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Abstract

This paper establishes existence of subgame perfect equilibrium for a general class of sequential multi-lateral bargaining games. The only required hypothesis is that utility functions are continuous on the space of economic outcomes. In particular, no assumption on the space of feasible payoffs is needed. The result covers arbitrary and even time-varying bargaining protocols (acceptance rules), arbitrary specifications of patience or impatience (geometric, hyperbolic, or otherwise), externalities, multiple selves, and other-regarding preferences.

Keywords: bargaining, equilibrium existence, infinite-horizon games, subgame perfection

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1 Introduction

Distributional conflicts stand at the core of economics. From budget allocation negotiations among institutional agents or states to collective or individual wage agreements, from cost-sharing decisions for the financing of public goods to asset liquidations and bankruptcies, such conflicts give rise to rich strategic problems. *Bargaining* models and procedures are rightly viewed as the main tool for their study and resolution. A vast literature has addressed this issue by applying techniques from both cooperative and non-cooperative game theory, with the former providing important axiomatic characterizations of appealing solutions (e.g. Nash 1950, Kalai and Smorodinsky 1975, Thomson 1981) and the latter allowing for explicit procedural analyses taking into account the timing of offers (e.g. Rubinstein 1982, Shaked and Sutton 1984, Binmore 1987).

A particularly important milestone was set by the canonical model of sequential bilateral bargaining (Stahl 1972, Rubinstein 1982), which can be viewed as a link between these two approaches, showing that with small enough frictions the equilibrium of the strategic game approximates the cooperative Nash bargaining solution (Nash 1950). Accordingly, this model has enjoyed widespread popularity and the insights arising from bilateral strategic bargaining have been applied to models of increasing generality, e.g. allowing for more than two bargaining partners. The basic structure of such models specifies a procedure by which a player makes a proposal and the rest of the players collectively decide whether to accept or reject it; in case of rejection another player gets to make a new proposal. Specific models vary in many dimensions, ranging from the order of proposals to the characteristics of individual utilities. For more than two players, a particularly crucial element is the collective acceptance rule. The most widespread rule found in the literature is unanimity (e.g. Haller 1986, Herrero 1989), but majority voting is a natural alternative (Eraslan and Merlo 2002), and many other rules are conceivable (Kalandrakis 2004). The identity of the next proposer after a rejection can also give rise to model differences, with a fixed cyclical order being a popular alternative, but some natural alternatives being equally plausible. For instance, the order might be endogenous, with the first player to reject the previous offer becoming the next proposer (Selten 1981, Chatterjee, Dutta, Ray, and Sengupta 1993, Ray and Vohra 1999). Finally, differences and asymmetries in utility functions (e.g., discount factors) remain important elements affecting equilibrium predictions.

An important characteristic of most non-cooperative bargaining models is that the potential horizon is infinite, so as to not impose artificial, exogenous constraints on the problem (last period effects). Also, potential proposals are naturally from a continuum, e.g. the division of a resource. Hence, bargaining games are large games. Even under perfect information, the existence of (subgame perfect) equilibria has always been an issue in this literature. Significant progress has been made in the recent decades (see the literature review below), especially when restricting to particular subclasses of games or to stationary strategies. Important equilibrium existence results have been provided by Merlo and Wilson (1995), Banks and Duggan (2000), Kultti and Vartiainen (2010), and Herings and Predtetchinski (2015); all those results consider the unanimity rule, make explicit assumptions on the set of feasible payoffs (e.g., convexity), and focus on stationary environments.

The objective of this paper is to present a general existence theorem for multilateral sequential bargaining, encompassing a large class of bargaining problems. The main result allows unanimity bargaining but also many other acceptance rules, like majority voting, veto rights, or dictatorial arrangements. Indeed, the acceptance rules need not be constant over time. The order or proposals also allows for many possibilities, including any exogenous order (cyclical or not) but also endogenous procedures as selecting the first player casting a negative vote as the next proposer, in case of rejection. Hence, the bargaining problems studied here have no stationary structure other than the fact that the bargaining partners remain the same in all rounds. Actually, even that requirement can be relaxed, since time-dependent acceptance rules allow to declare certain players "dummies" in given periods.

This work departs from the previous literature in two main respects. The first is that the framework extends beyond stationary environments, in that bargaining rules may be time-dependent. Accordingly, the analysis does not focus on stationary strategies but rather considers the existence of subgame perfect equilibrium in arbitrary pure strategies. The second departure is that the existence result relies on one elementary assumption only: continuity of payoffs on the set of feasible economic *outcomes*, i.e. on actual allocations. In particular, and in contrast to previous existence results, no direct assumptions are imposed on the sets of payoffs, and likewise no direct assumptions on the mapping from strategy profiles to bargaining outcomes are made. The result's hypotheses are hence stated on the actual primitives of the model (outcomes and payoff functions), and are easier to verify than properties of the space of feasible payoffs. This is important, because, contrary to convexity or other properties of the set of feasible payoffs, continuity of the payoff functions is often straightforward in applications. For instance, our result applies immediately to such utility functions as those capturing hyperbolic or quasi-hyperbolic discounting (Strotz 1956, Laibson 1997) or other-regarding preferences (Fehr and Schmidt 1999, Bolton and Ockenfels 2000).

It should be remarked that the continuity assumption is an elementary one. Topologically, we take advantage of the fact that the space of instantaneous allocations (vectors of shares) is compact in the Euclidean topology and construct appropriate topologies on the space of economic outcomes based on the Euclidean one. Specifically, what is assumed is that, for each fixed period t, payoff functions are continuous (in the Euclidean sense) with respect to the shares allocated to agents. Additionally, it is assumed that payoffs are "continuous at infinity," in the sense that payoff differences which accrue sufficiently far in the future become negligible in comparison to present payoff differences. This property corresponds to the assumption in, e.g., Rubinstein (1982, Assumption A4) and parallels the notion of continuity at infinity introduced by Fudenberg and Levine (1983). In those works, however, continuity at infinity was stated as an additional condition, without an explicitly topological foundation. As a byproduct of our analysis (and a fact of independent interest), however, we prove that continuity at infinity is *not* an additional condition. Rather, we show that payoff functions are continuous with respect to the natural topologies on the space of bargaining outcomes (allocations and times at which they obtain) *if and only if* both conditions mentioned above hold.

