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The consumption-investment decision of a prospect theory household: A two-period model with an endogenous second period reference level *

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Abstract

In this paper we analyze the two-period consumption-investment decision of a household with prospect theory preferences and an endogenous second period reference level which captures habit persistence in consumption and in the current consumption reference level. In particular, we examine three types of household depending on how the household's current consumption reference level relates to a given threshold which is equal to the average discounted endowment income. The first type of household has a relatively low reference level (less ambitious household) and can avoid relative consumption losses in both periods. The second type of household (balanced household) always consumes exactly its reference levels. The third type of household has a relatively high reference level (more ambitious household) and cannot avoid to incur relative consumption losses, either now or in the future. Note that these households may act very differently from one another and thus there will often be a diversity of behavior. For all three types we examine how the household reacts to changes in: income (e.g., income fall caused by recession or taxation of endowment income), persistence to consumption, the first period reference level and the degree of loss aversion. Among others we find that the household increases its exposure to risky assets in good economic times if it is less ambitious and in bad economic times if it is more ambitious. We also find that in some cases more income can lead to less happiness. In addition, the less ambitious household and the more ambitious household with a higher time preference will be less happy with a rising persistence in consumption while the more ambitious household with a lower time preference will be happier if it sticks more to its consumption habits. Finally, the household will be happiest for the lowest possible current consumption reference level, i.e., not comparing at all will lead to the highest level of happiness.

Keywords: prospect theory, loss aversion, consumption-savings decision, portfolio allocation, happiness, income effects

JEL classification: G02, G11, E20

1 Introduction

One of the most important decisions households face is consumption today versus consumption in the future. Households transfer current consumption into the future by allocating their savings into different types of assets some of which are riskier than others. These decisions are done with the knowledge that the future is risky. The expected utility theory (EUT) has been the cornerstone model for exploring these household decisions. This research deviates from the EUT model and explores, in a two-period model, the behavior of households which are characterized by reference dependent preferences (Kahneman and Tversky, 1979; Tversky and Kahneman, 1992) and by habit persistence (Abel, 1990; Campbell and Cochrane, 1999; Constantinides, 1990; Flavin and Nakagawa, 2008; Pagel, 2017) when deciding on consumption, savings, and the portfolio allocation of savings. We explore the factors that influence a household's consumption, savings and portfolio decisions when the second period reference level is assumed to depend on first period consumption and the first period consumption reference level. Households have been observed to show a habit for consumption that persists into the future, and hence a habit persistence model combined with prospect theory preferences will provide new insights on such important life cycle decisions.

By incorporating prospect theory type of preferences and habit persistence we will be able to address a number of issues on consumption and risk taking behavior that have not been explored in the literature previously. How does a household make intertemporal decisions under these two behavioral traits? Does the optimal solution depend on avoiding relative losses or not? Does the optimal choice depend on whether the household is sufficiently loss averse? Is the choice dependent on the household being less or more ambitious on targets? How do the second period reference level, consumption, risk taking, and happiness change when the first period reference level changes? Do the responses depend on the household's level of ambition? What impact does the habit persistence in consumption have on consumption and portfolio choice? How will a household react to sudden income changes? Do happiness, current consumption and risk taking always increase when income increases? This paper will attempt to shed some light on the above questions.

The first reference levels ever used in economic research were developed by Stone (1954) and Geary (1950). The Stone-Geary utility preferences involve *reference dependent utility on subsistence levels of consumption* and thus subsistence levels can be considered as a special type of reference points. Under such preferences households derive utility from consumption in excess of a subsistence level. Savings and portfolio choices with subsistence consumption have been explored by Achury et al. (2012).[1] They use a Stone-Geary expected utility model to explain the empirical observations that rich people show a higher savings rate, higher holdings

[1]Merton (1969, 1971) used HARA preferences to examine savings and portfolio allocations in an infinite horizon expected utility model. Achury et al. (2012) added subsistence and also habit persistence to Merton's CRRA utility function (a subset of HARA preferences).

of risky assets as a fraction of personal wealth, and face a higher volatility in consumption.

Another model that has been used is *habit persistence*. This model assumes that households derive satisfaction from consumption relative to a reference level which in turn depends on past consumption levels. Thus current consumption affects not only a household's current marginal utility but also its marginal utility in the next period, which may explain why the more a household consumes today the more it will want to consume tomorrow. The macroeconomics and finance literature uses habit persistence models to explain many puzzles, e.g., the equity premium puzzle (Abel, 1990; Constantinides, 1990; Campbell and Cochrane, 1999), excess consumption smoothing (Lettau and Uhlig, 2000) and many business cycle patterns (Boldrin et al., 2001; Christiano et al., 2005).

Reference levels are also used to compare one's own consumption levels to others (Falk and Knell, 2004; Hlouskova, Fortin and Tsigaris, 2017). Many households are influenced by the *self-enhancement* motive while others are determined by the *self-improvement* motive. The self-enhancement motive applies when people want to feel they are better than their peers and set their references at low levels possibly reflecting the wealth of poorer people. Others with a high reference level place importance to the self-improvement motive and compare themselves with the ones who are more successful. Hlouskova, Fortin and Tsigaris (2017) use a two-period life-cycle model with a sufficiently loss averse household to investigate the impact of these psychological traits on consumption, savings, portfolio decisions, as well as on welfare. They find that the optimal solution depends on whether the household's present value of the consumption reference levels is below, equal to, or above the present value of its endowment income. When reference levels are below the endowment income the authors associate this with the self-enhancement motive. Under this motive the household wants to avoid relative losses in consumption in any present or future state of nature (good or bad). Hence the degree of loss aversion does not affect optimal first period consumption and risky asset holdings. When reference levels are equal to the endowment income this is linked to the belonging motive (i.e., the sufficiently loss averse household belonging to a similar social class). They find that the sufficiently loss averse household's first period consumption is the exogenous reference consumption level and such households avoid playing the stock market. Finally, reference levels above the endowment income are connected with the self-improvement motive. Households with such high reference levels cannot avoid to consume below the reference level, either now or in the future. In this case loss aversion affects consumption and risky investment negatively. The current study differs from Hlouskova, Fortin and Tsigaris (2017) in that it incorporates habit persistence into the household's behavior.

Close to our work is also a recent paper by van Bilsen et al. (2017) who investigate optimal consumption and portfolio choice paths of a loss averse household with an endogenous reference level. The uncertainty arises from risky assets and it is assumed that the time is continuous. Mainly due to loss aversion, the household's behavior is geared towards protecting itself against bad states of nature to avoid or to reduce losses. Consumption choices are found to adjust slowly to financial shocks. In addition, welfare losses are found to be substantial given consumption and portfolio selections are suboptimal. Curatola (2015) also analyzes optimal consumption-savings decisions of a loss averse household with a time varying reference level in a continuous-time framework and finds that a loss averse household can consume below the reference level in bad economic times. This is done in order to invest in risky assets and increase the likelihood that in the future consumption exceeds its reference level. This behavioral approach can explain why investors increase their exposure to risky assets during financial crises. In contrast, standard habit persistence models do not allow consumption to be below the reference level. Our research complements the work by van Bilsen et al. (2017) and Curatola (2015) in that it provides additional insights: as our model is a two-period life-cycle model we can derive closed-form solutions which allow us to conduct comparative static analysis to detect why certain adjustments happen and also to conduct a welfare analysis.

In this paper, we find closed-form solutions for consumption and risk taking of a loss averse household whose endogenous second period reference level depends on current consumption (habit persistence) and on reference consumption. Households who have a relatively low first period reference level are more conservative (less ambitious), which allows them to achieve relative gains in both periods in both states of nature. Households who have a relatively high first period reference level and a low discount factor are more adventurous (more ambitious) and will thus face relative losses in the bad state of nature in the second period while they will achieve relative gains in the first period and in the good state of nature in the second period. On the other hand more ambitious households who value future consumption relatively more will have first period consumption below the reference level but will maintain future consumption in both states of nature above the endogenous second period reference level. We then conduct comparative statics and examine how these different types of households react to income changes, to changes in the first period reference level, to changes in loss aversion, and to changes in habit persistence.

The main difference with respect to Hlouskova, Fortin and Tsigaris (2017), henceforth called HFT, is that this study considers also habit persistence. An increase in the consumption habit persistence will reduce current consumption but stimulate risk taking for less ambitious households, reduce both current consumption and risk taking for more ambitious households with a high time preference, and stimulate both current consumption and risk taking for more ambitious households with a low time preference. In addition, we analyze income effects, which are closely related to the effects of income taxes. Another difference between this study and HFT is that the response of first and second period consumption of less ambitious households to a change of the first period reference level is ambiguous. Finally, unlike in HFT we also consider here a scarcity constraint on consumption, i.e., the consumption in both periods can not fall below a certain value.

Note that the household's first period reference level may be interpreted to equal the first period consumption of a reference household, the Joneses. Then *following the Joneses*[2] means that an increase of first period consumption of the Joneses will also trigger an increase of this household's first period consumption.[3] In HFT the less ambitious household and the more ambitious household with a high time preference (low discount factor) do follow the Joneses, while the more ambitious household with a low time preference (high discount factor) does not. In this study the behavior of the more ambitious household is similar, while that of the less ambitious household may be similar or different, depending on the household's time preference: for a lower time preference (larger discount factor) the household does follow the Joneses (like in HFT), while for a higher time preference it does not. The rest of the results are somewhat similar to HFT in terms of the impact of the exogenous parameters on the choice variables but differ in terms of magnitude.

Another interesting result that was not elaborated in HFT is the reaction of the choice variables of the household to income changes. When focusing, for instance, on risk taking then less ambitious households reduce risk taking when their income falls while more ambitious households increase risk taking when their income shrinks, which is consistent with the observation that investors increase their exposure to risky assets during financial crises (see Curatola, 2015). Finally, the same finding as in HFT is that the highest utility is achieved for the lowest current consumption reference level (while keeping everything else unchanged). Thus, not comparing at all (e.g., to others) leads to the highest level of happiness.

In the next section we present the model and lay out the methodology used to find the solutions. Section 3 presents the main results with a discussion and investigates the impact of income taxation. Finally, we offer some concluding remarks.

2 The two-period consumption-investment model

2.1 Model set-up

Consider a household who decides on current and future consumption within a two-period model. In the first period it decides how to allocate a non-stochastic exogenous income, $Y_1 > 0$, to current consumption, C_1, risk-free investment, m, and risky investment, $\alpha \geq 0$:

$$Y_1 = C_1 + m + \alpha = C_1 + S \tag{1}$$

Savings are composed of the risk-free investment and the risky investment, i.e., $S = m + \alpha$. The net of the dollar return $r_f > 0$ represents the yield from the safe asset. The risky asset yields a stochastic net of the dollar return r. We assume two states of nature, good and bad.

[2] See Clark et al. (2008) and Falk and Knell (2004), among others.

[3] This will work through the household's first period reference level which is equal to the Joneses' first period consumption.

The good state of nature occurs with probability p while the bad state of nature occurs with probability $1-p$. In the good state the risky asset yields net return r_g and in the bad state it yields net return r_b. Furthermore, it is assumed that $-1 < r_b < r_f < r_g$, $0 < p < 1$, and the expected return of the risky asset is greater than the return of the safe asset, namely $\mathbb{E}(r) = p\,r_g + (1-p)r_b > r_f$. In the second period (e.g., retirement years in a two-period life-cycle model) the household consumes

$$C_{2s} = Y_2 + (1+r_f)m + (1+r_s)\,\alpha$$

where $Y_2 \geq 0$ is the non-stochastic income in the second period (e.g., government pension income) and $s \in \{b, g\}$. Note that $C_{2g} \geq C_{2b}$ as $\alpha \geq 0$ and $r_g > r_b$, where C_{2g} is the second period household's consumption in the good state of nature and C_{2b} in the bad state of nature. The household is allowed to consume the non stochastic future income Y_2 in the first period, as long as consumption exceeds its scarcity constraint in either period (i.e., $C_1 \geq C_L \geq 0$ and $C_{2s} \geq (1+r_f)C_L$) and savings are negative. Hence, the household can partially borrow from the risk-free asset m against its future income. The gross return from total investments is equal to $(1+r_f)m + (1+r_s)\,\alpha$, $s \in \{b, g\}$. Based on this and (1) consumption in the second period for $s \in \{b, g\}$ is

$$C_{2s} = Y_2 + (1+r_f)(Y_1 - C_1) + (r_s - r_f)\,\alpha \tag{2}$$

Preferences are described by the following reference based utility function

$$U(C_1, \alpha) = V(C_1 - \bar{C}_1) + \delta\, V(C_2 - \bar{C}_2) \tag{3}$$

$\bar{C}_1$ is the first period exogenous consumption reference (or comparison) level, which can be viewed, for instance, as the first period consumption of the Joneses (a reference household to which our household compares to) or their income or, alternatively, as a fraction of this household's income. The first two types of reference level are examples of an external reference level, which relates to, e.g., people in the same neighborhood, region or country, or people with distinct demographic features, while the third one is an example of an internal reference level, which depends on, e.g., one's own income or one's own past consumption, see Clark et al. (2008). $\bar{C}_2$ is the second period *endogenous* reference level given such that

$$\bar{C}_2 = (1+r_f)\left[wC_1 + (1-w)\bar{C}_1\right] \tag{4}$$

where $w \in [0, 1]$. Note that the second period endogenous reference level depends on the first period consumption and the first period consumption reference level. The weight w shows the influence of the current consumption upon the future reference level. A higher w implies a stronger dependence between the future reference level and the current consumption level.

The weight w reflects thus the consumer's persistence to consumption habits. The weight $(1-w)$, on the other hand, determines the dependence of the second period consumption reference level on the first period consumption reference level. This can be seen as a habit persistence in consumption reference levels. The two weights are negatively related to each other, i.e., an increased habit persistence on current consumption implies a lower habit persistence on the first period reference level, and vice versa. The same habit-formation reference consumption level was used also in Fuhrer (2000). The assumption on the determination of the second period reference level is the main difference between this model and the one developed and analyzed in Hlouskova, Fortin and Tsigaris (2017) where the second period reference level was exogenous.

The δ is the discount factor, $0 < \delta < 1$, and will play an important role in the optimal solutions. A higher δ places more importance to the future relative to the presence, i.e., the household shows a lower time preference, while a smaller δ puts more weight to the presence, i.e., the household shows a higher time preference. The $V(\cdot)$ is a prospect theory (S-shaped) value function defined as

$$V(C_i - \bar{C}_i) = \left\{ \begin{array}{cc} \frac{(C_i-\bar{C}_i)^{1-\gamma}}{1-\gamma}, & C_i \geq \bar{C}_i \\ \\ -\lambda\frac{(\bar{C}_i-C_i)^{1-\gamma}}{1-\gamma}, & C_i < \bar{C}_i \end{array} \right\} \tag{5}$$

for $i \in \{1,2\}$, see Figure 1. Parameter $\lambda > 1$ represents the degree of loss aversion, while $\gamma \in (0,1)$ represents diminishing sensitivity to consumption. Consumption in excess of the reference level represents a *(relative) gain* of the magnitude $C_i - \bar{C}_i$, while consumption below the reference level represents a *(relative) loss* of the magnitude equal to $\bar{C}_i - C_i$. Note that the value function is non-differentiable at the consumption reference level and is steeper in the domain of losses than in the domain of gains. This implies that there is a higher dissatisfaction from a reduction in consumption when the household is in the domain of losses than dissatisfaction from the same size of decline in consumption when the household is in the domain of gains. Finally, the household is risk averse in the domain of relative gains (i.e., the value function is concave when consumption exceeds the reference level) and risk seeking in the domain of relative losses (i.e., the value function is convex when consumption is below its reference level).

The household maximizes the following expected utility as given by (3) and (5)

$$\begin{array}{rl} \text{Max}_{(\mathrm{C}_1,\alpha)}: & \mathbb{E}(U(C_1,\alpha)) = V(C_1 - \bar{C}_1) + \delta\,\mathbb{E}V(C_2 - \bar{C}_2) \\ \\ \text{such that}: & C_1 \geq C_L,\ C_{2g} \geq C_{2b} \geq (1+r_f)C_L,\ \alpha \geq 0 \text{ and} \\ & \bar{C}_2 = (1+r_f)\left[wC_1 + (1-w)\bar{C}_1\right] \end{array}$$

where C_L and $(1+r_f)C_L$ determine the minimum first and second period consumption levels,

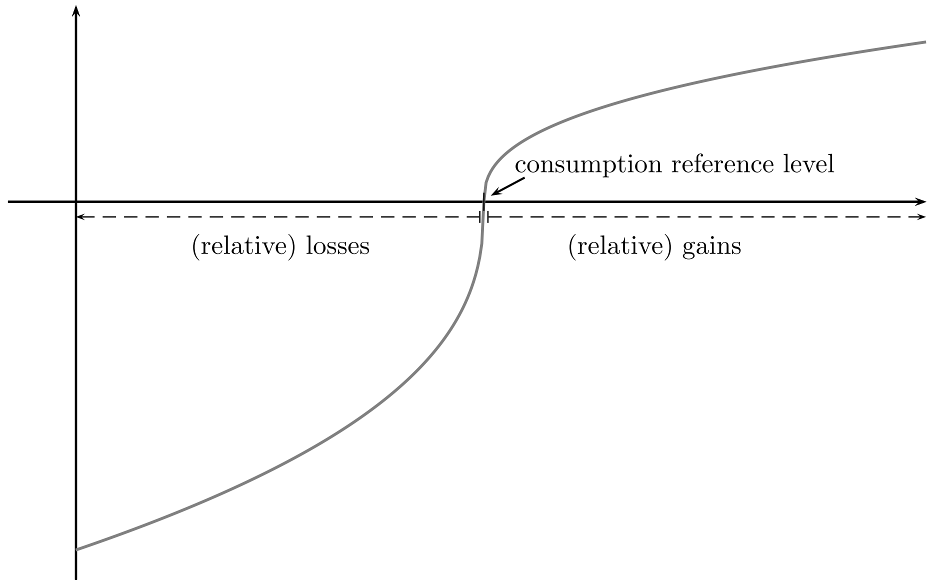

Figure 1: Prospect theory (S-shaped) value function

so that the household does not starve ($C_L \geq 0$). Based on this and (2) the household's maximization problem can be formulated as follows

$$\begin{array}{llll}
\mathrm{Max}_{(\mathrm{C}_1,\alpha)}: & \mathbb{E}(U(C_1,\alpha)) & = & V(C_1-\bar{C}_1) \\
& & + \;\; \delta\,\mathbb{E}\,V & \big((1+r_f)(Y_1-(1-w)\bar{C}_1)+Y_2-(1+r_f)(1+w)C_1+(r-r_f)\alpha\big) \\
& & & \\
\text{such that}: & C_L \leq & C_1 & \leq Y_1+\frac{Y_2}{1+r_f}-C_L-\frac{r_f-r_b}{1+r_f}\,\alpha \\
& & & \\
& 0 \leq & \alpha & \leq \frac{(1+r_f)(Y_1-2C_L)+Y_2}{r_f-r_b}
\end{array} \tag{6}$$

Note that the upper bound on C_1 follows from $C_{2b} \geq (1+r_f)C_L$ and the upper bound on α follows from the imposition of the upper bound on C_1, which is at the same time larger than or equal to C_L, i.e., $Y_1+\frac{Y_2}{1+r_f}-C_L-\frac{r_f-r_b}{1+r_f}\,\alpha \geq C_L$. Finally, the last inequality on α implies that

$$C_L \leq \frac{1}{2}\left(Y_1+\frac{Y_2}{1+r_f}\right) \tag{7}$$

which we will assume to hold. In addition we assume[4] that $C_L \leq \bar{C}_1$ and thus that

$$C_L \leq \min\left\{\frac{1}{2}\left(Y_1+\frac{Y_2}{1+r_f}\right),\bar{C}_1\right\} \tag{8}$$

[4]This is required for the feasibility of certain solutions.

2.2 Methodology

Prior to presenting the main results of the study we sketch the approach we chose to conduct the formal analysis, which requires the consideration of eight household consumption decision problems:

$$
\begin{array}{lllll}
\text{(P1)} & & \bar{C}_1 \leq C_1, & & \bar{C}_2 \leq C_{2b} \leq C_{2g} \\
\text{(P2)} & & \bar{C}_1 \leq C_1, & (1+r_f)C_L \leq C_{2b} \leq & \bar{C}_2 \leq C_{2g} \\
\text{(P3)} & & \bar{C}_1 \leq C_1, & (1+r_f)C_L \leq C_{2g} \leq & \bar{C}_2 \leq C_{2b} \\
\text{(P4)} & & \bar{C}_1 \leq C_1, & (1+r_f)C_L \leq C_{2b} \leq C_{2g} \leq & \bar{C}_2 \\
\text{(P5)} & C_L \leq C_1 \leq & \bar{C}_1, & & \bar{C}_2 \leq C_{2b} \leq C_{2g} \\
\text{(P6)} & C_L \leq C_1 \leq & \bar{C}_1, & (1+r_f)C_L \leq C_{2b} \leq & \bar{C}_2 \leq C_{2g} \\
\text{(P7)} & C_L \leq C_1 \leq & \bar{C}_1, & (1+r_f)C_L \leq C_{2g} \leq & \bar{C}_2 \leq C_{2b} \\
\text{(P8)} & C_L \leq C_1 \leq & \bar{C}_1, & (1+r_f)C_L \leq C_{2b} \leq C_{2g} \leq & \bar{C}_2
\end{array}
$$

The first four problems (P1)–(P4) assume that the household keeps current period consumption equal to or above the reference level experiencing a relative gain in the first period. In (P1) the household does not suffer from relative losses neither in the second period. In problems (P2)–(P4), however, there are relative losses: in (P2) the relative losses occur in the bad state of nature while in (P4) the relative losses are observed in both states of nature. Note that as $C_{2g} \geq C_{2b}$ any feasible solution for (P3) or (P7) satisfies $C_{2g} = C_{2b} = \bar{C}_2$, which implies that any solution feasible for (P3) is feasible also for (P1), (P2) and (P4), and any solution feasible for (P7) is feasible also for (P5), (P6) and (P8). Thus, problems (P3) and (P7) can be dropped from our analysis.

In the remaining problems (P5), (P6) and (P8) current consumption is below, or equal to, its reference level and thus the household experiences relative losses in the first period. In problem (P5) the household keeps future consumption above its reference level and suffers relative losses only in the first period. In (P6) there are losses if the bad state of nature occurs. In the last problem (P8) there are relative losses in both periods. In what follows we show that (P1), no losses, (P2), losses in the second period in the bad state of nature, and (P5), losses only in the first period, have optimal interior solutions, and for certain conditions based on the degree of loss aversion, the size of the current reference consumption level $\bar{C}_1$, and/or the size of the discount factor one of these solutions is the solution of our main problem (6). For higher first period consumption reference levels some of the problems have solutions at the border of the set of feasible solutions. These are problems (P4), (P6) and (P8), and their optimal solutions will not be explored in this paper.[5]

[5] Appendix B provides the optimal solution to all problems for a more general second period reference level $\bar{C}_2 = w_0 + w_1 C_1 + w_2 \bar{C}_1$, where $0 \leq w_0 \leq (1+r_f)Y_1 + Y_2$, $w_1, w_2 \geq 0$, $C_1 \geq C_{1L} \geq 0$ and $C_{2b} \geq C_{2L} \geq 0$. To reduce the complexity in the main text we use a simpler formulation for the determination of the second period reference level, namely: $w_0 = 0$, $w_1 = (1+r_f)w$, $w_2 = (1+r_f)(1-w)$, $w \in [0,1]$, $C_{1L} = C_L \geq 0$

We assume that a sufficiently loss averse household makes two sequential decisions. The household first decides whether it is the less ambitious type driven by the self-enhancement motive, whether it is balanced or whether it is more ambitious governed by the self-improvement motive, and only after figuring out its level of ambition the household makes its choices for current consumption and risk taking. We treat these motives as exogenous due to the household's psychological state of mind or due to its own income or the income of the Joneses to which it compares. The less ambitious household chooses its first period consumption reference level to be below some threshold level, the balanced household chooses its reference level to be equal to the threshold, and the more ambitious household chooses its first period reference consumption to be larger than the threshold. This threshold (separating less ambitious from more ambitious households) is equal to the average (half) of the present value of total income, i.e., $\frac{1}{2}\left(Y_1 + \frac{Y_2}{1+r_f}\right)$. Note that the balanced household with such a *neutral* first period reference level will consume exactly its reference consumption in both the first and the second period.[6] Depending on the level of ambition (and other characteristics), the sufficiently loss averse household will achieve relative gains in both periods, (P1), or current relative gains, relative gains in the good state of nature in second period but relative losses in the bad state of nature in the second period, (P2), or current relative losses and relative gains in the second period, (P5). Note finally that if the scarcity constraint exceeds the current consumption reference level (i.e., $C_L > \bar{C}_1$) then the household can achieve relative gains only in the first period[7] and thus the optimal solution is achieved only in either (P1) or (P2).

3 Main results

In Section 3.1, we show the optimal consumption and risky asset holdings to problem (6) for a *less ambitious* household. The household's solution is provided by problem (P1). The solution related to current consumption and risk taking exists for a sufficiently loss averse household with a relatively low reference level $\bar{C}_1$, namely below the threshold level, and is such that consumption exceeds the corresponding reference consumption in both periods across both states of nature. By being less ambitious the household selects consumption and risk taking in such a way as to avoid relative losses today and in the future. In addition, the household needs to be sufficiently loss averse for an optimal solution to exist in (P1), even though the loss aversion parameter does not explicitly appear in the optimal solution for current consumption and risk taking. Proposition 1 shows the closed-form solution to consumption and risk taking.

and $C_{2L} = (1 + r_f)C_L$. Note that the model in Hlouskova, Fortin and Tsigaris (2017), which dealt with an exogenous second period consumption reference level, is imbedded in this general model, namely, when $w_1 = w_2 = 0$ and thus $\bar{C}_2 = w_0$, and $C_L = 0$.

[6] In this case the second period reference level will be equal to the accumulated first period reference level, i.e., $\bar{C}_2 = (1 + r_f)\bar{C}_1 = \frac{1}{2}[(1 + r_f)Y_1 + Y_2]$.

[7] To avoid this we assume that $C_L \leq \bar{C}_1$, see (8).

In Section 3.2, we describe the optimal consumption and risk taking for a *balanced* household. This is a very special situation, where the household's first period reference level is equal to the average present value of its total wealth (*neutral* reference level) and hence consumption is exactly equal to its reference consumption in both periods. This can also be viewed as a comparison to a reference household with the same total wealth (comparison to *someone like me*).[8]

In Section 3.3 we show the optimal consumption and risky asset holdings to problem (6) for a *more ambitious* household. The solution is provided by problem (P2) or (P5). The optimal solution exists for a sufficiently loss averse household with a relatively high current reference level, namely above the threshold level, and is such that the optimal consumption is below its corresponding reference consumption in either the first or the second period. Proposition 2 shows the closed-form solution of (6) for a more ambitious household with a high time preference (or a sufficiently large probability of the good state of nature to occur), where the solution is the solution of problem (P2), while Proposition 3 presents the closed-form solution of (6) for a more ambitious household with a low time preference, where the solution is the solution of problem (P5). In the first case the household will achieve relative gains today and in the future in the good state of nature but will incur relative losses in the future in the bad state, while in the second case the household has to accept current relative losses but will achieve relative gains in both states in the future.

As we will show, the different types of households have very distinct solutions for current consumption and risk taking activity. Also their responses, as well as the responses of the indirect utility function (happiness), to exogenous changes in the loss aversion parameter, the first period reference level, the habit persistence and finally the income/wealth levels vary substantially.

In Sections 3.4 and 3.5 we summarize the income effects and other effects across the different types of household and investigate the impact of income taxation.

3.1 Low first period reference consumption: less ambitious households

In this section we consider a household with a relatively low first period reference consumption. This reference consumption is below a certain threshold[9] and is such that the household can consume above its reference levels in both the first and the second period, and may thus avoid any relative losses. We call a household with such a first period reference level *less ambitious*. This household's behavior is captured by problem (P1). Before proceeding further,

[8]See Clark et al. (2008).

