private sector absorbs all the risk (see Gordon, 1985; Bulow and Summers, 1984; Gordon and Wilson, 1989; and Ahsan and Tsigaris, 2009). This transfer removes the income effect of taxation and it also keeps the risk the investor faces the same as prior to the tax increase. Removing the income effect of capital income taxation would allow one to examine the pure substitution effect arising from changes in relative returns of the assets. Ahsan and Tsigaris (2009) found that the effect of a capital income tax increase, on investors with expected utility type of preferences under no risk sharing by the public sector, makes current consumption more attractive on the margin and risk taking less attractive and that the tax transfer is not sufficient to hold the investor on the same level of the utility as the pre-tax situation. This negative reaction could be an explanation of the behavioural tax biases found in experimental studies (Fochmann and Hemmerich, 2017). In this theoretical paper we do not incorporate potential tax aversion effects (e.g. Kallbekken et al., 2011), tax affinity effects (e.g., Djanali and Sheehan-Connor, 2012) nor tax salience effects (e.g., Chetty, Looney and Kroft, 2009) which could impact the findings of this research.

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APPENDIX

GLOSSARY

$$\bar{C}_1^L = Y_1 + \frac{1-\gamma}{1+\gamma\delta}\frac{Y_2-\bar{C}_2}{1+r_f} \tag{16}$$

$$\bar{C}_1^U = Y_1 + (1-\gamma)(Y_2-\bar{C}_2) \tag{17}$$

$$\bar{C}_2^L = Y_2 + \frac{(1-\gamma)\bar{M}_{\tau=0}}{\gamma+\bar{M}_{\tau=0}}(Y_1-\bar{C}_1) \tag{18}$$

$$\bar{C}_2^U = \frac{(1-\gamma)M}{\gamma(1+r_f)+M}(1+r_f)(Y_1-\bar{C}_1)+Y_2 \tag{19}$$

$$\tilde{\delta} = \frac{r_f-r_b}{r_g-r_b}\frac{1}{p\,(1+r_f)} \tag{20}$$

$$\tilde{\tilde{\delta}} = \frac{1}{\widetilde{M}}\left[\frac{1-2\gamma}{\gamma}r_f-1\right]^{\gamma} \tag{21}$$

$$\gamma_1 = \frac{r_f(1+r_f)\Omega}{Y_1+Y_2-\bar{C}_1-\bar{C}_2+r_f(1+r_f)\Omega} \tag{22}$$

$$\gamma_2 = \frac{r_f}{r_f+(1+r_f)(1+\delta)} \tag{23}$$

$$\gamma_3 = \frac{r_f\Omega}{r_f\Omega+(1+\delta)(Y_1+Y_2-\bar{C}_1-\bar{C}_2)} \tag{24}$$

$$\gamma_4 = 1-\frac{Y_2-\bar{C}_2}{\bar{C}_1-Y_1} \tag{25}$$

$$\gamma_5 = 1-\frac{Y_2-\bar{C}_2}{(1+r_f)(\bar{C}_1-Y_1)} \tag{26}$$

$$\gamma_6 = 1-\frac{Y_1-\bar{C}_1}{\bar{C}_2-Y_2} \tag{27}$$

$$\gamma_7 = 1-\frac{(1+r_f)(Y_1-\bar{C}_1)}{\bar{C}_2-Y_2} \tag{28}$$

$$\gamma_8 = \frac{(\bar{C}_1+\bar{C}_2-Y_1-Y_2)r_f}{(\bar{C}_1+\bar{C}_2-Y_1-Y_2)(1+r_f)+(\bar{C}_2-Y_2)r_f} \tag{29}$$

$$K_0 = \frac{(1-p)(r_f-r_b)}{p\,(r_g-r_f)} \tag{30}$$

$$K_\gamma = \frac{(1-p)(r_f-r_b)^{1-\gamma}}{p(r_g-r_f)^{1-\gamma}} \tag{31}$$

$$\bar{k}_2 = \left[\delta(1+\bar{r}_f)\,p\,\left(\frac{r_g-r_b}{r_f-r_b}\right)^{1-\gamma}\right]^{\frac{1}{\gamma}} \tag{32}$$

$$\bar{k} = \left[\delta(1+\bar{r}_f)\,(1-p)\,\left(\frac{r_g-r_b}{r_g-r_f}\right)^{1-\gamma}\right]^{\frac{1}{\gamma}} = \bar{k}_2K_\gamma^{1/\gamma} \tag{33}$$

$$M = \left[\delta(1+r_f)\,p\,\frac{r_g-r_b}{r_f-r_b}\right]^{\frac{1}{\gamma}}\frac{r_f-r_b+K_0^{\frac{1}{\gamma}}(r_g-r_f)}{r_g-r_b} = k_2\left(1+K_\gamma^{1/\gamma}\right) \tag{34}$$

$$\bar{M} = \left[\delta(1+\bar{r}_f)\, p\, \frac{r_g-r_b}{r_f-r_b}\right]^{\frac{1}{\gamma}} \frac{r_f-r_b+K_0^{\frac{1}{\gamma}}(r_g-r_f)}{r_g-r_b} = \bar{k}_2\left(1+K_\gamma^{1/\gamma}\right) \tag{35}$$

$$\bar{M}(\lambda) = \left[\delta(1+\bar{r}_f)\, p\, \frac{r_g-r_b}{r_f-r_b}\right]^{\frac{1}{\gamma}} \frac{(\lambda K_0)^{1/\gamma}(r_g-r_f)-(r_f-r_b)}{r_g-r_b}$$

$$= \bar{k}\left[\lambda^{1/\gamma} - \left(\frac{1}{K_\gamma}\right)^{1/\gamma}\right] = \bar{k}_2\left[\left(\lambda K_\gamma\right)^{1/\gamma} - 1\right] \tag{36}$$

$$\widetilde{M} = (1+r_f)\, p\, \frac{r_g-r_b}{r_f-r_b}\left[\frac{r_f-r_b+K_0^{\frac{1}{\gamma}}(r_g-r_f)}{r_g-r_b}\right]^{\gamma} \tag{37}$$

$$\bar{\Omega} = Y_1 - \bar{C}_1 + \frac{Y_2-\bar{C}_2}{1+\bar{r}_f}$$

$$\Omega = Y_1 - \bar{C}_1 + \frac{Y_2-\bar{C}_2}{1+r_f} \tag{38}$$

$$\bar{\tau} = 1 - \frac{\gamma}{(1-\gamma)r_f} \tag{39}$$

$$\bar{\tau}_1 = 1 + \frac{1}{r_f}\left[1 - \frac{1}{\delta p}\left(\frac{r_f-r_b}{r_g-r_b}\right)^{1-\gamma}\left(\frac{Y_2-\bar{C}_2}{Y_1-\bar{C}_1}\frac{1}{1+K_\gamma^{1/\gamma}}\right)^{\gamma}\right] \tag{40}$$

$$\bar{\tau}_2 = 1 - \frac{\gamma(1+M)}{(1-2\gamma)r_f} \tag{41}$$

$$\bar{\tau}_3 = 1 - \frac{\gamma[1+\delta(1+r_f)]}{(1-2\gamma)r_f} \tag{42}$$

$$\bar{\tau}_4 = 1 - \frac{1}{(1-2\gamma)r_f}\left[\gamma(1+\delta(1+r_f)) + \frac{(1-\gamma)r_f^2(\bar{C}_1-Y_1)}{Y_1+Y_2-\bar{C}_1-\bar{C}_2}\right] \tag{43}$$

$$\bar{\tau}_5 = 1 - \frac{\gamma}{(1-2\gamma)r_f} \tag{44}$$

$$\bar{\tau}_6 = 1 - \frac{\gamma(\bar{C}_1+\bar{C}_2-Y_1-Y_2)}{(1-2\gamma)(\bar{C}_2-Y_2)-(1-\gamma)(Y_1-\bar{C}_1)}\frac{1}{r_f} \tag{45}$$