Finally, it is worth noting that the existence result, which is based on recent formal results spelled out in Alós-Ferrer and Ritzberger (2016b, 2017a,b), comes with an algorithm which allows to actually identify subgame perfect equilibria for a given model, and which we will illustrate in the examples.

The plan of the paper is as follows. Section 2 contains a brief literature overview. Section 3 specifies the class of bargaining models to be studied. Section 4 states the main existence result and provides an intuitive explanation of its proof. Section 5 gives illustrations and discusses a few extensions of the result. Section 6 concludes. The formal proof of the theorem and the necessary topological constructions are relegated to the Appendix.

2 Related Literature

It would be an impossible task to review the extensive literature on bargaining models, even if one were to restrict to noncooperative models. This section merely attempts to provide a few pointers to key and recent developments in the area, to better put the main result in context.

The starting point of the literature is of course the model of sequential bilateral bargaining (Stahl 1972, Rubinstein 1982, Shaked and Sutton 1984, Binmore 1987). Models of multilateral ($n \geq 3$) bargaining were quick to follow, starting with Haller (1986) and Herrero (1989). Those authors studied multilateral bargaining games with a unanimity rule and found multiplicity of subgame perfect equilibria if either the bargaining partners are sufficiently patient or voting is simultaneous. At the same time, the link between cooperative and non-cooperative models remained a focal point. For instance, Hart and Mas-Colell (1996) presented a non-cooperative bargaining game applied to a coalitional game that yields the Shapley value and the Nash bargaining solution in special cases. Following on both developments, Krishna and Serrano (1996) recursively extended the bilateral bargaining model to a multilateral problem and established links to cooperative solutions.

Most general existence results for infinite-horizon multilateral bargaining have concentrated on subgame perfect equilibria in stationary strategies and the particular case of unanimity rules. Further, such results usually rely on explicit assumptions on the set of feasible payoffs. For instance, the seminal works of Merlo and Wilson (1995) and Banks and Duggan (2000) for the unanimity rule establish existence of (pure-strategy) stationary equilibria when the set of feasible payoffs is compact, convex, and comprehensive from below. More recently Kultti and

Vartiainen (2010) demonstrate that when the utility possibility set is compact, convex, and strictly comprehensive and the Pareto frontier is differentiable, all stationary subgame perfect equilibrium outcomes converge to the n-player Nash bargaining solution as the delay between proposals vanishes.

Further results along these lines have been recently obtained by Britz, Herings, and Predtetchinski (2010, 2014, 2015) and Herings and Predtetchinski (2015, 2016). Britz, Herings, and Predtetchinski (2010) study the convergence of stationary subgame perfect equilibrium payoffs as the cost of delay becomes negligible for multilateral sequential bargaining with action-independent proposers. Britz, Herings, and Predtetchinski (2014) provide an equilibrium existence result for stationary strategies under unanimity with action-dependent proposers. Herings and Predtetchinski (2015) show existence of stationary equilibria when feasible payoffs form a set that is closed and comprehensive from below and utility functions are bounded, also for the unanimity rule. Herings and Predtetchinski (2016) establish existence and uniqueness of equilibrium for unanimity bargaining in stationary strategies under monotonicity constraints.

However, Britz, Herings, and Predtetchinski (2015) show that with a stochastic selection of proposers and a random order of responders under unanimity subgame perfect equilibria in pure strategies may not exist. This latter result is important, as it establishes the limits of the approach. Random ordering of proposers and responders, even in perfect information games, may lead to a failure of equilibrium existence. This is an instance of the general observation by Luttmer and Mariotti (2003): even with perfect information and a finite horizon equilibrium existence may fail, if chance moves destroy continuity of payoff functions. Hence, the work presented here will accordingly concentrate on a deterministic settings. However, it should be mentioned that some positive results have been obtained for particular bargaining models with stochastic elements. For instance, Eraslan (2002) considers multilateral sequential bargaining when players differ with respect to their probability to become proposer and their discount factors, and characterizes the set of stationary subgame perfect equilibria. Eraslan and Merlo (2002) allow majority voting and that the surplus evolves stochastically; they find multiplicity of stationary subgame perfect equilibrium payoffs, which may not be efficient.

Extensions of the basic multilateral bargaining model have allowed for different (but typically fixed) agreement rules, as majority voting (Eraslan and Merlo 2002). Kalandrakis (2004, 2006) established existence of stationary subgame perfect equilibria for more general agreement rules. This development has also extended to the relation with the cooperative approach. For instance, Laruelle and Valenciano (2008) characterize an extension of Nash's bargaining solution for voting rules beyond unanimity. In the existence result presented below, all conceivable agreement rules are allowed, and additionally there is no requirement that the agreement rule should remain fixed over time.

Among the many additional extensions of the basic bargaining setting that have been explored in the literature, one should mentioned multi-issue bargaining. Assuming a unanimity rule, sequential bargaining about several issues has been studied by Inderst (2000), Busch and Horstmann (2002), and In and Serrano (2003, 2004). This extension can also be encompassed in our setting (see Section 5).