[9]Namely below the average present value of total wealth, see (11).

we introduce the following notation

$$\Omega = (1+r_f)Y_1 + Y_2 - 2\,(1+r_f)\,\bar{C}_1 \tag{9}$$

$$K_\gamma = \frac{(1-p)(r_f-r_b)^{1-\gamma}}{p\,(r_g-r_f)^{1-\gamma}} \tag{10}$$

$$\bar{C}_1^{U,P1} = \frac{1}{2}\left(Y_1 + \frac{Y_2}{1+r_f}\right) \tag{11}$$

$$\begin{aligned}\lambda^{P1-P2} = {} & \frac{\left[\left(\frac{r_f-r_b}{(1+r_f)(r_g-r_b+w(r_f-r_b))}\right)^{1-\gamma} + \delta p\right]\left[\Omega + (r_g-r_f)\alpha^{C_1=\bar{C}_1}_{C_{2b}=C_{2L}}\right]^{1-\gamma}}{\delta(1-p)\left[(1+r_f)(\bar{C}_1-C_L)\right]^{1-\gamma}} \\ & - \frac{\Omega^{1-\gamma}[(1+r_f)(1+w)+M]^\gamma}{\delta(1-p)(1+r_f)(1+w)\left[(1+r_f)(\bar{C}_1-C_L)\right]^{1-\gamma}} \quad \text{for} \quad \bar{C}_1 \le \bar{C}_1^{P1}\end{aligned} \tag{12}$$

$$\lambda^{P1-P5} = \left[\frac{k_2\left(1+K_\gamma^{\frac{1}{\gamma}}\right)}{(1+r_f)(1+w)}\right]^\gamma = \left[\frac{M}{(1+r_f)(1+w)}\right]^\gamma \tag{13}$$

$$k_2 = \left[\delta(1+r_f)(1+w)\,p\,\left(\frac{r_g-r_b}{r_f-r_b}\right)^{1-\gamma}\right]^{\frac{1}{\gamma}} \tag{14}$$

$$M = \left[\delta(1+r_f)(1+w)\,p\,\frac{r_g-r_b}{r_f-r_b}\right]^{\frac{1}{\gamma}} \frac{r_f-r_b+K_0^{\frac{1}{\gamma}}(r_g-r_f)}{r_g-r_b} \tag{15}$$

$$\alpha^{C_1=\bar{C}_1}_{C_{2b}=C_{2L}} = \frac{(1+r_f)(Y_1-\bar{C}_1-C_L)+Y_2}{r_f-r_b} \tag{16}$$

Note that $\bar{C}_1 < \bar{C}_1^{U,P1}$ is equivalent to $\Omega > 0$.[10] We present the optimal solution for first period consumption and risk taking of the less ambitious household in the following proposition.

Proposition 1 *Let $\bar{C}_1 < \bar{C}_1^{U,P1}$ and $\lambda > \max\left\{\lambda^{P1-P2}, \lambda^{P1-P5}\right\}$. Then problem (6) obtains*

[10]Note that HFT characterize the different types of household through Ω (being positive, equal to zero, or negative), while in this study we define the different types of household through their first period consumption reference levels (being smaller than, equal to, or larger than a threshold value), which we think makes more sense. However, we could equivalently describe our households through Ω.

a unique maximum at $(C_1^*, \alpha^*) = \left(C_1^{P1}, \alpha^{P1}\right)$*, where*

$$\begin{aligned} C_1^{P1} &= \bar{C}_1 + \frac{\Omega}{(1+r_f)(1+w)+M} \\ &= \frac{(1+r_f)Y_1 + Y_2 + [M - (1+r_f)(1-w)]\bar{C}_1}{(1+r_f)(1+w)+M} > \bar{C}_1 \end{aligned} \tag{17}$$

$$\alpha^{P1} = \frac{\left(1 - K_0^{\frac{1}{\gamma}}\right) M}{r_f - r_b + K_0^{\frac{1}{\gamma}}(r_g - r_f)} \left(C_1^{P1} - \bar{C}_1\right) > 0 \tag{18}$$

Proof. See Appendix B. ■

The future relative gains, or excess consumption, are given by:

$$\left.\begin{aligned} C_{2g}^{P1} - \bar{C}_2 = &\quad k_2 \tfrac{r_g - r_b}{r_f - r_b} \left(C_1^{P1} - \bar{C}_1\right) > 0 \\ C_{2b}^{P1} - \bar{C}_2 = &\quad k_2 K_0^{\frac{1}{\gamma}} \tfrac{r_g - r_b}{r_f - r_b} \left(C_1^{P1} - \bar{C}_1\right) > 0 \end{aligned}\right\} \tag{19}$$

Current relative gains, $C_1^{P1} - \bar{C}_1$, are driving both the investment in the financial market as well as future excess consumption, see (18) and (19). The higher the relative gains in the first period the higher the investment in the financial market and the higher the relative gains (excess consumption) in the future. Note that the household invests positively in the risky asset. Total savings, however, which include both risky and risk-free assets, may be either positive or negative. The household's consumption and risk taking does not directly depend on the degree of loss aversion; however, the household needs to be sufficiently loss averse.[11] Thus the optimal consumption in both periods as well as the relative consumption in both periods, risk taking and happiness are insensitive to changes in the degree of loss aversion.

The effect of an increase in the first period consumption reference level on current and future consumption cannot be determined a priori, see

$$\frac{dC_1^{P1}}{d\bar{C}_1} = \frac{\frac{M}{1+r_f} - 1 + w}{\frac{M}{1+r_f} + 1 + w} \begin{cases} > 0, & \text{if} \quad \delta > \bar{\delta} \\ = 0, & \text{if} \quad \delta = \bar{\delta} \\ < 0, & \text{if} \quad \delta < \bar{\delta} \end{cases} \tag{20}$$

where

$$\bar{\delta} = \left(\frac{1-w}{1+K_\gamma^{1/\gamma}}\right)^\gamma \left(\frac{r_f - r_b}{r_g - r_b}\right)^{1-\gamma} \frac{1}{p\,(1+w)(1+r_f)^{1-\gamma}} \tag{21}$$

[11] As shown in Proposition 1, the loss aversion parameter needs to be sufficiently large, namely $\lambda > \max\left\{\lambda^{P1-P2}, \lambda^{P1-P5}\right\}$, to guarantee that the utility of (P1) at its maximum exceeds the potential maximum of (P2) at its border, $\lambda > \lambda^{P1-P2}$, as well as the potential maximum of (P5) at its border, $\lambda > \lambda^{P1-P5}$. Note that problem (P1) is a concave programming problem and its unique maximum does not depend on λ.

It depends on the household's time preference, i.e., on its discount factor, as follows: a relatively high discount factor (large weight placed to the future) will cause current consumption to increase with increasing $\bar{C}_1$, while a relatively low discount factor (small weight place to the future) will cause current consumption to decrease.[12] However, the effect on optimal consumption in the second period is opposite: future consumption increases with a lower discount factor and shrinks with a higher discount factor. In addition, the sensitivity of second period consumption in the bad state to the first period reference consumption depends on the probability of the good state. Relative current and future consumption decreases with an increasing first period reference level and also risk taking decreases when the current consumption reference level increases. The latter happens because the increase in the current consumption reference level decreases the relative gains in the first period discouraging investment in the risky asset. Finally, an increase in the first period reference level will reduce the household's happiness and thus the highest possible level of happiness is achieved for the lowest possible current consumption reference level. This suggests that comparison does not make oneself happy, and indeed not comparing at all would be the best. Note that the sensitivity results with respect to the first period reference level are similar (in terms of sign) to the ones when the second period consumption reference level is exogenous (see Hlouskova, Fortin and Tsigaris, 2017), except for the sensitivity of first and second period consumption: if the second period reference level is exogenous then first period consumption always increases, and second period consumption in both states of nature always decreases, with a rising first period reference level.

As stated earlier habit persistence in consumption is determined by the parameter w. An increase in w reduces optimal first period consumption (and thus also the first period relative consumption) and the level of happiness, while it increases the investment in the risky asset. The effect of an increase in w on the second period reference level, however, is not unambiguous. It depends on the curvature, γ, the discount factor, δ, and on the level of habit persistence in consumption, w, itself. If the household is rather risk averse ($\gamma > 0.5$), however, then the effect of habit persistence on the second period reference level is always positive. Also the effect of w on the second period consumption in the bad state can be either positive or negative. Namely the second period consumption in the bad state increases with increasing habit persistence in the first period consumption when w is below a certain threshold and it decreases with increasing habit persistence in the first period consumption when w exceeds the threshold.[13] On the other hand, the impact of w on the second period consumption in the good state is always positive. Note that as habit persistence in consumption, w, relates negatively to habit persistence in the current consumption reference level, $1-w$, the reported dependencies hold with the opposite sign for habit persistence in the first period reference

[12] Note, however, that a larger persistence in consumption reduces the threshold of the discount factor, see (21), which makes it more plausible that first period reference consumption encourages current consumption.

[13] This threshold is a function of the parameters describing the financial market and on the curvature.

level.

Current consumption depends positively on income, i.e., it depends positively on both first period and second period income.[14] An increase in the first period income, as in good economic times, will increase current consumption by $(1+r_f)/[(1+r_f)(1+w)+M)]$, while an increase in the second period income (i.e., good future economic conditions) will increase current consumption by $1/[(1+r_f)(1+w)+M)]$. Note that the presence of habit persistence in consumption has reduced the impact of income upon current consumption relative to models without such a behavioral trait. Furthermore, an increase in income will increase second period consumption as well as the relative gains (excess consumption) in both periods, the second period reference level, the investment in the risky asset and the level of happiness. Note that a sudden reduction in income, caused by a recession or a loss of job (bad economic conditions) or by the introduction of an income tax, will cause the opposite effect and the household will thus reduce current consumption and risk taking. Note in addition that if the first period reference level is equal to a fraction of the present value of the total wealth, i.e., $\bar{C}_1 = c\left(Y_1 + \frac{Y_2}{1+r_f}\right)$ where $c \in \left(0, \frac{1}{2}\right)$,[15] then the sensitivity results will not change. This suggests that the direct income effect is stronger than the indirect effect of income through the first period consumption reference level. Table 1 summarizes the sensitivity results related to Proposition 1, which have been discussed above.

Finally, it can be shown that the expected utility evaluated at the optimal choices is determined by the relative gains in the first period:

$$(1-\gamma)\mathbb{E}\left(U\left(C_1^{P1}, \alpha^{P1}\right)\right) = \frac{[(1+r_f)(1+w)+M]^\gamma}{(1+r_f)(1+w)}\left(C_1^{P1} - \bar{C}_1\right)^{1-\gamma} \tag{22}$$

The household will be more happy with a rising income, while it will be less happy with a larger first period reference level (as the first period relative consumption decreases) and a higher persistence in current consumption, see Table 1.

	$C_1^* = C_1^{P1}$ and $\alpha^* = \alpha^{P1}$								
	dC_1^*	dC_{2g}^*	dC_{2b}^*	$d\alpha^*$	$d\bar{C}_2$	$d(C_1^* - \bar{C}_1)$	$d(C_{2g}^* - \bar{C}_2)$	$d(C_{2b}^* - \bar{C}_2)$	$d(\mathbb{E}(U(C_1^*, \alpha^*)))$
$d\lambda$	$=0$	$=0$	$=0$	$=0$	$=0$	$=0$	$=0$	$=0$	$=0$
$d\bar{C}_1$	$\gtreqless 0$	$\lesseqgtr 0$	$\lesseqgtr 0$	<0	>0	<0	<0	<0	<0
dw	<0	>0	$\lesseqgtr 0$	>0	$\lesseqgtr 0$	<0	>0	>0	<0
dY_i	>0	>0	>0	>0	>0	>0	>0	>0	>0

Table 1: Sensitivity results for the less ambitious household with respect to λ, $\bar{C}_1$, w and Y_i, $i = 1, 2$.

[14] We say that some quantity depends positively (negatively) on income, if it depends positively (negatively) on both first period income and second period income.

[15] The fraction needs to be less than one half such that the household is less ambitious.

3.2 Neutral first period reference consumption: balanced households

This special case applies when the household is neither less ambitious (see the previous section) nor more ambitious (see the following section). The household is *balanced* in the sense that it consumes exactly its reference levels, in both the first and the second period. This requires that the household's first period reference level is equal to the threshold separating less ambitious from more ambitious households. The reference consumption is thus equal to the average of the discounted income, i.e., $\bar{C}_1 = \bar{C}_1^{U,P1} = \frac{1}{2}\left(Y_1 + \frac{Y_2}{1+r_f}\right)$. We call this reference level the *neutral* first period reference consumption. Note that the neutral reference level depends explicitly on the household's exogenous income. Note, in addition, that if the household's total income coincides with the total income of some reference household then this current reference consumption can be viewed as an *external* reference consumption, as the household compares itself to *someone like itself.*

The following corollary describes the solution of the balanced household.

Corollary 1 *Let* $\bar{C}_1 = \bar{C}_1^{U,P1}$ *and* $\lambda > \max\left\{\lambda^{P1-P2}, \lambda^{P1-P5}\right\}$. *Then problem (6) obtains its unique maximum at* (C_1^*, α^*), *where*

$$\begin{aligned} C_1^* &= \frac{1}{2}\left(Y_1 + \frac{Y_2}{1+r_f}\right) = \bar{C}_1^{U,P1} \\ \alpha^* &= 0 \end{aligned}$$

Proof. See Appendix B. ■

The sufficiently loss averse balanced household will consume exactly its consumption reference level in the first period, which is equal to half the current value of total income. In addition, it will not invest in the financial market even though the expected return from the risky asset is greater than the return from the safe asset. This phenomenon can help to explain the equity premium puzzle as it indicates that the risk premium is not sufficient to induce the household to invest in the risky asset. The savings will thus consist only of the risk-free investment, which can be positive, zero or negative, based on how the first period income and the discounted second period income relate to each other:

$$S = m = \frac{1}{2}\left(Y_1 - \frac{Y_2}{1+r_f}\right)\begin{cases} > 0 & \text{if } Y_1 > \frac{Y_2}{1+r_f} \\ = 0 & \text{if } Y_1 = \frac{Y_2}{1+r_f} \\ < 0 & \text{if } Y_1 < \frac{Y_2}{1+r_f} \end{cases} \tag{23}$$

Note, in addition, that also in the second period in both states of nature the household consumes exactly its consumption reference level, i.e., $\bar{C}_2 = C_{2g} = C_{2b} = \frac{1}{2}[(1+r_f)Y_1 + Y_2] = (1+r_f)\bar{C}_1^{U,P1} = (1+r_f)\bar{C}_1$, which implies that this solution is feasible for all sub-problems (P1)–(P8) and thus can be considered a threshold solution, where the household achieves no relative gains and no relative losses in either period.

If the household's income increases either in the first period and/or in the second period, while other parameters remain unchanged, including $\bar{C}_1$, then the household's upper bound $\bar{C}_1^{U,P1}$ will also increase and as a result the household will become relatively less ambitious since now $\bar{C}_1 < \bar{C}_1^{U,P1}$. Thus, the household will be able to avoid relative losses in both periods. If on the other hand, the household's income falls unexpectedly, while other parameters remain unchanged, then this will reduce the household's threshold level $\bar{C}_1^{U,P1}$ and thus the first period reference level will be above this new upper bound $\bar{C}_1^{U,P1}$. As a result the household will become more ambitious in order to make up for the lost income. In this case its optimal consumption will be below the reference level either in the second period in the bad state of nature, problem (P2), or in the first period, problem (P5). We will discuss these cases in the next section.

Suppose the household has initially a current consumption reference level below the threshold level and hence is less ambitious. Then it is hit by a sudden reduction in income, e.g., due to a loss of job in bad economic times, which triggers a decrease of the threshold level such that the household's (constant) reference level is above the new threshold, and hence the household is more ambitious. This switch from the less ambitious (across the balanced) to the more ambitious type will change, for example, its sensitivity of risk taking with respect to income: while before the drop in income the household (which is less ambitious) takes on less risk with decreasing income, it will be eager to take on more risk with a decreasing income – with the hope to make up for the lost income – after the drop in income (when it will be more ambitious).[16]

Note that consumption in both periods (as well as the relative consumption in both periods), risk taking and happiness are unaffected by changes in the level of loss aversion, as well as by changes in the persistence level in current consumption.

3.3 High first period reference consumption: more ambitious households

If the first period reference level exceeds the threshold level which is equal to the average of the discounted income, i.e., if $\bar{C}_1 > \bar{C}_1^{U,P1} = \frac{1}{2}\left(Y_1 + \frac{Y_2}{1+r_f}\right)$, then the household cannot consume above its reference levels in both periods. In either the first or the second period the household will have to consume below its reference consumption, and thus will incur relative losses. A household with such a high first period reference level is called *more ambitious*. The optimal consumption of the more ambitious household will be either below its consumption reference level in the second period in the bad state of nature, problem (P2), or in the first period, problem (P5). Which case occurs, problem (P2) or (P5), depends on the household's time preference, i.e., on its discount factor, and on the probability of the good state to occur. If the sufficiently loss averse household is relatively time impatient and assigns a low weight to future consumption (i.e., it has a small discount factor, or a high time preference) then the

[16]See the sensitivity results in Tables 1 and 2.

optimal solution of (6) for optimal consumption and risk taking coincides with the optimal solution of problem (P2). In this problem the optimal consumption in the first period is above its reference level, as in problem (P1). However, in the second period the household cannot avoid relative losses in the bad state of nature. Proposition 2 provides the optimal solution for this case. This case also applies if the probability of the good state of nature is sufficiently large (irrespective of the household's time preference). On the other hand, if the discount factor is relatively large (i.e, future consumption is valued high), and the probability of the good state is not too high, then the sufficiently loss averse household will find a solution where first period consumption is below the first period reference level (suffering relative losses in the first period) but will keep future consumption above the endogenous reference level in both states of nature. The solution for this case is presented in Proposition 3. The first period reference level cannot be arbitrarily large, however. It needs to be smaller than a certain threshold, $\bar{C}_1^{U,P2}$.

To summarize, if the more ambitious household values first period consumption relatively high (lower discount factor), then it focuses on avoiding relative losses in the first period and thus first period consumption is above its reference level. If, however, the more ambitious household values second period consumption relatively high (larger discount factor), then it wants to prevent relative losses in the second period and consequently second period consumption exceeds its reference level. This is only true, however, if the probability of the good state is not too large. If it is larger than a certain threshold then only the first case applies, where relative losses occur in the second period in the bad state, irrespective of the household's time preference.[17]

[17]Note that for better readability we will often omit the information on the large (small) enough probability of the good state of nature in identifying the type of household, and simply call a household that finds it optimal solution in problem (P2) "more ambitious with a high time preference", and a household that finds its optimal solution in problem (P5) "more ambitious with a low time preference".

Before proceeding further, we introduce the following notation

$$\bar{C}_1^{U,P2} = \frac{\frac{r_g-r_b}{r_g-r_f}\left(Y_1+\frac{Y_2}{1+r_f}\right)-C_L}{1+2\,\frac{r_f-r_b}{r_g-r_f}} \tag{24}$$

$$\bar{C}_1^{U,P5} = \frac{Y_1+\frac{Y_2}{1+r_f}-(1+w)C_L}{1-w} \tag{25}$$

$$k = \left[\delta(1+r_f)(1+w)(1-p)\left(\frac{r_g-r_b}{r_g-r_f}\right)^{1-\gamma}\right]^{\frac{1}{\gamma}} \tag{26}$$

$$M(\lambda) = \left[\delta(1+r_f)(1+w)\,p\,\left(\frac{r_g-r_b}{r_f-r_b}\right)^{1-\gamma}\right]^{\frac{1}{\gamma}}\left[(\lambda K_\gamma)^{1/\gamma}-1\right] \tag{27}$$

$$C_L^U = \frac{r_g-r_b}{r_g-r_f}\left(Y_1+\frac{Y_2}{1+r_f}\right)-\left(1+2\,\frac{r_f-r_b}{r_g-r_f}\bar{C}_1\right) \tag{28}$$

$$\lambda^{P2} = \left[\frac{(1+r_f)(1+w)}{k}+\left(\frac{1}{K_\gamma}\right)^{1/\gamma}\right]^{\gamma}\left[1+\frac{\frac{r_f-r_b}{r_g-r_f}+\frac{1+r_f+k_2}{(1+r_f)(1+w)+k_2}}{C_{2L}^U-(1+r_f)C_L}(-\Omega)\right]^{\gamma} \tag{29}$$

$$\text{for}\;\; C_L<\frac{C_{2L}^U}{1+r_f}\;\;\text{and}\;\;\bar{C}_1^{U,P1}<\bar{C}_1<\bar{C}_1^{U,P2}$$

$$\lambda^{P2-P2} = \frac{\left[\left(\frac{r_f-r_b}{(1+r_f)\left(r_g-r_b+w(r_f-r_b)\right)}\right)^{1-\gamma}+\delta p\right]\left[\Omega+(r_g-r_f)\alpha_{C_{2b}=C_{2L}}^{C_1=\bar{C}_1}\right]^{1-\gamma}}{\delta(1-p)\left[\left((1+r_f)(\bar{C}_1-C_L)\right)^{1-\gamma}-\left(\frac{r_g-r_b}{r_g-r_f}\right)^{1-\gamma}(-\Omega)^{1-\gamma}\right]} \tag{30}$$

$$\lambda^{P4} = \frac{1}{(1+r_f)^{1-\gamma}w\,\delta}\left[\frac{w\left(Y_1+\frac{Y_2}{1+r_f}\right)+(1-w)\bar{C}_1-(1+w)C_L}{\alpha_{C_{2b}=C_{2L}}^{C_1=\bar{C}_1}-\alpha_{C_{2b}=C_{2L}}^{C_{2g}=\bar{C}_2}}\frac{1+r_f}{r_f-r_b}\right]^{\gamma} \tag{31}$$

$$\text{for}\;\;\bar{C}_1<\bar{C}_1^{U,P2}$$

$$\lambda^{P5} = \lambda^{P1-P5}\left[\frac{(1+w)(\bar{C}_1-C_L)}{(1-w)\left(\bar{C}_1^{U,P5}-\bar{C}_1\right)}\right]^{\gamma} \tag{32}$$

$$\lambda^{P2-P6} = \frac{\delta p(1+r_f)^2\left(\frac{r_g-r_b}{r_f-r_b}+w\right)^2(\bar{C}_1-C_L)^{1+\gamma}}{[1+\delta(1-p)(1+r_f)^{1-\gamma}w^2]\left[\Omega+(r_g-r_f)\alpha_{C_{2b}=C_{2L}}^{C_1=\bar{C}_1}\right]^{1+\gamma}} \tag{33}$$

$$\delta^{P2-P5} = \frac{1}{1-p}\left[\frac{r_g-r_f}{(1+r_f)(1+w)(r_g-r_b)}\right]^{1-\gamma} \tag{34}$$

$$\alpha_{C_{2b}=C_{2L}}^{C_{2g}=\bar{C}_2} = \frac{(1+r_f)(\bar{C}_1-C_L)+w\left[(1+r_f)(Y_1-\bar{C}_1-C_L)+Y_2\right]}{r_g-r_b+w\,(r_f-r_b)} \tag{35}$$

The optimal solution for first period consumption and risk taking is given in the next proposition.

Proposition 2 *Let* $\bar{C}_1^{P1} < \bar{C}_1 < \bar{C}_1^{U,P2}$, $\lambda > \max\left\{\lambda^{P2}, \lambda^{P2-P2}, \lambda^{P4}, \lambda^{P5}, \lambda^{P2-P6}\right\}$, $\delta \leq \delta^{P2-P5}$ *and* $C_L < C_L^U$. *Then problem (6) obtains a unique maximum at* $(C_1^*, \alpha^*) = \left(C_1^{P2}, \alpha^{P2}\right)$, *where*

$$C_1^{P2} = \bar{C}_1 - \frac{\Omega}{M(\lambda) - (1+r_f)(1+w)} > \bar{C}_1 \tag{36}$$

$$\alpha^{P2} = \frac{\left[\left(\frac{1}{K_0}\right)^{1/\gamma} + \lambda^{1/\gamma}\right] k}{r_g - r_f} \left(C_1^{P2} - \bar{C}_1\right) > 0 \tag{37}$$

Proof. See Appendix B. ■

Note that for a sufficiently large probability of the good state[18] the threshold value of the discount factor is larger than one ($\delta^{P2-P5} > 1$) and is thus not binding. In that case Proposition 2 applies, irrespective of the household's time preference. The reason is that the household is rather willing to accept a relative loss in the bad state of nature, which occurs with a small enough probability, than to face a relative loss in the first period, which occurs with certainty.

Future relative gains (in the good state of nature) and losses (in the bad state of nature) are given by

$$\left.\begin{array}{lll} C_{2g}^{P2} - \bar{C}_2 = & k\frac{r_g - r_b}{r_g - r_f}\left(\frac{1}{K_0}\right)^{\frac{1}{\gamma}}\left(C_1^{P2} - \bar{C}_1\right) & > 0 \\ \bar{C}_2 - C_{2b}^{P2} = & k\frac{r_g - r_b}{r_g - r_f}\lambda^{\frac{1}{\gamma}}\left(C_1^{P2} - \bar{C}_1\right) & > 0 \end{array}\right\} \tag{38}$$

In problem (P1) the loss aversion parameter does not affect the optimal choices but here loss aversion plays a significant role. An increase in the degree of loss aversion will result in a decline in the first period consumption, a decline in the future consumption in the good state of nature, and a decline in the endogenous second period consumption reference level, but will increase future consumption in the bad state of nature. An increase in loss aversion will also reduce relative gains in the good state of nature in the second period because of the decline in relative gains in the first period. In addition, an increase in loss aversion will reduce relative losses in the bad state of nature in the second period. Even though there are two opposite effects on relative losses in the second period arising from an increase in loss aversion it can be shown that the indirect effect from the decline in $C_1^{P2} - \bar{C}_1$ overpowers the direct impact from increasing the loss aversion parameter. Finally, an increase in loss aversion will reduce the exposure to the stock market and reduce the happiness level.

[18] Namely for $1 > p > 1 - \left[\frac{r_g - r_f}{(1+r_f)(1+w)(r_g - r_b)}\right]^{1-\gamma}$. Note that p must also be larger than $\frac{r_f - r_b}{r_g - r_b}$, which is implied by the assumption $\mathbb{E}(r) > r_f$.

Contrary to problem (P1), an increase in the first period reference level will increase first period consumption, see (36), which is in line with the assumption on preferring the presence to the future. Also it will increase second period consumption in the good state of nature, the second period reference level, and the investment in the financial market because the increase in $\bar{C}_1$ increases relative gains $C_1^{P2} - \bar{C}_1$. However, an increase in $\bar{C}_1$ will reduce future consumption in the bad state of nature. Relative gains of consumption in the first period will increase, and so will future relative gains in the good state of nature by having higher future relative losses in the bad state of nature. Similarly as in problem (P1), an increase in the first period reference level will decrease the level of happiness, i.e., not comparing at all makes one the happiest. Note that the sensitivities of the solutions (in terms of signs) with respect to loss aversion and the first period consumption reference level are the same as in the case of an exogenous second period reference level, as reported in Hlouskova, Fortin and Tsigaris (2017).

An increase in the habit persistence in consumption reduces the current consumption, the relative consumption in both periods, risk taking, as well as the happiness level. Finally, the increase in the habit persistence in current consumption reduces also the second period endogenous reference level, $\bar{C}_2$, and future consumption in the good state of nature, C_{2g}^{P2}, for a sufficiently large habit persistence level (where the threshold depends on the curvature parameter which is binding only for $\gamma \leq 0.5$), while it increases both $\bar{C}_2$ and C_{2g}^{P2} when the household is sufficiently loss averse and at the same time exhibits a lower level of habit persistence in consumption. Note that the opposite dynamics hold when we consider the effect of the habit persistence in the consumption *reference level*. Finally, note that the dynamics of the current consumption, current relative consumption, second period endogenous reference level and the happiness level with respect to the habit persistence are in line with the dynamics of the less ambitious households.

A change in income here has profoundly different effects from those related to the less ambitious household. An unexpected decrease in income, due to, e.g., a loss of job in bad economic times, will increase first period consumption, second period consumption in the good state of nature, investment in the financial market and also the endogenous second period consumption reference level. In addition, a decrease of income increases the relative consumption in both periods. On the other hand, the second period consumption in the bad state of nature will decrease when income decreases, and so will the happiness level. These effects are opposite (in terms of sign) with respect to those reported for the less ambitious household, with the exception of the future consumption in the bad state of nature and the happiness level, which both decrease with a falling income. The reason is probably related to the fact that the more ambitious household cannot consume above its consumption reference levels at all times while the less ambitious household can always do that. Total savings actually decrease with a falling income. Note finally that if the first period reference consumption level is equal to a fraction of the present value of total wealth, i.e., $\bar{C}_1 = c\left(Y_1 + \frac{Y_2}{1+r_f}\right)$ where $c > \frac{1}{2}$,[19] then the vast majority of sensitivity results become opposite in sign, including the happiness level. This suggests that the indirect effect of income through the first period consumption reference level is stronger than the direct income effect. Thus, in this case the happiness decreases with an increasing income, which is not entirely inconsistent with the literature which finds that as income moves beyond the levels associated with happiness, overall life satisfaction actually decreases, see Jebb et al. (2018).[20] All the sensitivities with respect to problem (P2), which we discussed above, are presented in Table 2.

Note, that the value of the expected utility at the optimum (the level of happiness) is determined by the relative gains in the first period, like in problem (P1):

$$(1-\gamma)\mathbb{E}\left(U\left(C_1^{P2}, \alpha^{P2}\right)\right) = -\left[\frac{k}{(1+r_f)(1+w)}\left(\lambda^{\frac{1}{\gamma}} - \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right) - 1\right]\left(C_1^{P2} - \bar{C}_1\right)^{1-\gamma} \tag{39}$$

The more ambitious household will be happier with a larger income, while it will be less happy with an increasing first period reference level (even though first period relative gains increase, but also relative losses in the second period in the bad state rise) and a higher persistence in current consumption. These effects are the same as those for the less ambitious household. In addition a larger degree of loss aversion affects the happiness negatively, see Table 2.

