SOCIAL CHOICE AND MARKET MECHANISM

Some properties of the model of perfect competition

Gerhard O. OROSEL

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After this paper was written I got the opportunity of discussing it with Professor Kenneth J. Arrow. In the discussion, which was very valuable to me, it turned out that my interpretation is not in accordance with Professor Arrow's idea, although Professor Arrow agrees that it is a possible interpretation of his book. The difference is that Professor Arrow has in mind varying sets of alternatives, e.g. due to varying production possibilities, rather than a fixed set of alternatives. The key point of this paper, the violation of condition 2', is still valid for both interpretations anyway.

Gerhard O. Orosel

C O N T E N T S

I. INTRODUCTION

In every society, no matter if it is a married couple, a country, or an international association, there are decisions to be made, decisions which will affect more than one member of the society, perhaps all of it. Everyone has to do with such decisions, active or passive, and it seems to be worthwhile to specify what conditions one wants to impose on social decisions. If one agrees that social decisions should somehow reflect individual values, the problem of a rule for social decisions becomes one of aggregating individual preference profiles.

In order to show that social decision making is not at all a simple problem and not a problem of just finding majorities two examples are given. In both a sciety, consisting of 5 individuals 1,...,5, has to give a ranking of 3 alternatives a, b, c, (e.g.: 3 candidates in elections). Each individual is assumed to have a pre-ordering on the set of alternatives. If a is preferred to b, b is written underneath a. The society has the following rule of decision making: In order to reach absolute majority there are two runs, in the first run every individual designates the alternative he prefers most. If no alternative got an absolute majority it comes to a second ballot, in which the individuals can vote for one of the alternatives which got the highest amount of votes in the first run. The society chooses the alternative which gets the majority in the second ballot.

Example 1:

1	2	3	4	5
a	a	b	b	c
c	c	c	c	a
b	b	a	a	b

The first run gives: two votes each for a and b, one for c; (if there is no strategic voting) i.e. the society prefers a to c and b to c.

The second ballot gives : a wins the elections, i.e. the social preference profile is a preferred to b and b preferred to c. ($a \succ b \succ c$).

Consider now alternative c. It would get absolute majority against each other alternative, i.e. it fulfils the Condorcet-Criterion. Nevertheless alternative c ranges as the last one in the social ranking. This shows that voting can give results which are intuitively not satisfying.

Example 2: A small change of example 1 gives the wellknown "voting paradoxon".

1	2	3	4	5
a	a	c	c	b
b	b	a	b	c
c	c	b	a	a

The above decision rule gives c preferred to a and a preferred to b (if there is no strategic voting).
Absolute majority prefers a to b (individuals 1,2 & 3)
Absolute majority prefers b to c (individuals 1,2 & 5)
Absolute majority prefers c to a (individuals 3,4 & 5)
Obviously this is an intransitive result.

There is a difference in those two examples: in example 1 there exists an alternative (c), which has a majority against each other alternative (i.e. it fulfils the Condorcet-Criterion), in example 2 this is not the case, each alternative beats the following one by majority.

II. ARROW'S THEOREM [1)]

1. Formulation of the problem

Let us start with some definitions and assumptions.

Definitions:

Alternatives: Let $A = \{x, y, \ldots, z\}$ be a set of alternatives.

Individuals: Denote the individuals of the society by $1, 2, \ldots, i, \ldots, n$.

Assumptions:

Preferences: For each individual i and any alternatives u and v, one and only one of the following holds:

(a) i prefers u to v

(b) i prefers v to u

(c) i is indifferent between u and v.

Transitivity: Both relations, preference and indifference, are assumed to be transitive.

Definitions:

Let $R = \{r^1, r^2, \ldots, r^m\}$ be the set of possible preference orderings for an individual.

By a profile of preference orderings for the individuals of the society we shall mean an n-tuple of orderings $(r_1, r_2, \ldots, r_n)$, where r_i is the preference ordering for the i-th individual.

The set of all possible profiles of preference orderings will be denoted by $R^n = R \times R \times \ldots \times R$.

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1) ARROW, K.J., Social Choice and Individual Values , Cowles Foundation Monograph 12, John Wiley and Sons, New York, 2nd ed., 1963.

Definition of a social welfare function:
A social welfare function is a general rule F, which associates to each profile of preference orderings (i.e. to each element of R^n) a preference ordering of the society itself (i.e. an element of R)

$$R^n \xrightarrow{F} R$$

I may emphasize that the problem is not to find a social preference ordering for a given set of individual preferences, but to find a general rule, which enables us to assign to each set of individual preferences exactly one social preference ordering.

2. Arrow's Conditions and Theorem*)

Condition 1

(a) The number of elements (alternatives) in A is greater than or equal to three.
(b) The social welfare function F is defined for all prossible profiles of individual orderings.
(c) There are at least two individuals.

Condition 2 (positive association of social and individual values)

If a welfare function asserts that x is preferred to y for a given profile of individual preferences, it shall assert the same when the profile is modified as follows:
(a) The individual paired comparisons between alternatives other than x are not changed, and
(b) Each individual paired comparison between x and any other alternative either remains unchanged or it is modified in x's favor.

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*) This formulatioh of the conditions and the theorem is given in Luce-Raiffa, "Games and Decisons", John Wiley & Sons, New York, 1957.

Condition 3 (independence of irrelevant alternatives)

Let A_1 be any subset of alternatives in A. If a profile of orderings is modefied in such a manner that each individual's paired comparisons among the alternatives of A_1 are left invariant, the social orderings resulting from the original and modified profiles of individual orderings should be identical for the alternatives in A_1.

If A_1 is simply any two-element set, then condition 3 reduces to:
If two profiles are such that each individual's paired comparisons between two alternatives x and y, say, are identical in both profiles, the society's ordering of x versus y should be identical for both profiles.

This means that the society's ordering of any pair of alternatives depends only on the individual orderings of just that pair and does not depend on individual orderings of other alternatives.

Condition 4 (citizens sovereignty)

For each pair of alternatives x and y, there exists a profile of individual orderings such that society prefers x to y.

This condition can be expressed by stating that the society's preference ordering should not be "imposed".

Condition 5 (non-dictatorship)

There is no individual with the property that whenever he prefers x to y, for any x and y, society does likewise, regardless of thepreferences of other individuals.

Arrow's Impossibility Theorem:

There does not exist a function which fulfils all the conditions 1 to 5, i.e. the conditions 1 to 5 are inconsistent.

This states that every time four conditions are fulfilled, the fifth will certainly be violated.

In the second edition of his book "Social Choice and Individual Values" Arrow gives a somewhat stronger modification of his theorem. Instead of conditions 2 and 4 he uses the condition of Pareto-optimality which is a weaker condition, because the conditions 2 and 4 imply the condition of Pareto-optimality but not vice versa. Since already a set of weaker conditions is inconstistent, one arrives at a stronger theorem. More exactly, condition and theorem are:

Condition P (Pareto-optimality)

If x is preferred to y by all individuals 1,2,...,n, then the society prefers x to y, for all alternatives x and y.

Theorem

The conditions 1,3,5 and P are inconsistent, each function which fulfils any three of them will violate the one left.

Although all conditions are used in the proof of the theorem, in a sense transititvity and independence of irrelevant alternatives seem to be the most opposing ones, because transitivity means exactly that third alternatives are important. For example take $x \succ y$ and $y \succ z$. This determines $x \succ z$. Hence y is important in this case for the relation between x and z. Transitivity causes a loss of degrees of freedom, we cannot fix the relation for every pair of alternatives independently, hence we cannot fulfil condition 3. This contradiction cannot occur for individual preferences, because every individual can be regarded as his own dictator. In the case of an imposed preference ordering it does not occur, because the only difference to the ditatorship is that here the "dictator" (who in this case needs not to be a single person) is not a member of the society. After this formulation of Arrow's theorem and the short intuitive interpretation of its proof some remarks on the conditions may be useful.

III. SOME REMARKS ON ARROW'S CONDITIONS

Condition 1 demands that the welfare function is a general rule, applicable to all cases, which logically can occur. Restricting the domain of the welfare function by taking into account only a subset of all possible profiles of individual orderings of the alternatives can help out of the trouble. This for instance is the case if preferences are "single-peaked". But looking for a general rule, one has to consider all conceivable profiles of individual preferences.