3 A General Model for Multi-Lateral Bargaining

The class of models studied here encompasses a wide variety of multi-lateral bargaining games, the bilateral case being nested. Their common feature is that offers are made by some proposer, who may be different each round, and then there is a sequential procedure for the decision on whether or not the proposal is implemented. As discussed above, the bargaining literature has often focused on a unanimity rule for the latter. Although this important case is of course covered, the framework presented here allows for many other procedures as well. For example, implementation could be decided by majority voting with simple or qualified majority, a veto mechanism where some or all partners may be able to block the proposal, or simply a dictatorial rule where a designated person has to agree. Furthermore, the decision procedure may change from one round to the next, as may the identity of the proposer. For instance, the first round may require unanimity for the implementation of the proposal, the second a 90 percent majority, the third an 80 percent majority, and so on until at some point the consent of one participant suffices for implementation. Of course, the bar could also move in the other direction, requiring a higher and higher majority as proposals get rejected. Finally, the decision procedure can depend on the result of the previous bargaining round: for instance, next round's proposer might be the first player to reject the previous proposal.

Formalizing this broad class of games requires some notation. There is a set of $n > 1$ players, denoted $I = \{1, \ldots, n\}$. Bargaining takes place over potentially infinitely many *rounds* indexed by $t = 1, 2, \ldots$ Each round t begins with a *proposal*

$$a^t = (a_1^t, \ldots, a_n^t) \in \Delta = \left\{ (a_1, \ldots, a_n) \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n} a_i = 1,\ a_i \geq 0\ \forall i \in I \right\}$$

which specifies the intended shares of the current surplus for all players (where w.l.o.g. the surplus is normalized to 1). All participants learn this proposal and then get to express their opinions or cast their votes sequentially. Votes take a 0–1 form, with 1 meaning acceptance and 0 indicating rejection. At every round t the votes cast by players moving from the second to the n-th position form a *voting profile* $b^t = (b_2^t, \ldots, b_n^t) \in B = \{0,1\}^{n-1}$ where, for notational convenience, the subscript indicates the order of play in that round and not the player's name.

Whether or not a proposal $a^t \in \Delta$ is actually implemented at (the end of) round t is determined by an *aggregation function* $\psi^t : B \to \{0, 1\}$. That is, given a voting profile b^t at stage t, the proposal a^t is accepted if $\psi^t\left(b^t\right) = 1$ and rejected if $\psi^t\left(b^t\right) = 0$. The only assumption on ψ^t

is that $\psi^t(0,\ldots,0)=0$ and $\psi^t(1,\ldots,1)=1$, that is, unanimous decisions are implemented. No other assumption is made.

Example 1. The unanimity rule is given by $\psi^t(b)=0$ for all $b\neq(1,\ldots,1)$. A strict majority rule would be given by $\psi^t(b)=1$ if and only if $\sum_{i=2}^n b_i>(n-1)/2$. A q-majority rule, $q\in[1/(n-1),1]$, would specify $\psi^t(b)=1$ if and only if $\sum_{i=2}^n b_i\geq q(n-1)$. One could for instance specify that ψ^t is an $f(t)$-majority rule, with $f:\{1,2,\ldots\}\to[1/n,1]$ a strictly decreasing function of t. In such an example, the acceptance threshold for a decisions would be lowered gradually over time, perhaps in an attempt to ensure a timely decision.

If round t's proposal is rejected, the game continues to round $t+1$. Once a proposal is accepted, the game ends. Potentially, the game can run forever (if proposals are always rejected).

The order of votes and the identity of the proposer at round t are determined by a *bargaining protocol* as follows. For each $t=1,2,\ldots$ and each $j=1,\ldots,n$, let $r_j^t:B^{t-1}\to I$ determine the order of play at t. That is, given the history of previous votes $\bar{b}\in B^{t-1}$, $r_1^t(\bar{b})\in I$ is the player acting as the proposer at t, hence choosing a^t, while players $r_2^t(\bar{b})$ to $r_n^t(\bar{b})$ are moving second to last, hence casting votes b_2^t to b_n^t, respectively. For instance, $r_1^t(\bar{b})=i$ means that player $i\in I$ gets to make a proposal, and $r_3^t(\bar{b})=j$ that player $j\in I$ is the second to cast her vote. The *bargaining protocol* is given by $\rho=\left(r^t\right)_{t=1}^\infty$, where $r^t=(r_1^t,\ldots,r_n^t):I\times B^{t-1}\to I$. The dependence of r^t on B^{t-1} allows to encompass protocols where the order of play depends on previous voting decisions, as Example 3 below illustrates. However, if in a particular example r^t is independent of the order of play and the votes in the previous period, we will simply write r_j^t rather than $r_j^t(\cdot)$.

Example 2. If $n=2$, the well-known bilateral bargaining game with alternating proposals (Rubinstein 1982, Shaked and Sutton 1984, Binmore 1987) is obtained by setting $r_1^t=1$ for all odd t, $r_1^t=2$ for all even t, and $\psi^t(1)=1$ and $\psi^t(0)=0$ for all t.

For $n>2$, a multi-lateral bargaining protocol with alternating proposers can be specified setting $r_1^t=t \mod n$. This could be combined with e.g. a unanimity rule or a simple majority rule as above to obtain standard examples.

Histories of play will just contain the previous offers a^t and previous voting decisions b^t. Note that there is no need to record the actual order of play within a given period t, since the bargaining protocol, which is part of the description of the game, allows to reconstruct that order from the previous voting decisions. The following example takes advantage of this fact to encompass an action-dependent voting order where the first player to reject the previous offer becomes the next proposer. The example makes transparent why the bargaining protocol needs to depend on all previous offers, because the name of "the first player to reject" at t depends on the order at t, which in turn depends on the offers at $t-1$.