LESS AMBITIOUS INVESTORS

Note that sufficient conditions for $\bar{\Omega} \geq 0$ for all τ are: $Y_1 - \bar{C}_1 + Y_2 - \bar{C}_2 \geq 0$ and $\bar{C}_2 > Y_2$ or $\Omega \geq 0$ and $\bar{C}_2 < Y_2$.

Let's assume at first that $\bar{\Omega} \geq 0$ or (equivalently) that $\bar{C}_1 \leq Y_1 + \frac{Y_2-\bar{C}_2}{1+\bar{r}_f}$. Then based on Hlouskova, Fortin and Tsigaris (2017) the solution is given by (9) and (10). It can be shown that

$$C_{2g}^* - \bar{C}_2 = \frac{r_g-r_b}{r_f-r_b}\ \bar{k}_2 \frac{\bar{\Omega}}{1+\frac{\bar{M}}{1+\bar{r}_f}} = \frac{r_g-r_b}{r_f-r_b}\ \bar{k}_2(C_1^* - \bar{C}_1) \tag{46}$$

$$C_{2b}^* - \bar{C}_2 = \bar{K}_0^{\frac{1}{\gamma}} \frac{r_g - r_b}{r_f - r_b} \ \bar{k}_2 \frac{\bar{\Omega}}{1+\frac{\bar{M}}{1+\bar{r}_f}} = \bar{K}_0^{\frac{1}{\gamma}} \frac{r_g - r_b}{r_f - r_b} \ \bar{k}_2 (C_1^* - \bar{C}_1) \tag{47}$$

and

$$(1-\gamma)\mathbb{E}\big(U(C_1^*, \alpha^*)\big) = \left(1+\frac{\bar{M}}{1+\bar{r}_f}\right)\left(\frac{\bar{\Omega}}{1+\frac{\bar{M}}{1+\bar{r}_f}}\right)^{1-\gamma} = \left(1+\frac{\bar{M}}{1+\bar{r}_f}\right)(C_1^* - \bar{C}_1)^{1-\gamma} = \bar{\Omega}^{1-\gamma}\left(1+\frac{\bar{M}}{1+\bar{r}_f}\right)^{\gamma} \tag{48}$$

Note that the following holds

$$\frac{d\bar{\Omega}}{d\tau} = \frac{r_f}{(1+\bar{r}_f)^2}(Y_2 - \bar{C}_2)\begin{cases} > 0, & \text{for } \ \bar{C}_2 < Y_2 \\ = 0, & \text{for } \ \bar{C}_2 = Y_2 \\ < 0, & \text{for } \ \bar{C}_2 > Y_2 \end{cases} \tag{49}$$

$$\frac{d\bar{k}_2}{d\tau} = -\frac{r_f}{\gamma(1+\bar{r}_f)}\bar{k}_2 < 0 \tag{50}$$

$$\frac{d\bar{k}}{d\tau} = -\frac{r_f}{\gamma(1+\bar{r}_f)}\bar{k} < 0$$

$$\frac{d\bar{M}}{d\tau} = -\frac{r_f}{\gamma(1+\bar{r}_f)}\bar{M} < 0$$

$$\frac{d\left(\frac{\bar{M}}{1-\tau}\right)}{d\tau} = \frac{\bar{M}}{(1-\tau)^2}\left[1 - \frac{\bar{r}_f}{\gamma(1+\bar{r}_f)}\right]\begin{cases} < 0 & \text{for } \tau < \bar{\tau} \\ > 0 & \text{for } \tau > \bar{\tau} \end{cases} \tag{51}$$

$$\frac{d\left(\frac{\bar{k}}{1-\tau}\right)}{d\tau} = \frac{\bar{k}}{(1-\tau)^2}\left[1 - \frac{\bar{r}_f}{\gamma(1+\bar{r}_f)}\right] \tag{52}$$

$$\frac{d\left(\frac{\bar{M}}{1+\bar{r}_f}\right)}{d\tau} = \frac{d\bar{\lambda}^{1/\gamma}}{d\tau} = -\frac{(1-\gamma)r_f\bar{M}}{\gamma(1+\bar{r}_f)^2} = -\frac{1-\gamma}{\gamma}\frac{r_f}{1+\bar{r}_f}\bar{\lambda}^{1/\gamma} < 0 \tag{53}$$

$$\frac{d\left(\frac{\bar{k}}{1+\bar{r}_f}\right)}{d\tau} = -\frac{(1-\gamma)r_f\bar{k}}{\gamma(1+\bar{r}_f)^2} < 0$$

where $\bar{\tau} = 1 - \frac{\gamma}{(1-\gamma)r_f}$. For $\gamma \geq \frac{r_f}{1+r_f}$, i.e., for sufficiently risk averse investors, is $\bar{\tau} \leq 0$ and thus condition $\tau > \bar{\tau}$ is automatically satisfied. The inequalities in (51) hold under assumption that $r_f > 0$.

After some derivations we obtain the following

$$\frac{dC_1^*}{d\tau} = \frac{r_f}{(1+\bar{r}_f+\bar{M})^2}\left[\left(\bar{C}_1 - Y_1 + \frac{\bar{\Omega}}{\gamma}\right)\bar{M} + Y_2 - \bar{C}_2\right]$$
$$= \frac{r_f}{\gamma(1+\bar{r}_f)(1+\bar{r}_f+\bar{M})^2}\left\{\left[(1-\gamma)(1+\bar{r}_f)(Y_1 - \bar{C}_1) + Y_2 - \bar{C}_2\right]\bar{M} + \gamma(1+\bar{r}_f)(Y_2 - \bar{C}_2)\right\}$$
$$= \frac{r_f}{\gamma(1+\bar{r}_f)(1+\bar{r}_f+\bar{M})^2}\left\{(1-\gamma)(1+\bar{r}_f)(Y_1 - \bar{C}_1)\bar{M} + \left[\gamma(1+\bar{r}_f) + \bar{M}\right](Y_2 - \bar{C}_2)\right\} \tag{54}$$

$$\frac{d\alpha^*}{d\tau} = \frac{1-\bar{K}_0^{\frac{1}{\gamma}}}{r_f - r_b + \bar{K}_0^{\frac{1}{\gamma}}(r_g - r_f)} \left[\frac{d}{d\tau}\left(\frac{\bar{M}}{1-\tau}\right) \frac{\bar{\Omega}}{1+\frac{\bar{M}}{1+\bar{r}_f}} + \frac{\bar{M}}{1-\tau}\frac{d}{d\tau}(C_1^* - \bar{C}_1) \right]$$