Condition 2, though being, of course, a value judgement like all those conditions, is very reasonable and one can assume it to be widely accepted.

It has been pointed out already that condition 3 is perhaps the most crucial one. It does not allow for interdependence between different subsets, especially pairs, of alternatives. But rejecting such an interdependence for a "rational" individual does not necessarily lead to rejecting it for the society as well, since for the society it is not necessarily unreasonable to take into account all individuals' preference orderings of each pair of alternatives simultaneously. This point will be more clear in connection with the "Ordinal Difference Approach" given in the appendix to this paper. Simultaneous consideration of all preferences may lead to a better compromise between individual preferences and since the social preference ordering is somehow a compromise of individual preference orderings it cannot be rejected as "unrational".

Although cardinal utility and interpersonal comparisons of utilities violate condition 3, such comparisons and nearly all ways of putting in cardinal utility do violate condition 1 as well. The insertion of fixed weights for the individual influence on the SWF[1] is not regarded as an interpersonal comparison in this context. Since a function assignes to each element of its domain exactly one element of its range, the social preference ordering must not alter as long as the vector $(r_1,\ldots,r_n)$ R^n does not alter. Hence if each individual keeps his preference ordering, but changes his "intensities" of preference, i.e. his utility differences, this cannot affect the social preference ordering. The only way the function allows for considering utility differences is that it connects certain utility intensities with each $r^k \in R$ with this connection given once for all and valid in exactly the same way for all individuals. For example, one may assume, the preference intensity between x and y may depend on the number s in the sequence $x \succsim a_1 \succsim \ldots \succsim a_s \succsim y$. Such an assumption, although not too reasonable, would not violate condition 1. It has to be emphasized that the connection between each r^k and the utility differences it is assumed to reveal has to be general, i.e. a certain preference pattern r^k would, once for all, imply certain utility differences, no matter which individual has chosen this r^k. Obviously a weaker condition than condition 3 would do to exclude such unwanted connections of any r^k and utility intensities.

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1) The abbreviation SWF is used for social welfare function.

Conditions 4 and 5 don't need much interpretation. The case of an imposed preference pattern can be regarded as "dictatorship" of someone outside the society. It is not ruled out that the "dictator" has different preferences as a dictator than he had as a simple individual, i.e. he may take into account not only his own wants and values but also what he thinks the society's needs are. This and the fact that the person of the dictator may change in time, makes dictatorship seem a little less formidable. Of course, it is not possible for the society to elect a dictator for a certain time-period, if Arrow's conditions must not be violated. Since the elction of a dictator is a social choice problem with, in general, more than two alternatives and at least two individuals, the whole problem comes in again.

IV. THE MARKET MECHANISM AS A SOCIAL WELFARE FUNCTION

The market in a barter economy can be looked at as a mechanism which assigns to each given set of

utility functions for individual i, i=1,2,...,n, and
initial endowments of each individual i with each commodity j, j=1,2,...,m

at least one matrix (a_{ij}), where a_{ij} gives the amount of commodity j individual i gets in equilibrium. The market assigns also to each equilibrium a price vector, which is unique up to a scalar multiplication. In order to make things easier there is no production included in this model.

In this section the following will be proved: The market mechanism in a competitive barter economy can be used to construct a social welfare function. This social welfare function violates conditions 1, 3 and 4 and fulfils condition 2 and 5. Condition 2 is fulfilled only because its preconditions cannot occur. It can be shown that a condition, in its philisophy very close to condition 2, is violated by the market mechanism. The violation of condition 4 follows from the violation of condition 1 and is only a formal one.

First a social welfare function, induced by the market mechanism, must be constructed. For a social welfare function we need a complete preordering of the alternatives. Alternatives are now all possible allocations of the commodities, where possible means that for each commodity the sum of the individual quantities of this commodity is constant and equal to the initial endowment of the society with this commodity. This follows from the absence of production. Hence alternatives are matrices (x_{ij}), where x_{ij} is the amount of commodity j individual i gets in alternative x and $\sum_i x_{ij}$ must equal the fixed amount of commodity j in the whole society for all j-s. The market mechanism does not give a complete preordering of all those alternatives, it only singles out certain alternatives, namely equilibrium points. Hence every ordering, which distinguishes equilibrium points from non-equilibrium allocations is compatible with the market mechanism.

We will distinguish three sets:

Definition: Denote by Q the set of equilibrium points, by P the set of pareto-optimal points not in Q and by S the set of possible allocations not in P and not in Q, i.e. the set of all non pareto-optimal allocations.

From this definition it follows that the union of Q, P, and S is the set of all alternatives A, $Q \cup P \cup S = A$. Hence if all elements of the sets Q, P, and S are comparable by a relation, all possible alternatives are comparable by that relation. We will define now such a relation:

Definition: The relation W is a preference relation, which gives the following ordering of alternatives: each element of Q is strictly preferred to each element of both, P and S; each element of P is strictly preferred to each element of S and elements of the same set are indifferent.

It is easy to see that the relation W does not violate the requirements, put on a preference relation, reflexivity, transitivity, and completeness. Hence we have defined a preference relation, compatible with the market mechansim. We will take this relation W now as a social preference pattern and look at the market mechanism as a social welfare function, which gives us in cooperation with the relation W a preference ordering of all alternatives, i.e. possible allocations. The interesting question now is, which of Arrow's conditions are fulfilled and which are violated. As already stated, it will be proved that conditions 1, 3, and 4 are violated and that the fulfilment of condition 2 is only a formal one and not satisfactory.

Condition 1 is violated, because in the model of a competitive economy certain assumptions about preferences are made, e.g. convexity of the indifference curves, absence of a bliss point, absence of external effects.

Condition 2 states that whenever preferences between any pair of alternatives other than (x,y) for a fixed x remain constant and if the society's preference is $x \succ y$, changes in individual preferences concerning only pairs involving the fixed x must not alter the society's preference $x \succ y$, if the changes in the individual preferences are only in x's favour.

Hence it is a precondition of condition 2 that it is possible that preferences concerning solely pairs of alternatives including x can change with preferences concerning all other alternatives remaining constant. If we can show that this precondition can never be fulfilled then by logical reasons condition 2 can never be violated.

Debreu[1)] has proved that under certain assumptions, made in general equilibrium theory (especially continuity of the choice space), it is possible to represent the individuals' preferences by a continuos utility function u, where $u(a) > u(b)$ resp. $u(a) = u(b)$ if and only if $a \succ b$ resp. $a \sim b$.

Assume, an individual i used to prefer a to b and prefers now b to a (i.e. b stand now for the fixed x in condition 2). We will prove that there exists an alternative c (in fact an infinite set of such alternatives) such that a change from $a \succ b$ to $b \succ a$ implies a change from $a \succ c$ to $c \succ a$. The proof for changes from preference to indifference or from indifference to preference is analogous and given afterwards.

Since $a \succ_i b$, $u(a) > u(b)$, where u(y) is the utility for alternative y and y can be any alternative. Out of continuity it folows that there is an ε-neighbourhood of b in which for every alternative c $u(a) > u(c)$ is true and hence c is strictly

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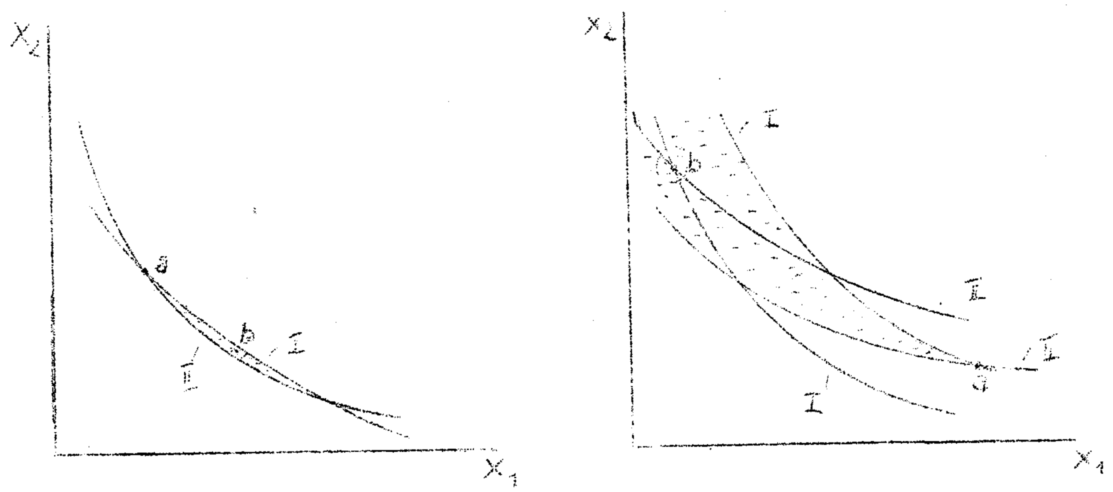
1) DEBREU, G., Theory of Value: An Axiomatic Analysis of Economic Equilibrium, Cowles Foundation, Monograph 17, John Wiley & Sons, New York, 1959.