Example 3. Let $r_j^1=j$ for all $j=1,\ldots n$ (which just means that players are named according to their order of play in the initial period). For each $t\geq 1$ and each $(b^1,\ldots,b^t)\in B^t$ with

$\psi^t(b^t) = 0$, let

$$r_1^{t+1}(b^1, \ldots, b^t) = \min \left\{ r_j^t(b^1, \ldots, b^{t-1}) \mid b_j^t = 0,\ j = 2, \ldots, n \right\}$$

and

$$r_j^{t+1}(b^1, \ldots, b^t) = \begin{cases} j - 1 & \text{if } j \leq r_1^{t+1}(b^1, \ldots, b^t) \\ j & \text{if } j > r_1^{t+1}(b^1, \ldots, b^t) \end{cases}$$

for all $j = 2, \ldots, n$. For $(b^1, \ldots, b^t) \in B^t$ with $\psi^t(b^t) = 1$, arbitrarily (and inconsequentially) fix $r^t(b^1, \ldots, b^t) = r^1$. This procedure assigns as proposer the first player to vote against the proposal in the previous period, and lets all other players vote in the fixed order derived from $\{1, \ldots, n\}$.

These specifications define a perfect information game among the n bargaining partners. A *play* in this game is a complete sequence of offers and voting profiles, from the beginning to eventual acceptance, including sequences of infinite length where no offer is ever accepted. The set of plays is given by the union

$$W = \left(\bigcup_{T=1}^{\infty} W^T \right) \bigcup W^\infty$$

where the plays that end after a finite number of T rounds with acceptance are

$$W^T = \left\{ \left(a^t, b^t\right)_{t=1}^T \mid \psi^T\left(b^T\right) = 1,\ \psi^t\left(b^t\right) = 0\ \forall t < T \right\}$$

and the infinite plays of perpetual disagreement are

$$W^\infty = \left\{ \left(a^t, b^t\right)_{t=1}^\infty \mid \psi^t\left(b^t\right) = 0\ \forall t \right\}.$$

The specifications above suffice to define the tree of this extensive form (see, e.g., Alós-Ferrer and Ritzberger 2016a). In particular, the *nodes* in the tree are those sets of plays that share a fixed initial segment (formal definitions are provided in the Appendix). Due to perfect information the players' *choices* are the (immediate) successor nodes of their decision points. That is, a player active at a node simply chooses among successor nodes, which represent their decisions (proposals or votes).

In general the set W of plays is the appropriate domain for the players' preferences. Yet, in the main part of the paper we take a "non-procedural" stance by assuming that players care only about the ultimate distribution of the surplus and about when agreement was reached—not about how. (See Section 5 below for extensions to a "procedural" approach.) In particular, it is assumed that the players' utility functions are defined on the set

$$Z = (\Delta \times \{1, 2, \ldots\}) \cup \{\infty\}$$

of *outcomes*. A pair $(a, t) \in \Delta \times \{1, 2, \ldots\}$ amounts to an agreement on the distribution $a \in \Delta$ at round $t = 1, 2, \ldots$; the outcome ∞ corresponds to perpetual rejections. Accordingly, the players' preferences are represented by *utility functions* (or payoff functions) $u_i : Z \to \mathbb{R}$ for all $i \in I$. In particular, no assumption on convexity of the set of feasible payoffs is needed (in contrast to Merlo and Wilson 1995, Banks and Duggan 2000, Britz, Herings, and Predtetchinski 2014).

To summarize, a *bargaining game* is a quadruple (I, ρ, ψ, u) consisting of a player set $I = \{1, \ldots, n\}$, a bargaining protocol $\rho = \left(r^t\right)_{t=1}^{\infty}$, a sequence of aggregation functions $\psi = \left(\psi^t\right)_{t=1}^{\infty}$, and utility functions $u = (u_i)_{i \in I} : Z \to \mathbb{R}^n$.

4 A General Existence Result

The main hypothesis for the existence theorem in this paper will be *continuity* of the utility functions. In the current setting, continuity has two parts (as in Rubinstein 1982, Assumption A4). First, it requires that for all $t = 1, 2, \ldots$ and every $\varepsilon > 0$ there is $\delta > 0$ such that $\|a - a'\| < \delta$ implies $|u_i(a, t) - u_i(a', t)| < \varepsilon$ for all $i \in I$. This is simply continuity on Δ (with respect to the topology induced by the Euclidean metric). Second, because of the possibility of perpetual disagreement, a "continuity at infinity" (Fudenberg and Levine 1983) is called for: For every $\varepsilon > 0$ there is T such that $|u_i(a, t) - u_i(\infty)| < \varepsilon$ for all $t \geq T$, all $a \in \Delta$, and all $i \in I$. Intuitively, this demands that agreements that are sufficiently late will make almost no difference.

Remark 1. Two remarks are in order. First, continuity at infinity is *not* an additional assumption here. Proposition B.1 in the appendix shows that, for the natural topology on the set Z, continuity of a function $u_i : Z \to \mathbb{R}$ holds if and only if the two properties stated above hold. This fact extends beyond the bargaining case and shows that continuity at infinity is not an additional condition but rather an integral part of continuity with respect to the appropriate topology.

Second, the present definition of continuity at infinity may appear slightly different from the one proposed by Fudenberg and Levine (1983). They required that for every $\varepsilon > 0$ there is T such that whenever two (pure) strategy profiles agree up to T, then the two strategy profiles yield payoffs within ε from each other. In particular, in their framework there was no outcome of eternal continuation corresponding to $\infty \in Z$. *With* such an outcome ∞ the two definitions in fact agree. For, given $\varepsilon > 0$ consider a strategy profile that rejects all proposals up to the T associated with ε by the Fudenberg-Levine criterion. Then there is a second strategy profile that is identical to the first up to T but thereafter always rejects, hence, yields the outcome ∞. If

$(a,t) \in Z$ denotes the outcome induced by the first strategy profile, then $|u_i(a,t) - u_i(\infty)| < \varepsilon$, as required. Conversely, given $\varepsilon/2 > 0$ consider two strategy profiles that reject all proposals up to the T associated with $\varepsilon/2$ by the present criterion. Then the payoffs to the two strategy profiles must both be within $\varepsilon/2$ from $u_i(\infty)$. By the triangle inequality it follows that the payoffs to the two strategy profiles must be within ε from each other. In short, for the present framework the two definitions coincide.

