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Perfect Quasi-Perfect Equilibrium*

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Abstract

In strategic-form games Selten's (1975) perfect equilibria are admissible. This is not true for extensive-form perfection. Quasi-perfect equilibria solves this problem using Selten's (1975) trembles to introduce a refinement of Nash equilibrium wherein each player puts infinitesimal weight on *other players's* strategies, but not her own. One might be sure of oneself, while (infinitesimally) unsure of others. However, also quasi-perfection itself is not without problems, precisely because it ignores future infinitesimal uncertainties in one's own play. We introduce a refinement; perfect quasi-perfect equilibrium, to capture the best of both concepts. Our idea is to force each player to consider infinitesimal deviations in her own future play, but to make them so unlikely that they are infinitely less likely than the combined likelihood of deviations by all other players. Our refinement uses only strategies that are neither weakly dominated in the strategic form nor in the agent normal form.

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1 Introduction

Perfect equilibrium was the first of many refinements based on "trembles", that is, perturbations of play, which attempt to cope with the problem of badly-behaved Nash equilibria. The idea of perfection in strategic-form games[1] is to put "infinitesimal weight" on all strategies so as to rule out such pathologies as inadmissable equilibria. Selten (1975) accomplished this for strategic form games by requiring each played strategy in a Nash equilibrium to be not just a best response to the equilibrium play of others, but also to a converging sequence of completely mixed nearby strategy profiles.

The task of defining away badly-behaved equilibria, difficult enough in strategic-form games, becomes even more so in extensive form games. In strategic-form games, that is, simultaneous-move games in the normal form, perfect equilibria are admissible, that is, weakly undominated. This is not true for extensive-form perfection, that is, perfection in the agent-normal form. Jean-Francois Mertens (1995, p. 379) made the following charming observation:

> There seems to be a prevalent opinion in the literature that extensive form perfect equilibria are probably preferable to sequential equilibria, by avoiding dominated choice. Even, e.g., Harsanyi and Selten (1988) write (p. 344): 'Moreover, as Kohlberg and Mertens (1982 [1986], pp. 4-5) have pointed out, unlike perfect equilibria, sequential equilibria often have the undesirable property of using dominated strategies': four different authors are involved, who should have known better (I could not believe it myself when I saw it printed).

In the simple game of Figure 1, the strategy for Player I of handing the move off to player II is part of an extensive-form (agent-normal form) perfect equilibrium, but it is nonetheless weakly dominated by player I's keeping the move for herself and then choosing the high-return payoff. In this equilibrium player I is more certain of player II's play than she is about her own future play. Mertens

[1] In memory of Lloyd Shapley

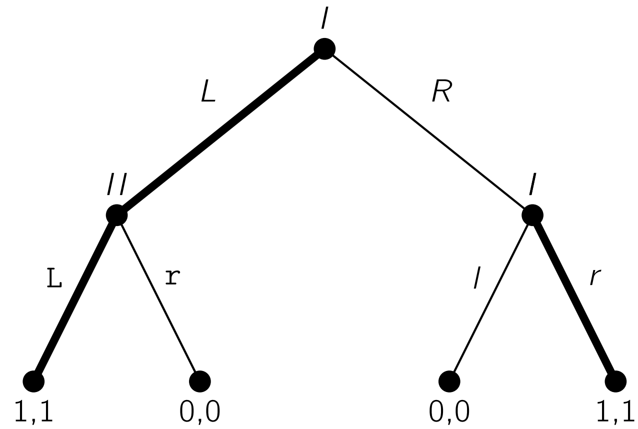

Figure 1: A perfect equilibrium which is not quasi-perfect.

concludes that van Damme's (1984) quasi-perfect equilibrium, a refinement of sequential equilibrium which is also normal-form perfect, may be a better concept. "..., by some irony of terminology, the 'quasi' -concept seems in fact far superior to the original unqualified perfection itself." (Mertens (1995), p. 380.)

Quasi-perfect equilibria uses Selten's (1975) trembles to introduce a refinement of Nash equilibrium wherein each player puts infinitesimal weight on *other players's* strategies, but not her own. One might be sure of oneself while (infinitesimally) unsure of others. Despite Mertens approbation, quasi-perfection is not itself without problems, precisely because it ignores future infinitesimal uncertainties in one's own play. Unlike van Damme (1984), we find the play described in Figure 2 (his Example 3) odd. After all, in world where mistakes are not completely impossible, why should the player risk to make a mistake in the future instead of insuring her most preferred payoff immediately?