$$= \frac{\alpha^*}{1-\tau}\left[1 - \frac{\bar{r}_f}{\gamma(1+\bar{r}_f)}\right] + \frac{r_f \alpha^*}{(1+\bar{r}_f+\bar{M})(1+\bar{r}_f)\bar{\Omega}}\left[\left(\bar{C}_1 - Y_1 + \frac{\bar{\Omega}}{\gamma}\right)\bar{M} + Y_2 - \bar{C}_2\right] \tag{55}$$

$$= \frac{\left(1-\bar{K}_0^{1/\gamma}\right)\bar{M}}{\gamma(1-\tau)\left[\bar{r}_f - \bar{r}_b + \bar{K}_0^{1/\gamma}(\bar{r}_g - \bar{r}_f)\right](1+\bar{r}_f+\bar{M})^2} \times$$
$$\times \left\{\left[\gamma(1+\bar{r}_f+\bar{M}) - (1-\gamma)\bar{r}_f(1+\bar{r}_f)\right](Y_1 - \bar{C}_1) + \left[\gamma(1+\bar{r}_f+\bar{M}) - (1-\gamma)\bar{r}_f\right](Y_2 - \bar{C}_2)\right\} \tag{56}$$

$$\frac{dC_{2g}^*}{d\tau} = \frac{d(C_{2g}^* - \bar{C}_2)}{d\tau} = \frac{r_g - r_b}{r_f - r_b}\left[\frac{d\bar{k}_2}{\tau}(C_1^* - \bar{C}_1) + \bar{k}_2 \frac{d(C_1^* - \bar{C}_1)}{d\tau}\right]$$
$$= -\frac{r_g - r_b}{r_f - r_b} \times \frac{r_f \bar{k}_2}{\gamma(1+\bar{r}_f+\bar{M})^2}\left[(1+\bar{r}_f+\gamma\bar{M})(Y_1 - \bar{C}_1) + (1-\gamma)(Y_2 - \bar{C}_2)\right] \tag{57}$$

$$\frac{dC_{2b}^*}{d\tau} = \frac{d(C_{2b}^* - \bar{C}_2)}{d\tau} = K_0^{1/\gamma}\frac{dC_{2g}^*}{d\tau} \tag{58}$$

$$\frac{d\mathbb{E}\left(U(C_1^*, \alpha^*)\right)}{d\tau} = \frac{r_f}{(1+\bar{r}_f)^{1+\gamma}(1+\bar{r}_f+\bar{M})^{1-\gamma}\bar{\Omega}^{\gamma}}\left[-\bar{M}(Y_1 - \bar{C}_1) + Y_2 - \bar{C}_2\right] \tag{59}$$

$$\frac{d\sigma_{C_2^*}}{d\tau} = -\frac{\left(1-\bar{K}_0^{1/\gamma}\right)\bar{r}_f\bar{M}}{\gamma(1-\tau)\left[r_f - r_b + K_0^{1/\gamma}(r_g - r_f)\right](1+\bar{r}_f+\bar{M})^2}$$
$$\times \left[(1+\bar{r}_f+\gamma\bar{M})(Y_1 - \bar{C}_1) + (1-\gamma)(Y_2 - \bar{C}_2)\right] \tag{60}$$

Note that (57), (58) and (60) imply that $\frac{dC_{2g}^*}{d\tau}$, $\frac{dC_{2b}^*}{d\tau}$ and $\frac{d\sigma_{C_2^*}}{d\tau}$ will be of the same sign.

Let us assume at first that $Y_1 > \bar{C}_1$ and $Y_2 > \bar{C}_2$ (case 1). It follows then from (54), (57) and (58) that $\frac{dC_1^*}{d\tau} > 0$, $\frac{dC_{2g}^*}{d\tau} < 0$ and $\frac{dC_{2b}^*}{d\tau} < 0$. Equation (56) implies that sufficient condition for $\frac{d\alpha^*}{d\tau} > 0$ is when $\tau > \bar{\tau} = 1 - \frac{\gamma}{(1-\gamma)r_f}$.[27] Note that for $\gamma \geq \frac{r_f}{1+r_f}$ is $\bar{\tau} \leq 0$ and thus $\tau > \bar{\tau}$ is automatically satisfied. This implies that for sufficiently risk-averse investors the investment in the risky asset increases when

27. Based on (56) sufficient condition for α^* to be increasing function in τ is that $\gamma(1+\bar{r}_f) - (1-\gamma)\bar{r}_f(1+\bar{r}_f) > 0$.

capital income tax increases. For less risk-averse investors, i.e., when $\gamma < \frac{r_f}{1+r_f}$, the investment in the risky asset increases with increasing τ when τ exceeds its threshold value $\bar{\tau}$. On the other hand, (56) implies that sufficient condition for $\frac{d\alpha^*}{d\tau} < 0$ is when $\tau < \bar{\tau}_2 = 1 - \frac{\gamma(1+M)}{(1-2\gamma)r_f}$.[28] Note that $\bar{\tau}_2 > 0$ when $\gamma < \frac{r_f}{1+2r_f}$ and $\delta < \tilde{\tilde{\delta}} = \frac{1}{\tilde{M}}\left[\frac{1-2\gamma}{\gamma} r_f - 1\right]^{\gamma}$.

Note finally, that the following holds for $\mathbb{E}\big(U(C_1^*, \alpha^*)\big)$

$$\frac{d\mathbb{E}\big(U(C_1^*,\alpha^*)\big)}{d\tau} \begin{cases} < 0 & for\ \ \tau < \bar{\tau}_1 \\ = 0 & for\ \ \tau = \bar{\tau}_1 \\ > 0 & for\ \ \tau > \bar{\tau}_1 \end{cases} \tag{61}$$

where $\bar{\tau}_1 = 1 + \frac{1}{r_f}\left[1 - \frac{1}{\delta p}\left(\frac{r_f - r_b}{r_g - r_b}\right)^{1-\gamma}\left(\frac{Y_2 - \bar{C}_2}{Y_1 - \bar{C}_1}\frac{1}{1+K_\gamma^{1/\gamma}}\right)^{\gamma}\right]$. Thus, the indirect utility is decreasing (with increasing tax rate) when $\tau < \bar{\tau}_1$ and is increasing (with increasing τ) when $\tau > \bar{\tau}_1$. Note that if $\bar{\tau}_1 < 0$, which could happen for very small δ, i.e., in the case of high time preference, then the indirect utility is increasing function of τ. The summary of results on the sensitivity analysis can be found in Table 1.