Before proceeding further let us introduce the following notation

$$\lambda^{P5-P2} = \left[\frac{\frac{k}{(1+r_f)(1+w)}\frac{1}{K_\gamma^{1/\gamma}} + 1}{\frac{k}{(1+r_f)(1+w)} - 1}\right]^\gamma \quad \text{for} \quad \delta > \delta^{P2-P5} \tag{40}$$

The next proposition shows the case where the household is again more ambitious (i.e., it cannot avoid relative losses at all times) but values future consumption higher (i.e., has a larger discount factor) than the household described by Proposition 2. This is why it strives to avoid relative losses in the second period but has to accept them in the first period. For this to hold, the probability of the good state of nature must be small enough. If it is larger,[21] then the household can avoid relative losses in the first period but has to accept them in the second period in the bad state (which occurs with a small enough probability), i.e., it always solves problem (P2), irrespective of its time preference.

Proposition 3 *Let* $\bar{C}_1^{U,P1} < \bar{C}_1 < \bar{C}_1^{U,P2}$, $\lambda > \max\left\{\lambda^{P1-P5}, \lambda^{P2}, \lambda^{P2-P2}, \lambda^{P5}, \lambda^{P5-P2}, \lambda^{P2-P6}\right\}$ *and* $\delta > \delta^{P2-P5}$. *Then problem (6) obtains a unique maximum at* $(C_1^*, \alpha^*) = \left(C_1^{P5}, \alpha^{P5}\right)$

[19] The fraction needs to be larger than one half such that the household is more ambitious.

[20] Jebb et al. (2018) find that the ideal income point when money no longer increases an individual's happiness is \$95,000 for overall satisfaction with life, and \$60,000 to \$75,000 for emotional well-being. They use a collection of survey responses from over 1.7 million people spanning 164 countries.

[21] For the precise threshold see Footnote 18.

where

$$
\begin{aligned}
C_1^{P5} &= \bar{C}_1 - \frac{\lambda^{1/\gamma}}{\lambda^{1/\gamma} - (\lambda^{P1-P5})^{1/\gamma}} \times \frac{-\Omega}{(1+r_f)(1+w)} \\
&= \frac{\lambda^{1/\gamma}\left[Y_1 + \frac{Y_2}{1+r_f} - (1-w)\bar{C}_1\right] - \left(\lambda^{P1-P5}\right)^{1/\gamma}(1+w)\bar{C}_1}{\left[\lambda^{1/\gamma} - (\lambda^{P1-P5})^{1/\gamma}\right](1+w)} < \bar{C}_1 \qquad (41)
\end{aligned}
$$

$$
\alpha^{P5} = \frac{1 - K_0^{1/\gamma}}{r_f - r_b + K_0^{1/\gamma}(r_g - r_f)} \times \frac{\left(\lambda^{P1-P5}\right)^{1/\gamma}}{\lambda^{1/\gamma} - (\lambda^{P1-P5})^{1/\gamma}} \times (-\Omega) > 0 \qquad (42)
$$

Proof. See Appendix B. ■

Here, too, the degree of loss aversion enters the solution, as in Proposition 2. However, loss aversion has a different impact on the consumption pattern. An increase in loss aversion will increase the first period consumption (and thus reduce the relative consumption losses in the first period) as well as the second period consumption reference level. On the other hand an increase in loss aversion will reduce the second period consumption in the good state of nature and also the relative reference consumption in both states of nature. Higher loss aversion will also reduce investment in the financial market and the happiness level, like in problem (P2). Finally, the second period consumption in the bad state of nature will be reduced with higher loss aversion if the habit persistence in consumption is sufficiently low, it will be enhanced if the habit persistence in consumption is sufficiently large, and it will remain the same if the habit persistence in consumption equals some threshold depending on the parameters describing the financial market.[22] Note that in the case with an exogenous second period consumption reference level C_{2b}^{P5} is decreasing with an increasing λ, while the other effects remain the same (in terms of sign), see Hlouskova, Fortin and Tsigaris (2017).

The effect of the first period reference level on the first period consumption is negative, while it is positive for the second period consumption in the good state, and mixed[23] for the second period consumption in the bad state. Also the effect of the first period reference level on the second period endogenous reference level is not unambiguous. For sufficiently loss averse households the effect of an increasing first period reference level is positive, while for households with a smaller loss aversion it is negative. Moreover, an increasing first period reference level enhances current relative losses and future relative gains in both states of nature. Also risk taking increases with a higher first period reference level. Finally, the happiness level shrinks when the first period consumption reference increases. Note that the sensitivities with respect to the first period reference level coincide (in terms of signs) with those related to an exogenous second period reference level, see Hlouskova, Fortin and Tsigaris

[22]This threshold value is equal to $\frac{r_g - r_b}{r_f - r_b}\,\frac{K_0^{1/\gamma}}{1-K_0^{1/\gamma}}$, and it is smaller than one, i.e., binding, only for a sufficiently large probability of the good state of nature, p.

[23]The effect is positive if the household is sufficiently loss averse.

(2017).

The effect of habit persistence in consumption, w, on the second period reference level is negative, while its effect on the investment in the risky asset, relative gains in the second period as well as on the happiness level is positive. The effects on consumption in the first and second periods and on relative losses in the first period cannot be determined unambiguously. For sufficiently loss averse households, however, the effect of w on the first period consumption is positive (and hence negative on current relative losses), while it is negative for the second period consumption in both states of nature (and vice versa).

Regarding the sensitivity analysis with respect to income, an increase in income increases first period consumption, the second period reference level and the happiness level, while it decreases risk taking and relative consumption in both periods. The income effect on consumption in the second period (in the good and in the bad states) cannot be determined unambiguously, it can be either positive or negative. The effect is positive if the household is sufficiently loss averse, while it is negative if the household is not that loss averse.[24] These income effects are partially different from those related to the less ambitious household: while the effect on the first period consumption is the same (in terms of sign), the effect on risk taking is opposite, and the effect on second period consumption can be either the same (if the household is sufficiently loss averse) or opposite. The reason for the difference is that the more ambitious household cannot consume above its reference levels at all times and incurs relative losses in the first period. Consequently, an extra amount of income is rather used to increase consumption in the first period, in order to decrease relative losses in the first period, than to increase consumption in the second period when it is anyway above the reference level. In addition, the less ambitious household increases its risk taking in the financial market with increasing income, while the more ambitious household reduces its risk taking. Finally, the main difference between the more ambitious household with a lower time preference, (P5), and the more ambitious household with a higher time preference, (P2), is that the household with a lower time preference increases its first period consumption and the second period reference consumption when income increases while the household with a higher time preference decreases its first period consumption and the second period reference consumption with a growing income. Note finally that if the first period reference consumption level is equal to a fraction of the present value of the total wealth, i.e., $\bar{C}_1 = c\left(Y_1 + \frac{Y_2}{1+r_f}\right)$ where $c > \frac{1}{2}$,[25] then, as in the case with a more ambitious household with a higher time preference, the majority of sensitivity results become opposite in sign. This suggests again that the indirect effect of income through the first period consumption reference level is stronger than the

[24] Note again that an unexpected reduction in income can change the household's degree of ambition. Let the household be originally less ambitious. Then a drop of income (while keeping the first period reference level constant) will also decrease the threshold $\bar{C}_1^{U,P1}$, which may change the household to be more ambitious, as the first period reference level might then exceed its threshold ($\bar{C}_1 > \bar{C}_1^{U,P1}$) and the household will face losses in the first period.

[25] The fraction needs to be larger than one half such that the household is more ambitious.

direct income effect. Thus, in this case the happiness will again decrease with an increasing income.

Table 2 summarizes the sensitivity results related to the solutions of problem (P5), which we discussed above.

	$C_1^* = C_1^{P2}$ and $\alpha^* = \alpha^{P2}$								
	dC_1^*	dC_{2g}^*	dC_{2b}^*	$d\alpha^*$	$d\bar{C}_2$	$d(C_1^* - \bar{C}_1)$	$d(C_{2g}^* - \bar{C}_2)$	$d(\bar{C}_2 - C_{2b}^*)$	$d(\mathbb{E}(U(C_1^*, \alpha^*)))$
$d\lambda$	< 0	< 0	> 0	< 0	< 0	< 0	< 0	< 0	< 0
$d\bar{C}_1$	> 0	> 0	< 0	> 0	> 0	> 0	> 0	> 0	< 0
dw	< 0	$\lesseqgtr 0$	> 0	< 0	$\lesseqgtr 0$	< 0	< 0	< 0	< 0
dY_i	< 0	< 0	> 0	< 0	< 0	< 0	< 0	< 0	> 0
	$C_1^* = C_1^{P5}$ and $\alpha^* = \alpha^{P5}$								
	dC_1^*	dC_{2g}^*	dC_{2b}^*	$d\alpha^*$	$d\bar{C}_2$	$d(\bar{C}_1 - C_1^*)$	$d(C_{2g}^* - \bar{C}_2)$	$d(C_{2b}^* - \bar{C}_2)$	$d(\mathbb{E}(U(C_1^*, \alpha^*)))$
$d\lambda$	> 0	< 0	$\lesseqgtr 0$	< 0	> 0	< 0	< 0	< 0	< 0
$d\bar{C}_1$	< 0	> 0	$\gtreqless 0$	> 0	$\gtreqless 0$	> 0	> 0	> 0	< 0
dw	$\gtreqless 0$	$\lesseqgtr 0$	$\lesseqgtr 0$	> 0	< 0	$\lesseqgtr 0$	> 0	> 0	> 0
dY_i	> 0	$\gtreqless 0$	$\gtreqless 0$	< 0	> 0	< 0	< 0	< 0	> 0

Table 2: Sensitivity results for the more ambitious household with respect to λ, $\bar{C}_1$, w, Y_i, $i = 1, 2$.

Finally, the value of the expected utility at the optimum can be determined by the relative losses in the first period:

$$\begin{aligned}(1-\gamma)\mathbb{E}\left(U\left(C_1^{P5}, \alpha^{P5}\right)\right) &= -\left[\frac{-\Omega}{(1+r_f)(1+w)}\right]^{1-\gamma}\left[\lambda^{1/\gamma} - (\lambda^{P1-P5})^{1/\gamma}\right]^{\gamma} \\ &= -\lambda\left[1 - \left(\frac{\lambda^{P1-P5}}{\lambda}\right)^{1/\gamma}\right](\bar{C}_1 - C_1^{P5})^{1-\gamma} \qquad (43)\end{aligned}$$

The effect of income on happiness is positive, while loss aversion and the first period reference level impact the level of happiness negatively, see Table 2. These results are the same (in terms of signs) as those for problem (P2). The only difference (between the two more ambitious households differing in the rate of time preference) is in the effect of the persistence of current period consumption on the happiness level. It is positive for the more ambitious household with a lower time preference, while it is negative for the more ambitious household with a higher time preference. The positive effect of income on happiness and the negative effect of the first period reference level are also the same as for the less ambitious household.

Before proceeding further let us introduce the following notation:

$$\lambda^{P2-P6}_{\bar{C}_1=\bar{C}_1^{U,P2}} = \frac{(1+r_f)p}{\frac{1}{\delta}-\frac{1}{\delta^{P2-P6}}}\left(\frac{r_g-r_b}{r_f-r_b}+w\right)^{1-\gamma} \quad (44)$$

$$\delta(\lambda)^{P2-P5}_{\bar{C}_1=\bar{C}_1^{U,P2}} = \delta^{P2-P5}\left[1-\left(\frac{\lambda^{P1-P5}}{\lambda}\right)^{1/\gamma}\right]^{\gamma} \quad (45)$$

$$\delta^{P2-P6} = \frac{\left[\frac{r_g-r_b}{r_f-r_b}\frac{Y_1-\bar{C}_1-C_L+\frac{Y_2}{1+r_f}}{\bar{C}_1^{U,P2}-C_L}-w\right]^{\gamma}}{(1-p)\,w} \quad (46)$$

The following corollary describes the results when the current consumption reference level reaches its upper bound. Note that, similarly as in Corollary 1, the first period reference level is a function of the household's income, but also its minimum consumption in the first period and the (risk-free and risky) financial returns are part of the solution.

Corollary 2 *Let* $\bar{C}_1 = \bar{C}_1^{U,P2}$, $\lambda > \max\left\{\lambda^{P2-P4}, \lambda^{P5}, \lambda^{P2-P6}_{\bar{C}_1=\bar{C}_1^{U,P2}}\right\}$ *and* $\delta < \min\left\{\delta(\lambda)^{P2-P5}_{\bar{C}_1=\bar{C}_1^{U,P2}}, \delta^{P2-P6}\right\}$. *Then problem (6) obtains a unique maximum at* (C_1^*, α^*), *where*

$$C_1^* = \bar{C}_1 = \bar{C}_1^{U,P2}$$

$$\alpha^* = \frac{(1+r_f)\left(Y_1-\bar{C}_1^{U,P2}-C_L\right)+Y_2}{r_f-r_b} > 0$$

Proof. See Appendix B. ■

Note that in this case, $C_1 = \bar{C}_1 = \bar{C}_1^{U,P2}$, $C_{2g} = \bar{C}_2 = (1+r_f)\bar{C}_1^{U,P2}$ and $C_{2b} = (1+r_f)C_L$. The solution implies that the household can only consume its minimum level in the second period in the bad state, and it consumes exactly the reference level in the first period and in the second period in the good state. This household, like the other ones but unlike the balanced household, engages positively in the stock market. Any further increase in the first period reference level, while keeping the other parameters constant (such as income, the scarcity constraint and/or returns of the safe and risky assets)[26], i.e., $\bar{C}_1 > \bar{C}_1^{U,P2}$, will result in a state where the household faces relative losses either in the second period (under both states of nature) while keeping gains in the first period, (P4), or where it faces relative losses only in the first period while keeping sure gains in the second period (in both states of nature), (P5).[27]

[26] Or any reduction in income while keeping the other parameters (such as the first period reference level, the scarcity constraint and/or returns of the safe and risky assets) constant.

[27] The solutions actually depend on the following threshold levels for the current reference consumption, namely on $\bar{C}_1^{U,P2} < \bar{C}_1^{U,P4} < \bar{C}_1^{U,P5} < \bar{C}_1^{U,P6}$. For $\bar{C}_1^{U,P2} < \bar{C}_1 \leq \bar{C}_1^{U,P4}$ the household that values future

3.4 Income and other effects – a summary

In this section we summarize the income effects and other effects on happiness, (relative) consumption and risk taking across less ambitious and more ambitious households.

Income effects

All other things equal, more income, as in good economic times, is good: for both less ambitious and more ambitious sufficiently loss averse households with a constant first period consumption reference level a higher income increases the household's happiness. For the less ambitious household, which always consumes above its reference levels, this works through higher current relative gains, see (22). For the more ambitious household which places less weight to the future and thus faces relative losses in the bad state of nature in the second period, this works through smaller current relative gains, see (39),[28] while for the more ambitious household which places more weight to the future and thus faces relative losses in the first period, this works through a reduction of the current relative losses, see (43). Note that the relation "more income – more happiness" is not true for the more ambitious household, when its first period reference level depends on income, e.g., when it coincides with a fraction (that exceeds one half) of the present value of its wealth. In this case more income will imply a larger first period consumption reference level and will actually lead to less happiness, which is due to the indirect effect through the first period reference level outweighing the direct effect of income.

In addition, the less ambitious household increases its investment in the risky asset with good economic times as income increases, while the more ambitious household decreases its exposure to risky financial markets in good economic times and increases its exposure in bad economic times.[29] See Tables 1 and 2 for the described sensitivity results with respect to the first and second period income.

Note that the effects of an increase of income, as discussed above, are the same as the effects of a decrease of the tax rate when we assume a taxation of income endowment. See Section 3.5 for more details.

consumption more, faces relative losses only in the current period, (P5), while the household that values current consumption more achieves relative gains only in the current period, (P4). For $\bar{C}_1^{U,P4} < \bar{C}_1 \leq \bar{C}_1^{U,P5}$ the household faces relative losses only in the current period, (P5). For even higher reference consumption, $\bar{C}_1^{U,P5} < \bar{C}_1 \leq \bar{C}_1^{U,P6}$, the household achieves relative gains only in the good state in the second period, (P6), and finally for the largest values of reference consumption, namely $\bar{C}_1 > \bar{C}_1^{U,P6}$, the household faces relative losses in both periods, (P8).

[28]Note that these smaller relative gains in the first period go hand in hand with smaller relative losses in the second period in the bad state.

[29]Technically speaking, we must consider the effects of larger first and second period income separately. As they are always the same (in terms of signs), however, we simply talk about the effects of income.

Note finally that an increase of current income increases savings for both less and more ambitious households.[30] The effect of second period income on savings is opposite to the effect of second period income on current consumption for both types of households. Thus, savings are discouraged for less ambitious households and for more ambitious households with a low time preference (that achieve relative gains in the second period), while they are encouraged for more ambitious households with a high time preference (that face relative losses in the bad state of nature in the second period).

Effects of the first period consumption reference level

Ceteris paribus, a higher first period consumption reference level is bad: both the less ambitious and the more ambitious households will be less happy with a larger first period reference level, i.e., a higher comparison level decreases happiness. Thus, the household seems to be happiest when it does not compare itself to anybody at all. For the less ambitious household being less happy with a rising first period reference level works through lower relative gains in both periods as the relative gains shrink with an increasing first period reference level. However, for the more ambitious household, this works only through larger relative losses, enhanced by the penalty on losses, as both relative gains and losses increase with an increasing first period consumption reference level.

The reaction of the less ambitious household's consumption to an increase in the first period reference level is ambiguous: A household with a smaller weight placed to the future will decrease its current consumption and increase its future consumption while the opposite happens for a household with a larger weight placed to the future. In the case of a more ambitious household with a higher time preference, the current consumption as well as the second period consumption in the good state of nature increase with increasing $\bar{C}_1$ while the second period consumption in the bad state shrinks. On the other hand, in the case of a more ambitious household with a lower time preference, the increase of the first period reference level will cause a reduction in the current consumption but an increase in the future consumption.[31] Thus, for instance, if $\bar{C}_1$ is equal to the consumption of a reference household (the Joneses) then our household is following the Joneses[32] when it is either less ambitious with a low time preference or when it is more ambitious with a high time preference. Note that in the case of an exogenous second period consumption reference level, see Hlouskova,

[30] Only more ambitious households with a low time preference need to be sufficiently loss averse. If the degree of loss aversion of these households, (P5), is not too large then savings will decrease with an increasing current income. This may happen for households with a small persistence to current consumption.

[31] This holds also for the second period consumption in the bad state of nature, when the household is sufficiently loss averse.

[32] The household is *following the Joneses* when the increase, or decrease, of the first period consumption of a reference household (the Joneses) impacts this household such that its current consumption will change in the same way as the one of the Joneses, i.e., it will increase if the current consumption of the Joneses increases and vice versa. Note that in this context the household's first period reference consumption is equal to the current consumption of the Joneses.

Fortin and Tsigaris (2017), the less ambitious household follows the Joneses irrespective of its time preference.

A larger first period reference level also implies a larger second period reference level, except for the more ambitious household with a low rate of time preference (high discount factor), where the effect can be positive or negative. Finally, the effect of a rising first period reference level upon the household's investment in the risky asset is negative for the less ambitious household, and positive for the more ambitious household. Thus, the risk taking decreases for less ambitious households with an increasing current consumption reference level while it increases for more ambitious households when the current reference level increases. See Tables 1 and 2 for the sensitivity results with respect to the first period consumption reference level.

Effects of loss aversion

All other things equal, the degree of loss aversion does not have any effects on the less ambitious household's happiness, nor on its consumption or its investment in the risky asset.[33] On the other hand, the more ambitious household is less happy with an increasing level of loss aversion, which is triggered solely by shrinking relative gains (whenever there are gains). An increasing level of loss aversion shows opposite effects (in terms of signs) on the second period reference level, for different time preferences. The effect is negative for a high time preference, and it is positive for a low time preference. Technically speaking, this works through the impact of loss aversion on first period consumption (which is negative for a high time preference, and positive for a low time preference). Finally, a higher degree of loss aversion implies a lower investment in the risky asset for the more ambitious household, which is what one would probably expect. See Tables 1 and 2 for the sensitivity results with respect to loss aversion.

Effects of the habit persistence in consumption

The effect of persistence in consumption on happiness depends on whether the household's first period optimal consumption is above or below the reference level. Whatever is smaller – either consumption or the reference level – should be followed more intensely (in the formation of the second period reference level) in order to increase happiness. If first period consumption is above the reference level, then increasing habit persistence in consumption makes the household less happy while increasing persistence in the consumption reference level makes it happier. Thus the household should intensify its persistence on the consumption target. This situation applies to the less ambitious household and the more ambitious household with a higher time preference. On the other hand, if first period consumption is below the reference

[33] The assumption on the degree of loss aversion to be sufficiently large is to guarantee that the maximum of (P1) exceeds the potential maxima of (P2) and (P5).

level, then growing habit persistence on consumption leads to more happiness while increasing persistence on the consumption reference level results in less happiness. Hence the household should stick more to its consumption habits. This applies to the more ambitious household with a lower time preference.

For the less ambitious household the decrease in happiness materializes only through a decline of the first period consumption (or, equivalently, through a decline of the first period relative gains), as the second period relative gains actually increase with an increasing persistence in consumption, see Table 1. On the other hand, for the more ambitious household with a sufficiently small discount factor the decrease in happiness is triggered by a decrease of both the first period relative gains and the second period relative gains when the good state of nature occurs. Finally, for the more ambitious household with a sufficiently large discount factor the increase in happiness is caused by a decrease of the relative losses in the first period (for a sufficiently loss averse household) as well as by an increase of the relative gains in the second period.

3.5 Implications for income taxation

Our analysis has important implications in terms of how a household responds to income reductions due to the impact of taxation of endowment income.[34] In fact an increase in the tax rate is equivalent to a decrease in income in the model without taxes. The effects of taxation will depend on whether the household has a low or a high first period consumption reference level, hence on the type of household. Suppose suddenly income is taxed and the household is less ambitious such that it only experiences relative gains. Then increased taxation will reduce current consumption, future consumption in both states of nature, risk taking, second period reference level, relative gains and happiness.

Suppose, then, the household is more ambitious with a high time preference (i.e., it values more current consumption) and thus experiences relative losses in the bad state of nature in the second period. Increased taxation of income in this case will increase current consumption, risk taking, consumption in the good state of nature, the second period reference level, current relative gains, second period relative gains in the good state of nature and second period relative losses in the bad state of nature, which is opposite to the response of a less ambitious household towards taxation of income. On the other hand, increased taxation will reduce consumption in the bad state of nature as well as happiness.

Suppose, further, the household is more ambitious with a low time preference (i.e., it values more future consumption) and is thus willing to experience relative losses in the first period, then increased taxation will discourage current consumption but stimulate risk taking while the direction of future consumption is ambiguous and happiness will decrease. In terms

[34] The corresponding model set-up is the same as presented by (6), only income is replaced by after-tax income in all formulations, propositions and corollaries. I.e., Y_1 is replaced by $(1-\tau)Y_1$ and Y_2 is replaced by $(1-\tau)Y_2$, where $\tau \in (0,1)$ is the tax rate of income.

of relative gains and losses increased taxation of income will increase relative losses in the first period as well as relative gains in the future.

Finally, if the household is at the threshold level, i.e., if it is balanced, then a sudden increase in taxation (while keeping all other parameters unchanged) will reduce the present value of after-tax income and thus the threshold level of the current reference consumption $\bar{C}_1^{U,P1}$ will shrink, which in turn makes the household more ambitious.

For the discussion of the effects of income taxes on savings it is reasonable to assume that the tax rates on current and future income are not independent. For simplicity we assume that they are the same. Then a higher income tax (which induces lower disposable income) discourages savings for the more ambitious household with a high time preference, and it also discourages savings for the less ambitious household and the more ambitious household with a low time preference provided second period income is small enough.[35] On the other hand, if second period income is larger than the threshold then a higher tax rate stimulates savings for the less ambitious household and the more ambitious household with a low time preference. Note that if the household has a sufficiently large persistence in current consumption and the second period is the retirement period then more plausible is the case when second period income does not exceed its threshold.[36]

A particularly interesting result is the impact of taxation on risk taking. Taxation of income will discourage risk taking for less ambitious households, while for more ambitious households, irrespective of their rate of time preference, taxation will increase risk taking. Finally, taxation makes a household less happy irrespective of its first period reference level.

$d\tau$	dC_1^*	dC_{2g}^*	dC_{2b}^*	$d\alpha^*$	$d\bar{C}_2$	$d\lvert C_1^* - \bar{C}_1\rvert$	$d\lvert C_{2g}^* - \bar{C}_2\rvert$	$d\lvert C_{2b}^* - \bar{C}_2\rvert$	dS^*	dU^*
$\bar{C}_1 < \bar{C}_1^{U,P1}$	< 0	< 0	< 0	< 0	< 0	< 0	< 0	< 0	$\lesseqgtr 0$	< 0
$\bar{C}_1 > \bar{C}_1^{U,P1}$, $\delta \leq \delta^{P2-P5}$	> 0	> 0	< 0	> 0	> 0	> 0	> 0	> 0	< 0	< 0
$\bar{C}_1 > \bar{C}_1^{U,P1}$, $\delta > \delta^{P2-P5}$	< 0	$\lesseqgtr 0$	$\lesseqgtr 0$	> 0	< 0	> 0	> 0	> 0	$\lesseqgtr 0$	< 0

Table 3: Sensitivity results for less ambitious and more ambitious households with respect to the income tax.

4 Concluding remarks

In this paper we analyze the two-period consumption-investment decision of a household with prospect theory preferences and an endogenous second period reference level which captures habit persistence in consumption and in the current consumption reference level. We find that the optimal solution of a sufficiently loss averse household depends on how its first period consumption reference level relates to a given threshold which is equal to the average

[35] In the latter case the household, in addition, needs to be sufficiently loss averse.

[36] The threshold is given by $(1 + r_f)wY_1 + \tilde{k}$, where $\tilde{k} \geq 0$.

discounted endowment income. The reference level may be below, equal to, or above this threshold and hence households can be of three types. These three types are characterized by how their optimal consumption relates to their reference consumption. First there are households with a relatively low reference level (less ambitious households), which can avoid relative consumption losses in both periods. This means that they always consume above their reference levels. Second there are balanced households with a neutral reference level, which always consume exactly their reference levels. This type of household, however, is very special and can only occur when its first period reference level is equal to the average of the discounted income. Third there are households with a relatively high reference level (more ambitious households), which cannot avoid to incur relative consumption losses, either now or in the future. More precisely, a more ambitious household with a lower discount factor (high time preference) will face relative losses in the second period in the bad state of nature while a more ambitious household with a higher discount factor (low time preference) incurs relative losses in the first period. Note that the three types of household sometimes act very differently from one another and thus there is a diversity of behavior resulting from the different levels of comparison.

We observe the following effects of habit persistence in consumption. A less ambitious household will be less happy with a rising persistence in consumption, but at the same time it will be happier with a rising persistence in the reference consumption. Hence it is better to stick to one's exogenously given consumption target than to one's consumption habits. The same applies to the more ambitious household with a high time preference. However, the situation is reverse for the more ambitious household with a low time preference: it will be happier if it sticks more to its consumption habits than to its target, i.e., if it intensifies its consumption habits. In addition clinging to one's consumption habits decreases current consumption for the less ambitious household and the more ambitious household with a relatively high time preference, while evidence is mixed for the more ambitious household with a relatively low time preference.

It is always true that more income is better, i.e., the larger the income, the happier the household – provided the first period reference level does not depend on income. However, if the reference level depends on income in the sense that it is equal to a fraction of the present value of total wealth and the household is more ambitious, then a higher income reduces happiness. This is due to the fact that in this case the indirect effect of income (through the first period reference level) outweighs the direct effect of income. We also observe that less ambitious households increase their exposure to risky assets during good economic times (i.e., when their income increases) while the more ambitious households increase their exposure to risky assets during bad economic times (i.e., when their income decreases).

Finally, we obtain the same findings as in Hlouskova, Fortin and Tsigaris (2017) related to the dependence of happiness upon the current consumption reference level: the highest utility is achieved for the lowest current consumption reference level (while keeping everything else unchanged). Thus, not comparing at all (e.g., to others) leads to the highest level of happiness.

We also discuss the effects of taxation of endowment income: increasing the tax rate in a model with income taxes is actually equivalent to decreasing income in a model without taxes. An interesting extension would certainly be to examine the household's optimal consumption-investment behavior if also capital income is taxed.