A middle ground would be to identify a refinement that requires "perfection" in the agent-normal form and yet rules out the behavior described in Figure 1. We introduce such a refinement here — *perfect quasi-perfect equilibrium (pqpe).* Our idea is to force each player to consider infinitesimal deviations in his/her own future play, but to make them so unlikely that they are infinitely less likely than the combined likelihood of deviations by all other

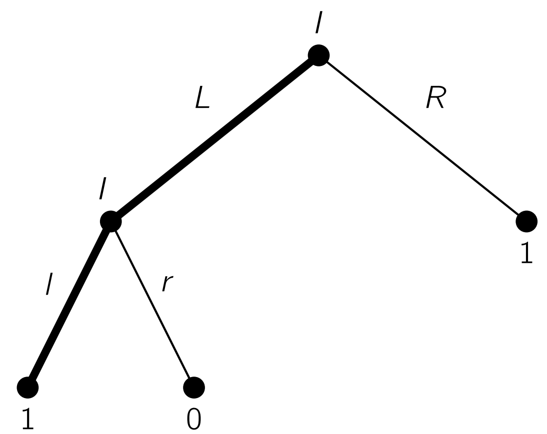

Figure 2: A strange quasi-perfect equilibrium.

players. This idea rules out trembles common to all players, since player 2 must believe that trembles on his own future play are arbitrarily smaller than are player 1's trembles, while player 1 must believe just the reverse. That is, with respect to player 1's future play, (player 1) must believe these deviations are infinitely less likely than do the other players in the game.

Perfect quasi-perfect equilibrium will be seen to refine quasi-perfect equilibrium. However, the converse will not be true; that is, there are quasi-perfect equilibria that are not perfect quasi-perfect. For example the equilibrium highlighted in Figure 2 is quasi-perfect, but not perfect quasi-perfect.

One virtue of this intuitive equilibrium refinement is that *pqpe* are un-weakly-dominated in both the agent-normal form and in the strategic form. As far as we know, this is the only equilibrium concept found so far that has both of these properties. Quasi-perfect equilibria can be weakly dominated in the agent-normal form, e.g. the highlighted equilibrium in the game of Figure 2. On the other hand, extensive-form perfect equilibria may be weakly dominated in the strategic form, as one can see in Figure 1.

Perfect equilibrium and quasi-perfect equilibrium seem to be very close. Yet van Damme (1984) points out that neither includes the other. This distinction is highlighted by the game of Figure 3, in which the set of extensive-form perfect equilibrium strategy profiles is entirely disjoint from the set of quasi-perfect strategy profiles. No strategy profile which is extensive-form perfect can be quasi-perfect, and vice versa. Thus, *pqpe* cannot refine perfect equilibrium. In fact, there cannot be a common refinement of both perfect and quasi-perfect equilibrium.[2]

2 Definitions

We use the notation of Blume and Zame (1994) except that we refer throughout to individuals as $i \in \mathcal{I}$, the set of choices at information set h is denoted C_h and the set of local strategies at h we denote by B_h. We will make use of the agent-normal form, where agents are labeled by the information sets they control; agent h chooses at information set h for player i, such that $h \in H_i$. Her strategies are probability distributions over the choices in C_h, that is, elements of B_h. By $i(h)$ we denote the player to which agent h belongs, that is $h \in H_{i(h)}$. In addition, we use the following additional notation:

- We use b's for behavior and local strategies. We let B_i denote the set of player i's behavior strategies and $B = \prod_{i \in \mathcal{I}} B_i$; $B_i = \prod_{h \in H_i} B_h$; $b^i = (b^h)_{h \in H_i}$
- $\succeq_i$: a transitive and reflexive relation on H_i, for all $i \in I$, such that $h \succeq_i h'$ iff $h, h' \in H_i$ and h comes after some node in h' in the game tree, that is, if there is a path from the root to h that crosses h'.
- We write $h \succ_i h'$ iff $h \succeq_i h'$ and $h \neq h'$.
- We define $S(h) = \{h' \in H_{i(h)} | h' \succ_{i(h)} h\}$ and call it the set of successors of h.
- We define $W(h) = \{h' \in H_{i(h)} | h' \succeq_{i(h)} h\}$ and call it the set of weak successors of h. Thus $S(h) \cup \{h\} = W(h)$.