Let us assume now that $\bar{C}_2 > Y_2$ and $Y_1 + Y_2 - \bar{C}_1 - \bar{C}_2 > 0$ (case 3). Then (54) implies that $\frac{dC_1^*}{d\tau} < 0$ when

$$(1-\gamma)(1+\bar{r}_f)(Y_1 - \bar{C}_1)\bar{M} + \big[\gamma(1+\bar{r}_f) + \bar{M}\big](Y_2 - \bar{C}_2) < 0$$

and thus when

$$\frac{(1-\gamma)\bar{M}}{\gamma(1+\bar{r}_f)+\bar{M}}(1+\bar{r}_f)(Y_1 - \bar{C}_1) + Y_2 < \bar{C}_2 < Y_1 - \bar{C}_1 + Y_2 \tag{62}$$

Based on (53), sufficient condition for (62) becomes

$$\bar{C}_2^U = \frac{(1-\gamma)M}{\gamma(1+r_f)+M}(1+r_f)(Y_1 - \bar{C}_1) + Y_2 < \bar{C}_2 < Y_1 - \bar{C}_1 + Y_2 \tag{63}$$

Note that condition (63) is feasible if $r_f M < \gamma(1+r_f)(1+M)$ which holds for $\gamma > \frac{r_f}{1+r_f}$. On the other hand, $\frac{dC_1^*}{d\tau} > 0$ when

$$Y_2 < \bar{C}_2 < \frac{(1-\gamma)\bar{M}}{\gamma(1+\bar{r}_f)+\bar{M}}(1+\bar{r}_f)(Y_1 - \bar{C}_1) + Y_2$$

sufficient condition for which is (when using (53))

$$Y_2 < \bar{C}_2 < \frac{(1-\gamma)\bar{M}_{\tau=0}}{\gamma+\bar{M}_{\tau=0}}(Y_1 - \bar{C}_1) + Y_2 = \bar{C}_2^L$$

Note that for $\bar{C}_1 < Y_1$ and $\bar{C}_2 > Y_2$ the second period consumption remains

28. Based on (56) sufficient condition for α^* to be decreasing function in τ is that $\gamma(1+\bar{r}_f+M) - (1-\gamma)\bar{r}_f < 0$.

decreasing function in τ under both states of nature, as $(1+\bar{r}_f+\gamma\bar{M})(Y_1-\bar{C}_1)+(1-\gamma)(Y_2-\bar{C}_2)=(1+\bar{r}_f)(Y_1-\bar{C}_1)+Y_2-\bar{C}_2+\gamma[\bar{C}_2-Y_2+\bar{M}(Y_1-\bar{C}_1)]>0$ and thus based on (57) is $\frac{dC_{2b}^*}{d\tau}<0$ and $\frac{dC_{2g}^*}{d\tau}<0$. (59) implies that for $\bar{C}_1<Y_1$ and $\bar{C}_2>Y_2$ is the indirect utility (happiness) decreasing function in τ. Finally, based on (56) it can be shown that the sufficient condition for $\frac{d\alpha^*}{d\tau}>0$ is that $\gamma>\frac{r_f(1+r_f)\Omega}{Y_1+Y_2-\bar{C}_1-\bar{C}_2+r_f(1+r_f)\Omega}=\gamma_1$ and that sufficient condition for $\frac{d\alpha^*}{d\tau}<0$ is that $\gamma(1+\bar{r}_f+\bar{M})-(1-\gamma)\bar{r}_f<0$. This can be achieved by assuming that $\delta<\frac{r_f-r_b}{r_g-r_b}\frac{1}{p\,(1+r_f)}$ (as then $\bar{M}<\delta(1+r_f)$), $\tau<1-\frac{\gamma[1+\delta(1+r_f)]}{(1-2\gamma)r_f}=\bar{\tau}_3$ and $\gamma<\gamma_2=\frac{r_f}{r_f+(1+r_f)(1+\delta)}$.

Let $\bar{C}_1>Y_1$ and $\Omega>0$ (case 2). Then, based on (54), is $\frac{dC_1^*}{d\tau}>0$. In addition, (57) implies also that $\frac{dC_{2g}^*}{d\tau}>0$ when

$$Y_1+\frac{1-\gamma}{1+\bar{r}_f+\gamma\bar{M}}\,(Y_2-\bar{C}_2)<\bar{C}_1<Y_1+\frac{Y_2-\bar{C}_2}{1+r_f} \tag{64}$$

Based on (64) the sufficient condition for $\frac{dC_{2g}^*}{d\tau}>0$, and thus also for $\frac{dC_{2b}^*}{d\tau}>0$ and $\frac{d\sigma_{C_2^*}}{d\tau}>0$, is

$$\bar{C}_1^U=Y_1+(1-\gamma)(Y_2-\bar{C}_2)<\bar{C}_1<Y_1+\frac{Y_2-\bar{C}_2}{1+r_f}\quad and\quad \gamma>\frac{r_f}{1+r_f} \tag{65}$$

On the other hand, $\frac{dC_{2g}^*}{d\tau}<0$ when

$$\bar{C}_1<Y_1+\frac{1-\gamma}{1+\bar{r}_f+\gamma\bar{M}}\,(Y_2-\bar{C}_2) \tag{66}$$

Based on (66) sufficient condition for $\frac{dC_{2g}^*}{d\tau}<0$, and thus also for $\frac{dC_{2b}^*}{d\tau}<0$ and $\frac{d\sigma_{C_2^*}}{d\tau}<0$, is[29]

$$\bar{C}_1<\bar{C}_1^L=Y_1+\frac{1-\gamma}{1+\gamma\delta}\frac{Y_2-\bar{C}_2}{1+r_f}\quad and\quad \delta<\tilde{\delta}=\frac{r_f-r_b}{r_g-r_b}\frac{1}{p\,(1+r_f)} \tag{67}$$

(56) implies that $\frac{d\alpha^*}{d\tau}>0$ when

$$\left[\gamma(1+\bar{r}_f+\bar{M})-(1-\gamma)\bar{r}_f\right](Y_1+Y_2-\bar{C}_1-\bar{C}_2)+(1-\gamma)\bar{r}_f^2(\bar{C}_1-Y_1)>0 \tag{68}$$

Sufficient condition for (68) is $\gamma(1+\bar{r}_f+\bar{M})-(1-\gamma)\bar{r}_f\geq 0$ which holds for sufficiently risk averse investors, namely when $\gamma\geq\frac{r_f}{1+2r_f}$. On the other hand

29. Note that $\tilde{\delta}$, as defined in (67), was determined such that $\delta(1+r_f)p\frac{r_g-r_b}{r_f-r_b}<1$, see (34), and thus $\bar{M}<M<\delta(1+r_f)$.

(56) also implies that $\frac{d\alpha^*}{d\tau} < 0$ when

$$\left[\gamma(1+\bar{r}_f+\bar{M})-(1-\gamma)\bar{r}_f\right](Y_1+Y_2-\bar{C}_1-\bar{C}_2)+(1-\gamma)\bar{r}_f^2(\bar{C}_1-Y_1)<0 \quad (69)$$

Note that $\bar{M} < \delta(1+r_f)$ when $\delta < \tilde{\delta} = \frac{r_f-r_b}{r_g-r_b}\frac{1}{p\,(1+r_f)}$, which implies that sufficient condition for (69) is

$$\frac{(1-\gamma)r_f^2(\bar{C}_1-Y_1)}{Y_1+Y_2-\bar{C}_1-\bar{C}_2} < (1-2\gamma)(1-\tau)r_f-\gamma[1+\delta(1+r_f)]$$

which, for $\gamma < 0.5$, boils down to

$$\tau < 1-\frac{1}{(1-2\gamma)r_f}\left[\gamma(1+\delta(1+r_f))+\frac{(1-\gamma)r_f^2(\bar{C}_1-Y_1)}{Y_1+Y_2-\bar{C}_1-\bar{C}_2}\right]=\bar{\tau}_4 \quad (70)$$

Note in addition that $\bar{\tau}_4 > 0$ for $\gamma < \frac{r_f\Omega}{r_f\Omega+(1+\delta)(Y_1+Y_2-\bar{C}_1-\bar{C}_2)} = \gamma_3$.