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Appendix A: Optimization problems

Before proceeding further, we introduce (or re-introduce) the following notation

$$\begin{aligned}
\Omega &= (1+r_f)(Y_1-\bar{C}_1)+Y_2-\left[w_0+(w_1+w_2)\bar{C}_1\right] \\
&= (1+r_f)Y_1+Y_2-w_0-(1+r_f+w_1+w_2)\bar{C}_1
\end{aligned}$$

$$\begin{aligned}
K_\gamma &= \frac{(1-p)(r_f-r_b)^{1-\gamma}}{p\,(r_g-r_f)^{1-\gamma}} \\
k &= \left[\delta(1+r_f+w_1)\,(1-p)\,\left(\frac{r_g-r_b}{r_g-r_f}\right)^{1-\gamma}\right]^{\frac{1}{\gamma}},\ \frac{dk}{dr_b}<0,\ \frac{dk}{dr_g}<0 && (47) \\
k_2 &= \left[\delta(1+r_f+w_1)\,p\,\left(\frac{r_g-r_b}{r_f-r_b}\right)^{1-\gamma}\right]^{\frac{1}{\gamma}} = \frac{k}{K_\gamma^{1/\gamma}},\ \frac{dk_2}{dr_b}>0,\ \frac{dk_2}{dr_g}>0 \\
M &= \left[\delta(1+r_f+w_1)\,p\,\frac{r_g-r_b}{r_f-r_b}\right]^{\frac{1}{\gamma}}\frac{r_f-r_b+K_0^{\frac{1}{\gamma}}(r_g-r_f)}{r_g-r_b} \\
&= k\left[1+\left(\frac{1}{K_\gamma}\right)^{1/\gamma}\right] = k_2\left(1+K_\gamma^{1/\gamma}\right) \\
&= (1+r_f+w_1)\left(\lambda^{P1-P5}\right)^{1/\gamma},\quad \frac{dM}{dr_b}>0,\ \frac{dM}{dr_g}>0 \\
M(\lambda) &= k\left[\lambda^{1/\gamma}-\left(\frac{1}{K_\gamma}\right)^{1/\gamma}\right] = k_2\left[(\lambda K_\gamma)^{1/\gamma}-1\right],\ \frac{dM(\lambda)}{dr_b}<0,\ \frac{dM(\lambda)}{dr_g}<0 \\
\bar{C}_1^{U,P1} &= \frac{(1+r_f)Y_1+Y_2-w_0}{1+r_f+w_1+w_2} \\
\bar{C}_1^{U,P2} &= \frac{(r_g-r_b)\left[(1+r_f)Y_1+Y_2\right]-(r_f-r_b)w_0-(r_g-r_f)C_{2L}}{(r_g-r_b)(1+r_f)+(r_f-r_b)(w_1+w_2)} \\
\bar{C}_1^{L,P4} &= \frac{(1+r_f+w_1)C_{2L}-w_1[(1+r_f)Y_1+Y_2]-(1+r_f)w_0}{(1+r_f)w_2} && (48) \\
\bar{C}_1^{U,P4} &= Y_1+\frac{Y_2-C_{2L}}{1+r_f} && (49) \\
\bar{C}_1^{U,P5} &= \frac{(1+r_f)Y_1+Y_2-w_0-(1+r_f+w_1)C_{1L}}{w_2} && (50) \\
\bar{C}_1^{U,P6} &= \frac{1}{w_2}\left[\frac{r_g-r_b}{r_f-r_b}\left((1+r_f)(Y_1-C_{1L})+Y_2-C_{2L}\right)-w_0-w_1C_{1L}+C_{2L}\right] && (51) \\
\bar{C}_1^{U,P6,(iii)} &= \frac{1}{w_2}\left[\frac{r_g-r_b}{r_f-r_b}\left((1+r_f)(Y_1-C_{1L})+Y_2-C_{2L}\right)-w_0-(1+r_f+w_1)C_{1L}\right] && (52)
\end{aligned}$$

$$C_{2L}^U = \frac{r_g - r_b}{r_g - r_f}\left[(1+r_f)(Y_1 - \bar{C}_1) + Y_2\right] - \frac{r_f - r_b}{r_g - r_f}\left[w_0 + (w_1 + w_2)\bar{C}_1\right]$$

$$\begin{aligned}
\lambda^{P2} &= \left[\frac{\frac{1+r_f+w_1}{k} + \left(\frac{1}{K_\gamma}\right)^{1/\gamma} - \frac{(-\Omega)}{w_0+(w_1+w_2)\bar{C}_1 - C_{2L}}\frac{w_1}{k}}{1 - \frac{(-\Omega)}{w_0+(w_1+w_2)\bar{C}_1-C_{2L}}\frac{r_g-r_b}{r_g-r_f}}\right]^\gamma \\
&= \left[\frac{1+r_f+w_1}{k} + \left(\frac{1}{K_\gamma}\right)^{1/\gamma}\right]^\gamma \\
&\quad \times \left[\frac{\frac{1}{1+r+k_2+w_1}\left[((1+r_f)(Y_1-\bar{C}_1)+Y_2)w_1 - (w_0+(w_1+w_2)\bar{C}_1)(1+r_f+k_2)\right] - C_{2L}}{C_{2L}^U - C_{2L}}\right]^\gamma \\
&= \left[\frac{1+r_f+w_1}{k} + \left(\frac{1}{K_\gamma}\right)^{1/\gamma}\right]^\gamma \left[1 + \frac{\frac{r_f-r_b}{r_g-r_f} + \frac{1+r_f+k_2}{1+r_f+k_2+w_1}}{C_{2L}^U - C_{2L}}(-\Omega)\right]^\gamma \\
&\quad \text{for } \; C_{2L} < C_{2L}^U \; \text{ and } \; \bar{C}_1^{U,P1} < \bar{C}_1 < \bar{C}_1^{U,P2}
\end{aligned}$$

$$\begin{aligned}
\lambda^{P1-P2} &= \frac{\left[\left(\frac{r_f-r_b}{(r_g-r_b)(1+r_f)+w_1(r_f-r_b)}\right)^{1-\gamma} + \delta p\right]\left[\Omega + (r_g - r_f)\alpha_{C_{2b}=C_{2L}}^{C_1=\bar{C}_1}\right]^{1-\gamma}}{\delta(1-p)\left[w_0 + (w_1+w_2)\bar{C}_1 - C_{2L}\right]^{1-\gamma}} \\
&\quad - \frac{\Omega^{1-\gamma}(1+r_f+w_1+M)^\gamma}{\delta(1-p)(1+r_f+w_1)\left[w_0+(w_1+w_2)\bar{C}_1 - C_{2L}\right]^{1-\gamma}} \quad \text{for } \; \bar{C}_1 \le \bar{C}_1^{P1}
\end{aligned}$$

$$\lambda^{P2-P2} = \frac{\left[\left(\frac{r_f-r_b}{(r_g-r_b)(1+r_f)+w_1(r_f-r_b)}\right)^{1-\gamma} + \delta p\right]\left[\Omega + (r_g - r_f)\alpha_{C_{2b}=C_{2L}}^{C_1=\bar{C}_1}\right]^{1-\gamma}}{\delta(1-p)\left[\left(w_0 + (w_1+w_2)\bar{C}_1 - C_{2L}\right)^{1-\gamma} - \left(\frac{r_g-r_b}{r_g-r_f}\right)^{1-\gamma}(-\Omega)^{1-\gamma}\right]}$$

$$\lambda^{P2-P4} = \frac{1}{\delta p}\left[\frac{r_f - r_b}{(1+r_f)(r_g-r_b) + w_1(r_f-r_b)}\right]^{1-\gamma}$$

$$\begin{aligned}
\lambda^{P4} &= \frac{1}{w_1\,\delta}\left[\frac{w_0 + w_1\left(Y_1 - \bar{C}_1 + \frac{Y_2 - C_{2L}}{1+r_f}\right) + (w_1+w_2)\bar{C}_1 - C_{2L}}{\alpha_{C_{2b}=C_{2L}}^{C_1=\bar{C}_1} - \alpha_{C_{2b}=C_{2L}}^{C_{2g}=\bar{C}_2}}\frac{1+r_f}{r_f-r_b}\right]^\gamma \\
&\quad \text{for } \; \bar{C}_1 < \bar{C}_1^{U,P2}
\end{aligned}$$

$$
\begin{aligned}
\lambda^{P5} &= \lambda^{P1-P5}\left[\frac{(1+r_f+w_1)(\bar{C}_1-C_{1L})}{w_2\left(\bar{C}_1^{U,P5}-\bar{C}_1\right)}\right]^{\gamma} \\
\lambda^{P1-P5} &= \left[\frac{k_2\left(1+K_{\gamma}^{\frac{1}{\gamma}}\right)}{1+r_f+w_1}\right]^{\gamma} = \left[\frac{k\left(1+\left(\frac{1}{K_{\gamma}}\right)^{\frac{1}{\gamma}}\right)}{1+r_f+w_1}\right]^{\gamma} = \left(\frac{M}{1+r_f+w_1}\right)^{\gamma} \\
\lambda^{P5-P2} &= \left[\frac{\frac{k}{1+r_f+w_1}\frac{1}{K_{\gamma}^{1/\gamma}}+1}{\frac{k}{1+r_f+w_1}-1}\right]^{\gamma} \quad \text{for} \quad \frac{k}{1+r_f+w_1}>1 \quad \text{or} \quad \text{when} \quad \delta>\delta^{P2-P5} \qquad (53) \\
\lambda^{P2-P6} &= \frac{\frac{\delta p\left(\frac{(1+r_f)(r_g-r_b)}{r_f-r_b}+w_1\right)^2}{\left[\Omega+(r_g-r_f)\alpha_{C_{2b}=C_{2L}}^{C_1=\bar{C}_1}\right]^{1+\gamma}}}{\frac{1}{(\bar{C}_1-C_{1L})^{1+\gamma}}+\frac{\delta(1-p)w_1^2}{\left[w_0+(w_1+w_2)\bar{C}_1-C_{2L}\right]^{1+\gamma}}}
\end{aligned}
$$

$$
\begin{aligned}
\delta^{P2-P5} &= \frac{1}{1-p}\left[\frac{r_g-r_f}{(1+r_f+w_1)(r_g-r_b)}\right]^{1-\gamma} \\
\delta(\lambda)_{\bar{C}_1=\bar{C}_1^{U,P2}}^{P2-P5} &= \delta^{P2-P5}\left[1-\left(\frac{\lambda^{P1-P5}}{\lambda}\right)^{1/\gamma}\right]^{\gamma} \\
\alpha_{C_{2b}=C_{2L}}^{C_{2g}=\bar{C}_2} &= \frac{(1+r_f+w_1)\left[w_0+(w_1+w_2)\bar{C}_1-C_{2L}\right]+w_1\Omega}{(1+r_f)(r_g-r_b)+w_1(r_f-r_b)} \\
&= \frac{(1+r_f)\left(w_0+w_2\bar{C}_1-C_{2L}\right)+w_1\left[(1+r_f)Y_1+Y_2-C_{2L}\right]}{(1+r_f)(r_g-r_b)+w_1(r_f-r_b)} \\
&= \frac{(1+r_f)\left(w_0+w_2\bar{C}_1\right)+w_1\left[(1+r_f)Y_1+Y_2\right]-(1+r_f+w_1)C_{2L}}{(1+r_f)(r_g-r_b)+w_1(r_f-r_b)} \qquad (54) \\
&= \frac{(1+r_f)\left[w_0+(w_1+w_2)\bar{C}_1-C_{2L}\right]+w_1\left[(1+r_f)(Y_1-\bar{C}_1)+Y_2-C_{2L}\right]}{(1+r_f)(r_g-r_b)+w_1(r_f-r_b)}
\end{aligned}
$$

$$
\alpha_{C_{2b}=C_{2L}}^{C_1=\bar{C}_1} = \frac{(1+r_f)(Y_1-\bar{C}_1)+Y_2-C_{2L}}{r_f-r_b} \qquad (55)
$$

$$
\alpha_{C_{2b}=C_{2L}}^{C_1=C_{1L}} = \frac{(1+r_f)(Y_1-C_{1L})+Y_2-C_{2L}}{r_f-r_b} \qquad (56)
$$

$$
\alpha_{C_{2b}=\bar{C}_2}^{C_1=\bar{C}_1} = \frac{\Omega}{r_f-r_b} = \frac{(1+r_f)Y_1+Y_2-w_0-(1+r_f+w_1+w_2)\bar{C}_1}{r_f-r_b} \qquad (57)
$$

$$
\alpha_{C_{2g}=\bar{C}_2}^{C_1=\bar{C}_1} = \frac{-\Omega}{r_g-r_f} \qquad (58)
$$

$$\alpha^{C_1=C_{1L}}_{C_{2g}=\bar{C}_2} = \frac{w_0 + (1+r_f+w_1)C_{1L} + w_2\bar{C}_1 - (1+r_f)Y_1 - Y_2}{r_g - r_f} = \frac{-\Omega - (1+r_f+w_1)(\bar{C}_1 - C_{1L})}{r_g - r_f} \tag{59}$$

$$\alpha^{C_1=C_{1L}}_{C_{2b}=\bar{C}_2} = \frac{(1+r_f)Y_1 + Y_2 - w_0 - (1+r_f+w_1)C_{1L} - w_2\bar{C}_1}{r_f - r_b} = \frac{\Omega + (1+r_f+w_1)(\bar{C}_1 - C_{1L})}{r_f - r_b} \tag{60}$$

$$\alpha^{C_{2b}=\bar{C}_2}_{C_{2b}=C_{2L}} = \frac{1}{r_f - r_b}\left[(1+r_f)Y_1 + Y_2 - C_{2L} + \frac{1+r_f}{w_1}\left(w_0 + w_2\bar{C}_1 - C_{2L}\right)\right] \tag{61}$$

The following lemma summarizes the basic properties of the threshold values for the reference level $\bar{C}_1$ and for the risky investment α.

Lemma 1 (i) $C^U_{2L} \geq \frac{(w_1+w_2)[(1+r_f)Y_1+Y_2]+(1+r_f)\,w_0}{1+r_f+w_1+w_2} \iff \bar{C}_1 \leq \bar{C}^{U,P1}_1$ *and* $C^U_{2L} < \frac{(w_1+w_2)[(1+r_f)Y_1+Y_2]+(1+r_f)\,w_0}{1+r_f+w_1+w_2} \iff \bar{C}_1 > \bar{C}^{U,P1}_1$.

(ii) *If* $C_{2L} \leq \frac{1+r_f}{1+r_f+w_1}\left[w_0 + w_1\left(Y_1 + \frac{Y_2}{1+r_f}\right) + w_2\bar{C}_1\right]$ *then* $\alpha^{C_{2g}=\bar{C}_2}_{C_{2b}=C_{2L}} \geq 0$. *However,* $C_{2L} \leq w_0 + w_1C_{1L} + w_2\bar{C}_1$ *is sufficient for* $\alpha^{C_{2g}=\bar{C}_2}_{C_{2b}=C_{2L}}$ *to be nonnegative.*

(iii) *If* $\bar{C}_1 < \bar{C}^{U,P2}_1$ *then* $\alpha^{C_1=\bar{C}_1}_{C_{2g}=\bar{C}_2} < \alpha^{C_{2g}=\bar{C}_2}_{C_{2b}=C_{2L}} < \alpha^{C_1=\bar{C}_1}_{C_{2b}=C_{2L}}$.
If $\bar{C}_1 = \bar{C}^{U,P2}_1$ *then* $\alpha^{C_1=\bar{C}_1}_{C_{2g}=\bar{C}_2} = \alpha^{C_{2g}=\bar{C}_2}_{C_{2b}=C_{2L}} = \alpha^{C_1=\bar{C}_1}_{C_{2b}=C_{2L}}$.

(iv) *If* $\bar{C}_1 = \bar{C}^{U,P2}_1$ *then* $w_0 + (w_1+w_2)\bar{C}^{U,P2}_1 - C_{2L} = (r_g - r_b)\,\alpha^{C_1=\bar{C}_1}_{C_{2b}=C_{2L}}$.

(v) $\alpha^{C_{2g}=\bar{C}_2}_{C_{2b}=C_{2L}} \leq \alpha^{C_1=C_{1L}}_{C_{2b}=C_{2L}}$.

(vi) $\bar{C}_1 < \bar{C}^{U,P5}_1 \iff \alpha^{C_1=C_{1L}}_{C_{2g}=\bar{C}_2} < 0$ *and* $\alpha^{C_1=C_{1L}}_{C_{2b}=\bar{C}_2} > 0$,
$\bar{C}_1 < \bar{C}^{U,P4}_1 \iff \alpha^{C_1=\bar{C}_1}_{C_{2b}=C_{2L}} \geq 0$ *and*
$\bar{C}_1 \geq \bar{C}^{L,P4}_1 \iff \alpha^{C_{2g}=\bar{C}_2}_{C_{2b}=C_{2L}} \geq 0.$

(vii) $\alpha^{C_1=C_{1L}}_{C_{2g}=\bar{C}_2} \leq \alpha^{C_{2g}=\bar{C}_2}_{C_{2b}=C_{2L}}$ *and* $\alpha^{C_1=C_{1L}}_{C_{2b}=C_{2L}} \geq \alpha^{C_1=C_{1L}}_{C_{2b}=\bar{C}_2}$.

(viii) $\bar{C}^{U,P1}_1 < \bar{C}^{U,P2}_1 < \bar{C}^{U,P4}_1 \iff C_{2L} < \frac{(w_1+w_2)\left[(1+r_f)Y_1+Y_2\right]+(1+r_f)w_0}{1+r_f+w_1+w_2}$,
$\bar{C}^{U,P1}_1 = \bar{C}^{U,P2}_1 = \bar{C}^{U,P4}_1 \iff C_{2L} = \frac{(w_1+w_2)\left[(1+r_f)Y_1+Y_2\right]+(1+r_f)w_0}{1+r_f+w_1+w_2}$ *and*
$\bar{C}^{U,P4}_1 < \bar{C}^{U,P2}_1 < \bar{C}^{U,P1}_1 \iff C_{2L} > \frac{(w_1+w_2)\left[(1+r_f)Y_1+Y_2\right]+(1+r_f)w_0}{1+r_f+w_1+w_2}$.

(ix) $(1+r_f)C_{1L} + C_{2L} \leq (1+r_f)Y_1 + Y_2$ *implies that* $\alpha^{C_1=C_{1L}}_{C_{2b}=C_{2L}} \geq 0$ *and* $\bar{C}^{U,P5}_1 \leq \bar{C}^{U,P6}_1$.

(x) $C_{1L} \leq \bar{C}^{U,P1}_1$ *(assumption (62))* $\iff w_0 + (w_1+w_2)C_{1L} \leq \frac{(w_1+w_1)[(1+r_f)Y_1+Y_2]+(1+r_f)\,w_0}{1+r_f+w_1+w_2}$.
In addition, if $C_{1L} \leq \bar{C}^{U,P1}_1$ *then* $\bar{C}^{U,P1}_1 \leq \bar{C}^{U,P5}_1$.

(xi) $C_{1L} < \frac{(1+r_f-w_2)\left(Y_1+\frac{Y_2}{1+r_f}\right)-w_0+\frac{w_2}{1+r_f}C_{2L}}{1+r_f+w_1} \iff \bar{C}^{U,P4}_1 \leq \bar{C}^{U,P5}_1$.

(xii) *Let* $C_{1L} = C_L \geq 0$, $C_{2L} = \frac{C_L}{1+r_f}$, $w_0 = 0$, $w_1 = (1+r_f)w$ *and* $w_2 = (1+r_f)(1-w)$ *where* $w \in [0,1]$. *Then* $\bar{C}_1^{U,P2} \leq \bar{C}_1^{U,P5}$.

There are eight cases to consider:

(P1) $C_1 \geq \bar{C}_1$, $\bar{C}_2 \leq C_{2b} \leq C_{2g}$

(P2) $C_1 \geq \bar{C}_1$, $C_{2L} \leq C_{2b} \leq \bar{C}_2 \leq C_{2g}$

(P3) $C_1 \geq \bar{C}_1$, $C_{2L} \leq C_{2g} \leq \bar{C}_2 \leq C_{2b}$

(P4) $C_1 \geq \bar{C}_1$, $C_{2L} \leq C_{2b} \leq C_{2g} \leq \bar{C}_2$

(P5) $C_{1L} \leq C_1 \leq \bar{C}_1$, $\bar{C}_2 \leq C_{2b} \leq C_{2g}$

(P6) $C_{1L} \leq C_1 \leq \bar{C}_1$, $C_{2L} \leq C_{2b} \leq \bar{C}_2 \leq C_{2g}$

(P7) $C_{1L} \leq C_1 \leq \bar{C}_1$, $C_{2L} \leq C_{2g} \leq \bar{C}_2 \leq C_{2b}$

(P8) $C_{1L} \leq C_1 \leq \bar{C}_1$, $C_{2L} \leq C_{2b} \leq C_{2g} \leq \bar{C}_2$

Let the following assumptions hold

$$\begin{aligned} C_{1L} &\leq \min\left\{\frac{(1+r_f)Y_1 + Y_2 - w_0}{1+r_f+w_1+w_2} = \bar{C}_1^{U,P1}, \bar{C}_1\right\} && (62) \\ C_{2L} &\leq (1+r_f)C_{1L} && (63) \\ \max\{C_{1L}, C_{2L}\} &\leq \bar{C}_1 && (64) \end{aligned}$$

Then the corresponding problems for $\bar{C}_1 \leq \bar{C}_1^{U,P2}$ are

Problem (P1):

$$\left.\begin{array}{llll} \mathrm{Max}_{(C_1,\alpha)}: & \mathbb{E}(U(C_1,\alpha)) = & \frac{(C_1-\bar{C}_1)^{1-\gamma}}{1-\gamma} & +\delta p \frac{\left[(1+r_f)Y_1+Y_2-w_0-w_2\bar{C}_1-(1+r_f+w_1)C_1+(r_g-r_f)\alpha\right]^{1-\gamma}}{1-\gamma} \\ & & & +\delta(1-p)\frac{\left[(1+r_f)Y_1+Y_2-w_0-w_2\bar{C}_1-(1+r_f+w_1)C_1-(r_f-r_b)\alpha\right]^{1-\gamma}}{1-\gamma} \\ \text{such that}: & & \bar{C}_1 \leq C_1 & \leq \frac{1}{1+r_f+w_1}\left[(1+r_f)Y_1+Y_2-w_0-w_2\bar{C}_1-(r_f-r_b)\alpha\right] \\ & & 0 \leq \alpha & \leq \frac{\Omega}{r_f-r_b} = \alpha^{C_1=\bar{C}_1}_{C_{2b}=\bar{C}_2} \end{array}\right\} \quad \text{(P1)}$$

Note that feasible solutions exist only when $C_{1L} \leq \bar{C}_1 \leq \bar{C}_1^{U,P1}$. Note in addition that assumptions (62) and (63) imply that $C_{2L} \leq \bar{C}_2$, namely,

$$C_{2L} \leq w_0 + (w_1+w_2)C_{1L} \leq w_0 + w_1 C_1 + w_2\bar{C}_1 = \bar{C}_2$$

This applies to all problems (P1)–(P8).

Problem (P2):

$$\begin{aligned}\mathrm{Max}_{(\mathrm{C}_1,\alpha)}:\mathbb{E}(U(C_1,\alpha)) = \tfrac{(C_1-\bar{C}_1)^{1-\gamma}}{1-\gamma} \quad &+\delta p\tfrac{\left[(1+r_f)Y_1+Y_2-w_0-w_2\bar{C}_1-(1+r_f+w_1)C_1+(r_g-r_f)\alpha\right]^{1-\gamma}}{1-\gamma}\\ &-\lambda\delta(1-p)\tfrac{\left[(1+r_f+w_1)C_1+(r_f-r_b)\alpha-(1+r_f)Y_1-Y_2+w_0+w_2\bar{C}_1\right]^{1-\gamma}}{1-\gamma}\end{aligned} \tag{P2}$$

such that

$$\begin{aligned}\tfrac{1}{1+r_f+w_1}\left[(1+r_f)Y_1+Y_2-w_0-w_2\bar{C}_1-(r_f-r_b)\alpha\right] &\le C_1\\ \tfrac{1}{1+r_f+w_1}\left[(1+r_f)Y_1+Y_2-w_0-w_2\bar{C}_1+(r_g-r_f)\alpha\right] &\ge C_1\\ \bar{C}_1 &\le C_1 \quad \le Y_1+\tfrac{1}{1+r_f}\left[Y_2-C_{2L}-(r_f-r_b)\alpha\right]\\ \max\left\{0, \tfrac{-\Omega}{r_g-r_f}=\alpha^{C_1=\bar{C}_1}_{C_{2g}=\bar{C}_2}\right\} &\le \alpha \quad \le \tfrac{(1+r_f)(Y_1-\bar{C}_1)+Y_2-C_{2L}}{r_f-r_b}=\alpha^{C_1=\bar{C}_1}_{C_{2b}=C_{2L}}\end{aligned}$$

Feasible solutions exist only if $\bar{C}_1 \le \min\left\{\bar{C}_1^{U,P2}, Y_1+\frac{Y_2-C_{2L}}{1+r_f}\right\}$.[37] Note that for $C_{2L} < \frac{(w_1+w_2)\left[(1+r_f)Y_1+Y_2\right]+(1+r_f)w_0}{1+r_f+w_1+w_2}$, which follows from assumptions (62) and (63), see Lemma 1-(x), is $\bar{C}_1^{U,P2} < Y_1+\frac{Y_2-C_{2L}}{1+r_f}$ and $\bar{C}_1^{U,P1} \le \bar{C}_1^{U,P2}$, see Lemma 1-(viii). Thus, the necessary conditions for feasibility of (P2) are: $C_{1L} \le \bar{C}_1 \le \bar{C}_1^{U,P2}$.

Note that in the case of **problem (P3)** $C_{2b}=C_{2g}=\bar{C}_2$ as $C_{2b} \le C_{2g}$. This and (4) imply that the only feasible solution for $\Omega \ge 0$ is $\left(C_1=\bar{C}_1+\frac{\Omega}{1+r_f+w_1},\ \alpha=0\right)$ which is also feasible for (P1)–(P4). There is no feasible solution for $\Omega<0$, i.e., when $\bar{C}_1 > \bar{C}_1^{U,P1}$.

Problem (P4):

$$\left.\begin{aligned}\mathrm{Max}_{(\mathrm{C}_1,\alpha)}:\quad &\mathbb{E}(U(C_1,\alpha)) =\\ &\tfrac{(C_1-\bar{C}_1)^{1-\gamma}}{1-\gamma}-\lambda\delta p\tfrac{((1+r_f+w_1)C_1-(r_g-r_f)\alpha-(1+r_f)Y_1-Y_2+w_0+w_2\bar{C}_1)^{1-\gamma}}{1-\gamma}\\ &-\lambda\delta(1-p)\tfrac{((1+r_f+w_1)C_1+(r_f-r_b)\alpha-(1+r_f)Y_1-Y_2+w_0+w_2\bar{C}_1)^{1-\gamma}}{1-\gamma}\\ \text{such that}:\quad &C_1 \ge \tfrac{1}{1+r_f+w_1}\left[(1+r_f)Y_1+Y_2-w_0-w_2\bar{C}_1+(r_g-r_f)\alpha\right]\\ &\bar{C}_1 \le C_1 \le Y_1+\tfrac{1}{1+r_f}\left[Y_2-C_{2L}-(r_f-r_b)\alpha\right]\\ &0 \le \alpha \le \alpha^{C_{2g}=\bar{C}_2}_{C_{2b}=C_{2L}}\end{aligned}\right\} \tag{P4}$$

[37]Constraint $\bar{C}_1 \le \bar{C}_1^{U,P2}$ follows from $\alpha^{C_1=\bar{C}_1}_{C_{2g}=\bar{C}_2} \le \alpha^{C_1=\bar{C}_1}_{C_{2b}=C_{2L}}$ and constraint $\bar{C}_1 \le Y_1+\frac{Y_2-C_{2L}}{1+r_f}$ follows from $0 \le \alpha^{C_1=\bar{C}_1}_{C_{2b}=C_{2L}}$.

where $\alpha_{C_{2b}=C_{2L}}^{C_{2g}=\bar{C}_2}$ is given by (54). Note that a necessary condition for the feasibility of (P4) is $\alpha_{C_{2b}=C_{2L}}^{C_{2g}=\bar{C}_2} \geq 0$ which is equivalent to

$$\bar{C}_1 \geq \frac{(1+r_f+w_1)C_{2L} - w_1[(1+r_f)Y_1+Y_2] - (1+r_f)w_0}{(1+r_f)w_2} = \bar{C}_1^{L,P4}$$

see Lemma 1-(vi), or to

$$C_{2L} \leq \frac{1+r_f}{1+r_f+w_1}\left[w_0 + w_1\left(Y_1 + \frac{Y_2}{1+r_f}\right) + w_2\bar{C}_1\right]$$

Problem (P5):

$$\left.\begin{array}{ll}
\mathrm{Max}_{(\mathrm{C}_1,\alpha)}: & \mathbb{E}(U(C_1,\alpha)) = \\
& -\lambda\frac{(\bar{C}_1-C_1)^{1-\gamma}}{1-\gamma} + \delta p\frac{\left[(1+r_f)Y_1+Y_2-(1+r_f+w_1)C_1-w_0-w_2\bar{C}_1+(r_g-r_f)\alpha\right]^{1-\gamma}}{1-\gamma} \\
& +\delta(1-p)\frac{\left[(1+r_f)Y_1+Y_2-(1+r_f+w_1)C_1-w_0-w_2\bar{C}_1-(r_f-r_b)\alpha\right]^{1-\gamma}}{1-\gamma} \\
\text{such that}: & C_{1L} \leq C_1 \leq \min\left\{\frac{1}{1+r_f+w_1}\left[(1+r_f)Y_1+Y_2-w_0-w_2\bar{C}_1-(r_f-r_b)\alpha\right], \bar{C}_1\right\} \\
& 0 \leq \alpha \leq \frac{(1+r_f)Y_1+Y_2-w_0-w_2\bar{C}_1-(1+r_f+w_1)C_{1L}}{r_f-r_b} = \alpha_{C_{2b}=\bar{C}_2}^{C_1=C_{1L}}
\end{array}\right\} \quad \text{(P5)}$$

Note that (P5) is feasible only when $\bar{C}_1 \leq \bar{C}_1^{U,P5}$, see Lemma 1-(vi).