less preferred than a, if a $\succsim_i$ b. For the same reason in the case b $\succsim_i$ a with u(b) $>$ u(a) there is an ε'-neighbourhood of b, in which ervery alternative is strictly preferred to a. If you now take $\varepsilon'' = \min(\varepsilon, \varepsilon')$, then for each c out of the ε''-neighbourhood of b the change from a $\succsim_i$ b to b $\succsim_i$ a implies a change from a $\succsim_i$ c to c $\succsim_i$ a. Hence it is not possible for individual preferences that only such pairs of alternatives change which include a fixed alternative x. This result follows from the continuity of the set of alternatives which together with other assumptions (e.g. that each individual has convex indifference curves) assures the continuity of the utility function.

Let us illustrate the proof in two ways. First, let the epsilons be real numbers in the open intervall (0,100), indicating percentage figures. Let the difference between the endowment-vectors of the individual i in the alternatives a and b α and β be $\delta = (d_1, d_2, \ldots, d_m)$, where d_j shows the amount of commodity j individual i gets less in alternative b compared with alternative a. Hence $\delta = \alpha - \beta$ and $\alpha = \beta + \delta$. Since externalities are excluded, a $\succsim_i$ b means $\alpha \succsim_i \beta$. Continuity now implies that $\alpha \succsim_i (\beta + \varepsilon\delta)$ for a small enough ε. Intuitively this means that, if an individual prefers a commodity bundle α to another one β, he will also prefer to every commodity bundle on the convex linear combination of α and β which corresponds to percentage numbers. For the same reason in the case $\beta \succsim_i \alpha$, $\beta + \varepsilon'\delta \succsim_i \alpha$ must hold for a small enough ε'. Hence γ with $\gamma = \beta + \varepsilon''\delta$ and $0 < \varepsilon'' \leq \min(\varepsilon, \varepsilon')$ represents alternatives c different from b, which change in individual i's preferences with respect to a in exactly the same way as b does.

The second illustration is a graphical one. Continuos indifference curves imply that no alternative, i.e. in this case a vector (x_1, x_2) can change solely, if all points in the positive quadrant belong to the set of possible alternatives (perfect divisibility). This is immediately seen on a graphical representation. Indifference curve I belongs to the preference pattern, where $a \succ_i b$ and indifference curve II belongs to $b \succ_i a$. They intersect, because preferences have changed. It is easy to see that there will always be a neighborhood of b (in the figure for example the marked circle) in which all alternatives rise in the individual preference pattern if b rises. In the case of the two graphical representations it is the whole dashed region between the two indifference curves.



We have to prove that there exists always an alternative c changing in individual preferences relative to a, if in a pair (a, b) the alternative b rises. Two cases are left, the case that the change is from $a \succ b$ to $a \sim b$ and the case that the change is from $a \succ b$ to $b \succ a$. Since the proof is the same for both cases, we will proof it only for the second one. $a \sim b$

implies $u(a) = u(b)$. Since $u(y)$ is a continuos function and (absence of a bliss point) in every neighborhood of b (the inside of a hypersphere in the Euklidean space R^m of the m commodities) there exists a c such that $u(a) > u(c)$, hence $a \succ c$, e.g. a c, where the individual gets a little less of at least one commodity and of all other commodities the same amount like in b. But from $b \succ a$, equivalent to $u(b) > u(a)$ follows that there exists an ε-eighborhood of b with $u(c) > u(a)$, hence $c \succ a$ for all c out of the ε-neighborhood of b. Since $a \sim b$ implied that in every neighborhood of b exists a c with $a \succ c$, such a c must be in the ε-neighborhood of b. But $b \succ a$ implies that for all c out of this neighborhood $c \succ a$ is true, so it must be true for the existing c, for which $a \succ c$ was true as long as $a \sim b$ was the case. Hence there exists c such that preferences concerning a pair (a,c) cannot stay fixed if preferences concerning a pair (a,b) change. It follows that the preconditions of Arrow's condition 2 cannot be fulfilled, which makes it impossible to violate condition 2.

Let us replace condition 2 by condition 2'.

<u>Condition 2'</u>:

(i) For given individual preferences the society prefers strictly x to y. If now individual preferences change that way that some individuals, who used to prefer strictly y to x, prefer now strictly x to y, all other individuals having unchanged preferences, the society must still prefer x to y, at least weakly in its new preference pattern.

(ii) For a given set of individual preference patterns the society is indifferent between x and y. Individual preferences change now such that some individuals, who used to prefer y to x, prefer now x to y and for every alternative z different from y which those individuals ordered

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1) which has a positive marginal utility.

$z \succsim y$, they order now $z \quad y$. All other individuals do not change their preferences. Then the new social preferences, belonging to the altered set of individual preferences must not prefer y to x, hence it must order $x \succsim y$.

Perhaps an explanation of condition (ii) will be useful. Contrary to (i) we start with x and y indifferent to each other for the society. But for y in individual preferences we demand more than in (i): if it falls in comparison to x it must fall to every possible other alternative z, i.e. there must not exist a z, which falls in comparison to y for any individual.

It will be proved by example that both, (i) and (ii), are violated by the welfare function, defined above. The violation of (i) will be considered in a little more detail. For the proof of the violation of (ii) we only need to draw the indifference curves in the example a little different. In the following we first consider only part (i) of condition 2'.

Condition 2'(i) is a special case of condition 2 without (a). If conditions 2 and 3 are fulfilled then condition 2 without precondition (a) is also fulfilled.

<u>Proof</u>: Let the society prefer x to y and let now x rise in individual preferences. Changes in individual preferences not involving x may happen.

Take first a profile of individual preferences without changes in pairwise comparisons between alternatives not x, but the rise of x has taken place. Since condition 2 holds the society prefers x to y.

Let now the individuals change their preferences concerning alternatives not x. If condition 3 holds, the social ordering of (x,y) must be the same, since (x,y) has not changed in individual preferences. Hence the society still prefers x to y.

Let x be an equilibrium point for a given set of utility functions I and given endowments. Let y be a Pareto-optimal point which is not an equilibrium point at utility functions I. Then it is possible as will be proved that x ceases to be an equilibrium point and y becomes one, if preferences of some individuals change from $y \succ_k x$ to $x \succ_k y$ with preferences of all other individuals remaining the same. Using the social preference ordering, induced by the market and defined in the above way, this means that a rise of an alternative x in some individual preference-patterns with other individuals' preferences remaining constant leads to a change in the social preference from $x \succ y$ to $y \succ x$, hence condition 2' is violated.

Let us first analize in more detail under which conditions this can happen. Then we will give a graphical example.

In an equilibrium point each individual maximizes his utility, if prices are taken as given. No one wants to buy or to sell at given prices. Each consumption point, which the individual would prefer, is too high in costs for the individual to afford it.

Let us introduce the following notations:

p^x	price vector for equilibrium point x	column vectors
p^y	price vector for equilibrium point y (if such a price vector making y to an equilibrium point exists)	
w^i	initial endowment for individual i	row vectors
z^i	consumption vector of individual i in the alternative (= distribution) $z \in A$ with **A** being the set of possible alternatives.	
$u(z^i)$	utility of z^i for individual i.	

If x is an equilibrium point then $z^i p^x > x^i p^x$ if $u(z^i) > u(x^i)$ for all i. I.e. $u(z^i) - u(x^i) > 0 \Longrightarrow (z^i - x^i)p^x > 0$. And in order to be compatible with a given initial endowment: $x^i p^x = w^i p^x \quad \forall\, i$.