[2] Another example can be found in Mertens (1995).

3 Existence of PQPE

3.1 Trembles

The definition of *pqpe* requires an elaboration of trembles that allow for two different players to disagree on the trembles assigned to each others' equilibrium choices. A strategy profile $b = (b^h)_{h\in H} = (b^i)_{i\in\mathcal{I}}$ together with the fixed probabilities for the moves of nature induces a probability distribution $p(b)$ over terminal nodes. For each player, $U_i(b) = \mathrm{Exp}_{p(b)}[u_i] = \sum_{z\in Z} u_i(z)p(b)(z)$ is the expected utility if the behavior strategy profile b is played. Let $Z(h)$ denote the set of terminal nodes that follow information set $h \in H_i$. If $p(b)(Z(h)) > 0$, then the expected continuation payoff for player i from h on if the profile b is played is

$$U_{ih}(b) = \sum_{z\in Z(h)} u_i(z)\frac{p(b)(z)}{p(b)(Z(h))}.$$

Note that in this case, perfect recall implies that the conditional distribution $p(b)(z)/p(b)(Z(h))$ for $z \in Z(h)$ does not depend on $(b^{\hat{h}})_{\hat{h}\in H_i\setminus W(h)}$, as long as b^i does not avoid h (which would imply $p(b)(Z(h)) = 0$).

Definition 1. *A* perturbation *is a function* $\eta : C \to]0,1[$ *such that*

$$\eta^h := \sum_{c\in C_h} \eta(c) < 1.$$

Here, we let $C := \bigcup_{h\in H} C_h$.

Notation 1. *Let* $b^h \in B_h$ *and* η *be a perturbation. Denote by* b^h_η *the following distribution:*

$$b^h_\eta(c) := (1-\eta^h)b^h(c) + \eta(c), \text{ for } c \in C_h.$$

Definition 2. *Let* η *and* ξ *be two perturbations. The mixed strategy profile* $\left(b^h\right)_{h\in H}$ *is an* (η,ξ)-perfect equilibrium *of the agent normal form, if for each* $h \in H$*, we have that* b^h *is a best reply to* $\left((b^{h'}_\eta)_{h'\in H_{-i(h)}}, (b^{h''}_\xi)_{h''\in H_{i(h)}}\right)$.

An (η,ξ)-perfect equilibrium is an equilibrium of the agent normal form such that each agent adds in her mind η-trembles to the strategies of agents of other players', while adding trembles of size ξ for agents that belong to the same player as herself.

Obviously, extensive-form perfect equilibria are precisely those that are the limit as $\eta \to 0$ of (η,η)-perfect equilibria.

Lemma 1. *For all pairs of perturbations (η,ξ) an (η,ξ)-perfect equilibrium exists.*

To prove this lemma we define an auxiliary game $G(\eta,\xi)$ for a pair (η,ξ) of perturbations as follows: The players are the players of the agent normal form of the given original extensive form game, the pure action set of agent h is the set C_h of pure choices at the information set h, but with payoffs constructed from perturbed-strategy beliefs. For pure local strategy profile $(c_{\hat{h}})_{\hat{h}\in H}$, agent h receives the payoff corresponding to the following behavioral strategy profile in the original extensive form game: Agents $\hat{h} \notin H_{i(h)}$ play $c_{\hat{h}}$ with probability $1-\eta^{\hat{h}}+\eta(c_{\hat{h}})$, and choice $a_{\hat{h}} \in C_{\hat{h}} \setminus \{c_{\hat{h}}\}$ with probability $\eta(a_{\hat{h}})$. Agents $k \in H_{i(h)}$ play c_k with probability $1-\xi^k+\xi(c_k)$ and choice $a_k \in C_k \setminus \{c_k\}$ with probability $\xi(a_k)$, while agent h herself plays c_h with probability 1.