Problem (P6):

$$\left.\begin{array}{ll}
\mathrm{Max}_{(\mathrm{C}_1,\alpha)}: & \mathbb{E}(U(C_1,\alpha)) = \\
& -\lambda\frac{(\bar{C}_1-C_1)^{1-\gamma}}{1-\gamma} + \delta p\frac{((1+r_f)Y_1+Y_2-w_0-w_2\bar{C}_1-(1+r_f+w_1)C_1+(r_g-r_f)\alpha)^{1-\gamma}}{1-\gamma} \\
& -\lambda\delta(1-p)\frac{((1+r_f+w_1)C_1+(r_f-r_b)\alpha-(1+r_f)Y_1-Y_2+w_0+w_2\bar{C}_1)^{1-\gamma}}{1-\gamma} \\
\text{such that}: & C_1 \leq \frac{1}{1+r_f+w_1}\left[(1+r_f)Y_1+Y_2-w_0-w_2\bar{C}_1+(r_g-r_f)\alpha\right] \\
& C_1 \geq \frac{1}{1+r_f+w_1}\left[(1+r_f)Y_1+Y_2-w_0-w_2\bar{C}_1-(r_f-r_b)\alpha\right] \\
& C_{1L} \leq C_1 \leq \min\left\{Y_1 + \frac{1}{1+r_f}\left[Y_2-C_{2L}-(r_f-r_b)\alpha\right], \bar{C}_1\right\} \\
& \max\left\{0, \alpha_{C_{2g}=\bar{C}_2}^{C_1=C_{1L}}, \alpha_{C_{2b}=\bar{C}_2}^{C_1=\bar{C}_1}\right\} \leq \alpha \leq \min\left\{\alpha_{C_{2b}=C_{2L}}^{C_1=C_{1L}}, \alpha_{C_{2b}=C_{2L}}^{C_{2b}=\bar{C}_2}\right\}
\end{array}\right\} \quad \text{(P6)}$$

where $\alpha_{C_{2g}=\bar{C}_2}^{C_1=C_{1L}}$ is given by (59), $\alpha_{C_{2b}=\bar{C}_2}^{C_1=\bar{C}_1}$ is given by (57), $\alpha_{C_{2b}=C_{2L}}^{C_1=C_{1L}}$ is given by (56) and $\alpha_{C_{2b}=C_{2L}}^{C_{2b}=\bar{C}_2}$ is given by (61). Assumptions (62) and (63) imply that $\alpha_{C_{2b}=C_{2L}}^{C_1=C_{1L}} \leq \alpha_{C_{2b}=C_{2L}}^{C_{2b}=\bar{C}_2}$ and

Lemma 1-(ix) gives that the upper bound on α is nonnegative, i.e., $\alpha^{C_1=C_{1L}}_{C_{2b}=C_{2L}} \geq 0$.

Note that $\alpha^{C_1=\bar{C}_1}_{C_{2b}=\bar{C}_2} \leq \alpha^{C_1=C_{1L}}_{C_{2b}=C_{2L}}$ holds if $\bar{C}_1 \geq \frac{(1+r_f)C_{1L}+C_{2L}-w_0}{1+r_f+w_1+w_2}$ and $\alpha^{C_1=C_{1L}}_{C_{2g}=\bar{C}_2} \leq \alpha^{C_1=C_{1L}}_{C_{2b}=C_{2L}}$ if $\bar{C}_1 \leq \bar{C}_1^{U,P6}$ which is defined by (51). Thus, the set of feasible solutions for (P6) is non-empty for

$$\frac{(1+r_f)C_{1L}+C_{2L}-w_0}{1+r_f+w_1+w_2} \leq \bar{C}_1 \leq \bar{C}_1^{U,P6}$$

Note finally that for $\bar{C}_1 \leq \min\left\{\bar{C}_1^{U,P2}, \bar{C}_1^{U,P5}\right\}$ is $\alpha^{C_1=C_{1L}}_{C_{2g}=\bar{C}_2} \leq 0$, see Lemma 1-(vi), and for $\bar{C}_1 \leq \bar{C}_1^{U,P1}$ is $\alpha^{C_1=\bar{C}_1}_{C_{2b}=\bar{C}_2} \geq 0$. Thus the set of feasible solutions for α is as follows

$$\begin{array}{llll}
\text{if } C_{1L} & \leq \bar{C}_1 \leq \bar{C}_1^{U,P1} & \text{then} & \alpha^{C_1=\bar{C}_1}_{C_{2b}=\bar{C}_2} \leq \alpha \leq \alpha^{C_1=C_{1L}}_{C_{2b}=C_{2L}} \\
\text{if } \bar{C}_1^{U,P1} & \leq \bar{C}_1 \leq \min\left\{\bar{C}_1^{U,P2}, \bar{C}_1^{U,P5}\right\} & \text{then} & 0 \leq \alpha \leq \alpha^{C_1=C_{1L}}_{C_{2b}=C_{2L}}
\end{array}$$

Problem (P7): Similarly as in (P3), $C_{2b} = C_{2g} = \bar{C}_2$ as $C_{2b} \leq C_{2g}$ and thus the only feasible solution for $\bar{C}_1 \geq \bar{C}_1^{U,P1}$ (i.e., $\Omega \leq 0$) is $\left(C_1 = \bar{C}_1 + \frac{\Omega}{1+r_f+w_1}, \, \alpha = 0\right)$ if $C_{1L} \leq \bar{C}_1 + \frac{\Omega}{1+r_f+w_1}$. Note that this point is feasible also for (P5)–(P8). There is no feasible solution for $\bar{C}_1 < \bar{C}_1^{U,P1}$ ($\Omega > 0$).

Problem (P8):

$$\left.\begin{array}{ll}
\mathrm{Max}_{(\mathrm{C}_1,\alpha)}: & \mathbb{E}(U(C_1,\alpha)) = -\lambda\frac{(\bar{C}_1-C_1)^{1-\gamma}}{1-\gamma} - \lambda\delta p\frac{\left[-\Omega+(1+r_f+w_1)(C_1-\bar{C}_1)-(r_g-r_f)\alpha\right]^{1-\gamma}}{1-\gamma} \\
 & -\lambda\delta(1-p)\frac{\left[-\Omega+(1+r_f+w_1)(C_1-\bar{C}_1)+(r_f-r_b)\alpha\right]^{1-\gamma}}{1-\gamma} \\
\text{such that}: & C_1 \geq \max\left\{\frac{1}{1+r_f+w_1}\left[(1+r_f)Y_1+Y_2-w_0-w_2\bar{C}_1+(r_g-r_f)\alpha\right], C_{1L}\right\} \\
 & C_1 \leq \min\left\{Y_1+\frac{1}{1+r_f}\left[Y_2-C_{2L}-(r_f-r_b)\alpha\right], \bar{C}_1\right\} \\
 & 0 \leq \alpha \leq \min\left\{\alpha^{C_1=C_{1L}}_{C_{2b}=C_{2L}}, \alpha^{C_{2g}=\bar{C}_2}_{C_{2b}=C_{2L}}\right\} \\
 & 0 \leq \alpha \leq \frac{-\Omega}{r_g-r_f}
\end{array}\right\} \quad \text{(P8)}$$

Note that the last inequality holds only if $\Omega \leq 0$, i.e., when $\bar{C}_1 \geq \bar{C}_1^{U,P1}$. Thus, based on this and Lemma 1-(vi), (P8) is feasible for

$$\bar{C}_1 \geq \max\left\{\bar{C}_1^{L,P4}, \bar{C}_1^{U,P1}\right\}$$

As problems (P3) and (P7) are imbedded in other problems we can exclude them from our analysis.

Appendix B

Problem (P1). Let $\bar{C}_1 \leq \frac{(1+r_f)Y_1+Y_2-w_0}{1+r_f+w_1+w_2} = \bar{C}_1^{U,P1}$, i.e., $(\Omega \geq 0)$ and $C_1 \leq \bar{C}_1 + \frac{\Omega}{1+r_f+w_1}$.[38] At first, we solve the concave programming problem (P1) as an unconstrained problem, i.e., we solve two equations in two unknown variables C_1 and α, namely $\frac{d\mathbb{E}(U)}{dC_1} = 0$ and $\frac{d\mathbb{E}(U)}{d\alpha} = 0$ $(\nabla\mathbb{E}(U) = 0)$, obtain the optimum solution (C_1^{P1}, α^{P1}) and finally verify that $C_{2b}^{P1} \geq \bar{C}_2$ and $C_{2g}^{P1} \geq \bar{C}_2$, $C_1^{P1} \geq \bar{C}_1$ and $0 \leq \alpha^{P1} \leq \frac{\Omega}{r_f - r_b}$, i.e., that the solution is also feasible.

The first order conditions are

$$\left.\begin{aligned}
\frac{1}{1+r_f+w_1}\frac{d\mathbb{E}(U)}{dC_1} =& \frac{(C_1-\bar{C}_1)^{-\gamma}}{1+r_f+w_1} - \delta p\left[\Omega - (1+r_f+w_1)(C_1-\bar{C}_1) + (r_g - r_f)\alpha\right]^{-\gamma} \\
&-\delta(1-p)\left[\Omega - (1+r_f+w_1)(C_1-\bar{C}_1) - (r_f - r_b)\alpha\right]^{-\gamma} = 0 \\
\frac{d\mathbb{E}(U)}{d\alpha} =& \delta p\left[\Omega - (1+r_f+w_1)(C_1-\bar{C}_1) + (r_g - r_f)\alpha\right]^{-\gamma}(r_g - r_f) \\
&-\delta(1-p)\left[\Omega - (1+r_f+w_1)(C_1-\bar{C}_1) - (r_f - r_b)\alpha\right]^{-\gamma}(r_f - r_b) = 0
\end{aligned}\right\} \tag{65}$$

$\frac{d\mathbb{E}(U)}{d\alpha} = 0$ from (65) implies the following

$$\begin{aligned}
&p\left[\Omega - (1+r_f+w_1)(C_1-\bar{C}_1) - (r_f - r_b)\alpha\right]^{\gamma}(r_g - r_f) \\
&= (1-p)\left[\Omega - (1+r_f+w_1)(C_1-\bar{C}_1) + (r_g - r_f)\alpha\right]^{\gamma}(r_f - r_b)
\end{aligned}$$

which after using the definition of K_γ, as given by (10), gives

$$K_0^{-\frac{1}{\gamma}}\left[\Omega - (1+r_f+w_1)(C_1-\bar{C}_1) - (r_f - r_b)\alpha\right] = \Omega - (1+r_f+w_1)(C_1-\bar{C}_1) + (r_g - r_f)\alpha$$

This implies that

$$\alpha = \frac{1 - K_0^{\frac{1}{\gamma}}}{r_f - r_b + K_0^{\frac{1}{\gamma}}(r_g - r_f)}\left[\Omega - (1+r_f+w_1)(C_1-\bar{C}_1)\right] \tag{66}$$

[38] There is no feasible solution for (P1) if $\bar{C}_1 > \bar{C}_1^{U,P1}$ or if $C_1 > \bar{C}_1 + \frac{\Omega}{1+r_f+w_1}$. If the latter inequality holds then $C_{2b} < \bar{C}_2$ which is infeasible in (P1).

If we plug the last expression for α into the C_1 part of the FOC in (65) we obtain

$$\begin{aligned}
&\frac{(C_1-\bar{C}_1)^{-\gamma}}{\delta(1+r_f+w_1)} \\
&= p\left[\Omega-(1+r_f+w_1)(C_1-\bar{C}_1)+\frac{\left(1-K_0^{\frac{1}{\gamma}}\right)(r_g-r_f)}{r_f-r_b+K_0^{\frac{1}{\gamma}}(r_g-r_f)}\left(\Omega-(1+r_f+w_1)(C_1-\bar{C}_1)\right)\right]^{-\gamma} \\
&+(1-p)\left[\Omega-(1+r_f+w_1)(C_1-\bar{C}_1)-\frac{\left(1-K_0^{\frac{1}{\gamma}}\right)(r_f-r_b)}{r_f-r_b+K_0^{\frac{1}{\gamma}}(r_g-r_f)}\left(\Omega-(1+r_f+w_1)(C_1-\bar{C}_1)\right)\right]^{-\gamma} \\
&= \left[\frac{r_f-r_b+K_0^{\frac{1}{\gamma}}(r_g-r_f)}{\left(\Omega-(1+r_f+w_1)(C_1-\bar{C}_1)\right)(r_g-r_b)}\right]^{\gamma} p\,\frac{r_g-r_b}{r_f-r_b}
\end{aligned}$$

under the assumption that $\Omega-(1+r_f+w_1)(C_1-\bar{C}_1)>0$ which is equivalent to $C_1 < \bar{C}_1+\frac{\Omega}{1+r_f+w_1}$.[39]

After some simplifications we obtain

$$\begin{aligned}
\Omega-(1+r_f+w_1)(C_1-\bar{C}_1) &= (C_1-\bar{C}_1)\frac{r_f-r_b+K_0^{\frac{1}{\gamma}}(r_g-r_f)}{r_g-r_b}\left[\delta(1+r_f+w_1)p\,\frac{r_g-r_b}{r_f-r_b}\right]^{\frac{1}{\gamma}} \\
&= (C_1-\bar{C}_1)M \qquad (67)
\end{aligned}$$

which gives

$$C_1=\bar{C}_1+\frac{\Omega}{1+r_f+w_1+M}=C_1^{P1}$$

as given by (17). Note that (17) and the assumption $\Omega \geq 0$ imply that $C_1^{P1} \geq \bar{C}_1$. In addition, after plugging C_1^{P1} into (66) we obtain α^{P1} as given in (18). It is also easy to verify that upper bounds on C_1^{P1} and α^{P1}, as given in (P1), are satisfied. Note finally that $\alpha^{P1} \geq 0$ as $K_0 < 1$ which follows from $\mathbb{E}(r) > r_f$.

Using (65), it is easy to verify that $\frac{d^2\mathbb{E}(U)}{dC_1^2} < 0$, $\frac{d^2\mathbb{E}(U)}{d\alpha^2} < 0$, and $\nabla^2\mathbb{E}(U(C_1,C_2)) = \frac{d^2\mathbb{E}(U)}{dC_1^2}\frac{d^2\mathbb{E}(U)}{d\alpha^2}-\left(\frac{d^2\mathbb{E}(U)}{dC_1 d\alpha}\right)^2 > 0$ and thus problem (P1) is a concave programming problem and (C_1^{P1},α^{P1}) is its unique global maximum.

Finally, C_{2g}^{P1} and C_{2b}^{P1} can be written as

$$C_{2g}^{P1} = w_0+(w_1+w_2)\bar{C}_1+\left[w_1+k_2\frac{r_g-r_b}{r_f-r_b}\right]\frac{\Omega}{1+r_f+w_1+M} \qquad (68)$$

$$C_{2b}^{P1} = w_0+(w_1+w_2)\bar{C}_1+\left[w_1+k_2K_0^{\frac{1}{\gamma}}\frac{r_g-r_b}{r_f-r_b}\right]\frac{\Omega}{1+r_f+w_1+M} \qquad (69)$$

[39] Note that the optimal solution satisfies this inequality.

and thus

$$
\begin{aligned}
C_{2g}^{P1} - \bar{C}_2 &= k_2 \frac{r_g - r_b}{r_f - r_b} \times \frac{\Omega}{1 + r_f + w_1 + M} \geq 0 \\
C_{2b}^{P1} - \bar{C}_2 &= k_2 K_0^{\frac{1}{\gamma}} \frac{r_g - r_b}{r_f - r_b} \times \frac{\Omega}{1 + r_f + w_1 + M} \geq 0
\end{aligned}
$$

It can be shown that

$$
\begin{aligned}
(1-\gamma)\mathbb{E}(U(C_1^{P1}, \alpha^{P1})) &= \frac{\Omega^{1-\gamma}}{1 + r_f + w_1}(1 + r_f + w_1 + M)^{\gamma} \\
&= \left(\frac{\Omega}{1 + r_f + w_1}\right)^{1-\gamma} \left[1 + \left(\lambda^{P1-P5}\right)^{1/\gamma}\right]^{\gamma} \qquad (70)
\end{aligned}
$$

where λ^{P1-P5} is given by (13).

As we have already mentioned the only feasible solution for $C_1 = \bar{C}_1 + \frac{\Omega}{1+r_f+w_1}$ is $\left(C_1 = \bar{C}_1 + \frac{\Omega}{1+r_f+w_1}, \alpha = 0\right)$ with

$$
(1-\gamma)\mathbb{E}\left(U\left(\bar{C}_1 + \frac{\Omega}{1 + r_f + w_1}, 0\right)\right) = \left(\frac{\Omega}{1 + r_f + w_1}\right)^{1-\gamma} \quad \text{for} \;\; \bar{C}_1 \leq \bar{C}_1^{U,P1} \qquad (71)
$$

which is below the value of the expected utility function at $(C_1^{P1}, \alpha_1^{P1})$ as $M > 0$ and $w_1 > 0$. Thus, the maximum of (P1) is reached at (C_1^{P1}, α^{P1}). Note that as $C_{2g} = C_{2b} = \bar{C}_2$ for $\left(C_1 = \bar{C}_1 + \frac{\Omega}{1+r_f+w_1}, \alpha = 0\right)$ then this point is feasible also for problems (P2)–(P4) if $\bar{C}_1 \leq \bar{C}_1^{U,P1}$ (i.e., $\Omega \geq 0$) and is feasible also for problems (P5)–(P8) if $\bar{C}_1 > \bar{C}_1^{U,P1}$ (i.e., $\Omega < 0$) where

$$
(1-\gamma)\mathbb{E}\left(U\left(C_1 = \bar{C}_1 + \frac{\Omega}{1 + r_f + w_1}, \alpha = 0\right)\right) = \left(\frac{-\Omega}{1 + r_f + w_1}\right)^{1-\gamma} \quad \text{for} \;\; \bar{C}_1 > \bar{C}_1^{U,P1} \qquad (72)
$$

for $C_{1L} \leq \frac{(1+r_f)Y_1 + Y_2 - w_0 - w_2\bar{C}_1}{1+r_f+w_1}$ and $\bar{C}_1 \leq \frac{(1+r_f)Y_1 + Y_2 - w_0}{w_2}$.[40] Note in addition that for $\bar{C}_1 = \bar{C}_1^{U,P1}$ (i.e., $\Omega = 0$) is $\left(C_1^{P1} = \bar{C}_1, \alpha^{P1} = 0\right)$ which is feasible for all problems (P1)–(P8).

Sensitivity analysis.

Equations (68) and (69) imply that

$$
\frac{dC_{2s}^{P1}}{d\bar{C}_1} = k_2 D_s + (1 + r_f) w_2 \qquad (73)
$$

[40]To guarantee that the upper bound on C_{1L} is nonnegative.

where $s \in \{b, g\}$ and

$$D_s = \begin{cases} (w_1+w_2)\left(1+K_\gamma^{1/\gamma}\right) - \frac{r_g-r_b}{r_f-r_b}(1+r_f+w_1+w_2), & \text{for } s=g \\ = -\frac{(w_1+w_2)(r_g-r_f)\left(1-K_0^{1/\gamma}\right)+(1+r_f)(r_g-r_b)}{r_f-r_b} < 0 & \\ & \\ (w_1+w_2)\left(1+K_\gamma^{1/\gamma}\right) - \frac{r_g-r_b}{r_f-r_b}(1+r_f+w_1+w_2)K_0^{1/\gamma}, & \text{for } s=b \\ = (w_1+w_2)\left(1-K_0^{1/\gamma}\right) - (1+r_f)\left(K_0^{1/\gamma}+K_\gamma^{1/\gamma}\right) & \end{cases} \tag{74}$$

Equations (73) and (74) imply that for $s = g$

$$\frac{dC_{2g}^{P1}}{d\bar{C}_1} \begin{cases} < 0, & \text{for } k_2 > \frac{(1+r_f)w_2}{-D_g} \Leftrightarrow \delta > \bar{\delta}_g \\ & \\ = 0, & \text{for } k_2 = \frac{(1+r_f)w_2}{-D_g} \Leftrightarrow \delta = \bar{\delta}_g \\ & \\ > 0, & \text{for } k_2 < \frac{(1+r_f)w_2}{-D_g} \Leftrightarrow \delta < \bar{\delta}_g \end{cases} \tag{75}$$

where

$$\bar{\delta}_g = \left[\frac{(1+r_f)w_2}{-D_g}\right]^\gamma \left(\frac{r_f-r_b}{r_g-r_b}\right)^{1-\gamma} \frac{1}{(1+r_f+w_1)p} = \left[\frac{w_2}{\frac{(w_1+w_2)(r_g-r_f)}{(1+r_f)(r_g-r_b)}+1}\right]^\gamma \left(\frac{r_f-r_b}{r_g-r_b}\right) \frac{1}{(1+r_f+w_1)p}$$

Equations (73) and (74) imply that for $i = b$

$$\frac{dC_{2b}^{P1}}{d\bar{C}_1} \begin{cases} < 0, & \text{for } \left(k_2 > \frac{(1+r_f)w_2}{-D_b} \text{ and } D_b < 0\right) \Leftrightarrow \left(\delta > \bar{\delta}_b \text{ and } D_b < 0\right) \\ & \\ = 0, & \text{for } \left(k_2 = \frac{(1+r_f)w_2}{-D_b} \text{ and } D_b < 0\right) \Leftrightarrow \left(\delta = \bar{\delta}_b \text{ and } D_b < 0\right) \\ & \\ > 0, & \text{for } D_b > 0 \text{ or} \\ & \text{for } \left(k_2 < \frac{(1+r_f)w_2}{-D_b} \text{ and } D_b < 0\right) \Leftrightarrow \left(\delta < \bar{\delta}_b \text{ and } D_b < 0\right) \end{cases} \tag{76}$$

where

$$\bar{\delta}_b = \left[\frac{(1+r_f)w_2}{-D_b}\right]^\gamma \left(\frac{r_f-r_b}{r_g-r_b}\right)^{1-\gamma} \frac{1}{(1+r_f+w_1)p}$$

for $D_b < 0$ which is equivalent to $w_1 + w_2 < (1+r_f)\frac{K_0^{1/\gamma}+K_\gamma^{1/\gamma}}{1-K_0^{1/\gamma}}$. Note that the following holds when $w_1 + w_2 = 1 + r_f$

$$D_b \begin{cases} < 0, & \text{for } p < \bar{p} \\ = 0, & \text{for } p = \bar{p} \\ > 0 & \text{for } p > \bar{p} \end{cases} \tag{77}$$

where

$$\bar{p} = \frac{(r_f - r_b)^{1-\gamma}[r_g - r_f + 2(r_f - r_b)]^\gamma}{(r_f - r_b)^{1-\gamma}[r_g - r_f + 2(r_f - r_b)]^\gamma + r_g - r_f}$$

Under this condition, i.e., when $w_1 = (1 + r_f)w$, $w_2 = (1 + r_f)(1 - w)$ and $w \in [0, 1]$, it can also be shown that for sufficiently large w are the threshold values for δ, namely $\bar{\bar{\delta}}_b$ and $\bar{\bar{\delta}}_g$ smaller than one, i.e., both C_{2b}^{P1} and C_{2g}^{P1} can be both increasing or decreasing with respect to $\bar{C}_1$, see (76) and (75). However, based on (77) this applies to C_{2b}^{P1} only for $p < \bar{p}$. For a sufficiently large p ($p > \bar{p}$) is C_{2b}^{P1} increasing with respect to $\bar{C}_1$.

Problem (P2). Let $\bar{C}_1 \leq \bar{C}_1^{U,P2}$.

We proceed in the following way: At first we solve problem (P2) as an unconstrained problem i.e., we solve $\nabla\mathbb{E}(U) = 0$, so that the FOC are satisfied, obtain the unique solution (C_1^{P2}, α^{P2}), verify that the objective function of (P2) is concave at (C_1^{P2}, α^{P2}) and that the solution is also feasible. As the utility function is differentiable at the domain under consideration, (C_1^{P2}, α^{P2}) is the only local extremum (namely a local maximum) and if the objective function at the border of (P2) does not exceed its value at (C_1^{P2}, α^{P2}), then this point is also a global maximum of (P2) when $\lambda > \lambda^{P2}$ and $\bar{C}_1^{U,P1} \leq \bar{C}_1 \leq \bar{C}_1^{U,P2}$. For $\bar{C}_1 \leq \bar{C}_1^{U,P1}$ is the maximum reached at the border.

The first order conditions are

$$\left.\begin{aligned}
\frac{d\mathbb{E}(U)}{dC_1} = \;& (C_1 - \bar{C}_1)^{-\gamma} \\
& -\delta p \left[\Omega - (1 + r_f + w_1)(C_1 - \bar{C}_1) + (r_g - r_f)\alpha\right]^{-\gamma}(1 + r_f + w_1) \\
& -\lambda\delta(1 - p)\left[(1 + r_f + w_1)(C_1 - \bar{C}_1) - \Omega + (r_f - r_b)\alpha\right]^{-\gamma}(1 + r_f + w_1) = 0 \\
\\
\frac{d\mathbb{E}(U)}{d\alpha} = \;& \delta p \left[\Omega - (1 + r_f + w_1)(C_1 - \bar{C}_1) + (r_g - r_f)\alpha\right]^{-\gamma}(r_g - r_f) \\
& -\lambda\delta(1 - p)\left[(1 + r_f + w_1)(C_1 - \bar{C}_1) - \Omega + (r_f - r_b)\alpha\right]^{-\gamma}(r_f - r_b) = 0
\end{aligned}\right\} \quad (78)$$

$\frac{d\mathbb{E}(U)}{d\alpha} = 0$ from (78) implies the following

$$\begin{aligned}
& p\left[(1 + r_f + w_1)(C_1 - \bar{C}_1) - \Omega + (r_f - r_b)\right]^\gamma (r_g - r_f) \\
& = \lambda(1 - p)\left[\Omega - (1 + r_f + w_1)(C_1 - \bar{C}_1) + (r_g - r_f)\alpha\right]^\gamma (r_f - r_b)
\end{aligned}$$

which gives

$$\left(\frac{1}{K_0}\right)^{\frac{1}{\gamma}}\left[(1 + r_f + w_1)(C_1 - \bar{C}_1) - \Omega + (r_f - r_b)\alpha\right] = \lambda^{\frac{1}{\gamma}}\left[\Omega - (1 + r_f + w_1)(C_1 - \bar{C}_1) + (r_g - r_f)\alpha\right]$$

This implies that

$$\begin{aligned}
\alpha &= \frac{\lambda^{\frac{1}{\gamma}} + \left(\frac{1}{K_0}\right)^{\frac{1}{\gamma}}}{\lambda^{\frac{1}{\gamma}}(r_g - r_f) - \left(\frac{1}{K_0}\right)^{\frac{1}{\gamma}}(r_f - r_b)} \left[(1 + r_f + w_1)(C_1 - \bar{C}_1) - \Omega\right] \\
&= \frac{\lambda^{\frac{1}{\gamma}} + \left(\frac{1}{K_0}\right)^{\frac{1}{\gamma}}}{\left(\lambda^{\frac{1}{\gamma}} - \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right)(r_g - r_f)} \left[(1 + r_f + w_1)(C_1 - \bar{C}_1) - \Omega\right] \qquad (79)
\end{aligned}$$

for $\lambda > \frac{1}{K_\gamma}$. If we plug the last expression for α into the C_1 part of the FOC in (78) we obtain

$$\begin{aligned}
\frac{(C_1 - \bar{C}_1)^{-\gamma}}{\delta(1 + r_f + w_1)} &= \left[w_0 + w_2\bar{C}_1 - (1 + r_f)Y_1 - Y_2 + (1 + r_f + w_1)C_1\right]^{-\gamma} \\
&\times \left[p\left(\frac{\left(\frac{1}{K_0}\right)^{\frac{1}{\gamma}} + \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}}{\lambda^{\frac{1}{\gamma}} - \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}}\right)^{-\gamma} + \lambda(1 - p)\left(1 + \frac{\lambda^{\frac{1}{\gamma}} + \left(\frac{1}{K_0}\right)^{\frac{1}{\gamma}}}{\lambda^{\frac{1}{\gamma}} - \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}}\frac{r_f - r_b}{r_g - r_f}\right)^{-\gamma}\right]
\end{aligned}$$

After some simplifications we obtain

$$\begin{aligned}
&(1 + r_f + w_1)(C_1 - \bar{C}_1) - \Omega \\
&\qquad = (C_1 - \bar{C}_1)\left[\delta(1 + r_f + w_1)(1 - p)\left(\frac{r_g - r_b}{r_g - r_f}\right)^{1-\gamma}\right]^{\frac{1}{\gamma}}\left[\lambda^{\frac{1}{\gamma}} - \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right] \\
&\qquad = (C_1 - \bar{C}_1)M(\lambda) \qquad (80)
\end{aligned}$$

which gives

$$C_1 = C_1^{P2} \equiv \bar{C}_1 + \frac{(-\Omega)}{M(\lambda) - 1 - r_f - w_1}$$

In addition, after plugging C_1^{P2} into (79) we obtain

$$\alpha = \alpha^{P2} \equiv \frac{\left[\left(\frac{1}{K_0}\right)^{1/\gamma} + \lambda^{1/\gamma}\right]k}{r_g - r_f}\left(C_1^{P2} - \bar{C}_1\right)$$

Note that $C_1^{P2} > \bar{C}_1$ if $\Omega < 0$ and $\lambda > \left[\frac{1+r_f+w_1}{k} + \left(\frac{1}{K_\gamma}\right)^{1/\gamma}\right]^\gamma$ (i.e., $M(\lambda) > 1 + r_f + w_1$) or if $\Omega > 0$ and $\lambda < \left[\frac{1+r_f+w_1}{k} + \left(\frac{1}{K_\gamma}\right)^{1/\gamma}\right]^\gamma$ (i.e., $M(\lambda) < 1 + r_f + w_1$).