Let some individuals $\{k\}$ change from a set of individual preference patterns I to such a set II, where in I $y^k \succ_k x^k$ (which is equivalent to $y \succ_k x$, because there is the assumption of absence of externalities). Hence in I $u(y^k) - u(x^k) > 0$ and let in II k prefer x to y, which gives $u(x^k) - u(y^k) > 0$ in II. All individuals not in $\{k\}$ have unaltered preferences in I and II.

In order that y becomes an equilibrium point the following necessary conditions which are given for an illustration must be fulfilled:

(i) If there exists at least one individual k with $x \succ_k y$ in II, there exist individuals k with $y \succ_k x$ respectively $u(y^k) - u(x^k) > 0$. This is necessary, because any equilibrium point lies on the Pareto-optimal surface. This is welknown and can easily be shown by the implication $u(z^i) - u(yi) > 0 \Longrightarrow (z^i - y^i)p^y > 0$. Since there is no production, $\sum_i z^i = \sum_i y^i$.

Hence $\quad_i (z^i - y^i) = (0,\dots,0)$ and $\left[\sum_i (z^i - y^i)\right] p^y = \sum_i (z^i - y^i)p^y = 0$.

Hence $(z^i - y^i)p^y$ cannot be positve for all i and from $(z^i - y^i)p^y \leq 0$ follows $u(z^i) - u(y^i) \leq 0$, if y is an equilibrium point. And there must exist an s with $u(z^s) - u(y^s) < 0$, i.e. s prefers y to z, if there exists a t, such that $(z^t - y^t)p^y > 0$, which makes it possible that t prefers z to y and y is an equilibrium point . Hence, if y is an equilibrium point, either all individuals are indifferent between y and an alternative z or if there exists individuals preferring z to y, then there must exist individuals preferring y to z. Hence the equilibrium is Pareto optimal.

(ii) Individuals with unaltered preferences are maximizing their utilities in x with prices p^x and also in y with prices p^y, i.e. the price ration of any pair of goods equals their marginal rate of substitution. (In both, x and y, preferred consumption points are too expensive at prices p^x resp. p^y).

(iii) For all individuals k with altered preferences the following must hold:

$(y^k - x^k)p^x > 0$ because $u(y^k) - u(x^k) > 0$ in I.

$(x^k - y^k)p^y > 0$ because $u(x^k) - u(y^k) > 0$ in II.

(iv) For all individuals k with altered preferences the following must hold:
For any alternative z^k, k prefers to y^k at preferences I and can afford at prices p^y, $z^k \succ_k y^k$ in I must change to $y^k \succ_k z^k$ in II. This is necessary because $(z^k - y^k)p^y \leqq 0$ (i.e. k can afford z) implies $u(z^k) - u(y^k) \leqq 0$ for all z, if y is an equilibrium point. Since y was not an equilibrium point in the set of preference patterns I, there existed a z^k for some k with $u(z^k) - u(y^k) > 0$ and $(z^k - y^k)p^y \leqq 0$. Hence if y was not an equilibrium point in I, there always exists at least one alternative z with $z \succ_k y$ in I and $y \succ_k z$ in II. By transitivity it follows from $y \succ_k x$ in I and $x \succ_k y$ in II that $z \succ_k x$ in I and $x \succ_k z$ in II. The rather strange result is the necessary condition that y can only become an equilibrium point (by above defined changes in individual preferences), if

x rises so strongly in k's preferences that $x^k \succ_k z^k$ for all consumption points z^k k can afford at prices p^y and his initial endowment w^k. It also shows that y must rise in individual preferences relative to a z. If one such z^k is still preferred to x^k in II, y can never be an equilibrium point, because $z^k \succ_k x^k \succ_k y^k$ and hence $z^k \succ_k y^k$ and k can afford z^k.

(v) x and y must be compatible with the initial endowment w, i.e.

$$w^i p^x = x^i p^x \text{ and } x^i p^y = y^i p^y \text{ for all i.}$$

(vi) Since there is no production and x, y, z are possible alternatives

$$\sum_i w^i = \sum_i x^i = \sum_i y^i = \sum_i z^i.$$

There is still the possibility that the conditions (i) to (vi) are inconsistent and hence equilibrium points can never change this way, but an example can be shown, which fulfills them all.

There are two commodities a and b and three groups of individuals Each group has the same size and identical utility functions. Group 2 and 3 don't alter their preferences, whereas group 1 does change its preferences from $y \succ_k x$ to $x \succ_k y$. Because of this change y becomes an equilibrium point, which is the paradoxon. This is shown in the graphical representation on page 2 . First we must do the computations in order to show that no individual violates its budget constraint and that the sum of the commodities equals the sum of the initial endowments, which must be the case because there is no production.

Initial endowment w = (4,4) for all individuals

prices: $p^x = \binom{2}{1}$ $\qquad p^y = \binom{1}{2}$

$\sum_i w = (12,12)$ if we assume that there is 1 individual in each group.

$wp^x = (4\ 4)\binom{2}{1} = 12 \qquad wp^y = (4\ 4)\binom{1}{2} = 12$

equilibria \ quantities	a_1	a_2	a_3	b_1	b_2	b_3
x	3,5	3,5	5	5	5	2
y	5	2	5	3,5	5	3,5

(1): $(3{,}5\ \ 5)\binom{2}{1} = 7+5 = 12 \qquad (5\ \ 3{,}5)\binom{1}{2} = 5+7 = 12$

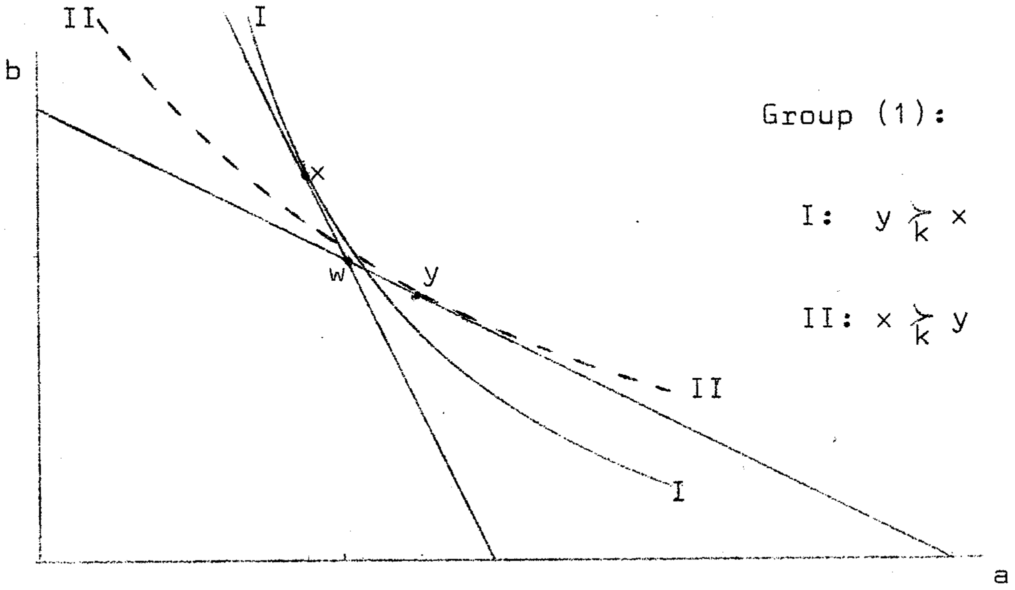
(2): $(3{,}5\ \ 5)\binom{2}{1} = 7+5 = 12 \qquad (2\ \ 5)\binom{1}{2} = 2+10 = 12$

(3): $(5\ \ 2)\binom{2}{1} = 10+2 = 12 \qquad (5\ \ 3{,}5)\binom{1}{2} = 5+7 = 12$