Impatience, as implied e.g. by geometric discounting and bounded utility functions, is a sufficient condition for continuity at infinity. The reason is that discounting will drive all payoffs to zero as time goes on, which must then also be the utility from perpetual disagreement. But impatience is a strictly stronger condition than continuity at infinity. The following example illustrates this point, that without continuity at infinity subgame perfect equilibria may not exist, and that the present set-up covers highly non-stationary cases.

Example 4. Rumor has it that some committees are kept busy negotiating for the sole purpose of avoiding to be dissolved, even though nobody wishes to negotiate forever. This can be expressed by specifying $u_i(a,t) = 1 - \delta^{t-1}$ for some $\delta \in (0,1)$ and all $t = 1, 2, \ldots$, and $u_i(\infty) = 0$ for all $i \in I$. These utility functions fail continuity at infinity. Let $n = 2$, alternating offers as given by $r_1^t = t \mod 2$, and $\psi^t(0) = 0$ and $\psi^t(1) = 1$. This game has no subgame perfect equilibrium. For, suppose that there is an equilibrium that ends with agreement in round t. Then the responder r_2^t can do better by rejecting now and accepting two rounds later. If there were an equilibrium with perpetual disagreement, then at every finite t the responder r_2^t could do better by accepting immediately. If instead $u_i(\infty)$ were set to 1 for all $i \in I$, restoring continuity at infinity and existence of equilibrium, the committee would negotiate forever.

A *subgame perfect equilibrium* is a Nash equilibrium that induces a Nash equilibrium in every subgame. Since bargaining games have perfect information, a new subgame begins at every move (non-terminal node) of the tree. The following is the main result of the present paper.

Theorem 1. *Every bargaining game with continuous payoff functions has a subgame perfect equilibrium.*

Due to the generality of the class of games studied here the details of the proof of this theorem are somewhat involved and therefore relegated to the Appendix. Still, the next section provides an intuitive tour of the main ideas that drive the proof. (Readers who are not interested in the construction may skip it.)

4.1 Structure of the Proof

The proof consists of four steps. The first, and most laborious, is to endow the set Z of outcomes and the set W of plays with compact separated (Hausdorff) topologies in such a way that the mapping from plays to outcomes is a continuous function $\varphi : W \to Z$. The purpose is to turn the utility functions $u_i : Z \to \mathbb{R}$ that are defined on Z into continuous functions $u_i \circ \varphi : W \to \mathbb{R}$, now defined on plays.

The set of plays W is the infinite union of the sets of plays of length T, W^T, and the set of plays of infinite length W^∞. However, topologies derived from unions are notorious for being badly behaved, and hence we consider an alternative approach. The topology on Z is derived from considering the auxiliary set $\Delta \times \{1, 2, \ldots, \infty\}$. The simplex Δ carries the natural relative Euclidean topology. The extended natural numbers $\{1, 2, \ldots, \infty\}$ are equipped with a standard one-point compactification topology, known as the Fort topology with distinguished point ∞ (see, e.g., Steen and Seebach, Jr. 1978, p. 52). This is the topology for which the open sets are those whose complements are either finite or contain the point ∞. (Equivalently, a subset is closed if it is either finite or contains ∞.) The topology on the product $\Delta \times \{1, 2, \ldots, \infty\}$ is then the product topology (the coarsest topology that makes both projections continuous). To obtain a topology on Z from that construction, all elements of $\Delta \times \{1, 2, \ldots, \infty\}$ with second coordinate ∞ are identified to a single equivalence class. The resulting quotient set $\tilde{Z}$ is endowed with the identification topology, the finest topology that makes the identification map π continuous, where π is defined by $\pi((a, t)) = (a, t)$ if $t \neq \infty$ and $\pi((a, t)) = [\infty]$ otherwise. The end result corresponds to a one-point compactification of the space $\Delta \times \{1, 2, \ldots\}$. Since $\tilde{Z}$ and the original set Z of outcomes are isomorphic, this induces a topology on Z. And this topology can be shown to be compact and separated. (Compactness of $\tilde{Z}$, hence Z, is directly inherited from compactness of $\Delta \times \{1, 2, \ldots, \infty\}$; to show that it is separated takes some work.)

As for the set W of plays, this is first embedded into the space

$$\Omega^\infty = [(\Delta \cup \{*\}) \times (B \cup \{**\})]^\infty$$

of infinite sequences, where $*$ and $**$ denote dummy options, essentially signaling that an offer has been previously accepted. Again, Δ carries the relative Euclidean topology, which remains compact and separated if the single, discretely separated point $*$ is appended. The finite set $B \cup \{**\}$ is endowed with the discrete topology and is trivially compact and separated. By Tychonoff's theorem the infinite product Ω^∞ is also compact (and Hausdorff is easily established). Yet, this set is too big. Plays correspond only to infinite sequences in the subset $\bar{W} \subseteq \Omega^\infty$ where once an offer is accepted, play actually ends. This is characterized by the following three conditions.