The expected payoff of agent h in the mixed extension of $G(\eta,\xi)$ for mixed strategy profile $(\sigma^{\hat{h}})_{\hat{h}\in H}$ is the same as the expected payoff to agent h from the following profile of strategies in the agent normal form of the original game: agents $\hat{h} \notin H_{i(h)}$ play $a_{\hat{h}} \in A_{\hat{h}}$ with probability

$$\sigma^{\hat{h}}_{\eta}(a_{\hat{h}}) := (1-\eta^{\hat{h}})\sigma^{\hat{h}}(a_{\hat{h}}) + \eta(a_{\hat{h}}), \tag{1}$$

and agents $k \in H_{i(h)}$ play $a_k \in A_k$ with probability

$$\sigma^{\hat{h}}_{\xi}(a_k) := (1-\xi^k)\sigma^k(a_k) + \xi(a_k), \tag{2}$$

while agent h herself plays $a_h \in A_h$ with probability $\sigma^h(a_h)$.

In other words, if the profile $(b^{h'})_{h'\in H}$ is played in $G(\eta,\xi)$, agent h receives the same payoff as if the behavior profile $\left((b^{h'}_{\eta})_{h'\in H_{-i(h)}}, (b^{h''}_{\xi})_{h''\in H_{i(h)}\setminus\{h\}}, b^h\right)$ is played in the original game.

Proof of Lemma 1. Each $G(\eta, \xi)$ as defined above is a normal form game with finitely many players and finitely many actions, and hence has some Nash equilibrium. $\square$

3.2 Quasi-best replies

Given a behavior strategy profile $b = (b^j)_{j \in I}$, some information set $h \in H_i$ and a behavior strategy $\hat{b}^i$ of player i, we denote by $b/_h\hat{b}^i$ the behavior strategy profile, where players different from i play the same as in the profile b, i plays according to $\hat{b}^i$ on information sets $h' \in W(h)$ (that is, from h onwards), and according to b^i on all other of her information sets (that is, sets $h" \in H_i \setminus W(h)$).

Definition 3. *Let b be a completely mixed strategy profile and $h \in H_i$. Then, we call $\bar{b}^h \in B_h$ a* quasi-best reply *to b at h, if there are $\hat{b}^i = (\hat{b}^{h'})_{h' \in H_i} \in B_i$, with $\hat{b}^h = \bar{b}^h$ such that $U_{ih}(b/_h\hat{b}^i) = \max_{\tilde{b}^i \in B_i} U_{ih}(b/_h\tilde{b}^i)$.*

A useful lemma of van Damme (1984) points out that exactly those local strategies that put weight only on pure quasi-best replies to a strategy profile b are themselves quasi-best replies to b.

3.3 Existence of PQPE

Now we begin the discussion of *pqpe* by introducing several perturbed equilibrium concepts.

Definition 4. *Let η be a perturbation. A behavioral strategy profile b is an* η-quasi-perfect equilibrium *if for each $h \in H$, b^h is a quasi-best reply to the perturbed strategy profile b_η.*

Now we proceed to take limits of perturbations. Fix a perturbation η. To express the idea that the self-perturbations are infinitesimal relative to other-pertubations, we first take ξ to 0.

Definition 5. *A* perfected η-quasi-perfect equilibrium *is a limit as* ξ *approaches* 0 *of* (η, ξ)*-perfect equilibria. More precisely,* b *is a perfected* η*-quasi-perfect equilibrium if there are a sequence of perturbations* $\xi_n \to 0$ *and a sequence of* (η, ξ_n)*-perfect equilibria* b_n*, such that* $b_n \to b$*.*

Our existence theorem is based on the following two results, which are interesting in their own right. The proof of Theorem 2 is in the appendix.

Theorem 1. *For every finite extensive form game with perfect recall and every perturbation* η *a perfected* η*-quasi-perfect equilibrium exists.*

Proof of Theorem 1. Choose a sequence of perturbations $\xi_n \to 0$ and for each n an equilibrium b_n of $G(\eta, \xi_n)$. Since the strategy profiles of those games are elements of a fixed compact space, there is a convergent subsequence $(b_{n_m})_{m\in\mathbb{N}}$ converging, for $m \to \infty$ to a strategy profile b. By abuse of notation, we let $(b_m)_{m\in\mathbb{N}}$ denote now this subsequence converging to b. □

Theorem 2. *For every finite extensive form game with perfect recall and every perturbation* η*, every perfected* η*-quasi-perfect equilibrium is an* η*-quasi-perfect equilibrium.*

Now comes the definition of a *pqpe*.