What remains to be shown is when the expected utility function is strictly concave at (C_1^{P2}, α^{P2}). For this to hold it is sufficient to show that the following holds at (C_1^{P2}, α^{P2}):

$\frac{d^2\mathbb{E}(U)}{d\alpha^2} < 0$ and $D \equiv \nabla^2\mathbb{E}(U(C_1, C_2)) = \frac{d^2\mathbb{E}(U)}{dC_1^2}\frac{d^2\mathbb{E}(U)}{d\alpha^2} - \left(\frac{d^2\mathbb{E}(U)}{dC_1 d\alpha}\right)^2 > 0$. Note that

$$C_{2g}^{P2} - \bar{C}_2 = \frac{k(-\Omega)}{M(\lambda) - 1 - r_f - w_1}\frac{r_g - r_b}{r_g - r_f}\left(\frac{1}{K_0}\right)^{\frac{1}{\gamma}} \tag{81}$$

$$\bar{C}_2 - C_{2b}^{P2} = \frac{k(-\Omega)}{M(\lambda) - 1 - r_f - w_1}\frac{r_g - r_b}{r_g - r_f}\lambda^{\frac{1}{\gamma}} \tag{82}$$

which are positive for either $\bar{C}_1^{U,P1} < \bar{C}_1 \leq \bar{C}_1^{U,P2}$ $(\Omega < 0)$ and $\lambda > \left[\frac{1+r_f+w_1}{k} + \left(\frac{1}{K_\gamma}\right)^{1/\gamma}\right]^\gamma$ or for $\bar{C}_1 < \bar{C}_1^{U,P1}$ $(\Omega > 0)$ and $\lambda < \left[\frac{1+r_f+w_1}{k} + \left(\frac{1}{K_\gamma}\right)^{1/\gamma}\right]^\gamma$.[41] Thus, $\bar{C}_2 - C_{2b}^{P2} = (K_0\lambda)^{\frac{1}{\gamma}}(C_{2g}^{P2} - \bar{C}_2)$. Using (78), (81) and (82) we obtain the following

$$\begin{aligned}\frac{1}{\gamma}\frac{d^2\mathbb{E}(U)}{dC_1^2}|_{(C_1,\alpha)=(C_1^{P2},\alpha^{P2})} &= \left[\frac{-\Omega}{M(\lambda) - 1 - r_f - w_1}\right]^{-1-\gamma} \\ &\times\left[-1 + \frac{1+r_f+w_1}{k}\left(\frac{r_g - r_f}{r_g - r_b}\right)^2\left(\lambda^{-\frac{1}{\gamma}} - K_0^{\frac{1}{\gamma}}\frac{r_f - r_b}{r_g - r_f}\right)\right] \end{aligned}\tag{83}$$

$$\begin{aligned}\frac{1}{\gamma}\frac{d^2\mathbb{E}(U)}{d\alpha^2}|_{(C_1,\alpha)=(C_1^{P2},\alpha^{P2})} &= \frac{(r_f - r_b)^2}{k(1+r_f)}\left(\frac{r_g - r_f}{r_g - r_b}\right)^2\left[\frac{-\Omega}{M(\lambda) - 1 - r_f - w_1}\right]^{-1-\gamma}\left(\lambda^{-\frac{1}{\gamma}} - K_\gamma^{\frac{1}{\gamma}}\right) \end{aligned}\tag{84}$$

$$\frac{1}{\gamma}\frac{d^2\mathbb{E}(U)}{dC_1 d\alpha}|_{(C_1,\alpha)=(C_1^{P2},\alpha^{P2})} = \frac{r_f - r_b}{k}\left(\frac{r_g - r_f}{r_g - r_b}\right)^2\left[\frac{-\Omega}{M(\lambda) - 1 - r_f - w_1}\right]^{-1-\gamma}\left(\lambda^{-\frac{1}{\gamma}} + K_0^{\frac{1}{\gamma}}\right)$$

Note that (84) and $\lambda > \frac{1}{K_\gamma}$ imply that $\frac{d^2\mathbb{E}(U)}{d\alpha^2}|_{(C_1,\alpha)=(C_1^{P2},\alpha^{P2})} < 0$. In addition,

$$\begin{aligned}\frac{1}{\gamma^2}\left[\frac{-\Omega}{M(\lambda) - 1 - r_f - w_1}\right]^{2(1+\gamma)} D &= \left[-1 + \frac{1+r_f+w_1}{k}\left(\frac{r_g - r_f}{r_g - r_b}\right)^2\left(\lambda^{-\frac{1}{\gamma}} - \frac{r_f - r_b}{r_g - r_f}K_0^{\frac{1}{\gamma}}\right)\right] \\ &\times\frac{r_f - r_b}{k(1+r_f+w_1)}\left(\frac{r_g - r_f}{r_g - r_b}\right)^2\left[(r_f - r_b)\lambda^{-\frac{1}{\gamma}} - (r_g - r_f)K_0^{\frac{1}{\gamma}}\right] \\ &- \left(\frac{r_f - r_b}{k}\right)^2\left(\frac{r_g - r_f}{r_g - r_b}\right)^4\left(\lambda^{-\frac{1}{\gamma}} + K_0^{\frac{1}{\gamma}}\right)^2\end{aligned}$$

and thus

$$\frac{1}{\gamma^2}\left[\frac{-\Omega}{M(\lambda) - 1 - r_f - w_1}\right]^{2(1+\gamma)}\left(\frac{r_g - r_b}{r_g - r_f}\right)^2\frac{k}{r_f - r_b}D = \frac{1}{1+r_f+w_1}\left[(r_g - r_f)K_0^{\frac{1}{\gamma}} - (r_f - r_b)\lambda^{-\frac{1}{\gamma}}\right]$$

[41] For $\bar{C}_1 = \bar{C}_1^{U,P1}$ $(\Omega = 0)$ is $C_1^{P2} = \bar{C}_1$, $C_{2g}^{P2} = C_{2b}^{P2} = \bar{C}_2 = w_0 + (w_1 + w_2)\bar{C}_1$ and $\alpha^{P2} = 0$.

$$
\begin{aligned}
&+ \quad \frac{1}{k}\left(\frac{r_g - r_f}{r_g - r_b}\right)^2 \left(\lambda^{-\frac{1}{\gamma}} - \frac{r_f - r_b}{r_g - r_f} K_0^{\frac{1}{\gamma}}\right)\left[(r_f - r_b)\lambda^{-\frac{1}{\gamma}} - (r_g - r_f) K_0^{\frac{1}{\gamma}}\right] \\
&- \quad \frac{r_f - r_b}{k}\left(\frac{r_g - r_f}{r_g - r_b}\right)^2 \left(\lambda^{-\frac{1}{\gamma}} + K_0^{\frac{1}{\gamma}}\right)^2
\end{aligned}
$$

After some derivations we obtain

$$
\begin{aligned}
\frac{1}{\gamma^2}\left[\frac{-\Omega}{M(\lambda) - 1 - r_f - w_1}\right]^{2(1+\gamma)} \left(\frac{r_g - r_b}{r_g - r_f}\right)^2 \frac{k}{r_f - r_b} D \quad &= \quad \frac{1}{1 + r_f + w_1}\left[(r_g - r_f) K_0^{\frac{1}{\gamma}} - (r_f - r_b)\lambda^{-\frac{1}{\gamma}}\right] \\
&\quad - \frac{r_g - r_f}{k}\lambda^{-\frac{1}{\gamma}} K_0^{\frac{1}{\gamma}}
\end{aligned}
$$

Now it can be easily shown that if $\bar{C}_1^{U,P1} < \bar{C}_1 \le \bar{C}_1^{U,P2}$ and $\lambda > \left[\frac{1+r_f+w_1}{k} + \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right]^\gamma$ then $D > 0$ and thus the local maximum is reached at $\left(C_1^{P2}, \alpha^{P2}\right)$. There is no local maximum for $\bar{C}_1 < \bar{C}_1^{U,P1}$ and thus in this case the maximum occurs at the border.

Thus, for analyzing the feasibility of $\left(C_1^{P2}, \alpha^{P2}\right)$ we assume that $\bar{C}_1^{U,P1} < \bar{C}_1 \le \bar{C}_1^{U,P2}$ and $C_{2L} \le \frac{[(w_1+w_2)(1+r_f)Y_1+Y_2]+(1+r_f)w_0}{1+r_f+w_1+w_2}$ (see Lemma 1-(viii)). Note that (81) and (82) imply that $C_{2g}^{P2} > \bar{C}_2$ and $C_{2b}^{P2} < \bar{C}_2$ only when $M(\lambda) - 1 - r_f - w_1 > 0$ which holds for $\lambda > \left[\frac{1+r_f+w_1}{k} + \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right]^\gamma$. In addition, $C_{2L} \le C_{2L}^U$ and $\lambda \ge \lambda^{P2}$ imply that $C_{2b}^{P2} \ge C_{2L}$. In more detail, $C_{2b}^{P2} = (1 + r_f)(Y_1 - C_1^{P2}) + Y_2 - (r_f - r_b)\alpha^{P2} \ge C_{2L}$ if

$$
\begin{aligned}
\lambda^{1/\gamma}\left[\frac{w_0 + (w_1 + w_2)\bar{C}_1 - C_{2L}}{-\Omega} - \frac{r_g - r_b}{r_g - r_f}\right] \quad &\ge \quad \frac{w_0 + (w_1 + w_2)\bar{C}_1 - C_{2L}}{-\Omega}\left[\frac{1 + r_f + w_1}{k} + \left(\frac{1}{K_\gamma}\right)^{1/\gamma}\right] \\
&\quad - \frac{w_1}{k} \qquad (85)
\end{aligned}
$$

For $C_{2L} < C_{2L}^U$ is the left-hand-side in (85) positive and thus (85) holds for $\lambda \ge \lambda^{P2}$. If, on the other hand, $C_{2L} > C_{2L}^U$ then the left-hand-side in (85) is negative and there are two possibilities for the right-hand-side: if it is non-negative then $\lambda^{1/\gamma}$ is non-positive and thus infeasible; if, on the other hand, the right-hand-side in (85) is non-positive then

$$
\lambda^{1/\gamma} \le \frac{-\Omega\frac{w_1}{k} + \left[C_{2L} - w_0 - (w_1 + w_2)\bar{C}_1\right]\left[\frac{1+r_f+w_1}{k} + \left(\frac{1}{K_\gamma}\right)^{1/\gamma}\right]}{\frac{r_g - r_b}{r_g - r_f}(-\Omega) + C_{2L} - w_0 - (w_1 + w_2)\bar{C}_1} = \left(\lambda^{P2}\right)^{1/\gamma}
$$

and it can be shown that the right-hand-side of the last inequality is below $\frac{1+r_f+w_1}{k} + \left(\frac{1}{K_\gamma}\right)^{1/\gamma}$ which then contradicts the other feasibility assumption, namely $C_{2g}^{P2} \ge \bar{C}_2$ and $C_{2b}^{P2} < \bar{C}_2$ that imply $\lambda > \left[\frac{1+r_f+w_1}{k} + \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right]^\gamma$.

If these conditions are not satisfied then the maximum will be reached at the border. Note in addition that

$$
\begin{aligned}
(1-\gamma)\mathbb{E}\left(U\left(C_1^{P2},\alpha^{P2}\right)\right) &= \left[\frac{-\Omega}{M(\lambda)-1-r_f-w_1}\right]^{1-\gamma}\left[1+\frac{k}{1+r_f+w_1}\left(\left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}-\lambda^{\frac{1}{\gamma}}\right)\right] \\
&= -\frac{(-\Omega)^{1-\gamma}}{1+r_f+w_1}\left[M(\lambda)-1-r_f-w_1\right]^{\gamma} \qquad (86)
\end{aligned}
$$

What remains to show is that feasible solutions at the border do not exceed the expected utility at (C_1^{P2},α^{P2}), where (P2) obtains its local maximum for $\bar{C}_1^{U,P1}<\bar{C}_1\leq\bar{C}_1^{U,P2}$ as well as to analyze feasible solutions at the border for $\bar{C}_1\leq\bar{C}_1^{U,P1}$. The feasible solutions at the border that come into consideration are: (i) $C_{2g}=\bar{C}_2$, (ii) $C_{2b}=\bar{C}_2$, (iii) $C_{2b}=C_{2L}$ and (iv) $C_1=\bar{C}_1$.

Note that for $\bar{C}_1=\bar{C}_1^{U,P1}$, i.e., $\Omega=0$, is $C_1^{P2}=\bar{C}_1^{U,P1}$ and $\alpha^{P2}=0$. As this is the inflation point then the maximum in this case will be reached at the border.

Case (i). $C_{2g}=\bar{C}_2$ when

$$
C_1=\frac{1}{1+r_f+w_1}\left[(1+r_f)Y_1+Y_2-w_0-w_2\bar{C}_1+(r_g-r_f)\alpha\right]=\bar{C}_1+\frac{\Omega+(r_g-r_f)\alpha}{1+r_f+w_1}
$$

and $\max\left\{0,\frac{-\Omega}{r_g-r_f}\right\}\leq\alpha\leq\alpha_{C_{2b}=C_{2L}}^{C_{2g}=\bar{C}_2}$ where $\alpha_{C_{2b}=C_{2L}}^{C_{2g}=\bar{C}_2}$ is given by (54).

Note that condition $\alpha\geq\frac{-\Omega}{r_g-r_f}$ follows from $C_1\geq\bar{C}_1$ and the upper bound on α follows from $C_{2b}\geq C_{2L}$. It can be seen that

$$
(1-\gamma)\mathbb{E}\left(U\left(\bar{C}_1+\frac{\Omega+(r_g-r_f)\alpha}{1+r_f+w_1},\alpha\right)\right)=\left(\frac{\Omega+(r_g-r_f)\alpha}{1+r_f+w_1}\right)^{1-\gamma}-\lambda\delta(1-p)(r_g-r_b)^{1-\gamma}\alpha^{1-\gamma} \qquad (87)
$$

Let $\Omega\geq 0$, i.e., $\bar{C}_1\leq\bar{C}_1^{U,P1}$. Then the potential maximum occurs either at $\alpha=0$ or $\alpha=\alpha_{C_{2b}=C_{2L}}^{C_{2g}=\bar{C}_2}$ or at the stationary point of the function given by (87) which can be easily derived and has the value $\alpha=\bar{\alpha}\equiv\frac{k\lambda^{\frac{1}{\gamma}}}{1+r_f+w_1-k\lambda^{\frac{1}{\gamma}}}\frac{\Omega}{r_g-r_f}$ when $\lambda<\left(\frac{1+r_f+w_1}{k}\right)^{\gamma}$ and is infeasible for $\lambda>\left(\frac{1+r_f+w_1}{k}\right)^{\gamma}$. Thus, for $\lambda<\left(\frac{1+r_f+w_1}{k}\right)^{\gamma}$

$$
\begin{aligned}
(1-\gamma)\mathbb{E}\left(U\left(C_1=\bar{C}_1+\frac{\Omega+(r_g-r_f)\bar{\alpha}}{1+r_f+w_1},\bar{\alpha}\right)\right) &= \frac{\Omega^{1-\gamma}}{1+r_f+w_1}\left(1+r_f+w_1-k\lambda^{\frac{1}{\gamma}}\right)^{\gamma} \\
\leq\left(\frac{\Omega}{1+r_f+w_1}\right)^{1-\gamma} &= (1-\gamma)\mathbb{E}\left(U\left(\bar{C}_1+\frac{\Omega}{1+r_f+w_1},0\right)\right)
\end{aligned}
$$

therefore point $\left(C_1=\bar{C}_1+\frac{\Omega+(r_g-r_f)\bar{\alpha}}{1+r_f+w_1},\bar{\alpha}\right)$ can not be a maximum of the main problem (6) as its utility function is below the utility function of $\left(\bar{C}_1+\frac{\Omega}{1+r_f+w_1},0\right)$ which is feasible for (P1) and also corresponds to the case when $\alpha=0$.

The end-point $\alpha = \alpha^{C_{2g}=\bar{C}_2}_{C_{2b}=C_{2L}}$, which gives $C_{2b} = C_{2L}$, is dealt with in case (iii) where it was shown that it can not be the point of the maxima.

Let $\Omega < 0$, i.e., $\bar{C}_1^{U,P1} < \bar{C}_1 \le \bar{C}_1^{U,P2}$. Then $\frac{-\Omega}{r_g - r_f} \le \alpha^{C_{2g}=\bar{C}_2}_{C_{2b}=C_{2L}}$ only if $C_{2L} \le C_{2L}^U$. The potential maximum occurs either at $\alpha = \frac{-\Omega}{r_g-r_f}$ or at $\alpha = \alpha^{C_{2g}=\bar{C}_2}_{C_{2b}=C_{2L}}$ or at the stationary point $\alpha = \bar{\alpha} = \frac{k\lambda^{\frac{1}{\gamma}}}{k\lambda^{\frac{1}{\gamma}}-1-r_f-w_1}\frac{(-\Omega)}{r_g-r_f}$. As

$$\lim_{\alpha \to^+ \frac{-\Omega}{r_g-r_f}} \frac{d\mathbb{E}\left(U\left(\bar{C}_1 + \frac{\Omega+(r_g-r_f)\alpha}{1+r_f+w_1}\alpha,\, \alpha\right)\right)}{d\alpha} = +\infty$$

then the maximum can not occur at $\alpha = \frac{-\Omega}{r_g-r_f}$. Regarding the stationary point $\bar{\alpha}$, note that the value of the utility for $\lambda > \left(\frac{1+r_f+w_1}{k}\right)^{\gamma}$ is

$$(1-\gamma)\mathbb{E}\left(U\left(\bar{C}_1 + \frac{\Omega + (r_g - r_f)\bar{\alpha}}{1 + r_f + w_1}, \bar{\alpha}\right)\right) \quad = \quad -\frac{(-\Omega)^{1-\gamma}}{1 + r_f + w_1}\left(k\lambda^{\frac{1}{\gamma}} - 1 - r_f - w_1\right)^{\gamma} \tag{88}$$

It is easy to see that the utility at this stationary point is below the utility at (C_1^{P2}, α^{P2}), as given by (86). It can be also verified that the objective function as given by (87) obtains its local maximum at $\alpha = \bar{\alpha}$. Finally, $\bar{\alpha}$ is feasible if $\bar{\alpha} \le \alpha^{C_{2g}=\bar{C}_2}_{C_{2b}=C_{2L}}$ which holds if

$$\lambda \ge \left[\frac{1 + r_f + w_1}{k} \times \frac{C_{2L}^U - C_{2L} + \left(\frac{1+r_f}{1+r_f+w_1} + \frac{r_f-r_b}{r_g-r_f}\right)(-\Omega)}{C_{2L}^U - C_{2L}}\right]^{\gamma} \equiv \lambda^{P2}_{(i)}$$

As $\lambda^{P2}_{(i)} < \lambda^{P2}$ then for $\lambda \ge \lambda^{P2} > \lambda^{P2}_{(i)}$ is $\bar{\alpha}$ feasible and is the point of the maximum for (87) which does not exceed the value of the objective function at $\left(C_1^{P2}, \alpha^{P2}\right)$.

Case (ii). Any feasible solution of this case is also a feasible solution of (P1).

Case (iii). $C_{2b} = C_{2L}$ when $C_1 = Y_1 + \frac{Y_2 - C_{2L}}{1+r_f} - \frac{r_f - r_b}{1+r_f}\alpha$ and

$$\alpha^{C_{2g}=\bar{C}_2}_{C_{2b}=C_{2L}} \le \alpha \le \frac{(1 + r_f)(Y_1 - \bar{C}_1) + Y_2 - C_{2L}}{r_f - r_b} = \alpha^{C_1=\bar{C}_1}_{C_{2b}=C_{2L}}$$

where $\alpha^{C_{2g}=\bar{C}_2}_{C_{2b}=C_{2L}}$ is given by (54). Note that the lower bound on α is below its upper bound only if $\bar{C}_1 \le \bar{C}_1^{U,P2}$.

It can be seen that

$$
\begin{aligned}
& (1-\gamma)\mathbb{E}\left(U\left(Y_1+\frac{Y_2-C_{2L}}{1+r_f}-\frac{r_f-r_b}{1+r_f}\,\alpha,\,\alpha\right)\right)=\left[Y_1-\bar{C}_1+\frac{Y_2-C_{2L}}{1+r_f}-\frac{r_f-r_b}{1+r_f}\,\alpha\right]^{1-\gamma}\\
+ \quad & \delta p\left[\frac{1+r_f+w_1}{1+r_f}C_{2L}-w_0-w_1\left(Y_1+\frac{Y_2}{1+r_f}\right)-w_2\bar{C}_1+\left(r_g-r_b+\frac{r_f-r_b}{1+r_f}w_1\right)\alpha\right]^{1-\gamma}\\
- \quad & \lambda\delta(1-p)\left[w_0+w_1\left(Y_1+\frac{Y_2}{1+r_f}\right)+w_2\bar{C}_1-\frac{1+r_f+w_1}{1+r_f}C_{2L}-\frac{w_1(r_f-r_b)}{1+r_f}\,\alpha\right]^{1-\gamma} \qquad (89)
\end{aligned}
$$

Based on (89) we obtain for $\bar{C}_1 \leq \bar{C}_1^{U,P1}$ and $\alpha \in \left[\alpha_{C_{2b}=C_{2L}}^{C_{2g}=\bar{C}_2}, \alpha_{C_{2b}=C_{2L}}^{C_1=\bar{C}_1}\right]$

$$
\begin{aligned}
& (1-\gamma)\mathbb{E}\left(U\left(Y_1+\frac{Y_2-C_{2L}}{1+r_f}-\frac{r_f-r_b}{1+r_f}\,\alpha,\,\alpha\right)\right)\leq\left[Y_1-\bar{C}_1+\frac{Y_2-C_{2L}}{1+r_f}-\frac{r_f-r_b}{1+r_f}\,\alpha_{C_{2b}=C_{2L}}^{C_{2g}=\bar{C}_2}\right]^{1-\gamma}\\
+ \quad & \delta p\left[\frac{1+r_f+w_1}{1+r_f}C_{2L}-w_0-w_1\left(Y_1+\frac{Y_2}{1+r_f}\right)-w_2\bar{C}_1+\left(r_g-r_b+\frac{r_f-r_b}{1+r_f}w_1\right)\alpha_{C_{2b}=C_{2L}}^{C_1=\bar{C}_1}\right]^{1-\gamma}\\
- \quad & \lambda\delta(1-p)\left[w_0+w_1\left(Y_1+\frac{Y_2}{1+r_f}\right)+w_2\bar{C}_1-\frac{1+r_f+w_1}{1+r_f}C_{2L}-\frac{w_1(r_f-r_b)}{1+r_f}\,\alpha_{C_{2b}=C_{2L}}^{C_1=\bar{C}_1}\right]^{1-\gamma}\\
= \quad & \left[\frac{r_f-r_b}{(1+r_f)(r_g-r_b)+w_1(r_f-r_b)}\right]^{1-\gamma}\left[\Omega+(r_g-r_f)\alpha_{C_{2b}=C_{2L}}^{C_1=\bar{C}_1}\right]^{1-\gamma}\\
& +\delta p\left[\Omega+(r_g-r_f)\alpha_{C_{2b}=C_{2L}}^{C_1=\bar{C}_1}\right]^{1-\gamma}-\lambda\delta(1-p)\left[w_0+(w_1+w_2)\bar{C}_1-C_{2L}\right]^{1-\gamma} \qquad (90)
\end{aligned}
$$

and thus if the right-hand-side of (90) is below $(1-\gamma)\mathbb{E}\left(U\left(C_1^{P1},\alpha^{P1}\right)\right)$ then also $\mathbb{E}\left(U\left(Y_1+\frac{Y_2-C_{2L}}{1+r_f}-\frac{r_f-r_b}{1+r_f}\,\alpha,\,\alpha\right)\right)<\mathbb{E}\left(U\left(C_1^{P1},\alpha^{P1}\right)\right)$ for $\bar{C}_1 \leq \bar{C}_1^{U,P1}$, $\alpha \in \left[\alpha_{C_{2b}=C_{2L}}^{C_{2g}=\bar{C}_2}, \alpha_{C_{2b}=C_{2L}}^{C_1=\bar{C}_1}\right]$ and for

$$
\begin{aligned}
\lambda \quad > \quad & \frac{\left[\left(\frac{r_f-r_b}{(r_g-r_b)(1+r_f)+w_1(r_f-r_b)}\right)^{1-\gamma}+\delta p\right]\left[\Omega+(r_g-r_f)\alpha_{C_{2b}=C_{2L}}^{C_1=\bar{C}_1}\right]^{1-\gamma}}{\delta(1-p)\left[w_0+(w_1+w_2)\bar{C}_1-C_{2L}\right]^{1-\gamma}}\\
& -\frac{\Omega^{1-\gamma}(1+r_f+w_1+M)^{\gamma}}{\delta(1-p)(1+r_f+w_1)\left[w_0+(w_1+w_2)\bar{C}_1-C_{2L}\right]^{1-\gamma}}=\lambda^{P1-P2} \qquad (91)
\end{aligned}
$$

Note that these are sufficient conditions for the expected utility of case (iii) not to exceed the maximum of (P1). As we could not find an explicit solution (maximum) of (89) we tackled it by imposing a lower bound on λ, which is in line with our goal to deal with loss averse households.

In a similar way it can be shown that for $\bar{C}_1^{U,P1} < \bar{C}_1 < \bar{C}_1^{U,P2}$ is $\mathbb{E}\left(U\left(Y_1+\frac{Y_2-C_{2L}}{1+r_f}-\frac{r_f-r_b}{1+r_f}\,\alpha,\,\alpha\right)\right)<$

$\mathbb{E}\left(U\left(C_1^{P2}, \alpha^{P2}\right)\right)$ for all $\alpha \in \left[\alpha_{C_{2b}=C_{2L}}^{C_{2g}=\bar{C}_2}, \alpha_{C_{2b}=C_{2L}}^{C_1=\bar{C}_1}\right]$ when[42]

$$\lambda \;>\; \frac{\left[\left(\frac{r_f-r_b}{(r_g-r_b)(1+r_f)+w_1(r_f-r_b)}\right)^{1-\gamma}+\delta p\right]\left[\Omega+(r_g-r_f)\alpha_{C_{2b}=C_{2L}}^{C_1=\bar{C}_1}\right]^{1-\gamma}}{\delta(1-p)\left[\left(w_0+(w_1+w_2)\bar{C}_1-C_{2L}\right)^{1-\gamma}-\left(\frac{r_g-r_b}{r_g-r_f}\right)^{1-\gamma}(-\Omega)^{1-\gamma}\right]} = \lambda^{P2-P2} \quad (92)$$

Case (iv). $C_1 = \bar{C}_1$ and thus

$$(1-\gamma)\mathbb{E}\left(U\left(\bar{C}_1,\, \alpha\right)\right) \;=\; \delta p\left[\Omega+(r_g-r_f)\alpha\right]^{1-\gamma}-\lambda\delta(1-p)\left[-\Omega+(r_f-r_b)\alpha\right]^{1-\gamma} \quad (93)$$

Let $\Omega \geq 0$, i.e., $\bar{C}_1 \leq \bar{C}_1^{U,P1}$. Then the set of feasible solutions is: $\frac{\Omega}{r_f-r_b} \leq \alpha \leq \frac{(1+r_f)(Y_1-\bar{C}_1)+Y_2-C_{2L}}{r_f-r_b}$. The left end point $\alpha = \frac{\Omega}{r_f-r_b}$ follows from $C_{2b} = \bar{C}_2$ and is thus feasible also for (P1) while the right end point $\alpha = \frac{(1+r_f)(Y_1-\bar{C}_1)+Y_2-C_{2L}}{r_f-r_b}$ is implied by $C_{2b} = C_{2L}$ and thus was tackled in case (iii). What remains to investigate is the stationary point $\bar{\alpha} = \frac{1+\lambda^{1/\gamma}K_0^{1/\gamma}}{1-\lambda^{1/\gamma}K_\gamma^{1/\gamma}}\frac{\Omega}{r_f-r_b}$ for $\lambda < \frac{1}{K_\gamma}$ so $\bar{\alpha} \geq 0$. It can be shown that $\bar{\alpha}$ is the point of maxima only when $\lambda > \frac{1}{K_\gamma}$ which contradicts the feasibility. Thus, $\bar{\alpha}$ can not be the point of local maxima.