(S.1) $\tilde{a}^1 \in \Delta$,

(S.2) $\tilde{a}^t = * \Leftrightarrow \tilde{b}^t = **$, for all $t = 1, 2, \ldots$,

(S.3) $\left(\tilde{a}^t, \tilde{b}^t\right) \in \Delta \times B \Leftrightarrow \psi^s\left(\tilde{b}^s\right) = 0 \, \forall s = 1, \ldots, t-1$ for all $t = 1, 2, \ldots$

That is, (S.1), they begin with offers; (S.2), dummy options occur always in both coordinates if at all; and, (S.3), a new bargaining round starts if and only if all previous proposals have been rejected. Call sequences in $\bar{W}$ *bargaining sequences.*

The relevant set, hence, is the set $\bar{W}$ of bargaining sequences. The proof establishes that this set is closed in the set Ω^∞ of all sequences, which in turn implies that it is compact in the relative topology. This topology then defines the appropriate topology for the analysis, because the set of plays can be fully identified with $\bar{W}$ as follows. Consider the map $\Lambda : W \to \bar{W}$ defined by

$$\Lambda(w) = \left((a^1, b^1), \ldots, (a^T, b^T), (*, **), \ldots, (*, **), \ldots\right)$$

if $w = \left(a^t, b^t\right)_{t=1}^T \in W^T$, $T \neq \infty$, and $\Lambda(w) = w$ if $w \in W^\infty$. Showing that Λ is bijective establishes that the set $\bar{W}$ of bargaining sequences and the set W of plays are isomorphic. Hence, the relative topology on $\bar{W}$ can be used as the compact and separated topology on W.

Once W and Z are endowed with compact separated topologies as described, one can turn to the map assigning outcomes to plays, $\varphi : W \to Z$, which is defined by $\varphi(w) = \left(a^T, T\right)$ if $w = \left(a^t, b^t\right)_{t=1}^T \in W^T$, $T \neq \infty$, and $\varphi(w) = \infty$ otherwise. The first step of the proof is completed by showing that φ is continuous, resulting in continuous payoff functions $u_i \circ \varphi$ on a compact separated space of plays.

With this preparation in place, it is now possible to invoke the general existence theorem by Alós-Ferrer and Ritzberger (2016b, Theorem 1; 2016a, Theorem 7.4). This is an abstract result establishing existence of subgame perfect equilibria for arbitrarily large perfect information games, provided the space of plays is compact and separated and the payoff functions defined on the space of plays are continuous with respect to the topology on plays.[1]

The existence theorem requires three additional hypotheses, and hence the three remaining steps amount to checking them. The second step establishes that the perfect information game is *well-behaved.* This means showing that the set of non-terminal nodes at a given "distance" from the root is partitioned into finitely many cells, each of which consists of decision points of a single player, whose unions are closed in the topology on plays. This is essentially straightforward in the current setting, because at the voting stages only finitely many players move. The only subtlety occurs when a proposal is made, because there are terminal nodes at the same distance from the root as the proposers' nodes.

The third step verifies that all nodes of the tree are closed as sets of plays, which intuitively is necessary for the players' optimization problems to be well-defined. The fourth step, finally, establishes that the assignment of immediate predecessors of nodes in the tree constitutes an open map, i.e. takes open sets to open sets. This is equivalent to the assignment of immediate successors being lower hemi-continuous, and essentially allows to "paste" the solutions of individual optimization problems together.

[1] Theorem 1 of Alós-Ferrer and Ritzberger (2016b) becomes a characterization when the separation axiom (Hausdorff) is strengthened to perfectly normal; see Alós-Ferrer and Ritzberger (2017b). Therefore, it is as general as any (topological) existence theorem for perfect information games can become.

Once these four preparations are in place, all hypotheses of the general existence theorem by Alós-Ferrer and Ritzberger (2016b, Theorem 1; 2016a, Theorem 7.4) are verified and the existence of a subgame perfect equilibrium follows.

5 Extensions and Illustrations

This section illustrates the algorithm which underlies the proof of Theorem 1 and offers a few possible extensions of the main result.

5.1 The Algorithm

The proof of Theorem 1 invokes the abstract existence theorem by Alós-Ferrer and Ritzberger (2016b, Theorem 1; 2016a, Theorem 7.4). The proof of the latter is based on an algorithm that iterates the players' expectations about what later players will do until, in the limit, expectations are correct and behavior is optimal with respect to these expectations.

The algorithm works as follows. Players start naïvely, that is, when they decide, they pick a play, as if they had full control of all other players moving afterwards. This is the first step. In the second step players develop some anticipation and now reoptimize under the constraints generated by what later players have done in the first step. Hence, they become "smarter" and foresee the choices (from the first step) of later players. This is repeated in the third step. Players now reoptimize under the constraints generated by the choices of later players from the second step, and so on. This iteration has a limit, which is a set, though. The existence theorem mentioned above shows that from this set strategy profiles can be selected which in turn form subgame perfect equilibria. If there is a unique equilibrium, the limit set is a singleton and the algorithm delivers the equilibrium directly. This can be nicely illustrated by bargaining games with a unique equilibrium, in particular by the classical bilateral alternating-proposer model of Rubinstein (1982).

Example 5. More concretely, let $n = 2$, $r_1^t = t \mod 2$, $r_2^t = 3 - r_1^t$, and $\psi^t(0) = 0$, $\psi^t(1) = 1$, for all $t = 1, 2, \ldots$, that is, players take turns in making offers and the game ends once the responder has accepted. For simplicity let payoffs be given by $u_i(a, T) = \delta^{T-1} a_i$ for some common discount factor $\delta \in (0, 1)$, for all $a \in \Delta$ and all $T = 1, 2, \ldots$, and by $u_i(\infty) = 0$, for all $i \in I$. Discounting is from one bargaining round to the next.

Obviously, Theorem 1 covers this basic example, which we use now to illustrate the algorithm. The first step works as follows. All proposers ask everything for themselves, $c_1 = 1$, on the assumption that they can force the responders to accept. All responders who are offered more than δ will accept, because they earn more than by rejecting and asking everything for themselves next round. All responders who are offered less than δ will reject, on account of making an accepted counteroffer next period that allocates the whole surplus to them. Therefore, the "critical offer" (from the proposer's viewpoint) for the second step is $c_2 = 1 - \delta$. This is *critical* because it is the only one at which the responder may choose both Yes (1) and No (0). Since all offers give 1 to the proposer and 0 to the responder, by $\delta < 1$ all offers effectively lead to perpetual rejections and payoffs $u_i(\infty) = 0$ for $i = 1, 2$.