(x): $\sum_i (a_i b_i) = (3{,}5 + 3{,}5 + 5,\ 5 + 5 + 2) = (12,12) = \sum_i w$

(y): $\sum_i (a_i b_i) = (5 + 2 + 5,\ 3{,}5 + 5 + 3{,}5) = (12,12) = \sum_i w$

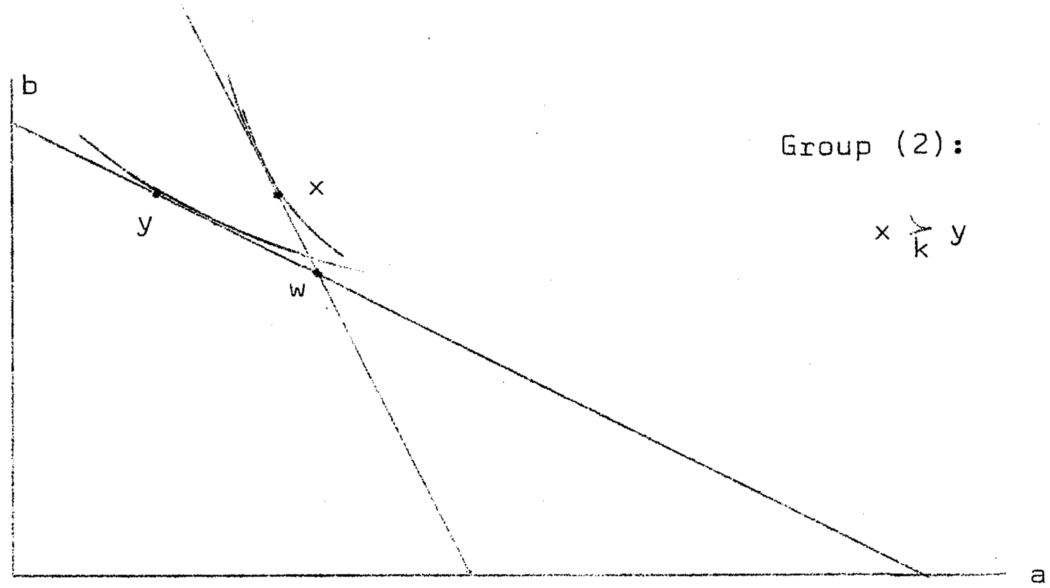
Hence x and y are equilibria, if the indifference corves touch the budget lines in the allocation points of x resp. y.



Group (1):

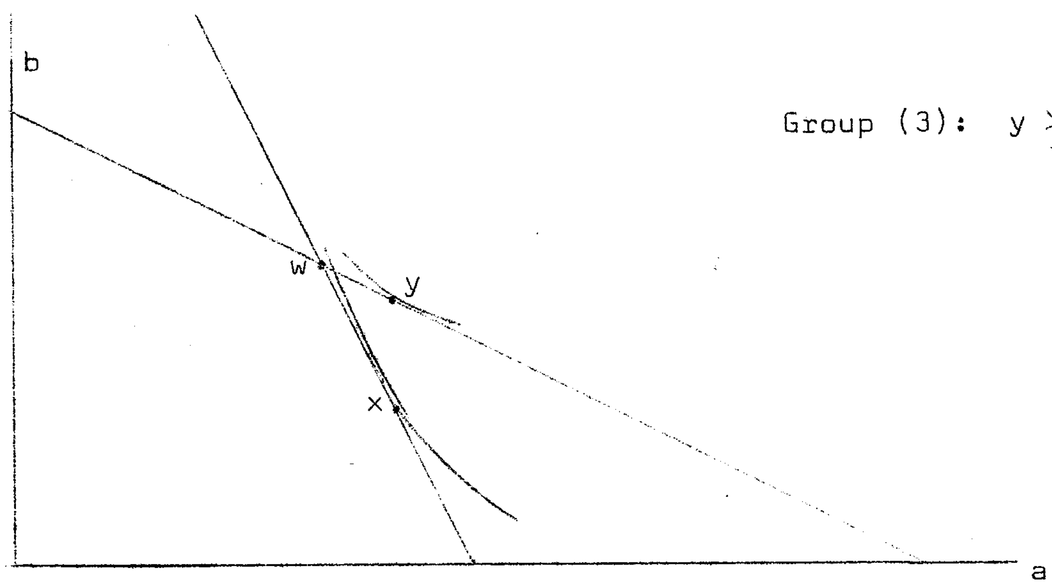
I: $y \succ_k x$

II: $x \succ_k y$



Group (2):

$x \succ_k y$



Group (3): $y \succ_l x$

Denote by $\left.\dfrac{\partial u^i/\partial a}{\partial u^i/\partial b}\right|_{\substack{a=\bar{a}\\ b=\bar{b}}}^{j}$ $\qquad i=h,k,l \qquad j=I,II$

the ratio of the marginal utilities of the commodities a and b (hence the marginal rate of substitution) for individual i at its endowment $a = \bar{a}$, $b = \bar{b}$ and preference pattern j. Then in the example the following holds:

(1) $$2 = \left.\frac{\partial u^k/\partial a}{\partial u^k/\partial b}\right|_{\substack{a=3,5\\ b=5}}^{I} \neq \left.\frac{\partial u^k/\partial a}{\partial u^k/\partial b}\right|_{\substack{a=3,5\\ b=5}}^{II}$$

$$1/2 = \left.\frac{\partial u^k/\partial a}{\partial u^k/\partial b}\right|_{\substack{a=5\\ b=3,5}}^{II} \neq \left.\frac{\partial u^k/\partial a}{\partial u^k/\partial b}\right|_{\substack{a=5\\ b=3,5}}^{I}$$

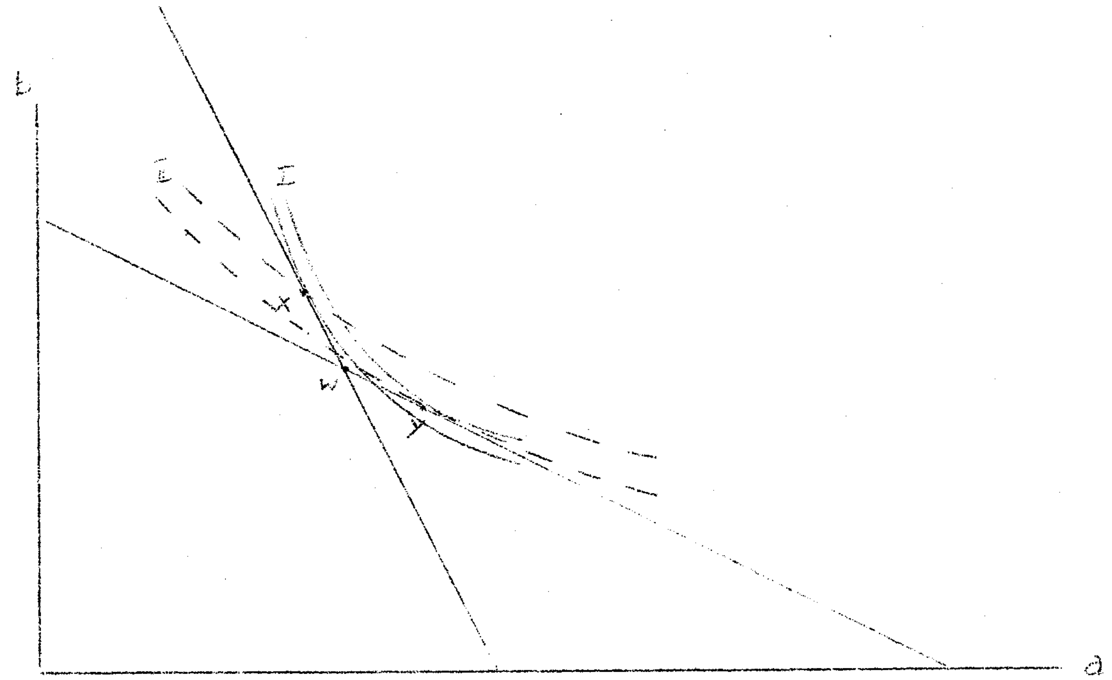
(2) $$2 = \left.\frac{\partial u^k/\partial a}{\partial u^k/\partial b}\right|_{\substack{a=3,5\\ b=5}}^{j} \qquad 1/2 = \left.\frac{\partial u^k/\partial a}{\partial u^k/\partial b}\right|_{\substack{a=2\\ b=5}}^{j} \qquad j=I,II$$

(3) $$2 = \left.\frac{\partial u^l/\partial a}{\partial u^l/\partial b}\right|_{\substack{a=5\\ b=2}}^{j} \qquad 1/2 = \left.\frac{\partial u^l/\partial a}{\partial u^l/\partial b}\right|_{\substack{a=5\\ b=3,5}}^{j} \qquad j=I,II$$

Since at allocation x at preferences II and at allocation y at preferences I the marginal rate of substitution of individuals k differ from that rate of individuals not k (hence individuals k or l) x is not pareto-optimal at preferences II and y is not pareto-optimal at preferences I.