MORE AMBITIOUS INVESTORS

We assume that λ is sufficiently large[30] and that $\bar{\Omega} < 0$ or (equivalently) that $\bar{C}_1 > Y_1 + \frac{Y_2-\bar{C}_2}{1+\bar{r}_f}$. Let introduce the following notation We refer to the investor with these specifications and $\delta \le \bar{\delta}$, where $\bar{\delta}$ is given by (11), as the more ambitious investor with a higher time preference. On the other hand, if $\delta > \bar{\delta}$ while everything else is kept unchanged, then we refer to this investor as the investor with lower time preference. Note however that threshold value for δ, $\bar{\delta}$, is an increasing function in τ where $\bar{\delta}_{\min} = \bar{\delta}(0) = \frac{1}{1-p}\left[\frac{r_g-r_f}{(1+r_f)(r_g-r_b)}\right]^{1-\gamma}$ is its minimum value. Thus, if $\delta \le \bar{\delta}_{\min}$ then the investor will be of this type (i.e., investor with a higher time preference) for any tax rate τ. If, however, δ exceeds this threshold, i.e., $\delta > \bar{\delta}_{\min}$, and thus if $\bar{\delta}_{\min} < 1$, which holds for sufficiently small p such that

$$\frac{r_f-r_b}{r_g-r_b} < p < 1-\left[\frac{r_g-r_f}{(1+r_f)(r_g-r_b)}\right]^{1-\gamma} \quad (71)$$

where the lower bound follows from $\mathbb{E}(r) > r_f$,[31] then this investor is at first the investor with lower time preference for all tax rates such that $\tau \in (0,\tilde{\tau})$, where $\delta = \bar{\delta}(\tilde{\tau})$, as then $\delta > \bar{\delta}(\tau)$. When τ exceeds $\tilde{\tau}$, i.e., when $\tau \in [\tilde{\tau}, 1)$, then the investor becomes the investor with higher time preference as then $\delta \le \bar{\delta}(\tau)$.

30. For more details regarding the assumption on loss aversion, see Proposition 3 in Hlouskova, Fortin and Tsigaris (2017).

31. Note finally that upper bound of p in (71) exceeds its lower bound when following holds: $r_g > \frac{r_f(1+r_f)^{\frac{1}{\gamma}-1}}{(1+r_f)^{\frac{1}{\gamma}-1}-1}$ and $r_b > r_g-(1+r_f)^{\frac{1}{\gamma}}(r_g-r_f)$.

Note that the case of the investor with lower time preference; i.e., when $\delta > \bar{\delta}$, is feasible only when $\bar{\delta}(\tau) = \frac{1}{1-p}\left[\frac{r_g - r_f}{(1+\bar{r}_f)(r_g - r_b)}\right]^{1-\gamma} < 1$ which holds if

$$\tau < \frac{1}{r_f}\left[1 + r_f - \frac{r_g - r_f}{(1-p)^{\frac{1}{1-\gamma}}(r_g - r_b)}\right] = \tau^U \tag{72}$$

and thus if $\tau^U > 0$. The last inequality is guaranteed if probability p of the good state is sufficiently small such that condition (71) is satisfied. Note that condition $\mathbb{E}(r) > r_f$, i.e., that $p > \frac{r_f - r_b}{r_g - r_b}$, implies that $\tau^U < 1$. Thus, based on this, is $\delta < 1 \leq \bar{\delta}(\tau)$ for $\tau \geq \tau^U$ and the investor behaves as the investor with the higher time preference.

MORE AMBITIOUS INVESTORS WITH A HIGHER TIME PREFERENCE

We assume that $\delta \leq \bar{\delta}$. Then the following holds

$$\gamma \frac{d\bar{M}(\lambda)}{d\tau} = -\frac{r_f}{1+\bar{r}_f}\bar{M}(\lambda) < 0$$

$$\frac{d}{d\tau}\frac{\bar{M}(\lambda)}{1+\bar{r}_f} = -\frac{1-\gamma}{\gamma}\frac{r_f}{(1+\bar{r}_f)^2}\bar{M}(\lambda) < 0 \tag{73}$$

where $\bar{M}(\lambda)$ is given by (36).

Based on Hlouskova, Fortin and Tsigaris (2017), the solution for sufficiently loss averse investor is given as by (12) and (13) and

$$C_{2g}^* - \bar{C}_2 = \frac{r_g - r_b}{r_g - r_f}\frac{\bar{k}(-\bar{\Omega})}{\frac{\bar{M}(\lambda)}{1+\bar{r}_f} - 1}\left(\frac{1}{K_0}\right)^{\frac{1}{\gamma}} \tag{74}$$

$$\bar{C}_2 - C_{2b}^* = \frac{r_g - r_b}{r_g - r_f}\frac{\bar{k}(-\bar{\Omega})}{\frac{\bar{M}(\lambda)}{1+\bar{r}_f} - 1}\lambda^{\frac{1}{\gamma}} = (\lambda K_0)^{1/\gamma}\left(C_{2g}^* - \bar{C}_2\right) \tag{75}$$

Note in addition that

$$(1-\gamma)\mathbb{E}\big(U(C_1^*, \alpha^*)\big) = \left[\frac{-\bar{\Omega}}{\frac{\bar{M}(\lambda)}{1+\bar{r}_f} - 1}\right]^{1-\gamma}\left[1 + \frac{\bar{k}}{1+\bar{r}_f}\left(\left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}} - \lambda^{\frac{1}{\gamma}}\right)\right]$$

$$= -(-\bar{\Omega})^{1-\gamma}\left[\frac{\bar{M}(\lambda)}{1+\bar{r}_f} - 1\right]^{\gamma} \tag{76}$$