Definition 6. *A behavior strategy profile* b *is a* perfect quasi-perfect equilibrium *if there are sequences of perturbations* $\eta_n \to 0$ *and perfected* η_n*-quasi-perfect equilibria* b_n*, such that* $b_n \to b$*.*

Simply put, a perfect quasi-perfect equilibrium is a limit as η approaches 0 of perfected η-quasi-perfect equilibria. The main existence result is:

Theorem 3. *For every finite extensive form game with perfect recall a perfect quasi-perfect equilibrium exists.*

Proof. Choose a sequence of perturbations $\eta_n \to 0$ and for each n a perfected η_n-quasi-perfect equilibrium b_n of the game. Since the strategy profiles of those games are elements of a fixed compact space, there is a convergent subsequence of $(b_{n_m})_{m\in\mathbb{N}}$ converging to a strategy profile b. The existence of this subsequence proves the theorem. □

4 PQPE Among Its Friends

Recall the following definition from van Damme (1984):

Definition 7. *Let $\epsilon > 0$. A completely mixed strategy profile b is an* ϵ-quasi-perfect equilibrium *if for each $h \in H$, if $c \in C_h$ is not a quasi-best response to b, then $b^h(c) \leq \epsilon$.*

If b is an η-quasi-perfect equilibrium, then b_η is an ϵ-quasi-perfect equilibrium for any $\epsilon \geq \max_{c \in C} \eta(c)$. In the other direction, given some ϵ-quasi-perfect equilibrium $\hat{b}$, one can find a perturbation η and a strategy profile b that is an η-quasi-perfect equilibrium such that $b_\eta = \hat{b}$, with $\epsilon \geq \max_{c \in C} \eta(c)$.

Perfect quasi-perfect equilibrium is a refinement of quasi-perfect equilibrium.

Theorem 4. *For every finite extensive form game with perfect recall, every* pqpe *is a quasi-perfect equilibrium.*

The converse is not true. In Figure 2, Ll is quasi-perfect but not perfect quasi-perfect. The only *pqpe* is Rl.

Proof. Let b be a perfect quasi-perfect equilibrium. By the definition and by Theorem 2 we have a sequence of η_n-quasi-perfect equilibria b_n, which converge to b, with perturbations η_n that converge to 0. As remarked above, b_n, being an η_n-quasi-perfect equilibrium, implies that b_{n,η_n} is an ϵ_n-quasi-perfect equilibrium for any $\epsilon_n := \max_{c \in C} \eta_n(c)$. As η_n converges to 0, ϵ_n converges to 0 as well. As η_n converges to 0, and b_n converges to b, it follows that b_{n,η_n} converges to b. Hence we found a sequence of ϵ_n-quasi-perfect equilibria converging to b, for a sequence of ϵ_n converging to 0. Van Damme's (1984) Proposition 2 implies that b is a quasi-perfect equilibrium. $\square$

One might hope that *pqpe* refines other equilibrium concepts as well. Obviously it is a refinement of sequential equilibrium, since quasi-perfection refines sequentiality. One might hope that it refines extensive form perfection as well. Indeed, generically it does because for generic games, all sequential equilibria are extensive-form perfect; see Blume and Zame (1994). Strictly speaking, however, it does not. Figure 3 displays an extensive form game wherein the sets of quasi-perfect equilibria and extensive form perfect equilibria are disjoint. Since every *pqpe* is a quasi-perfect equilibrium, no *pqpe* in this game is extensive form perfect.

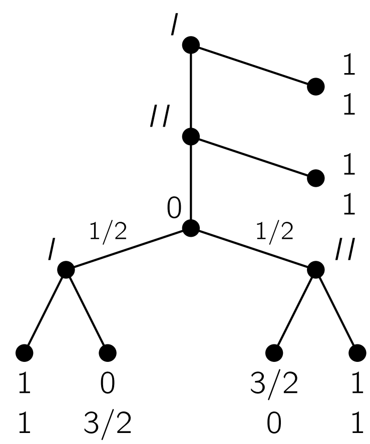

Figure 3: A game with disjoint perfect and quasi-perfect equilibrium sets.