Therefore there cannot exist prices which make those points x, respectively y to equilibrium points at preferences II, resp. I, because pareto-optimality is a necessary condition for an equilibrium point.

We still have to show that part (ii) of condition 2' is violated by our welfare function. We can use the same example, leaving everything unchanged for group 2 and group 3. Preferences of group 1 must now be such that y is as well as x an equilibrium point at preferences I, hence the society is indifferent between x and y. If the new indifference curve through y, implied by preference pattern II, touches the old one in y and lies nearer to the origin in every point else for each z not y the following holds: z $\succsim_k$ a in preference pattern I implies z $\succsim_k$ y in preference pattern II. Obviously it is possible that two strictly convex curves touch each other in a point. Since in the example everything exept the indifference curves of group 1 is the same as in the previous example, only the graphical representation of group 1 is given. Obviously part (ii) of condition 2' is violated.



The dotted curves are the indifference curves at preferences II.

At preferences I: $y \underset{k}{\succ} x$

At preferences II: $x \underset{k}{\succ} y$

$$2 = \left.\frac{\partial u^k/\partial a}{\partial u^k/\partial b}\right|^{I}_{\substack{a=3,5\\ b=5}} \neq \left.\frac{\partial u^k/\partial a}{\partial u^k/\partial b}\right|^{II}_{\substack{a=3,5\\ b=5}}$$

hence there cannot exist prices making x to an equilibrium point at preferences II, because the marginal substitution rate between a and b is equal to 2 for individuals not k. Therefore x cannot be pareto-optimal. Hence it cannot be an equilibrium point.

$$1/2 = \left.\frac{\partial u^k/\partial a}{\partial u^k/\partial b}\right|^{I}_{\substack{a=5\\ b=3,5}} = \left.\frac{\partial u^k/\partial a}{\partial u^k/\partial b}\right|^{II}_{\substack{a=5\\ b=3,5}}$$

The results reached so far , show that the market mechanism has unsatisfying properties, which are immanent in the theoretical model, no matter if this model is an acceptable approximation of reality or not.

Condition 3 (independence of irrelevant alternatives) is violated by the welfare function, we have defined in the above way. This can be shown easily. Take the set of equilibrium points $Q = \{q_1, q_2, \ldots, q_s\}$. The society is indifferent between all q_i's, i=1,...,s. Hence, if condition 3 holds the society must keep on to be indifferent between all q_i's as long as no individual changes his preference-ordering of the q_i's. Take an allocation x, which is not an equilibrium point, i.e. $x \notin Q$, and has the property that an individual k can afford x^k (i.e. his commodity vector in allocation x) at equilibrium prices belonging to a certain equilibrium point, say $q_t \in Q$. Hence k must order $q_t \succsim_k x$. Assume that k alters his preferences to $x \succsim_k q_t$, but keeps his preferences concerning all q_i's $\in Q$ constant. If all individuals other than k don't alter their preferences, the society must not alter its preferences concerning the q_i's$\in Q$, i.e. all q_i's still must be indifferent to the society, in order to fulfil condition 3, in this case the irrelevance of x. Hence, since q_t is not an equilibrium point any more, no $q_i \in Q$ can continue to be an equilibrium point. But this is not necessarily the case, as the following graphical example shows:

There are two groups of equal size with identical utility functions in each group.
For simplicity assume one individual in each group.

$w = (4,3) \qquad \sum_i a = 8 \qquad \sum_i b = 6$

$p^{q_1} = \binom{2}{3} \qquad wp^{q_1} = (4,3)\binom{3}{2} = 18$

$p^{q_2} = \binom{2}{2} \qquad wp^{q_2} = (4,3)\binom{2}{2} = 14$

	a_1	a_2	b_1	b_2
q_1	3	5	4,5	1,5
q_2	3,5	4,5	3,5	2,5

$$(a_1 b_1)\binom{3}{2} = (3 \;\; 4{,}5)\binom{3}{2} = 18$$

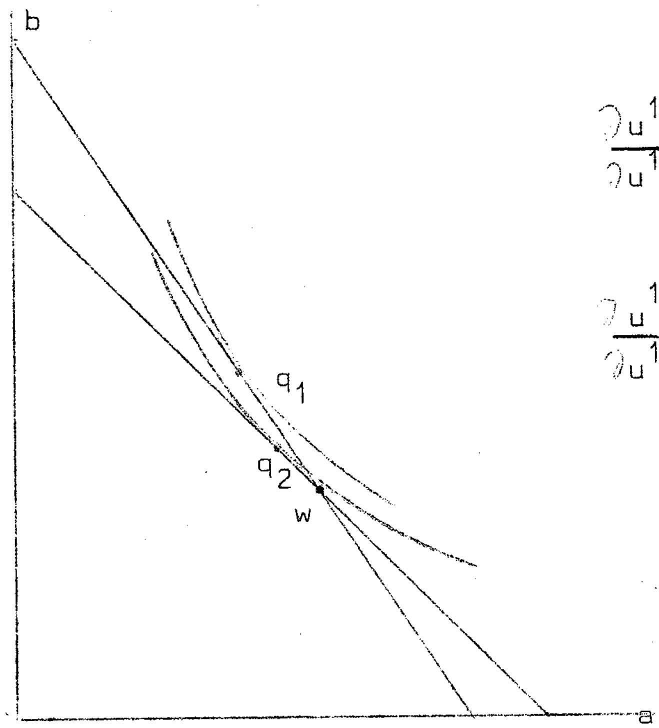
$$(a_1 b_1)\binom{2}{2} = (3{,}5 \;\; 3{,}5)\binom{2}{2} = 14$$

$$(a_2 b_2)\binom{3}{2} = (5 \;\; 1{,}5)\binom{3}{2} = 18$$

$$(a_2 b_2)\binom{2}{2} = (4{,}5 \;\; 2{,}5)\binom{2}{2} = 14$$

$$\sum_i a' = 8 \qquad \sum_i b' = 6$$

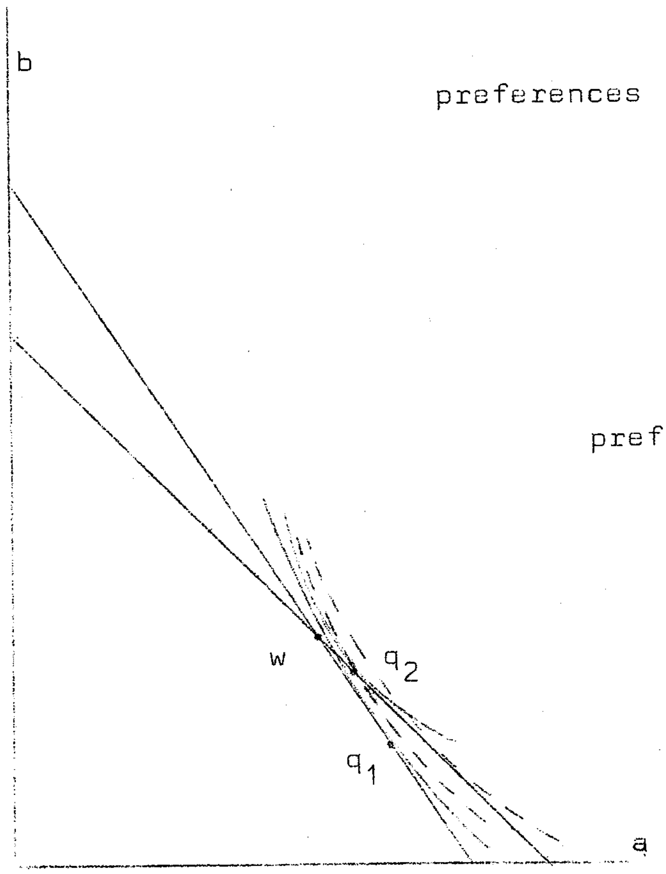
q_1 and q_2 are equilibria. Group 2 alters its preferences in such a way that the indifference curve through q_1 is still valid, but instead of that through q_2 preferences are now shown by the dotted lines. Obviously q_1 is still an equilibrium, but q_2 is not an equilibrium any more. Hence q_1 and q_2 are not equivalent to the society any more which violates condition 3 because no individual has altered its preferences concerning q_1 and q_2



$$\left.\frac{\partial u^1/\partial a}{\partial u^1/\partial b}\right|^{j}_{q_1} = \frac{3}{2} \qquad j=I,II$$

$$\left.\frac{\partial u^1/\partial a}{\partial u^1/\partial b}\right|^{j}_{q_2} = 1$$