The following holds

$$\frac{dC_1^*}{d\tau} = \frac{r_f}{\gamma(1+\bar{r}_f)[\bar{M}(\lambda) - 1 - \bar{r}_f]^2}\Big\{\big[(1-\gamma)(1+\bar{r}_f)(\bar{C}_1 - Y_1) + \bar{C}_2 - Y_2\big]\bar{M}(\lambda) + \gamma(1+\bar{r}_f)(Y_2 - \bar{C}_2)\Big\}$$

$$= \frac{r_f}{\gamma(1+\bar{r}_f)[\bar{M}(\lambda)-1-\bar{r}_f]^2}\{(1-\gamma)(1+\bar{r}_f)(\bar{C}_1 - Y_1)\bar{M}(\lambda) + [\bar{M}(\lambda) - \gamma(1+\bar{r}_f)](\bar{C}_2 - Y_2)\} \quad (77)$$

$$\frac{d\alpha^*}{d\tau} = \frac{\left(\frac{1}{\bar{K}_0}\right)^{1/\gamma} + \lambda^{1/\gamma}}{r_g - r_f}\left[\frac{d}{d\tau}\left(\frac{\bar{k}}{1-\tau}\right)\frac{-\bar{\Omega}}{\frac{\bar{M}(\lambda)}{1+\bar{r}_f}-1} + \frac{\bar{k}}{1-\tau}\frac{d}{d\tau}(C_1^* - \bar{C}_1)\right]$$

$$= \frac{\left[\left(\frac{1}{\bar{K}_0}\right)^{1/\gamma} + \lambda^{1/\gamma}\right]\bar{k}}{\gamma(1-\tau)(\bar{r}_g - \bar{r}_f)\left[\bar{M}(\lambda)-1-\bar{r}_f\right]^2} \times$$

$$\times\left\{\left[\gamma(\bar{M}(\lambda) - 1 - \bar{r}_f) + (1-\gamma)\bar{r}_f(1+\bar{r}_f)\right](\bar{C}_1 - Y_1) + \left[\gamma(\bar{M}(\lambda) - 1 - \bar{r}_f) + (1-\gamma)\bar{r}_f\right](\bar{C}_2 - Y_2)\right\} \quad (78)$$

$$\frac{dC_{2g}^*}{d\tau} = \frac{d\left(C_{2g}^* - \bar{C}_2\right)}{d\tau}$$
$$= \frac{r_g - r_b}{r_f - r_b}\frac{1}{K_0^{1/\gamma}}\left[\frac{d\bar{k}}{\tau}(C_1^* - \bar{C}_1) + \bar{k}\frac{dC_1^* - \bar{C}_1}{d\tau}\right]$$
$$= -\frac{r_g - r_b}{r_f - r_b}\left(\frac{1}{K_0}\right)^{1/\gamma}\frac{r_f\bar{k}}{\gamma\left[\bar{M}(\lambda)-1-\bar{r}_f\right]^2}$$

$$\times\left[(\gamma\bar{M}(\lambda) - 1 - \bar{r}_f)(\bar{C}_1 - Y_1) - (1-\gamma)(\bar{C}_2 - Y_2)\right] \quad (79)$$

$$\frac{dC_{2b}^*}{d\tau} = -\frac{d\left(\bar{C}_2 - C_{2b}^*\right)}{d\tau} = -(K_0\lambda)^{1/\gamma}\frac{dC_{2g}^*}{d\tau} \quad (80)$$

$$\frac{d\mathbb{E}\left(U(C_1^*, \alpha^*)\right)}{d\tau} = \frac{r_f}{(1+\bar{r}_f)^{1+\gamma}\left[\bar{M}(\lambda)-1-\bar{r}_f\right]^{1-\gamma}\bar{\Omega}^\gamma}\left[\bar{M}(\lambda)(\bar{C}_1 - Y_1) + \bar{C}_2 - Y_2\right] \quad (81)$$

$$\frac{d\sigma_{C_2^*}}{d\tau} = -\frac{\left[\left(\frac{1}{K_0}\right)^{1/\gamma} + \lambda^{1/\gamma}\right]\bar{r}_f\bar{k}}{\gamma(1-\tau)(r_g - r_f)\left[\bar{M}(\lambda)-1-\bar{r}_f\right]^2} \times$$

$$\times\left[\left(\gamma\bar{M}(\lambda) - 1 - \bar{r}_f\right)(\bar{C}_1 - Y_1) - (1-\gamma)(\bar{C}_2 - Y_2)\right] \quad (82)$$

(79), (80) and (82) imply that $\frac{dC_{2g}^*}{d\tau}$ and $\frac{d\sigma_{C_2^*}}{d\tau}$ will be of the same sign which will be opposite to the sign of $\frac{dC_{2b}^*}{d\tau}$.

Let $\bar{C}_1 > Y_1$ and $\bar{C}_2 > Y_2$ (case 1) and the investor is sufficiently loss averse. Then the following holds based on (77)–(82): $\frac{dC_1^*}{d\tau} > 0$, $\frac{d\alpha^*}{d\tau} > 0$, $\frac{dC_{2g}^*}{d\tau} < 0$, $\frac{dC_{2b}^*}{d\tau} >$

0, $\frac{d\mathbb{E}(U(C_1^*,\alpha^*))}{d\tau} > 0$ and $\frac{d\sigma_{C_2^*}}{d\tau} < 0$. The summary of results on the sensitivity analysis can be found in Table 1.

Let $\bar{C}_1 > Y_1$, $Y_1 + Y_2 - \bar{C}_1 < \bar{C}_2 < Y_2$ (case 2) and the investor is sufficiently loss averse. Then (77) implies that $\frac{dC_1^*}{d\tau} < 0$ when

$$(1-\gamma)(1+\bar{r}_f)(\bar{C}_1 - Y_1) + \bar{C}_2 - Y_2 < 0$$

and thus when

$$\tau > 1 + \frac{(1-\gamma)(\bar{C}_1 - Y_1) + \bar{C}_2 - Y_2}{(1-\gamma)r_f(\bar{C}_1 - Y_1)} = \tau_5 \tag{83}$$

Note that τ is feasible, i.e., $\tau_5 < 1$, when $\gamma > 1 - \frac{Y_2 - \bar{C}_2}{\bar{C}_1 - Y_1} = \gamma_4$. In addition for $\gamma > 1 - \frac{Y_2 - \bar{C}_2}{(1+r_f)(\bar{C}_1 - Y_1)} = \gamma_5$ is $\tau_5 < 0$ and thus $\gamma > \gamma_5$ is a sufficient condition for $\frac{dC_1^*}{d\tau} < 0$ (when investor is sufficiently loss averse). On the other hand, $\frac{dC_1^*}{d\tau} > 0$ when $\tau < \tau_5$ and $\gamma < \gamma_5$ (so that τ is feasible). Finally $\gamma < \gamma_4$ is a sufficient condition for $\frac{dC_1^*}{d\tau} > 0$. Regarding the risk taking, (78) implies that it increases with increasing τ, i.e., $\frac{d\alpha^*}{d\tau} > 0$. (79)–(82) imply that $\frac{dC_{2g}^*}{d\tau} < 0$, $\frac{d\sigma_{C_2^*}}{d\tau} < 0$, $\frac{dC_{2b}^*}{d\tau} > 0$ and $\frac{d\mathbb{E}(U(C_1^*,\alpha^*))}{d\tau} > 0$ for sufficiently loss averse investor.

Let $\bar{C}_1 < Y_1$, $\bar{C}_2 > (1+r_f)(Y_1 - \bar{C}_1) + Y_2$ (case 3) and the investor is sufficiently loss averse. Then (77)–(82) imply that $\frac{dC_1^*}{d\tau} > 0$, $\frac{d\alpha^*}{d\tau} > 0$. (79)–(82) imply that $\frac{dC_{2g}^*}{d\tau} > 0$, $\frac{d\sigma_{C_2^*}}{d\tau} > 0$, $\frac{dC_{2b}^*}{d\tau} < 0$ and $\frac{d\mathbb{E}(U(C_1^*,\alpha^*))}{d\tau} < 0$ for sufficiently loss averse investor.