Now turn to the second step. Since under first-step behavior all offers that allocate less than δ to the responder are rejected forever, in the second step all proposers offer δ to the responder and demand $c_2 = 1 - \delta$ for themselves. Hence, a responder, in a subgame after a proposal that allocates less than δ to her, cannot count on rejecting and asking 1 for herself next round, but must take into account that she can at best get $1 - \delta$ as next round's proposer, which is now worth $\delta(1 - \delta)$ to her. Therefore, she will now accept any offer that leaves the proposer with no more than $1 - \delta(1 - \delta) = 1 - \delta + \delta^2$. The critical offer for the third step is consequently $c_3 = 1 - \delta + \delta^2$. At this proposal the responder is indifferent between accepting and rejecting.

More generally, denote the critical offer from the τ-th step for the $(\tau+1)$-th step by c_τ. That is, any offer that allocates more than c_τ to the proposer (less than $1 - c_\tau$ to the responder) leads to perpetual rejections, and any offer that allocates less than c_τ to the proposer (more than $1-c_\tau$ to the responder) is accepted under τ-th step behavior; only at c_τ both acceptance and rejection are possible. Then in the $(\tau + 1)$-th step all proposals will allocate c_τ to the proposer and $1 - c_\tau$ to the responder. But a responder who is confronted with an offer that gives her less than $1 - c_\tau$ anticipates that by rejecting she will only be able to ask c_τ for herself next round, which is now worth δc_τ to her. (If the offer still gives her more than δc_τ, she will now accept.) Therefore, the critical offer at the $(\tau + 1)$-th step is

$$c_{\tau+1} = 1 - \delta c_\tau.$$

This is a difference equation with initial condition $c_1 = 1$ and the unique solution

$$c_{\tau+1} = \sum_{j=0}^{\tau} (-1)^j \delta^j.$$

It follows that under the algorithm the critical offers converge to

$$c_\infty = \sum_{j=0}^{\infty} (-1)^j \delta^j = \frac{1}{1+\delta},$$

and it follows from the algorithm that this must be the offer made by all proposers in any subgame perfect equilibrium; in particular, such an equilibrium is necessarily unique.

5.2 Hyperbolic Discounting and Multiple Selves

Simple textbook examples of bargaining games often use geometric discounting (and $u_i(\infty) = 0$) to ensure continuity at infinity. Theorem 1 makes no such assumptions. In fact, it is even consistent with time-inconsistent choices as those arising from "hyperbolic" discounting (Strotz 1956) where decision makers today discount the step from today to tomorrow more than the step from tomorrow to the day after tomorrow. Consider, for instance, the case of quasi-hyperbolic discounting (Laibson 1997), which captures this effect through the sequence of discount factors $1, \beta\delta, \beta\delta^2, \beta\delta^3, \ldots$ for $\beta, \delta \in (0,1)$. The only subtlety that arises for such "time-inconsistent" preferences is that players have to be split into agents—a different agent for each bargaining round. This is because players have different preferences in each consecutive bargaining round. More generally, dynamically inconsistent preferences are often analyzed by splitting an agent into a sequence of temporal selves making choices in a dynamic game; this has given rise to the literature on "multiple selves" (see, e.g., O'Donoghue and Rabin 1999, 2001, Fudenberg and Levine 2006, 2012). This poses no difficulty, because the underlying existence theorem invoked in the proof of Theorem 1 actually allows for infinitely many players (that is, agents or selves), provided only finitely many of them move at the same stage (round) of the game. Therefore, any kind of time preference can be accommodated, as long as continuity is preserved.

Example 6. Let $n = 3$, $r_j^t = (t+j-1) \bmod 3$ for $j = 1, 2, 3$, and $\psi^t(b^t) = 1$ if and only if $b_2^t + b_3^t = 2$ for all $b^t \in \{0,1\}^2$, for all $t = 1, 2, \ldots$ That is, three players take turns in making proposals and acceptance is by unanimity. Utility functions are given by $u_i(a, 1) = a_i$ and $u_i(a, T) = \beta\delta^{T-1} a_i$ for some $\beta, \delta \in (0,1)$ for $T = 2, 3, \ldots$, for all $a \in \Delta$ and all $i = 1, 2, 3$, i.e., the next round is discounted by $\beta\delta$ while later rounds $t > 1$ are discounted by $\beta\delta^{t-1}$. Theorem 1 (extended as explained above) applies, and hence the existence of subgame-perfect equilibrium is guaranteed. In this case, the equilibrium can be identified through standard arguments (Shaked and Sutton 1984).

Let $\bar{v}_i^t$ resp. $\underline{v}_i^t$ denote the supremum resp. the infimum of player i's equilibrium payoffs in any subgame starting in the t-th round, for $t = 1, 2, \ldots$ and $i = 1, 2, 3$. All subgames starting in the fourth round are identical to the game starting in the first round. Thus, by the usual argument (see Shaked and Sutton 1984), for $t = 3$ and player 3,

$$\bar{v}_3^3 = 1 - \beta\delta \left(\underline{v}_1^4 + \underline{v}_2^4\right) \text{ and } \underline{v}_3^3 = 1 - \beta\delta \left(\bar{v}_1^4 + \bar{v}_2^4\right),$$

and $\bar{v}_i^3 = \beta\delta\bar{v}_i^4$ and $\underline{v}_i^3 = \beta\delta\underline{v}_i^4$ for players $i = 1, 2$. Further, for $t = 2$,

$$\bar{v}_3^2 = \beta\delta - \beta^2\delta^2 \left(\underline{v}_1^4 + \underline{v}_2^4\right) \text{ and } \underline{v}_3^2 = \beta\delta - \beta^2\delta^2 \left(\bar{v}_1^4 + \bar{v}_2^4\right)$$