Group 1: $q_1 \succ_1 q_2$



preferences I $\begin{cases} \left.\dfrac{\partial u^2/\partial a}{\partial u^2/\partial b}\right|^{I}_{q_1} = \dfrac{3}{2} \\ \left.\dfrac{\partial u^2/\partial a}{\partial u^2/\partial b}\right|^{I}_{q_2} = 1 \end{cases}$

preferences II $\begin{cases} \left.\dfrac{\partial u^2/\partial a}{\partial u^2/\partial b}\right|^{II}_{q_1} = \dfrac{3}{2} \\ \left.\dfrac{\partial u^2/\partial a}{\partial u^2/\partial b}\right|^{II}_{q_2} \neq 1 \end{cases}$

Group 2: $q_2 \succ_2 q_1$

Condition 4 (not imposed) is obviously not fulfilled, because not every possible allocation can become an equilibrium point. Prices must be such that for each individual its initial and final endowment have the same value. Hence it is impossible for example that an individual gets the same amount of all but one commodity in its initial and final endowment and less of the one commodity in its final endowment. Therefore not all possible allocations can become equilibrium points. But since by the assumptions about the individual preferences it is excluded that any individual prefers an allocation, where it gets less of at least one commodity and only the same amount of all other commodities, this violation of condition 4 is rather just a formal one.[1)] It would be reasonable to replace it by Pareto-optimality, a condition which is wellknown to be fulfilled by the market mechansim. Another way to get rid of the violation would be to restrict the set A of alternatives to allocations, where no individual gets a smaller commodity vector.

Condition 5 (non-dictatorship) is fulfilled. If it were not, by the assumptions about individual utilities (no bliss-point), one individual would get all commodities of the society. This can only happen, if this individual already had all commodities at the initial endowment and in this case there is no market and no competitive economy.

Before discussing a statement of Arrow concerning the market mechanism as a social choice function, I may mention that the social welfare function, defined in this section, is very bad from a sensitivity point of view. An infinitely small deviation away from an equilibrium point has at least the same effect in the social ordering as a very large one and if the pareto-optimal surface is left, the society will prefer any point on this surface strictly to the point, which is infinitely near to an equilibrium point.

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1) see next page !

The results we have reached contradict Arrow, who states: "... the market mechanism also operates independently of irrelevant alternatives. If we alter the utility functions of individuals with respect to allocations which are socially infeasible, we do not alter the competitive equilibrium." And in a footnote to this passage: "... it must violate another condition, which is clearly that of Collective Rationality. This violation is the precisely wellknown intersection of community indifference curves." (K.J. Arrow, "Social Choice and Individual Values", 2nd edition, page 110).

[redacted]. The intersection of community indifference-curves implies that it is not possible to find a social preference pattern, which can be represented by indifference-curves, but it does not imply that we cannot find a social preference pattern at all. In a barter economy there is no use for community indifference curves, because there is no production and hence the amount of all commodities in the society is fixed. Hence the society cannot choose between different commodity vectors and therefore indifference curves are not needed. But even if we include production, we can still define society's preferences in the way we have done for a barter economy and those preferences will be transitive although they cannot be represented by indifference curves.

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1) The violation is an implication of the violation of condition 1 and probably should not be regarded as a violation of condition 4.

APPENDIX: THE "ORDINAL DIFFERENCE APPROACH"

In this appendix an approach is suggested, which tries to find a social ordering of the alternatives, which may be regarded as an "optimal compromise", if only ordinal utilities are allowed for. It is only a rough idea what is given in this appendix, and not a well worked out welfare function.

The idea is very simple: define a "difference" between any possible social preference pattern r and any possible individual preference pattern r^i, where this difference depends only on ordinal properties of the preferences. Hence we call it "ordinal difference" and write $d_i(r,r^i)$ with $r,r^i \in R$. It must be defined for every pair $(r,r^i) \in R^2$, where R is the set of all possible preference orderings of the alternatives. The next step is to define a function $f(d(r,r^i))$, where $d(r,r^i)$ is the vector $(d_1(r,r^1), d_2(r,r^2),\ldots, d_n(r,r^n))$, and find that $r \in R$, which minimizes $f(d(r,r^i))$. Hence we are looking for an r', which minimizes a function, e.g. the sum of the differences between the social and the individual preferences. That r' will be the social preference pattern.

Of course we can write $g(r,r^1,\ldots,r^n)$ for the function f or in a short notation $g(r, r^i)$ and the problem is to find an r' such that $g(r',r_i) \leqq g(r,r_i)$ for all $r \in R$. This shows already one of the main troubles: the minimum will in general not be unique. If there are only two such minima, the society can vote for one, but if there are more than two, all voting problems come in again.

Two procedures seem most obvious. First one may think of taking the minimizing r-s, call them solutions, as alternatives and start the whole thing again, minimizing now a function g_1 and iterating the process until one gets a unique solution or at most a pair of solutions, assuming that the process does converge. But even if it does, this procedure does not work, because we want only to take out one of the solutions and not to get a social preference ordering of all solutions. Using the "ordinal difference approach" would not give a good result, because then the way individuals order the r-s, which are not taken influence the outcome. But we only want the "best" r, not a quasi-ordering of all r's.

The second procedure, which seems to be obvious, is to choose one of the solutions by a random-process, because they are all "equally good" from the society's point of view. The trouble in this case is, that instead of a welfare function we have only a stochastic welfare correspondence. Also many people do not like random processes as decision rules and regard such behavior as irrational. Hence before using a random process one should try to find some other criteria and only if they are not available, draw a lottery. Such other criteria could be for example the Condorcet-criterion, relative majority or maximum amount of other solutions being in minority against the one finally taken. In all those cases the solutions are looked at as alternatives. The Condorcet-criterion, which says, take that alternative, which has the majority against all other alternatives, is intuitively the most acceptable one, but there may be no alternative fulfilling it. This is e.g. the case in the voting paradoxon.

Let us now disregard this problem and inquire our welfare correspondence with regard to Arrow's conditions. Clearly condition 1 is violated, if we use a random process. Since we have a general interdependence, condition 3 is violated. In the following sufficient conditions are given, that Arrow's conditions except condition 3 are fulfilled, if there is a unique solution.

1) $d(r,r^i)$ is a real number greater or equal to zero.

2) If r' and r^i are identical, then $d(r', r^i) = \min_r d(r,r^i)$. It is convenient to take this minimum equal to zero.

3) Let r^i and $\bar{r}^i$ be identical for all pairs of alternatives other than x.but different for at least one individual in at least one pair involving x. For every pair (x,y) $x \succsim y$ respectively $x \succ y$ in r^i shall imply $x \succsim y$ respectively $x \succ y$ in $\bar{r}^i$. If there exists a minimizing r for $g(r,r_i)$ with $x \succ y$, denote it by r', then there must exist an $\bar{r}'$, ranking $x \succ y$, such that $g(\bar{r},\bar{r}_i) < g(\tilde{r},r_i)$, where $\tilde{r}$ is any social preference profile, which ranks $y \succsim x$.

4) There is no such j that $g(r,r_i)$ is the minimum, if r is identical to r^j for all possible $(r^1,\ldots r^n)$.

5) $f(d(r,r^i)) > f(d(\bar{r},\bar{r}^i))$ if $d(r,r^i) \geq d(\bar{r},\bar{r}^i)$, hence the function f must have a higher value, if the ordinal differences are greater or equal for all individuals and strictly greater for at least one.

<u>Lemma:</u> If the function g has a unique minimum in r for all possible$(r^1,\ldots,r^n)$ and if 1) to 5) hold, a social welfare function, which assignes to each $(r^1,\ldots,r^n)$ the r, which minimizes $g(r,r_i)$, fulfills Arrow's conditions 1,2,4 and 5.

Proof:

(i) Condition 1: r is a preference ordering of all alternatives and it is unique by precondition. Hence condition 1 is fulfilled.