In the game of Figure 3, in *every* extensive-form perfect equilibrium there is at least one player who plays a strategy that is weakly dominated in the strategic form! *pqpe* do not use weakly dominated strategies, so none of these equilibria are extensive-form perfect. In fact, in the games of Figures 1 and 2 there has been no refinement of Nash equilibrium in the extensive form that does the obvious correct thing in both games, that is, choose Rr in Figure 1 and Rl in Figure 2. Perfect quasi-perfection does just this. Perfected η-quasi-perfection protects an agent from choosing strategies weakly dominated because of choices available to her own future agents. The limit of η-perturbations protects that agent against the perturbations of the agents of other players. Thus *pqpe* are un-weakly dominated in the agent-normal form. Weak domination does not occur in the strategic form because quasi-perfect equilibria are normal-form perfect, and Theorem 4 states that *pqpe* are quasi-perfect. We have the following theorem:

Theorem 5. *Every perfect quasi-perfect equilibrium is in strategies that are neither weakly dominated in the normal form nor weakly dominated in the agent normal form.*

The aforementioned relationships are summarized in the following table:

	Plays Strategies Dominated in the: Strategic Form	Agent-Normal Form
Ext. Form Perfect	Figure 1	XXX
Quasi-perfect	XXX	Figure 2
PQP	XXX	XXX

Finally, we note that, following Blume and Zame (1994), Hillas, Kao and Schiff (2017), and Pimienta and Shen (2013) have shown that for generic payoffs in any tree, the sets of quasi-perfect-, perfect-, and sequential equilibrium strategy profiles are identical. Since *pqpe* are quasi-perfect, it follows that for generic payoffs in any tree the set of *pqpe* strategy profiles is contained in the set of extensive-form perfect profiles.

5 Conclusion

In this paper we introduce a new extensive-form equilibrium refinement, perfect quasi-perfect equilibrium. Our purpose in doing this is not to change the practice of applied game theory, but to provide some additional insight into the nature of equilibrium refinements. Extensive-for perfect and quasi-perfect equilibrium each allow for some kind of inadmissibility: in the stratgic form for extensive perfection and in the agent-normal form for quasi-perfection. Perfect quasi-perfection guarantees admissibility in both game representations. From existing examples one might have conjectured the existence of games with no quasi-perfect equilibrium admissible in the agent-normal form. The existence of *pqpe* shows this not to be the case. The refinement "quasi-perfect and admissible" is coherent.

The new idea in perfect quasi-perfect equilibrium is that each player considers deviations by her future selves to be in a different class, infinitely less likely, than deviations by other players. This requires a more complicated treatment of trembles, since how each player thinks about herself and how others do is different, but it also enriches the range of modeling possibilities. For instance, we can imagine games in which some players share an identity, a common understanding of the world, and consequently those in the group view their groupmates differently than those out of the group, in exactly this way. Examples include models in which ethnic identification is important, and games in which groups of players are agents for some larger organization. We leave the exploration of this idea to future research.

Appendix

Proof of Theorem 2. Assume by contradiction that the statement of the theorem is wrong for the perfected η-quasi-perfect equilibrium b. Then, there is an agent h of some player i such that this agent does not play a quasi-best response to b_η, but each later moving agent $h' \in S(h)$ of player i does play a quasi-best response. Then, in particular, $(b^{h'})_{h' \in W(h)}$ is not a best reply to b_η for player i from information set h onwards. So, there is a $(\hat{b}^{h'})_{h' \in W(h)}$ such that

$$U_{ih}\left((b_\eta^{h''})_{h'' \in H \setminus W(h)}, (\hat{b}^{h'})_{h' \in W(h)}\right) > U_{ih}\left((b_\eta^{h''})_{h'' \in H \setminus W(h)}, (b^{h'})_{h' \in W(h)}\right).$$

Since we have assumed that local strategies of agents of player i that move after agent h do constitute quasi-best replies to b_η, Lemma 2 of van Damme 1984 together with the optimality principle does imply that replacing $(\hat{b}^{h'})_{h' \in S(h)}$ by $(b^{h'})_{h' \in S(h)}$, while keeping everything else fixed does not decrease the expected payoff for player i from h onwards:

$$U_{ih}\left((b_\eta^{h''})_{h'' \in H \setminus W(h)}, (b^{h'})_{h' \in S(h)}, \hat{b}^{h}\right) \geq U_{ih}\left((b_\eta^{h''})_{h'' \in H \setminus W(h)}, (\hat{b}^{h'})_{h' \in W(h)}\right).$$

This, together with the preceding inequality gives:

$$U_{ih}\left((b_\eta^{h''})_{h'' \in H \setminus W(h)}, (b^{h'})_{h' \in S(h)}, \hat{b}^{h}\right) > U_{ih}\left((b_\eta^{h''})_{h'' \in H \setminus W(h)}, (b^{h'})_{h' \in W(h)}\right).$$

Since b is a perfected η-quasi-perfect equilibrium, there are sequences of perturbations $(\xi_n) \to 0$ and (η, ξ_n)-perfect equilibria b_n, such that $b_n \to b$.