Let $\Omega < 0$, i.e., $\bar{C}_1^{U,P1} < \bar{C}_1 \leq \bar{C}_1^{U,P2}$. Then the set of feasible solutions is: $-\frac{\Omega}{r_g-r_f} \leq \alpha \leq \frac{(1+r_f)(Y_1-\bar{C}_1)+Y_2-C_{2L}}{r_f-r_b}$. The following can be easily shown

$$\lim_{\alpha \to^{+} \frac{-\Omega}{r_g-r_f}} \frac{d\mathbb{E}\left(U\left(\bar{C}_1,\, \alpha\right)\right)}{d\alpha} = +\infty \quad (94)$$

and

$$\left.\frac{d\mathbb{E}\left(U\left(\bar{C}_1,\, \alpha\right)\right)}{d\alpha}\right|_{\alpha=\bar{\alpha}} = 0$$

for

$$\bar{\alpha} = \frac{\lambda^{\frac{1}{\gamma}}+\left(\frac{1}{K_0}\right)^{\frac{1}{\gamma}}}{\lambda^{\frac{1}{\gamma}}-\left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}}\frac{(-\Omega)}{r_g-r_f}$$

and $\lambda > \frac{1}{K_\gamma}$. As

$$(1-\gamma)\mathbb{E}\left(U\left(\bar{C}_1,\, \bar{\alpha}\right)\right) = -\frac{k^\gamma(-\Omega)^{1-\gamma}}{1+r_f+w_1}\left[\lambda^{\frac{1}{\gamma}}-\left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right]^\gamma \quad (95)$$

[42] Note that for $\bar{C}_1 < \bar{C}_1^{U,P2}$ is the denominator in (92) strictly positive. Note in addition that the right-hand-side of (90) being below $-\frac{(-\Omega)^{1-\gamma}}{1+r_f+w_1}k^\gamma\lambda$ is sufficient for $\mathbb{E}\left(U\left(Y_1+\frac{Y_2-C_{2L}}{1+r_f}-\frac{r_f-r_b}{1+r_f}\,\alpha,\, \alpha\right)\right) < \mathbb{E}\left(U\left(C_1^{P2}, \alpha^{P2}\right)\right)$.

then based on this and (86) it can be shown that for $\lambda > \left[\frac{1+r_f+w_1}{k} + \left(\frac{1}{K_\gamma}\right)^{1/\gamma}\right]^\gamma$

$$(1-\gamma)\mathbb{E}\left(U\left(\bar{C}_1,\, \bar{\alpha}\right)\right) < (1-\gamma)\mathbb{E}\left(U\left(C_1^{P2},\, \alpha^{P2}\right)\right)$$

Note that the maximum can not be reached at $\alpha = \frac{-\Omega}{r_g - r_f}$, see (94).

Note that for $\bar{C}_1 = \bar{C}_1^{U,P2}$ there is the only feasible solution for (P2), namely $\left(C_1 = \bar{C}_1 = \bar{C}_1^{U,P2}, \alpha = \alpha^{C_1=\bar{C}_1}_{C_{2b}=C_{2L}}\right)$ as $\alpha^{C_1=\bar{C}_1}_{C_{2b}=C_{2L}} = \alpha^{C_1=\bar{C}_1}_{C_{2g}=\bar{C}_2}$. Note in addition that $C_{2g} = \bar{C}_2 = w_0 + (w_1+w_2)\bar{C}_1^{U,P2}$, $C_{2b} = C_{2L}$ and

$$\begin{aligned}(1-\gamma)\mathbb{E}\left(U\left(\bar{C}_1^{U,P2},\, \alpha^{C_1=\bar{C}_1}_{C_{2b}=C_{2L}}\right)\right) &= -\lambda\,\delta(1-p)\left[w_0 + (w_1+w_2)\bar{C}_1^{U,P2} - C_{2L}\right]^{1-\gamma} \\ &= -\lambda\,\delta(1-p)(r_g - r_b)^{1-\gamma}\left(\alpha^{C_1=\bar{C}_1}_{C_{2b}=C_{2L}}\right)^{1-\gamma} \qquad (96)\end{aligned}$$

Note that this point is feasible also for (P4)–(P8).

Summary for (P2):

- For $\bar{C}_1 \le \bar{C}_1^{U,P1}$ and $\lambda > \lambda^{P1-P2}$ problem (P1) exceeds (P2).
- For $\bar{C}_1^{U,P1} < \bar{C}_1 < \bar{C}_1^{U,P2}$ and $\lambda > \max\left\{\lambda^{P2}, \lambda^{P2-P2}\right\}$ is the maximum given by $\left(C_1 = C_1^{P2}, \alpha = \alpha^{P2}\right)$, see (36) and (37).
- For $\bar{C}_1 = \bar{C}_1^{U,P2}$ there is the only feasible solution for (P2), namely $\left(C_1 = \bar{C}_1^{U,P2}, \alpha = \alpha^{C_1=\bar{C}_1}_{C_{2b}=C_{2L}}\right)$.

Problem (P4). Let $\bar{C}_1 \le \bar{C}_1^{U,P2}$. As $\frac{d^2\mathbb{E}(U)}{d\alpha^2} > 0$ then there is no local interior maximum, which implies that the maximum will occur at the border. The cases under consideration are: (i) $C_{2g} = \bar{C}_2$, (ii) $C_{2b} = \bar{C}_2$, (iii) $C_{2b} = C_{2L}$ and (iv) $C_1 = \bar{C}_1$. Note that the point implied by case (i) is feasible for (P2) and the point implied by case (ii) is feasible for (P1).

Case (iii): $C_{2b} = C_{2L}$ when $C_1 = Y_1 + \frac{Y_2 - C_{2L}}{1+r_f} - \frac{r_f - r_b}{1+r_f}\alpha$ and

$$0 \le \alpha \le \alpha^{C_{2g}=\bar{C}_2}_{C_{2b}=C_{2L}} \quad \text{for} \quad \bar{C}_1^{L,P4} \le \bar{C}_1 \le \bar{C}_1^{U,P2} \qquad (97)$$

where $\alpha^{C_{2g}=\bar{C}_2}_{C_{2b}=C_{2L}}$ is given by (54). It can be then seen that

$$\begin{aligned}(1-\gamma)\mathbb{E}\left(U\left(Y_1 + \frac{Y_2 - C_{2L}}{1+r_f} - \frac{r_f - r_b}{1+r_f}\alpha,\, \alpha\right)\right) &= \left[Y_1 - \bar{C}_1 + \frac{Y_2 - C_{2L}}{1+r_f} - \frac{r_f - r_b}{1+r_f}\alpha\right]^{1-\gamma} \\ - \lambda\delta p\left[w_0 + w_1\left(Y_1 + \frac{Y_2}{1+r_f}\right) + w_2\bar{C}_1 - \frac{1+r_f+w_1}{1+r_f}C_{2L} - \left(r_g - r_b + \frac{r_f - r_b}{1+r_f}w_1\right)\alpha\right]^{1-\gamma} \\ - \lambda\delta(1-p)\left[w_0 + w_1\left(Y_1 + \frac{Y_2}{1+r_f}\right) + w_2\bar{C}_1 - \frac{1+r_f+w_1}{1+r_f}C_{2L} - \frac{w_1(r_f - r_b)}{1+r_f}\alpha\right]^{1-\gamma} \qquad (98)\end{aligned}$$

and thus

$$\frac{d\mathbb{E}\left(U\left(Y_1+\frac{Y_2-C_{2L}}{1+r_f}-\frac{r_f-r_b}{1+r_f},\,\alpha\right)\right)}{d\alpha} = -\left[Y_1-\bar{C}_1+\frac{Y_2-C_{2L}}{1+r_f}-\frac{r_f-r_b}{1+r_f}\,\alpha\right]^{-\gamma}\frac{r_f-r_b}{1+r_f}$$
$$+\quad \lambda\delta p\left[w_0+w_1\left(Y_1+\frac{Y_2}{1+r_f}\right)+w_2\bar{C}_1-\frac{1+r_f+w_1}{1+r_f}C_{2L}-\left(r_g-r_b+\frac{r_f-r_b}{1+r_f}w_1\right)\alpha\right]^{-\gamma}$$
$$\times\left[r_g-r_b+\frac{r_f-r_b}{1+r_f}w_1\right]$$
$$+\quad \lambda\delta(1-p)\left[w_0+w_1\left(Y_1+\frac{Y_2}{1+r_f}\right)+w_2\bar{C}_1-\frac{1+r_f+w_1}{1+r_f}C_{2L}-\frac{w_1(r_f-r_b)}{1+r_f}\,\alpha\right]^{-\gamma}\frac{w_1(r_f-r_b)}{1+r_f} \tag{99}$$
$$\geq\quad -\left(\frac{r_f-r_b}{1+r_f}\right)^{1-\gamma}\left(\alpha_{C_{2b}=C_{2L}}^{C_1=\bar{C}_1}-\alpha_{C_{2b}=C_{2L}}^{C_{2g}=\bar{C}_2}\right)^{-\gamma}$$
$$+\lambda\,\delta\left[w_0+w_1\left(Y_1-\bar{C}_1+\frac{Y_2-C_{2L}}{1+r_f}\right)+(w_1+w_2)\bar{C}_1-C_{2L}\right]^{-\gamma}\frac{w_1(r_f-r_b)}{1+r_f} \tag{100}$$

which is positive and thus the objective function (98) is increasing, for

$$\lambda > \frac{1}{w_1\,\delta}\left[\frac{w_0+w_1\left(Y_1-\bar{C}_1+\frac{Y_2-C_{2L}}{1+r_f}\right)+(w_1+w_2)\bar{C}_1-C_{2L}}{\alpha_{C_{2b}=C_{2L}}^{C_1=\bar{C}_1}-\alpha_{C_{2b}=C_{2L}}^{C_{2g}=\bar{C}_2}}\,\frac{1+r_f}{r_f-r_b}\right]^{\gamma}=\lambda^{P4} \tag{101}$$

This implies that the maximum of (98) is reached at $\alpha_{C_{2b}=C_{2L}}^{C_{2g}=\bar{C}_2}$ which is feasible for (P2) as well and thus is not the maximum of the main problem (6).

For $\bar{C}_1=\bar{C}_1^{U,P2}$ is $\alpha_{C_{2b}=C_{2L}}^{C_1=\bar{C}_1}=\alpha_{C_{2b}=C_{2L}}^{C_{2g}=\bar{C}_2}$ and thus the first derivative, as given by (100), can be written as

$$\frac{d\mathbb{E}\left(U\left(Y_1+\frac{Y_2-C_{2L}}{1+r_f}-\frac{r_f-r_b}{1+r_f},\,\alpha\right)\right)}{d\alpha} = -\left(\frac{r_f-r_b}{1+r_f}\right)^{1-\gamma}\left(\alpha_{C_{2b}=C_{2L}}^{C_1=\bar{C}_1}-\alpha\right)^{-\gamma}$$
$$+\quad \lambda\delta p\left(r_g-r_b+\frac{r_f-r_b}{1+r_f}w_1\right)^{1-\gamma}\left(\alpha_{C_{2b}=C_{2L}}^{C_1=\bar{C}_1}-\alpha\right)^{-\gamma}$$
$$+\quad \lambda\delta(1-p)w_1\left(\frac{r_f-r_b}{1+r_f}\right)^{1-\gamma}\left[\frac{(r_g-r_b)(1+r_f)}{r_f-r_b}\,\frac{\alpha_{C_{2b}=C_{2L}}^{C_1=\bar{C}_1}}{\alpha_{C_{2b}=C_{2L}}^{C_1=\bar{C}_1}-\alpha}+w_1\right]^{-\gamma}\left(\alpha_{C_{2b}=C_{2L}}^{C_1=\bar{C}_1}-\alpha\right)^{-\gamma}$$

which is positive and thus the objective function is increasing for

$$\lambda > \frac{1}{\delta}\times\frac{1}{p\left[\frac{(1+r_f)(r_g-r_b)}{r_f-r_b}+w_1\right]^{1-\gamma}+(1-p)w_1\left[\frac{(1+r_f)(r_g-r_b)}{r_f-r_b}\,\frac{\alpha_{C_{2b}=C_{2L}}^{C_1=\bar{C}_1}}{\alpha_{C_{2b}=C_{2L}}^{C_1=\bar{C}_1}-\alpha}+w_1\right]^{-\gamma}}=\lambda(\alpha)$$

As $\lambda(\alpha)$ is an increasing function in α then for $\alpha\in\left[0,\alpha_{C_{2b}=C_{2L}}^{C_1=\bar{C}_1}\right]$ is $\lambda(\alpha)\leq\lambda\left(\alpha_{C_{2b}=C_{2L}}^{C_1=\bar{C}_1}\right)$ and thus it is sufficient for the objective function (98) with $\bar{C}_1 = \bar{C}_1^{U,P2}$ to be increasing when $\lambda > \lambda\left(\alpha_{C_{2b}=C_{2L}}^{C_1=\bar{C}_1}\right)$, i.e., when

$$\lambda > \frac{1}{\delta p}\left[\frac{r_f - r_b}{(1+r_f)(r_g - r_b) + w_1(r_f - r_b)}\right]^{1-\gamma} = \lambda\left(\alpha_{C_{2b}=C_{2L}}^{C_1=\bar{C}_1}\right) = \lambda^{P2-P4} \tag{102}$$

Thus, for $\bar{C}_1 = \bar{C}_1^{U,P2}$ and $\lambda > \lambda^{P2-P4}$ the maximum of the objective function (98) is reached for $\alpha = \alpha_{C_{2b}=C_{2L}}^{C_1=\bar{C}_1}$ which is of the same value as the objective function of (P2) at its maximum, see (96).

Case (iv): $C_1 = \bar{C}_1$. Note that

$$\bar{C}_2 - C_{2g} = (1+r_f+w_1)\bar{C}_1 - (r_g - r_f)\alpha - (1+r_f)Y_1 - Y_2 + w_0 + w_2\bar{C}_1 = -\Omega - (r_g - r_f)\alpha \geq 0$$

for $\alpha \leq \frac{-\Omega}{r_g - r_f}$ for which is also $\bar{C}_2 - C_{2b}$ nonnegative and $C_{2b} \geq C_{2L}$ for $\alpha \leq \frac{(1+r_f)(Y_1-\bar{C}_1)+Y_2-C_{2L}}{r_f - r_b}$. This inequality can not be satisfied for $\Omega > 0$. When $\Omega = 0$ then the only feasible solution is $\left(\bar{C}_1^{U,P1}, 0\right)$ which is feasible also for (P1). Let $\bar{C}_1 > \bar{C}_1^{U,P1}$, i.e., $\Omega < 0$, and note that the objective function in this case is

$$(1-\gamma)\mathbb{E}(U) = -\lambda\delta p\left[-\Omega - (r_g - r_f)\alpha\right]^{1-\gamma} - \lambda\delta(1-p)\left[-\Omega + (r_f - r_b)\alpha\right]^{1-\gamma} \tag{103}$$

where

$$0 \leq \alpha \leq \frac{-\Omega}{r_g - r_f} = \alpha_{C_{2g}=\bar{C}_2}^{C_1=\bar{C}_1}$$

It can be seen that for $\bar{C}_1 > \bar{C}_1^{U,P1}$ and $\mathbb{E}(r) > r_f$ is the objective function (103) increasing in α, so the maximum occurs at $\left(\bar{C}_1, \alpha_{C_{2g}=\bar{C}_2}^{C_1=\bar{C}_1}\right)$. Note that point $\left(\bar{C}_1, \alpha_{C_{2g}=\bar{C}_2}^{C_1=\bar{C}_1}\right)$ is feasible for (P2). Thus, the only relevant case (in terms of comparisons to other potential candidates for maxima) in (P4) is case (iii).

Summary for (P4):

- For $\bar{C}_1^{L,P4} \leq \bar{C}_1 < \bar{C}_1^{U,P2}$ and $\lambda > \lambda^{P4}$ is (P4) exceeded by (P1) or (P2).
- For $\bar{C}_1 = \bar{C}_1^{U,P2}$ and $\lambda > \lambda^{P2-P4}$ the maximum of (P4) coincides with the maximum of (P2) as given by (96).

Problem (P5). Let $\bar{C}_1 \leq \bar{C}_1^{U,P2}$. Note that $C_{2b} \geq \bar{C}_2$ implies that $C_1 \leq \bar{C}_1 + \frac{\Omega}{1+r_f+w_1}$. It is easy to show that for any fixed $\tilde{C}_1$ such that $C_{1L} \leq \tilde{C}_1 \leq \min\left\{\bar{C}_1, \bar{C}_1 + \frac{\Omega}{1+r_f+w_1}\right\}$ is the expected utility of (P5) concave and its maximum is achieved at

$$\tilde{\alpha} = \frac{1 - K_0^{\frac{1}{\gamma}}}{r_f - r_b + K_0^{\frac{1}{\gamma}}(r_g - r_f)}\left[\Omega + (1+r_f+w_1)(\bar{C}_1 - \tilde{C}_1)\right] \tag{104}$$

Note that $(\tilde{C}_1, \tilde{\alpha})$ is feasible for (P5). Thus, the candidates for the maximum of (P5) are $(\tilde{C}_1, \tilde{\alpha})$ with $C_{1L} \leq \tilde{C}_1 \leq \min\left\{\bar{C}_1, \bar{C}_1 + \frac{\Omega}{1+r_f+w_1}\right\}$ and $\tilde{\alpha}$ given by (104). By plugging this point into the expected utility of (P5) we obtain (after some derivations)

$$(1-\gamma)\mathbb{E}(U) = -\lambda\left(\bar{C}_1 - \tilde{C}_1\right)^{1-\gamma} + \lambda^{P1-P5}\left[\frac{\Omega}{1+r_f+w_1} + \bar{C}_1 - \tilde{C}_1\right]^{1-\gamma}$$

where

$$\lambda^{P1-P5} = \left[\frac{k_2\left(1+K_\gamma^{\frac{1}{\gamma}}\right)}{1+r_f+w_1}\right]^\gamma = \left[\frac{k\left(1+\left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right)}{1+r_f+w_1}\right]^\gamma$$

It can be seen (after some derivations) that for $C_{1L} \leq \bar{C}_1 \leq \bar{C}_1^{U,P1}$ and any $\tilde{C}_1 \in \left[C_{1L}, \bar{C}_1\right]$ is the expected utility of (P5) given by (105) exceeded by the expected utility of (P1) at its maximum given by (70) for $\lambda \geq \lambda^{P1-P5}$, namely

$$\begin{aligned}
-\lambda\left(\bar{C}_1 - \tilde{C}_1\right)^{1-\gamma} + \lambda^{P1-P5}\left[\frac{\Omega}{1+r_f+w_1} + \bar{C}_1 - \tilde{C}_1\right]^{1-\gamma} &\leq -\left(\lambda - \lambda^{P1-P5}\right)\left(\bar{C}_1 - \tilde{C}_1\right)^{1-\gamma} \\
&\quad + \lambda^{P1-P5}\left(\frac{\Omega}{1+r_f+w_1}\right)^{1-\gamma} \\
&< \left(\frac{\Omega}{1+r_f+w_1}\right)^{1-\gamma}\left[1+\left(\lambda^{P1-P5}\right)^{1/\gamma}\right]^\gamma \\
&= (1-\gamma)\mathbb{E}\left(U\left(C_1^{P1}, \alpha^{P1}\right)\right)
\end{aligned}$$

Let $\bar{C}_1^{U,P1} < \bar{C}_1 \leq \bar{C}_1^{U,P2}$. As

$$\lim_{\tilde{C}_1 \to C_{1L}} \frac{d\mathbb{E}(U)}{d\tilde{C}_1} > 0 \quad \text{for} \quad \lambda > \lambda^{P1-P5}\left[\frac{(1+r_f+w_1)(\bar{C}_1 - C_{1L})}{w_2\left(\bar{C}_1^{U,P5} - \bar{C}_1\right)}\right]^\gamma = \lambda^{P5}$$

and

$$\lim_{\tilde{C}_1 \to \bar{C}_1 + \frac{\Omega}{1+r_f+w_1}} \frac{d\mathbb{E}(U)}{d\tilde{C}_1} = -\infty < 0$$

then for $\lambda > \lambda^{P5}$ is the maximum reached at the stationary point

$$\begin{aligned}
C_1^{P5} &= \bar{C}_1 - \frac{\lambda^{1/\gamma}}{\lambda^{1/\gamma} - (\lambda^{P1-P5})^{1/\gamma}} \times \frac{-\Omega}{1+r_f+w_1} \\
&= \frac{\lambda^{1/\gamma}\left[(1+r_f)Y_1 + Y_2 - w_0 - w_2\bar{C}_1\right] - (\lambda^{P1-P5})^{1/\gamma}(1+r_f+w_1)\bar{C}_1}{\left[\lambda^{1/\gamma} - (\lambda^{P1-P5})^{1/\gamma}\right](1+r_f+w_1)}
\end{aligned}$$

Note that $\lambda^{P5} > \lambda^{P1-P5}$. After plugging the stationary point C_1^{P5} into (104) we obtain

$$\alpha^{P5} = \frac{1 - K_0^{1/\gamma}}{r_f - r_b + K_0^{1/\gamma}(r_g - r_f)} \times \frac{(\lambda^{P1-P5})^{1/\gamma}}{\lambda^{1/\gamma} - (\lambda^{P1-P5})^{1/\gamma}} \times (-\Omega) \tag{105}$$

Thus the maximum of (P5) for $\lambda > \lambda^{P5}$ is reached at $\left(C_1^{P5}, \alpha^{P5}\right)$ with

$$(1-\gamma)\mathbb{E}\left(U\left(C_1^{P5}, \alpha^{P5}\right)\right) = -\left(\frac{-\Omega}{1 + r_f + w_1}\right)^{1-\gamma} \left[\lambda^{1/\gamma} - (\lambda^{P1-P5})^{1/\gamma}\right]^{\gamma} \tag{106}$$

It can be shown that for $\bar{C}_1^{U,P1} < \bar{C}_1 < \min\left\{\bar{C}_1^{U,P2}, \bar{C}_1^{U,P5}\right\}$ is the maximum of (P5) below the maximum of (P2), i.e.,

$$\begin{aligned}
(1-\gamma)\mathbb{E}\left(U\left(C_1^{P5}, \alpha^{P5}\right)\right) &= -\left(\frac{-\Omega}{1 + r_f + w_1}\right)^{1-\gamma} \left[\lambda^{1/\gamma} - (\lambda^{P1-P5})^{1/\gamma}\right]^{\gamma} \\
&< -\frac{(-\Omega)^{1-\gamma}}{1 + r_f + w_1}\left[M(\lambda) - 1 - r_f - w_1\right]^{\gamma} \\
&= (1-\gamma)\mathbb{E}\left(U\left(C_1^{P2}, \alpha^{P2}\right)\right)
\end{aligned} \tag{107}$$

see (86), when $\frac{k}{1+r_f+w_1} < 1$ as inequality (107) boils down to

$$\lambda^{1/\gamma} + 1 > \frac{k}{1 + r_f + w_1}\left(\lambda^{1/\gamma} + 1\right)$$

Thus, for $\frac{k}{1+r_f+w_1} = 1$ both objective functions have the same value and for $\frac{k}{1+r_f+w_1} > 1$ (P5) at its maximum exceeds (P2) at its maximum. Note finally that $\frac{k}{1+r_f+w_1} < 1$ when

$$\delta < \frac{1}{1-p}\left[\frac{r_g - r_f}{(1 + r_f + w_1)(r_g - r_b)}\right]^{1-\gamma} = \delta^{P2-P5}$$

$\frac{k}{1+r_f+w_1} = 1$ when $\delta = \delta^{P2-P5}$ and $\frac{k}{1+r_f+w_1} > 1$ when $\delta > \delta^{P2-P5}$.

Note that for $\bar{C}_1 = \bar{C}_1^{U,P2}$ is the maximum of (P5) as given by (106) for $\lambda > \lambda^{P5}$ below the maximum of (P2), which is the value of its objective function at its only feasible solution when

$$\begin{aligned}
(1-\gamma)\mathbb{E}\left(U\left(C_1^{P5}, \alpha^{P5}\right)\right) &= -\left(\frac{-\Omega}{1 + r_f + w_1}\right)^{1-\gamma} \left[\lambda^{1/\gamma} - (\lambda^{P1-P5})^{1/\gamma}\right]^{\gamma} \\
&\leq -\lambda\,\delta(1-p)\left[w_0 + (w_1 + w_2)\bar{C}_1^{U,P2} - C_{2L}\right]^{1-\gamma} \\
&= (1-\gamma)\mathbb{E}\left(U\left(\bar{C}_1^{U,P2}, \alpha^{C_1=\bar{C}_1}_{C_{2b}=C_{2L}}\right)\right)
\end{aligned} \tag{108}$$

This holds for any $\lambda > \lambda^{P5}$ and $\delta \leq \delta^{P2-P5}\left[1 - \left(\frac{\lambda^{P1-P5}}{\lambda}\right)^{1/\gamma}\right]^{\gamma} = \delta(\lambda)^{P2-P5}_{\bar{C}_1=\bar{C}_1^{U,P2}}$. On the other hand, the maximum of (P5) exceeds the maximum of (P2) when $\delta > \delta(\lambda)^{P2-P5}_{\bar{C}_1=\bar{C}_1^{U,P2}}$.

Summary for (P5):

- For $C_{1L} < \bar{C}_1 \leq \bar{C}_1^{U,P1}$ (P1) exceeds (P5) for $\lambda \geq \lambda^{P1-P5}$.
- The following holds for for $\bar{C}_1^{U,P1} < \bar{C}_1 < \min\left\{\bar{C}_1^{U,P2}, \bar{C}_1^{U,P5}\right\}$ and $\lambda > \max\left\{\lambda^{P2}, \lambda^{P2-P2}, \lambda^{P5}\right\}$:
  - (P2) at its maximum exceeds (P5) at its maximum when $\delta < \delta^{P2-P5}$,
  - (P5) at its maximum coincides with (P2) at its maximum when $\delta = \delta^{P2-P5}$ and
  - (P5) at its maximum exceeds (P2) at its maximum when $\delta > \delta^{P2-P5}$.
- The following holds for $\bar{C}_1 = \bar{C}_1^{U,P2} \leq \bar{C}_1^{U,P5}$ and $\lambda > \lambda^{P5}$:
  - (P2) at its maximum exceeds (P5) at its maximum when $\delta < \delta(\lambda)^{P2-P5}_{\bar{C}_1=\bar{C}_1^{U,P2}}$,
  - (P2) at its maximum coincides with (P5) at its maximum when $\delta = \delta(\lambda)^{P2-P5}_{\bar{C}_1=\bar{C}_1^{U,P2}}$ and
  - (P2) at its maximum is below (P5) at its maximum when $\delta > \delta(\lambda)^{P2-P5}_{\bar{C}_1=\bar{C}_1^{U,P2}}$.

Problem (P6). Let $\bar{C}_1 \leq \bar{C}_1^{U,P2}$. We show at first that there is no interior local maximum or minimum for (P6) which implies that the maximum will occur at the border of the set of feasible solutions for (P6). Then we check all potential feasible solutions at the border.