MORE AMBITIOUS INVESTORS WITH A LOWER TIME PREFERENCE

Now we assume that $\delta > \frac{1}{1-p}\left[\frac{r_g - r_f}{(1+\bar{r}_f)(r_g - r_b)}\right]^{1-\gamma} = \bar{\delta}$. Based on (Hlouskova, Fortin and Tsigaris, 2017) the solution is given by (14) and (15) and

$$C_{2g}^* - \bar{C}_2 = \frac{(1+\bar{r}_f)\,(-\bar{\Omega})\,\bar{\lambda}^{1/\gamma}}{\lambda^{1/\gamma} - \bar{\lambda}^{1/\gamma}}\left(1 + \frac{r_g - r_f}{r_f - r_b}\frac{1 - K_0^{1/\gamma}}{1 + K_\gamma^{1/\gamma}}\right) \tag{84}$$

$$C_{2b}^* - \bar{C}_2 = \frac{(1+\bar{r}_f)\,(-\bar{\Omega})\,\bar{\lambda}^{1/\gamma}}{\lambda^{1/\gamma} - \bar{\lambda}^{1/\gamma}} \times \frac{K_\gamma^{1/\gamma} + K_0^{1/\gamma}}{1 + K_\gamma^{1/\gamma}} \tag{85}$$

$$(1-\gamma)\mathbb{E}\big(U(C_1^*,\alpha^*)\big) = -(-\bar{\Omega})^{1-\gamma}\left(\frac{\lambda^{1/\gamma} - \bar{\lambda}^{1/\gamma}}{1+\bar{r}_f}\right)^{\gamma} \tag{86}$$

where

$$\bar{\lambda} = \left(\frac{\bar{M}}{1+\bar{r}_f}\right)^{\gamma} \tag{87}$$

The following holds

$$\frac{dC_1^*}{d\tau} = \frac{r_f}{\gamma(1+\bar{r}_f)^2}\frac{\lambda^{1/\gamma}}{\left(\lambda^{1/\gamma}-\bar{\lambda}^{1/\gamma}\right)^2}$$
$$\times\left\{\left[(1-\gamma)(\bar{C}_1 - Y_1) + \frac{\bar{C}_2-Y_2}{1+\bar{r}_f}\right]\bar{M} - \gamma(\bar{C}_2 - Y_2)\lambda^{1/\gamma}\right\} \tag{88}$$

$$\frac{d\alpha^*}{d\tau} = \frac{1-K_0^{1/\gamma}}{r_f-r_b+K_0^{1/\gamma}(r_g-r_f)} \times \frac{1}{\lambda^{1/\gamma}} \times \frac{d}{d\tau}\left[\frac{\bar{M}}{1-\tau}(\bar{C}_1 - C_1^*)\right]$$
$$= \frac{1-K_0^{1/\gamma}}{r_f-r_b+K_0^{1/\gamma}(r_g-r_f)} \times \frac{1}{\lambda^{1/\gamma}}$$
$$\times\left[\left(\frac{d}{d\tau}\frac{\bar{M}}{1-\tau}\right)(\bar{C}_1 - C_1^*) + \frac{\bar{M}}{1-\tau}\frac{d(\bar{C}_1-C_1^*)}{d\tau}\right]$$
$$= \frac{\left(1-K_0^{1/\gamma}\right)\bar{\lambda}^{1/\gamma}}{\gamma(r_f-r_b)\left(1+K_\gamma^{1/\gamma}\right)(1-\tau)^2(1+\bar{r}_f)\left[\lambda^{1/\gamma}-\bar{\lambda}^{1/\gamma}\right]^2}$$

$$\times\left\{[\gamma(1+\bar{r}_f) - \bar{r}_f](1+\bar{r}_f)(-\bar{\Omega})\left(\lambda^{1/\gamma} - \bar{\lambda}^{1/\gamma}\right) + \gamma\bar{r}_f(\bar{C}_2 - Y_2)\left(\lambda^{1/\gamma} - \bar{\lambda}^{1/\gamma}\right) - (1-\gamma)\bar{r}_f(1+\bar{r}_f)(-\bar{\Omega})\bar{\lambda}^{1/\gamma}\right\} \tag{89}$$

$$\frac{dC_{2g}^*}{d\tau} = -\frac{r_f\bar{\lambda}^{1/\gamma}}{\gamma\left(\lambda^{1/\gamma}-\bar{\lambda}^{1/\gamma}\right)^2}\left(1 + \frac{r_g-r_f}{r_f-r_b} \times \frac{1-K_0^{1/\gamma}}{1+K_\gamma^{1/\gamma}}\right)$$
$$\times\left\{\left[\bar{C}_1 - Y_1 + \frac{1-\gamma}{1+\bar{r}_f}(\bar{C}_2 - Y_2)\right]\left(\lambda^{1/\gamma} - \bar{\lambda}^{1/\gamma}\right) - (1-\gamma)\,\bar{\Omega}\,\bar{\lambda}^{1/\gamma}\right\} \tag{90}$$

$$\frac{dC_{2b}^*}{d\tau} = -\frac{r_f\bar{\lambda}^{1/\gamma}}{\gamma\left(\lambda^{1/\gamma}-\bar{\lambda}^{1/\gamma}\right)^2} \times \frac{K_\gamma^{1/\gamma}+K_0^{1/\gamma}}{1+K_\gamma^{1/\gamma}}$$
$$\times\left[(\bar{C}_1 - Y_1)\left(\lambda^{1/\gamma} - \gamma\bar{\lambda}^{1/\gamma}\right) + \frac{(1-\gamma)(\bar{C}_2-Y_2)}{1+\bar{r}_f}\lambda^{1/\gamma}\right]$$

$$= -\frac{r_f\bar{\lambda}^{1/\gamma}}{\gamma\left(\lambda^{1/\gamma}-\bar{\lambda}^{1/\gamma}\right)^2} \times \frac{K_\gamma^{1/\gamma}+K_0^{1/\gamma}}{1+K_\gamma^{1/\gamma}}$$
$$\times\left\{\left[\bar{C}_1 - Y_1 + \frac{1-\gamma}{1+\bar{r}_f}(\bar{C}_2 - Y_2)\right]\left(\lambda^{1/\gamma} - \bar{\lambda}^{1/\gamma}\right) - (1-\gamma)\,\bar{\Omega}\,\bar{\lambda}^{1/\gamma}\right\} \tag{91}$$

$$\frac{d\mathbb{E}\left(U(C_1^*,\alpha^*)\right)}{d\tau} = \frac{-r_f}{(1+\bar{r}_f)^2\left(\lambda^{1/\gamma}-\bar{\lambda}^{1/\gamma}\right)^{1-\gamma}\bar{\Omega}^{\gamma}}\left[(1+\bar{r}_f)(\bar{C}_1 - Y_1)\,\bar{\lambda}^{1/\gamma} + \right.$$

$(\bar{C}_2 - Y_2)\,\lambda^{1/\gamma}\Big]$

$$(92)$$

$$\frac{d\sigma_{C_2^*}}{d\tau} = -\frac{\left(1-K_0^{1/\gamma}\right)\bar{r}_f\bar{\lambda}^{1/\gamma}}{\gamma(r_f-r_b)\left(1+K_\gamma^{1/\gamma}\right)(1-\tau)(1+\bar{r}_f)\left(\lambda^{1/\gamma}-\bar{\lambda}^{1/\gamma}\right)^2}$$

$$\times\left[(1+\bar{r}_f)(\bar{C}_1 - Y_1)\left(\lambda^{1/\gamma} - \gamma\bar{\lambda}^{1/\gamma}\right) + (1-\gamma)(\bar{C}_2 - Y_2)\,\lambda^{1/\gamma}\right] \quad (93)$$

(90), (91) and (93) imply that $\frac{dC_{2g}^*}{d\tau}$, $\frac{dC_{2b}^*}{d\tau}$ and $\frac{d\sigma_{C_2^*}}{d\tau}$ will be of the same sign.