Past actions of player i do not matter for computing U_{ih} as long as they reach h with positive probability, and so we have as well, for every $m \in \mathbb{N}$:

$$U_{ih}\left((b_{\eta}^{h"})_{h" \in H_{-i}}, (b_{\xi_m}^{\overline{h}})_{\overline{h} \in H_i \setminus W(h)}, (b^{h'})_{h' \in S(h)}, \hat{b}^h\right)$$
$$> U_{ih}\left((b_{\eta}^{h"})_{h" \in H_{-i}}, (b_{\xi_m}^{\overline{h}})_{\overline{h} \in H_i \setminus W(h)}, (b^{h'})_{h' \in W(h)}\right).$$

For any agent $h' \in H$, $b_n^{h'} \to b^{h'}$, which implies

1. $b_{n,\eta}^{h"} \to b_{\eta}^{h"}$, for $h" \in H_{-i}$,
2. $b_{n,\xi_m}^{\overline{h}} \to b_{\xi_m}^{\overline{h}}$, for $\overline{h} \in H_i \setminus W(h)$ and every $m \in \mathbb{N}$,

Since $(\xi_n) \to 0$, we also have

3. $b_{n,\xi_n}^{h'} \to b^{h'}$, for $h' \in W(h)$, and
4. $b_n^h \to b^h$.

Therefore, for any $m \in \mathbb{N}$:

$$\left((b_{n,\eta}^{h"})_{h" \in H_{-i}}, (b_{n,\xi_m}^{\overline{h}})_{\overline{h} \in H_i \setminus W(h)}, (b_{n,\xi_n}^{h'})_{h' \in S(h)}, \hat{b}^h\right)$$
$$\longrightarrow \left((b_{\eta}^{h"})_{h" \in H_{-i}}, (b_{\xi_m}^{\overline{h}})_{\overline{h} \in H_i \setminus W(h)}, (b^{h'})_{h' \in S(h)}, \hat{b}^h\right),$$

and

$$\left((b_{n,\eta}^{h"})_{h" \in H_{-i}}, (b_{n,\xi_m}^{\overline{h}})_{\overline{h} \in H_i \setminus W(h)}, (b_{n,\xi_n}^{h'})_{h' \in S(h)}, b_n^h\right)$$
$$\longrightarrow \left((b_{\eta}^{h"})_{h" \in H_{-i}}, (b_{\xi_m}^{\overline{h}})_{\overline{h} \in H_i \setminus W(h)}, (b^{h'})_{h' \in S(h)}, b^h\right).$$

Hence, for n large enough, we have

$$U_{ih}\left((b^{h''}_{n,\eta})_{h''\in H_{-i}}, (b^{\overline{h}}_{n,\xi_m})_{\overline{h}\in H_i\setminus W(h)}, (b^{h'}_{n,\xi_n})_{h'\in S(h)}, \hat{b}^h\right)$$
$$> U_{ih}\left((b^{h''}_{n,\eta})_{h''\in H_{-i}}, (b^{\overline{h}}_{n,\xi_m})_{\overline{h}\in H_i\setminus W(h)}, (b^{h'}_{n,\xi_n})_{h'\in S(h)}, b^h_n\right).$$