(ii) Condition 2: its fulfillment is assured by (3).

(iii) Condition 4: 2) and 5) show, that in the case of unanimity the social preference pattern is identical to the individual ones. Hence each $r \in R$ can be reached and condition 4 is fulfilled.

(iv) Condition 5: dictatorship is excluded by (4).

Lemma: If there are more than one minimizing r-s, then every rule, which takes out one, e.g. a random mechanism, fulfills still condition 2, if (3) holds.

Proof: Since in (3) for the g's the strict inequality holds, all minimizing $\bar{r}'$ order $x \succ y$. Hence the lemma is true.

In this appendix it is not shown, that an ordinal difference d and a function g exist, which fulfill (1) to (5), nor it is shown, that it does not exist. The crucial condition is number (3), which corresponds to Arrow's condition 2.

It is interesting that the ordinal difference between r and r^i can be different from the difference between $\bar{r}$ and $\bar{r}^i$, where r is identical with $\bar{r}^i$ and $\bar{r}$ with r^i. Take for example a set of two alternatives, a and b. It is reasonable, that the difference is zero, if i orders a b and the society a b. But it is also reasonable, that the difference is greater than zero, if i orders $a \succsim_i b$ and the society $a \sim b$. [1)]

For the following example it is easy to verify that (1), (2), (4) and (5) are fulfilled, but it is doubtful, if (3) is fulfilled. Nevertheless the example will perhaps bring out the idea of the ordinal difference approach more clearly.

Define a real number δ_k for the k-th pair of alternatives, which is ordered $x \succsim y$ in R:

r	r^i	δ_k
$x \succ y$	$x \succsim_i y$	$k_1 = 0$
$x \sim y$	$x \sim_i y$	$k_1 = 0$
$x \sim y$	$y \succ_i x$	$k_2 > k_1$
$x \sim y$	$x \succ_i y$	$k_2 > k_1$
$x \succ y$	$y \succ_i x$	$k_3 > k_2$

In our example we will take the numbers 0,1,2 and 0,2,3 and 0,1,3 for k_1, k_2, k_3 respectively. We will give an argument, why k_2 should equal $k_3/2$, but let us first define d and g.

Give each individual i a weight for the k-th pair of alternatives: w_{ik}, for which the following must hold: All w_{ik} are non-negative and there does not exist a j such that $w_{ik} > \sum_{i \neq j} w_{ik} \; \forall k$.

Let $d_i = d_i(r,r^i) = \sum_{k=1}^{m} w_{ik}\delta_k$ and let g be the sum of the d's

1) Otherwise the following could happen: If e.g. all individuals but one are indifferent between x and y and one prefers x to y the society could be indifferent between x and y.

$$g(r,r^1,\ldots,r^n) = \sum_{i=1}^{n} \left(\sum_{k=1}^{m} w_{ik}\delta_k \right) = \sum_{i=1}^{n} d_i(r,r^i)$$

If the weights are the same for all individuals $w_{ik} = w_k$, we have equality, if the weights are the same for each pair of alternatives $(w_{ik} = w_i)$, we have neutrality (the labelling of the alternatives has no influence on the social preference pattern). If all $w_{ik} = w$, there is equality and neutrality in the society.

It seems to be most reasonable for k_2 to take it such that $k_2 = k_3 - k_2$, which gives $k_2 = k_3/2$. Since the society is indifferent in the case k_2 enters the function, it may in some cases use a random mechanism with probabilities 1/2 for its decisions. If k_2 equals $k_3/2$, k_2 equals the expected value of δ_k, which is (in the case of probabilities 1/2) $E = k_3/2+k_1/2 = = k_3/2$. As it is easy to see, adding a constant to all δ_k's or multiplying all δ_k's by a constant does not change the minimizing r. In the example for k_2 also values different from $(k_1+k_3)/2$ are used, but the outcome will be roughly the same in our example (the difference lies only in the nonuniqueness). In general, of course, it will be different for different values of the k's.

Example

In the example are three alternatives a, b, and c, which gives 13 possible preference patterns:

a	a	b	b	c	c	a	b	c	a-b	a-c	b-c	a-b-c
b	c	a	c	a	b	b-c	a-c	a-b	c	b	a	
c	b	c	a	b	a							

k_1, k_2, and k_3 are 0, 1, and 2 respectively:

1	2	3
a	c	b
b	b	a
c	a	c

individual preference patterns

d_1:	0	2	2	4	4	6	1	3	5	1	3	5	3
d_2:	6	4	4	2	2	0	5	3	1	5	3	1	3
d_3:	2	4	0	2	6	4	3	1	5	1	5	3	3
$\sum_{i=1}^{3} d_i$:	8	10	<u>6</u>	8	12	10	9	7	11	7	11	9	9

k_1, k_2, and k_3 are 0, 2, and 3 respectively:

d_1:	0	3	3	6	6	9	2	5	8	2	5	8	6
d_2:	9	6	6	3	3	0	8	5	2	8	5	2	6
d_3:	3	6	0	3	9	6	5	2	8	2	8	5	6
$\sum_{i=1}^{3} d_i$:	12	15	<u>9</u>	12	18	15	15	12	18	12	18	15	18

k_1, k_2, and k_3 are 0, 1, and 3 respectively:

d_1:	0	3	3	6	6	9	1	4	7	1	4	7	3
d_2:	9	6	6	3	3	0	7	4	1	7	4	1	3
d_3:	3	6	0	3	9	6	4	1	7	1	7	4	3
$\sum_{i=1}^{3} d_i$:	12	15	<u>9</u>	12	18	15	12	<u>9</u>	15	<u>9</u>	15	12	<u>9</u>

1	2	3
a	c	b
c	b	a
b	a	c

This is the case, where the voting paradoxon occurs. We are taking values 0, 1, and 2 for k_1, k_2 and k_3 respectively: d_2 and d_3 can be taken from the first part of the example.

d_1:	2	0	4	6	2	4	1	5	3	3	1	5	3
$\sum_{i=1}^{3} d_i$:	10	<u>8</u>	<u>8</u>	10	10	<u>8</u>	9	9	9	9	9	9	9

1	2	3	4
a	c	b	a
b	b	a	c
c	a	c	b

k_1, k_2, and k_3 are again 0, 1, and 2 respectively: the values for the d's can be taken from pagec38. For the subset of individuals 2, 3, and 4 the voting paradoxon occurs.

$\sum_{i=1}^{4} d_i$	<u>10</u>	<u>10</u>	<u>10</u>	14	14	14	<u>10</u>	12	14	<u>10</u>	12	14	10

If we add a fifth individual with preference pattern like individual 1:

$\sum_{i=1}^{5} d_i$	<u>10</u>	12	12	18	18	20	11	15	19	11	15	19	15

The example shows that in the case,where the voting paradoxon occurs, not only the set of solutions is quite big, but also the difference to the values of g for non-solutions is very small. This shows that the society has not enough common tastes for a clear decision. One thing very unsatisfying is, that nearly always one ordinal difference is zero. This would be better, if the sum of the squares of the differences were taken. But then people with "strange" preferences would have more influence than people with "ordinary" ones and probably Arrow's condition 2 would be violated.

Simple majority rule can be looked at as a degenerated case of the ordinal difference approach. Assume n voters and s alternatives. Each voter can designate the alternative he prefers most. Denote with a_j the number of votes for alternative j. Let d_i be a fixed sumber k, if the society chooses another alternative than that most preferred by individual i and zero otherwise. Let g be the sum of all d_i's, hence it is the sum of all k's, which gives $g = k(n - a_j)$, if the alternative j is taken. Since k and n are fixed numbers g is minimized, if a_j is maximized, hence g is minimized, if the alternative with the maximum amount of votes is chosen. But this is the simple majority rule. Obviously in this case the ordinal difference approach is degenerated, because only the first element of the preference pattern influences d_i and not the whole preference pattern.

Finally, I may point out that in the ordinal difference approach no cardinal utility and no interpersonal comparisons of utilities are involved, because $d(r, r^i)$ depends only on differences in the ordinal properties of the preferences and must not change, if those ordinal properties do not change. A change only in cardinal properties has no influence on $d(r, r^i)$.

The aim of this appendix was only to put the idea of a compromise between individual preferences on more formal grounds. It would need much more and much deeper work in order to show, if this approach can give good results or not.