Note from (88) that for sufficiently loss averse investor and $\bar{C}_2 > Y_2$ is $\frac{dC_1^*}{d\tau} < 0$ while $\frac{dC_1^*}{d\tau} > 0$ for $\bar{C}_2 < Y_2$. Equation (90) implies that $\frac{dC_{2g}^*}{d\tau} < 0$, and thus also $\frac{dC_{2b}^*}{d\tau} < 0$ and $\frac{d\sigma_{C_2^*}}{d\tau} < 0$, when

$$(1+\bar{r}_f)(\bar{C}_1 - Y_1) + (1-\gamma)(\bar{C}_2 - Y_2) > 0 \quad (94)$$

and thus when

$$-(1+\bar{r}_f)\,\bar{\Omega} > \gamma\,(\bar{C}_2 - Y_2)$$

The above inequalities hold for $\bar{C}_1 > Y_1$ and $\bar{C}_2 > Y_2$, or when $\bar{C}_2 < Y_2$ and $\bar{C}_1 + \bar{C}_2 - Y_1 - Y_2 > 0$. Let $\bar{C}_1 < Y_1$, $\bar{C}_2 > (1+r_f)(Y_1 - \bar{C}_1) + Y_2$ and the investor is sufficiently loss averse. Then (90) implies that $\frac{dC_{2g}^*}{d\tau} < 0$, when inequality (94) is satisfied, and thus when

$$\tau > 1 - \frac{\bar{C}_1 - Y_1 + (1-\gamma)(\bar{C}_2 - Y_2)}{r_f(Y_1 - \bar{C}_1)} = \tau_6 \quad (95)$$

Note that τ is feasible, i.e., $\tau_6 < 1$, when $\gamma < 1 - \frac{Y_1 - \bar{C}_1}{\bar{C}_2 - Y_2} = \gamma_6$. In addition for $\gamma < 1 - \frac{(1+r_f)(Y_1 - \bar{C}_1)}{\bar{C}_2 - Y_2} = \gamma_7$ is $\tau_6 < 0$ and thus $\gamma < \gamma_7$ is a sufficient condition for $\frac{dC_{2g}^*}{d\tau} < 0$ (when investor is sufficiently loss averse). On the other hand, $\frac{dC_{2g}^*}{d\tau} > 0$ when $\tau < \tau_6$ and $\gamma > \gamma_7$ (so that τ is feasible). Finally, $\gamma > \gamma_6$ is a sufficient condition for $\frac{dC_{2g}^*}{d\tau} > 0$.

(92) implies that the indirect utility (happiness) is a decreasing function in τ for $\bar{C}_1 > Y_1$ and $\bar{C}_2 > Y_2$ or when $\bar{C}_1 < Y_1$, $\bar{C}_2 > (1+r_f)(Y_1 - \bar{C}_1) + Y_2$ and the investor is sufficiently loss averse, namely $\lambda > \left[\frac{(1+r_f)(Y_1 - \bar{C}_1)}{\bar{C}_2 - Y_2}\right]^\gamma \bar{\lambda} = \bar{\lambda}_1$. On the other hand the investor's happiness is an increasing function in τ when $\bar{C}_1 > Y_1$, $Y_1 + Y_2 - \bar{C}_1 < \bar{C}_2 < Y_2$ and $\lambda > \bar{\lambda}_1$.

Finally, (89) implies that investment in the risky asset is an increasing function in τ, $\frac{d\alpha^*}{d\tau} > 0$, for sufficiently loss averse investor if

$$\gamma(1+\bar{r}_f) - \bar{r}_f > 0 \tag{96}$$

and $\bar{C}_1 > Y_1$ and $\bar{C}_2 > Y_2$, or if (96) holds and $\bar{C}_1 < Y_1$, $\bar{C}_2 > (1+r_f)(Y_1 - \bar{C}_1) + Y_2$. Sufficiently condition for (96) is $\gamma \geq \frac{r_f}{1+r_f}$. Sufficient condition for $\frac{d\alpha^*}{d\tau} < 0$ when $\bar{C}_1 > Y_1$ and $\bar{C}_2 > Y_2$ is $\gamma(1+\bar{r}_f) - (1-\gamma)\bar{r}_f < 0$, see (89), which is true for $\tau < 1 - \frac{\gamma}{(1-2\gamma)r_f}$. This condition is feasible for $\gamma < \frac{r_f}{1+2r_f}$. On the other hand, sufficient condition for $\frac{d\alpha^*}{d\tau} < 0$ when $\bar{C}_1 < Y_1$ and $\Omega < 0$ follows from

$$[\gamma - \bar{r}_f(1-\gamma)](\bar{C}_1 + \bar{C}_2 - Y_1 - Y_2) + \gamma\, \bar{r}_f(\bar{C}_2 - Y_2) < 0 \tag{97}$$

assuming that τ and γ are sufficiently small.[32] Note that for sufficiently small γ is (97) satisfied for

$$\tau < 1 - \frac{\gamma\,(\bar{C}_1 + \bar{C}_2 - Y_1 - Y_2)}{(1-2\gamma)(\bar{C}_2 - Y_2) - (1-\gamma)(Y_1 - \bar{C}_1)} \frac{1}{r_f} = \bar{\tau}_6$$

The feasibility is guaranteed by assuming

$$\gamma < \frac{(\bar{C}_1 + \bar{C}_2 - Y_1 - Y_2)r_f}{(\bar{C}_1 + \bar{C}_2 - Y_1 - Y_2)(1+r_f) + (\bar{C}_2 - Y_2)r_f} = \gamma_8$$

If, however, $\bar{C}_2 < Y_2$ and $\bar{C}_1 + \bar{C}_2 - Y_1 - Y_2 > 0$, then based on (89) is $\frac{d\alpha^*}{d\tau} < 0$ when $\gamma(1+\bar{r}_f) - \bar{r}_f < 0$, which holds for $\tau < 1 - \frac{\gamma}{(1-\gamma)r_f} = \bar{\tau}$ and $\gamma < \frac{r_f}{1+r_f}$. Sufficient condition for $\frac{d\alpha^*}{d\tau} > 0$, when $\bar{C}_2 < Y_2$ and $\bar{C}_1 + \bar{C}_2 - Y_1 - Y_2 > 0$, is $\tau > \bar{\tau}_6$.

32. Namely $\tau < 1 - \frac{\gamma}{(1-\gamma)r_f}$ and $\gamma < \frac{r_f}{1+r_f}$