Again, since past actions do not matter for computing U_{ih} as long as they are full support we have

$$U_{ih}\left((b^{h''}_{n,\eta})_{h''\in H_{-i}}, (b^{\overline{h}}_{n,\xi_m})_{\overline{h}\in H_i\setminus W(h)}, (b^{h'}_{n,\xi_n})_{h'\in S(h)}, \hat{b}^h\right)$$
$$= U_{ih}\left((b^{h''}_{n,\eta})_{h''\in H_{-i}}, (b^{\overline{h}}_{n,\xi_n})_{\overline{h}\in H_i\setminus W(h)}, (b^{h'}_{n,\xi_n})_{h'\in S(h)}, \hat{b}^h\right)$$

and

$$U_{ih}\left((b^{h''}_{n,\eta})_{h''\in H_{-i}}, (b^{\overline{h}}_{n,\xi_m})_{\overline{h}\in H_i\setminus W(h)}, (b^{h'}_{n,\xi_n})_{h'\in S(h)}, b^h_n\right)$$
$$= U_{ih}\left((b^{h''}_{n,\eta})_{h''\in H_{-i}}, (b^{\overline{h}}_{n,\xi_n})_{\overline{h}\in H_i\setminus W(h)}, (b^{h'}_{n,\xi_n})_{h'\in S(h)}, b^h_n\right).$$

Therefore, we get

$$U_{ih}\left((b^{h''}_{n,\eta})_{h''\in H_{-i}}, (b^{\overline{h}}_{n,\xi_n})_{\overline{h}\in H_i\setminus W(h)}, (b^{h'}_{n,\xi_n})_{h'\in S(h)}, \hat{b}^h\right)$$
$$> U_{ih}\left((b^{h''}_{n,\eta})_{h''\in H_{-i}}, (b^{\overline{h}}_{n,\xi_n})_{\overline{h}\in H_i\setminus W(h)}, (b^{h'}_{n,\xi_n})_{h'\in S(h)}, b^h_n\right).$$

This contradicts the fact that b_n is an (η, ξ_n)-perfect equilibrium, as agent h by playing b^h_n, is not best replying. □

Proof of Theorem 5. As remarked in van Damme (1984, p. 9), a quasi-perfect equilibrium of an extensive form game constitutes a perfect equilibrium of the corresponding normal form game. As is well-known, perfect equilibria in a normal form game are in strategies that are not weakly dominated in that normal form game. Hence Theorem 4 shows that perfect quasi-perfect equilibria are in strategies that are not weakly dominated in the corresponding normal form.

Let b be a perfect quasi-perfect equilibrium and $h \in H$. By definition, there is a sequence of perturbations $\eta_n \to 0$ and a sequence of perfected

η_n-quasi-perfect equilibria $b_n \to b$. In particular, $b_n^h \to b^h$. Hence there is some $m \in \mathbb{N}$ such that for all $n \geq m$ and for all $c \in C_h$: if $b^h(c) > 0$, then $b_n^h(c) > 0$. Since b_m is a perfected η_m-quasi-perfect equilibrium, there is a sequence of perturbations ξ_n and a sequence of (η_m, ξ_n)-perfect equilibria $b_{m,n} \to b_m$. Hence there is a $l \in \mathbb{N}$ such that for all $n \geq l$ and for all $c \in C_h$: if $b_m^h(c) > 0$, then $b_{m,n}^h(c) > 0$. Therefore, b_m^h is a best reply to the completely mixed strategy profile $\left((b_{\eta_m}^{h'})_{h' \in H_{-i(h)}}, (b_{\xi_l}^{h''})_{h'' \in H_{i(h)}}\right)$ in the agent normal form. As the support of b^h is contained in the support of b_m^h, b^h is a best reply to the same full support strategy profile in the agent normal form. Hence b^h is not weakly dominated in the agent normal form. $\square$

References

Blume, Lawrence and William Zame. 1994. "The algebraic geometry of perfect and sequential equilibrium." *Econometrica* 62(4): 783–94.

van Damme, Eric. 1984. "A relation between perfect equilibria in extensive form games and proper equilibrium in normal form games." *International Journal of Game Theory* 13(1): 1–13.

Hillas, J., T. Kao and A. Schiff. 2017. "On a theorem of Blume and Zame." unpublished, University of Auckland. https://drive.google.com/open?id=1uFXBTK4ACuMg3aTAp98W-2WQ70K8fc0c

Mertens, Jean-Francois. 1995. "Two examples of strategic equilibrium," *Games and Economic Behavior* 8(2): 378–388.

Pimienta, Carlos and Jian Shen. 2013. "On the equivalence between (quasi-)perfect and sequential equilibria." *International Journal of Game Theory* 43(1): 1–8.

Selten, Reinhard. 1975. "Reexamination of the perfectness concept for equilibrium points in extensive games." *International Journal of Game Theory* 4(1): 25–55.