The first order conditions are

$$\left.\begin{aligned}
\frac{d\mathbb{E}(U)}{dC_1} = {} & \lambda(\bar{C}_1 - C_1)^{-\gamma} - \delta p\left[\Omega - (1+r_f+w_1)(C_1-\bar{C}_1) + (r_g - r_f)\alpha\right]^{-\gamma}(1+r_f+w_1) \\
& -\lambda\delta(1-p)\left[(1+r_f+w_1)(C_1-\bar{C}_1) - \Omega + (r_f-r_b)\alpha\right]^{-\gamma}(1+r_f+w_1+w_1) \\
= {} & 0 \\
\frac{d\mathbb{E}(U)}{d\alpha} = {} & \delta p\left[\Omega - (1+r_f+w_1)(C_1-\bar{C}_1) + (r_g - r_f)\alpha\right]^{-\gamma}(r_g - r_f) \\
& -\lambda\delta(1-p)\left[(1+r_f+w_1)(C_1-\bar{C}_1) - \Omega + (r_f-r_b)\alpha\right]^{-\gamma}(r_f - r_b) = 0
\end{aligned}\right\} \tag{109}$$

$\frac{d\mathbb{E}(U)}{d\alpha} = 0$ from (109) implies the expression for α given by (79). Note that α is feasible only for $\Omega \leq 0$ where for $\Omega = 0$ there is only one feasible solution, namely $(\bar{C}_1, 0)$ which is feasible also for (P1). Thus, assume that $\Omega < 0$ and let's plug (79) into the C_1 part of the FOC in (109). After some simplifications we obtain

$$\lambda^{\frac{1}{\gamma}}\left[(1+r_f+w_1)(C_1-\bar{C}_1) - \Omega\right] = (\bar{C}_1 - C_1)M(\lambda)$$

which gives

$$C_1^+ = \bar{C}_1 + \frac{\Omega}{\frac{M(\lambda)}{\lambda^{\frac{1}{\gamma}}} + 1 + r_f + w_1} \tag{110}$$

In addition, after plugging C_1^+ from (110) into (79) we obtain

$$\alpha^+ = \frac{k}{r_g - r_f}\left[\left(\frac{1}{K_0}\right)^{\frac{1}{\gamma}} + \lambda^{\frac{1}{\gamma}}\right]\frac{-\Omega}{M(\lambda) + \lambda^{\frac{1}{\gamma}}(1 + r_f + w_1)}$$

Next we show that the expected utility function is indifferent at (C_1^+, α^+), namely, we show that at (C_1^+, α^+) are $\frac{d^2\mathbb{E}(U)}{d\alpha^2} < 0$, and $D_3 \equiv \nabla^2\mathbb{E}(U(C_1, C_2)) = \frac{d^2\mathbb{E}(U)}{dC_1^2}\frac{d^2\mathbb{E}(U)}{d\alpha^2} - \left(\frac{d^2\mathbb{E}(U)}{dC_1 d\alpha}\right)^2 < 0$. Note that

$$C_{2g}^+ - \bar{C}_2 = k\frac{-\Omega}{M(\lambda) + \lambda^{\frac{1}{\gamma}}(1 + r_f + w_1)}\frac{r_g - r_b}{r_g - r_f}\left(\frac{1}{K_0}\right)^{\frac{1}{\gamma}} \tag{111}$$

$$\bar{C}_2 - C_{2b}^+ = k\frac{-\Omega}{M(\lambda) + \lambda^{\frac{1}{\gamma}}(1 + r_f + w_1)}\frac{r_g - r_b}{r_g - r_f}\lambda^{\frac{1}{\gamma}} \tag{112}$$

and thus $\bar{C}_2 - (C_{2b}^+) = (K_0\lambda)^{\frac{1}{\gamma}}((C_{2g}^+) - \bar{C}_2)$. Using (109), (111) and (112) we obtain the following

$$\begin{aligned}
\frac{1}{\gamma}\frac{d^2\mathbb{E}(U)}{dC_1^2}|_{(C_1^+,\alpha^+)} &= \left[\frac{-\Omega}{M(\lambda) + \lambda^{\frac{1}{\gamma}}(1 + r_f + w_1)}\right]^{-1-\gamma} \\
&\quad \times \left[\lambda^{-\frac{1}{\gamma}} + \frac{1 + r_f + w_1}{k}\left(\frac{r_g - r_f}{r_g - r_b}\right)^2\left(\lambda^{-\frac{1}{\gamma}} - K_0^{\frac{1}{\gamma}}\frac{r_f - r_b}{r_g - r_f}\right)\right] \qquad (113)\\
\frac{1}{\gamma}\frac{d^2\mathbb{E}(U)}{d\alpha^2}|_{(C_1^+,\alpha^+)} &= \frac{(r_f - r_b)^2}{k\,(1 + r_f + w_1)}\left(\frac{r_g - r_f}{r_g - r_b}\right)^2\left[\frac{-\Omega}{M(\lambda) + \lambda^{\frac{1}{\gamma}}(1 + r_f + w_1)}\right]^{-1-\gamma}\left(\lambda^{-\frac{1}{\gamma}} - K_\gamma^{\frac{1}{\gamma}}\right) \\
&\qquad (114)\\
\frac{1}{\gamma}\frac{d^2\mathbb{E}(U)}{dC_1 d\alpha}|_{(C_1^+,\alpha^+)} &= \frac{r_f - r_b}{k}\left(\frac{r_g - r_f}{r_g - r_b}\right)^2\left[\frac{-\Omega}{M(\lambda) + \lambda^{\frac{1}{\gamma}}(1 + r_f + w_1)}\right]^{-1-\gamma}\left(\lambda^{-\frac{1}{\gamma}} + K_0^{\frac{1}{\gamma}}\right)
\end{aligned}$$

Note that (114) and $\lambda > \frac{1}{K_\gamma}$ implies that $\frac{d^2\mathbb{E}(U)}{d\alpha^2}|_{(C_1^+,\alpha^+)} < 0$. In addition,

$$\begin{aligned}
\frac{1}{\gamma^2}\left[\frac{-\Omega}{M(\lambda) + \lambda^{\frac{1}{\gamma}}(1 + r_f + w_1)}\right]^{2(1+\gamma)} D &= \left[\lambda^{-\frac{1}{\gamma}} + \frac{1 + r_f}{k}\left(\frac{r_g - r_f}{r_g - r_b}\right)^2\left(\lambda^{-\frac{1}{\gamma}} - \frac{r_f - r_b}{r_g - r_f}K_0^{\frac{1}{\gamma}}\right)\right] \\
&\quad \times \frac{(r_f - r_b)^2}{k(1 + r_f + w_1)}\left(\frac{r_g - r_f}{r_g - r_b}\right)^2\left[\lambda^{-\frac{1}{\gamma}} - K_\gamma^{\frac{1}{\gamma}}\right] \\
&\quad - \left(\frac{r_f - r_b}{k}\right)^2\left(\frac{r_g - r_f}{r_g - r_b}\right)^4\left(\lambda^{-\frac{1}{\gamma}} + K_0^{\frac{1}{\gamma}}\right)^2
\end{aligned}$$

where $D = \nabla^2 \mathbb{E}(U(C_1, C_2))_{(C_1^+, \alpha^+)} = \frac{d^2\mathbb{E}(U)}{dC_1^2}\frac{d^2\mathbb{E}(U)}{d\alpha^2} - \left(\frac{d^2\mathbb{E}(U)}{dC_1 d\alpha}\right)^2 |_{(C_1^+, \alpha^+)}$. Thus,

$$\begin{aligned} \frac{1}{\gamma^2}\left[\frac{-\Omega}{M(\lambda) + \lambda^{\frac{1}{\gamma}}(1 + r_f + w_1)}\right]^{2(1+\gamma)} \left(\frac{r_g - r_b}{r_g - r_f}\right)^2 \left(\frac{k}{r_f - r_b}\right)^2 D &= \frac{\lambda^{-\frac{1}{\gamma}}}{1 + r_f + w_1}\left[\lambda^{-\frac{1}{\gamma}} - K_\gamma^{\frac{1}{\gamma}}\right] \\ &\quad -\lambda^{-\frac{1}{\gamma}} K_\gamma^{\frac{1}{\gamma}} < 0 \end{aligned}$$

for $\lambda > \frac{1}{K_\gamma}$ which gives that $D = \nabla^2 \mathbb{E}(U(C_1, C_2)) = \frac{d^2\mathbb{E}(U)}{dC_1^2}\frac{d^2\mathbb{E}(U)}{d\alpha^2} - \left(\frac{d^2\mathbb{E}(U)}{dC_1 d\alpha}\right)^2 < 0$. From (114) it follows that $\frac{d^2\mathbb{E}(U)}{d\alpha^2}|_{(C_1^+, \alpha^+)} \geq 0$ for $\lambda \leq \frac{1}{K_\gamma}$ and thus (C_1^+, α_1^+) can not be a point of local maxima for any $\lambda > 1$ and thus the maximum will occur at the border.

The feasible solutions at the border for (P6) that come into consideration are given by: (i) $C_{2g} = \bar{C}_2$, (ii) $C_{2b} = \bar{C}_2$, (iii) $C_{2b} = C_{2L}$, (iv) $C_1 = \bar{C}_1$ and (v) $C_1 = C_{1L}$.

Case (i): $C_{2g} = \bar{C}_2$ when

$$C_1 = \frac{1}{1 + r_f + w_1}\left[(1 + r_f)Y_1 + Y_2 - w_0 - w_2\bar{C}_1 + (r_g - r_f)\alpha\right] = \bar{C}_1 + \frac{\Omega + (r_g - r_f)\alpha}{1 + r_f + w_1}$$

and

$$\max\left\{0, \alpha^{C_1 = C_{1L}}_{C_{2g} = \bar{C}_2}\right\} \leq \alpha \leq \min\left\{\alpha^{C_1 = \bar{C}_1}_{C_{2g} = \bar{C}_2} = \frac{-\Omega}{r_g - r_f},\ \alpha^{C_{2g} = \bar{C}_2}_{C_{2b} = C_{2L}}\right\}$$

Note that this case can occur only for $\Omega \leq 0$, i.e., for $\bar{C}_1 \geq \bar{C}_1^{U,P1}$,[43] where in the case of $\Omega = 0$ the only feasible solutions is $(\bar{C}_1, 0)$. Thus, in the following we assume that $\bar{C}_1^{U,P1} < \bar{C}_1 \leq \bar{C}_1^{U,P2}$. It can be seen that

$$\begin{aligned} (1-\gamma)\mathbb{E}\left(U\left(\bar{C}_1 + \frac{\Omega + (r_g - r_f)\alpha}{1 + r_f + w_1}, \alpha\right)\right) &= -\lambda\left[\frac{-\Omega - (r_g - r_f)\alpha}{1 + r_f + w_1}\right]^{1-\gamma} \\ &\quad -\lambda\delta(1-p)(r_g - r_b)^{1-\gamma}\alpha^{1-\gamma} \end{aligned}$$

is a convex function in α and thus its maximum is reached either at $\alpha = 0$, when $C_{1L} \leq \bar{C}_1 + \frac{\Omega}{1+r_f+w_1}$, or at $\alpha = \alpha^{C_1 = C_{1L}}_{C_{2g} = \bar{C}_2}$, when $\bar{C}_1 + \frac{\Omega}{1+r_f+w_1} \leq C_{1L} \leq \bar{C}_1$, or at $\alpha = \alpha^{C_1 = \bar{C}_1}_{C_{2g} = \bar{C}_2}$ or at $\alpha^{C_{2g} = \bar{C}_2}_{C_{2b} = C_{2L}}$. Thus, the potential candidates for the maximum in this case are

(a) $\left(C_1 = \bar{C}_1 + \frac{\Omega}{1+r_f+w_1}, \alpha = 0\right)$ when $C_{1L} \leq \bar{C}_1 + \frac{\Omega}{1+r_f+w_1}$ or

(b) $\left(C_1 = C_{1L}, \alpha = \alpha^{C_1 = C_{1L}}_{C_{2g} = \bar{C}_2}\right)$ when $\bar{C}_1 + \frac{\Omega}{1+r_f+w_1} \leq C_{1L} \leq \bar{C}_1$ and $\bar{C}_1^{U,P5} \leq \bar{C}_1 \leq \bar{C}_1^{U,P6}$ or

(c) $\left(C_1 = \bar{C}_1, \alpha = \alpha^{C_1 = \bar{C}_1}_{C_{2g} = \bar{C}_2}\right)$ when $\bar{C}_1^{U,P1} < \bar{C}_1 \leq \bar{C}_1^{U,P2}$ or

(d) $\left(C_1 = \bar{C}_1 + \frac{\Omega + (r_g - r_f)\alpha^{C_{2g} = \bar{C}_2}_{C_{2b} = C_{2L}}}{1+r_f+w_1}, \alpha = \alpha^{C_{2g} = \bar{C}_2}_{C_{2b} = C_{2L}}\right)$ when $\bar{C}_1^{U,P2} \leq \bar{C}_1 \leq \bar{C}_1^{U,P6}$

[43] Note that $\alpha^{C_{2g} = \bar{C}_2}_{C_{2b} = C_{2L}} \geq 0$ when $\bar{C}_1 \geq \bar{C}_1^{L,P4}$ and condition (63) guarantees that $C_{1L} \geq \bar{C}_1^{L,P4}$.

Note that the point in case (a) is feasible also for (P5), when $\bar{C}_1 \leq \min\left\{\bar{C}_1^{U,P2}, \bar{C}_1^{U,P5}\right\}$ and is also the only feasible solution for case (ii).

Based on Lemma 1-(vi), case (b) is relevant, i.e., $\alpha_{C_{2g}=\bar{C}_2}^{C_1=C_{1L}} \geq 0$, only for $\bar{C}_1^{U,P5} \leq \bar{C}_1 \leq \bar{C}_1^{U,P6}$ which we do not consider here. The point in case (c) is a feasible solution for (P2). Case (d) holds only for $\bar{C}_1^{U,P2} \leq \bar{C}_1 \leq \bar{C}_1^{U,P6}$ which is not considered here.

Case (iii). $C_{2b} = C_{2L}$ when $C_1 = Y_1 + \frac{Y_2 - C_{2L}}{1+r_f} - \frac{r_f - r_b}{1+r_f}\alpha$ for

$$\alpha_{C_{2b}=C_{2L}}^{C_1=\bar{C}_1} \leq \alpha \leq \alpha_{C_{2b}=C_{2L}}^{C_1=C_{1L}} \tag{115}$$

where $\alpha_{C_{2b}=C_{2L}}^{C_1=\bar{C}_1} = \frac{(1+r_f)(Y_1-\bar{C}_1)+Y_2-C_{2L}}{r_f-r_b}$ and $\alpha_{C_{2b}=C_{2L}}^{C_1=C_{1L}} = \frac{(1+r_f)(Y_1-C_{1L})+Y_2-C_{2L}}{r_f-r_b}$. Then

$$\begin{aligned}
&(1-\gamma)\mathbb{E}\left(U\left(Y_1 + \frac{Y_2 - C_{2L}}{1+r_f} - \frac{r_f - r_b}{1+r_f}\,\alpha,\, \alpha\right)\right) = -\lambda\left[\bar{C}_1 - Y_1 - \frac{Y_2 - C_{2L}}{1+r_f} + \frac{r_f - r_b}{1+r_f}\,\alpha\right]^{1-\gamma} \\
+ \quad & \delta p\left[\frac{1+r_f+w_1}{1+r_f}C_{2L} - w_0 - w_1\left(Y_1 + \frac{Y_2}{1+r_f}\right) - w_2\bar{C}_1 + \left(r_g - r_b + \frac{r_f - r_b}{1+r_f}w_1\right)\alpha\right]^{1-\gamma} \\
- \quad & \lambda\delta(1-p)\left[w_0 + w_1\left(Y_1 + \frac{Y_2}{1+r_f}\right) + w_2\bar{C}_1 - \frac{1+r_f+w_1}{1+r_f}C_{2L} - \frac{w_1(r_f - r_b)}{1+r_f}\,\alpha\right]^{1-\gamma}
\end{aligned} \tag{116}$$

Note that

$$\begin{aligned}
&\frac{1}{\gamma}\frac{d^2\mathbb{E}\left(U\left(Y_1 + \frac{Y_2 - C_{2L}}{1+r_f} - \frac{r_f - r_b}{1+r_f}\,\alpha,\, \alpha\right)\right)}{d\alpha^2} = \lambda\left[\bar{C}_1 - Y_1 - \frac{Y_2 - C_{2L}}{1+r_f} + \frac{r_f - r_b}{1+r_f}\,\alpha\right]^{-1-\gamma}\left(\frac{r_f - r_b}{1+r_f}\right)^2 \\
- \quad & \delta p\left[\frac{1+r_f+w_1}{1+r_f}C_{2L} - w_0 - w_1\left(Y_1 + \frac{Y_2}{1+r_f}\right) - w_2\bar{C}_1 + \left(r_g - r_b + \frac{r_f - r_b}{1+r_f}\,w_1\right)\alpha\right]^{-1-\gamma} \\
& \times\left(r_g - r_b + \frac{r_f - r_b}{1+r_f}\,w_1\right)^2 \\
+ \quad & \lambda\delta(1-p)\left[w_0 + w_1\left(Y_1 + \frac{Y_2}{1+r_f}\right) + w_2\bar{C}_1 - \frac{1+r_f+w_1}{1+r_f}C_{2L} - \frac{w_1(r_f - r_b)}{1+r_f}\,\alpha\right]^{-1-\gamma}\left(\frac{r_f - r_b}{1+r_f}\,w_1\right)^2
\end{aligned}$$

$$
\begin{aligned}
\geq\quad & \lambda\left[\bar{C}_1 - Y_1 - \frac{Y_2 - C_{2L}}{1+r_f} + \frac{r_f - r_b}{1+r_f}\,\alpha^{C_1=C_{1L}}_{C_{2b}=C_{2L}}\right]^{-1-\gamma}\left(\frac{r_f - r_b}{1+r_f}\right)^2 \\
-\quad & \delta p\left[\frac{1+r_f+w_1}{1+r_f}C_{2L} - w_0 - w_1\left(Y_1 + \frac{Y_2}{1+r_f}\right) - w_2\bar{C}_1 + \left(r_g - r_b + \frac{r_f - r_b}{1+r_f}\,w_1\right)\alpha^{C_1=\bar{C}_1}_{C_{2b}=C_{2L}}\right]^{-1-\gamma} \\
& \times\left(r_g - r_b + \frac{r_f - r_b}{1+r_f}\,w_1\right)^2 \\
+\quad & \lambda\delta(1-p)\left[w_0 + w_1\left(Y_1 + \frac{Y_2}{1+r_f}\right) + w_2\bar{C}_1 - \frac{1+r_f+w_1}{1+r_f}C_{2L} - \frac{w_1(r_f - r_b)}{1+r_f}\,\alpha^{C_1=\bar{C}_1}_{C_{2b}=C_{2L}}\right]^{-1-\gamma} \\
& \times\left(\frac{r_f - r_b}{1+r_f}\,w_1\right)^2 \\
=\quad & \lambda\left(\bar{C}_1 - C_{1L}\right)^{-1-\gamma}\left(\frac{r_f - r_b}{1+r_f}\right)^2 - \delta p\left[\Omega + (r_g - r_f)\alpha^{C_1=\bar{C}_1}_{C_{2b}=C_{2L}}\right]^{-1-\gamma}\left(r_g - r_b + \frac{r_f - r_b}{1+r_f}\,w_1\right)^2 \\
& +\lambda\delta(1-p)\left[w_0 + (w_1 + w_2)\bar{C}_1 - C_{2L}\right]^{-1-\gamma}\left(\frac{r_f - r_b}{1+r_f}\,w_1\right)^2 \qquad (117)
\end{aligned}
$$

It is sufficient for the objective function (116) to be convex if the right-hand-side of the last expression is positive which is guaranteed for

$$
\begin{aligned}
\lambda\quad > \quad & \frac{\dfrac{\delta p\left(\frac{(1+r_f)(r_g - r_b)}{r_f - r_b} + w_1\right)^2}{\left[\Omega + (r_g - r_f)\alpha^{C_1=\bar{C}_1}_{C_{2b}=C_{2L}}\right]^{1+\gamma}}}{\dfrac{1}{(\bar{C}_1 - C_{1L})^{1+\gamma}} + \dfrac{\delta(1-p)w_1^2}{\left[w_0 + (w_1 + w_2)\bar{C}_1 - C_{2L}\right]^{1+\gamma}}} \\
= \quad & \frac{\delta p\left[\frac{(r_g - r_b)(1+r_f)}{r_f - r_b} + w_1\right]^2}{\left[\Omega + (r_g - r_f)\alpha^{C_1=\bar{C}_1}_{C_{2b}=C_{2L}}\right]^{1+\gamma}} \times \frac{\left(\bar{C}_1 - C_{1L}\right)^{1+\gamma}\left[w_0 + (w_1 + w_2)\bar{C}_1 - C_{2L}\right]^{1+\gamma}}{\delta(1-p)w_1^2\left(\bar{C}_1 - C_{1L}\right)^{1+\gamma} + \left[w_0 + (w_1 + w_2)\bar{C}_1 - C_{2L}\right]^{1+\gamma}} \\
= \quad & \lambda^{P2-P6} \qquad (118)
\end{aligned}
$$

Thus the maximum occurs at one of the end points $\alpha = \alpha^{C_1=\bar{C}_1}_{C_{2b}=C_{2L}}$ or $\alpha = \alpha^{C_1=C_{1L}}_{C_{2b}=C_{2L}}$. Note that $\alpha = \alpha^{C_1=\bar{C}_1}_{C_{2b}=C_{2L}}$ is feasible for (P2) and $\alpha = \alpha^{C_1=C_{1L}}_{C_{2b}=C_{2L}}$ is tackled in case (v) where it is shown that it is exceeded by another point which is feasible for (P5).

For $\bar{C}_1 = \bar{C}_1^{U,P2}$ is the first derivative of the objective function (116) given by

$$
\begin{aligned}
& \frac{d\mathbb{E}\left(U\left(Y_1 + \frac{Y_2 - C_{2L}}{1+r_f} - \frac{r_f - r_b}{1+r_f},\,\alpha\right)\right)}{d\alpha} = -\lambda\left(\frac{r_f - r_b}{1+r_f}\right)^{1-\gamma}\left(\alpha - \alpha^{C_1=\bar{C}_1}_{C_{2b}=C_{2L}}\right)^{-\gamma} \\
+\quad & \delta p\left(r_g - r_b + \frac{r_f - r_b}{1+r_f}w_1\right)^{1-\gamma}\left(\alpha - \alpha^{C_1=\bar{C}_1}_{C_{2b}=C_{2L}}\right)^{-\gamma} \\
+\quad & \lambda\,\delta(1-p)w_1\left(\frac{r_f - r_b}{1+r_f}\right)^{1-\gamma}\left[\frac{(r_g - r_b)(1+r_f)}{r_f - r_b}\,\frac{\alpha^{C_1=\bar{C}_1}_{C_{2b}=C_{2L}}}{\alpha - \alpha^{C_1=\bar{C}_1}_{C_{2b}=C_{2L}}} - w_1\right]^{-\gamma}\left(\alpha - \alpha^{C_1=\bar{C}_1}_{C_{2b}=C_{2L}}\right)^{-\gamma}
\end{aligned}
$$

while using Lemma 1 (iv). Sufficient conditions for $\frac{d\mathbb{E}\left(U\left(Y_1+\frac{Y_2-C_{2L}}{1+r_f}-\frac{r_f-r_b}{1+r_f},\alpha\right)\right)}{d\alpha} < 0$ and thus for the objective function being decreasing are

$$\lambda > \frac{p}{\frac{1}{\delta}-\frac{1}{\delta^{P2-P6}}}\left[\frac{(1+r_f)(r_g-r_b)}{r_f-r_b}+w_1\right]^{1-\gamma} = \lambda^{P2-P6}_{\bar{C}_1=\bar{C}_1^{U,P2}}$$

and

$$\delta < \delta^{P2-P6} = \frac{1}{(1-p)w_1}\left(\frac{r_g-r_b}{\bar{C}_1^{U,P2}-C_{1L}}\,\alpha^{C_1=\bar{C}_1}_{C_{2b}=C_{2L}} - w_1\right)^{\gamma} \tag{119}$$

Thus, for $\bar{C}_1 = \bar{C}_1^{U,P2}$, $\delta < \delta^{P2-P6}$ and $\lambda > \lambda^{P2-P6}_{\bar{C}_1=\bar{C}_1^{U,P2}}$ the maximum of the objective function (116) is reached for $\alpha = \alpha^{C_1=\bar{C}_1}_{C_{2b}=C_{2L}}$ which is of the same value as the objective function of (P2) at its maximum, see (96).

In case (iv) any feasible solution is also feasible for (P2) and thus this case can occur only for $\bar{C}_1 \leq \bar{C}_1^{U,P2}$.

Case (v). $C_1 = C_{1L}$ for $\alpha^{C_1=C_{1L}}_{C_{2b}=\bar{C}_2} \leq \alpha \leq \alpha^{C_1=C_{1L}}_{C_{2b}=C_{2L}}$, see (56), (60) and Lemma 1-(vi). Then

$$\begin{aligned}
&(1-\gamma)\mathbb{E}\left(U\left(C_{1L},\alpha\right)\right) = -\lambda(\bar{C}_1-C_{1L})^{1-\gamma}\\
&+\delta p\left[(1+r_f)(Y_1-C_{1L})+Y_2-w_0-w_1C_{1L}-w_2\bar{C}_1+(r_g-r_f)\alpha\right]^{1-\gamma}\\
&-\lambda\delta(1-p)\left[w_0+w_1C_{1L}+w_2\bar{C}_1-(1+r_f)(Y_1-C_{1L})-Y_2+(r_f-r_b)\alpha\right]^{1-\gamma}
\end{aligned}$$

It can be shown that for $\bar{C}_1 \leq \bar{C}_1^{U,P2}$ and $\lambda \geq \frac{1}{K_\gamma}$ is the objective function of case (v) decreasing and thus the maximum is reached at $\alpha = \alpha^{C_1=C_{1L}}_{C_{2b}=\bar{C}_2}$ which is feasible also for (P5). The value of the expected utility function at this point is as follows

$$(1-\gamma)\mathbb{E}\left(U\left(C_{1L},\alpha^{C_1=C_{1L}}_{C_{2b}=\bar{C}_2}\right)\right) = -\lambda(\bar{C}_1-C_{1L})^{1-\gamma}+\delta p\left[\frac{r_g-r_b}{r_f-r_b}\,w_2\left(\bar{C}_1^{U,P5}-\bar{C}_1\right)\right]^{1-\gamma} \tag{120}$$

Summary for (P6): Let $\lambda > \frac{1}{K_\gamma}$. Then the following holds.

- For $\bar{C}_{1L} < \bar{C}_1 < \bar{C}_1^{U,P2}$ and $\lambda > \lambda^{P2-P6}$ (P2) exceeds (P6).
- For $\bar{C}_1 = \bar{C}_1^{U,P2}$, $\delta < \delta^{P2-P6}$ and $\lambda > \lambda^{P2-P6}_{\bar{C}_1=\bar{C}_1^{U,P2}}$ (P2) exceeds (P6).

Note that the above mentioned conditions are sufficient, not necessary.

Problem (P8). Let $\bar{C}_1^{U,P1} \leq \bar{C}_1 \leq \bar{C}_1^{U,P2}$. As the utility function of (P8) is convex a maximum will occur at the border of the set of feasible solutions. The feasible solutions at the border that come into consideration are: (i) $C_{2g} = \bar{C}_2$, (ii) $C_{2b} = \bar{C}_2 = C_{2g}$, (iii) $C_{2g} = C_{2L} = C_{2b}$, (iv) $C_{2b} = C_{2L}$, (v) $C_1 = \bar{C}_1$ and (vi) $C_1 = C_{1L}$.

Case (i) is feasible for (P6) and was already dealt with in the proof of (P6) in case (i) and thus $\bar{C}_1 \leq \bar{C}_1^{U,P6}$. The only feasible solution in case (ii) is $\left(C_1 = C_1^{P5} = \bar{C}_1 + \frac{\Omega}{1+r_f+w_1}, \alpha = \alpha^{P5} = 0\right)$ which is also feasible (and thus dealt with) for (P5) and (P6).

The only feasible solution in case (iii) when $C_{2g} = C_{2L} = C_{2b}$ is $\left(C_1 = Y_1 + \frac{Y_2 - C_{2L}}{1+r_f}, \alpha = 0\right)$ which is feasible only for $\bar{C}_1 \geq Y_1 + \frac{Y_2 - C_{2L}}{1+r_f}$.

Case (iv): $C_{2b} = C_{2L}$ when $C_1 = Y_1 + \frac{Y_2 - C_{2L}}{1+r_f} - \frac{r_f - r_b}{1+r_f}\alpha$ and

$$\max\left\{0, \alpha_{C_{2b}=C_{2L}}^{C_1=\bar{C}_1}\right\} \leq \alpha \leq \min\left\{\alpha_{C_{2b}=C_{2L}}^{C_{2g}=\bar{C}_2}, \alpha_{C_{2b}=C_{2L}}^{C_1=C_{1L}}\right\}$$

where $\alpha_{C_{2b}=C_{2L}}^{C_{2g}=\bar{C}_2}$ is given by (54), $\alpha_{C_{2b}=C_{2L}}^{C_1=\bar{C}_1} = \frac{(1+r_f)(Y_1-\bar{C}_1)+Y_2-C_{2L}}{r_f - r_b}$ and $\alpha_{C_{2b}=C_{2L}}^{C_1=C_{1L}} = \frac{(1+r_f)(Y_1-C_{1L})+Y_2-C_{2L}}{r_f - r_b}$. Note that for $\bar{C}_1 < \bar{C}_1^{U,P2}$ is $\alpha_{C_{2b}=C_{2L}}^{C_1=\bar{C}_1} > \alpha_{C_{2b}=C_{2L}}^{C_{2g}=\bar{C}_2}$ and thus case (iv) has no feasible solution. Note in addition that the only feasible solution for $\bar{C}_1 = \bar{C}_1^{U,P2}$ is $\alpha = \alpha_{C_{2b}=C_{2L}}^{C_1=\bar{C}_1}$ as $\alpha_{C_{2b}=C_{2L}}^{C_1=\bar{C}_1} = \alpha_{C_{2b}=C_{2L}}^{C_{2g}=\bar{C}_2}$ which coincides with the value of (P2) at its maximum, see (96).

If $C_1 = \bar{C}$, case (v), then any feasible solution will be feasible also for (P4). For more details in the proof see case (iv) of (P4).

If $C_1 = C_{1L}$, case (vi), then

$$\begin{aligned}(1-\gamma)\mathbb{E}(U(C_{1L},\alpha)) = & -\lambda(\bar{C}_1 - C_{1L})^{1-\gamma} - \lambda\delta p\left[-\Omega - (1+r_f+w_1)(\bar{C}_1 - C_{1L}) - (r_g - r_f)\alpha\right]^{1-\gamma} \\ & - \lambda\delta(1-p)\left[-\Omega - (1+r_f+w_1)(\bar{C}_1 - C_{1L}) + (r_f - r_b)\alpha\right]^{1-\gamma}\end{aligned} \tag{121}$$

for

$$0 \leq \alpha \leq \min\left\{\alpha_{C_{2g}=\bar{C}_2}^{C_1=C_{1L}}, \alpha_{C_{2b}=C_{2L}}^{C_1=C_{1L}}\right\}$$

where $\alpha_{C_{2g}=\bar{C}_2}^{C_1=C_{1L}}$ is given by (59) and $\alpha_{C_{2b}=C_{2L}}^{C_1=C_{1L}}$ is given by (56). Based on Lemma 1-(vi) is $\alpha_{C_{2g}=\bar{C}_2}^{C_1=C_{1L}} < 0$ and thus (121) is infeasible.