for player 3, while for player 2

$$\bar{v}_2^2 = 1 - \beta\delta\underline{v}_1^3 - \beta\delta\underline{v}_3^3 = 1 - \beta\delta + \beta^2\delta^2\left(\bar{v}_1^4 - \underline{v}_1^4\right) + \beta^2\delta^2\bar{v}_2^4 \text{ and}$$

$$\underline{v}_2^2 = 1 - \beta\delta\left(\bar{v}_1^3 + \bar{v}_3^3\right) = 1 - \beta\delta - \beta\delta\left(\bar{v}_1^4 - \underline{v}_1^4\right) + \beta^2\delta^2\underline{v}_2^4,$$

and $\bar{v}_1^2 = \beta^2\delta^2\bar{v}_1^4$ and $\underline{v}_1^2 = \beta^2\delta^2\underline{v}_1^4$ for player 1. Finally, for $t = 1$,

$$\bar{v}_3^1 = \beta^2\delta^2 - \beta^3\delta^3\left(\underline{v}_1^4 + \underline{v}_2^4\right) \text{ and } \underline{v}_3^1 = \beta^2\delta^2 - \beta^3\delta^3\left(\bar{v}_1^4 + \bar{v}_2^4\right)$$

for player 3, while for player 2

$$\bar{v}_2^1 = \beta\delta - \beta^2\delta^2 + \beta^3\delta^3\left(\bar{v}_1^4 - \underline{v}_1^4\right) + \beta^3\delta^3\bar{v}_2^4 \text{ and}$$

$$\underline{v}_2^1 = \beta\delta - \beta^2\delta^2 - \beta^3\delta^3\left(\bar{v}_1^4 - \underline{v}_1^4\right) + \beta^3\delta^3\underline{v}_2^4,$$

and for player 1

$$\bar{v}_1^1 = 1 - \beta\delta + \beta^3\delta^3\bar{v}_1^4 + \beta^3\delta^3\left(\bar{v}_1^4 - \underline{v}_1^4 + \bar{v}_2^4 - \underline{v}_2^4\right) \text{ and}$$

$$\underline{v}_1^1 = 1 - \beta\delta + \beta^3\delta^3\underline{v}_1^4 + \beta^3\delta^3\left(\bar{v}_1^4 - \underline{v}_1^4\right) - \beta^3\delta^3\left(\bar{v}_2^4 - \underline{v}_2^4\right).$$

Since all subgames beginning in the fourth round are identical to the game that starts in the first round, $\bar{v}_i^1 = \bar{v}_i^4$ and $\underline{v}_i^1 = \underline{v}_i^4$ for all $i = 1, 2, 3$. Solving this linear equation system yields

$$\bar{v}_i^1 = \underline{v}_i^1 = \frac{\beta^{i-1}\delta^{i-1}}{1 + \beta\delta + \beta^2\delta^2}$$

for all $i = 1, 2, 3$. Due to hyperbolic discounting player 3's agent in the first round accepts the offer $a_3^1 = \beta^2\delta^2/(1 + \beta\delta + \beta^2\delta^2)$ even though she would prefer to wait for the third round and ask $1/(1 + \beta\delta + \beta^2\delta^2)$ for herself, which she now values at $\beta\delta^2/(1 + \beta\delta + \beta^2\delta^2) > a_3^1$. But she has no control over her agent in the second round who accepts $a_2^2 = \beta\delta/(1 + \beta\delta + \beta^2\delta^2)$, which in the second round is as good to him as asking $1/(1 + \beta\delta + \beta^2\delta^2)$ in the third round. Therefore, player 3's agent in the first round has to accept an offer that is worth less to her than the discounted value of her own offer in the third round.

A special case of this example is $\beta = 1$, the standard case of geometric discounting. For that case the example shows that with a unanimity rule three-player bargaining yields a unique subgame perfect equilibrium.

5.3 Other-Regarding Preferences

The specification of utility functions for the general bargaining games studied in this paper only requires continuity of payoffs on the space of bargaining outcomes. There is no requirement whatsoever that payoffs depend only on the own coordinate of the allocation. In particular, it need not be the case that players care only about their share of the surplus, as was explicitly assumed e.g. in Rubinstein (1982, Assumption A1). This leaves room for explicitly incorporating externalities and other-regarding preferences, as in the models of Fehr and Schmidt (1999) or Bolton and Ockenfels (2000). That is, as a consequence of Theorem 1, equilibrium existence is guaranteed in any bargaining model where some or all players display other-regarding preferences as in those models.

More complex examples are also possible, where the preferences of players not only depend on the shares received by other players, but also do so through the actual utility that players derive from the shares. This is illustrated in the following example with three bargaining partners.

Example 7. Let $n = 3$, $r_j^t = (t + j - 1) \bmod 3$ for all $j = 1, 2, 3$, and $\psi^t(b^t) = 1$ if and only if $\sum_{j=2}^{3} b_j^t \geq 1$ for all $b^t \in \{0, 1\}^2$, for all $t = 1, 2, \ldots$ That is, players take turns in proposing and voting and the acceptance decision is by simple majority voting (since the proposal counts as a Yes-vote). Picture the three players in a triangle, with player 2 to the right of player 1 and player 3 to her left. The payoff functions display other-regarding preferences as follows.

$$u_i(a, T) = \delta^{T-1} a_i + \alpha u_{(i+1) \bmod 3}(a, T) - \alpha^2 u_{(i+2) \bmod 3}(a, T)$$

for some $\alpha, \delta \in (0, 1)$ for all $a \in \Delta$, all $T = 1, 2, \ldots$, and $i = 1, 2, 3$. That is, what matters to a player is not only her share of the surplus, but also how her neighbor to the right feels about the allocation (with weight α), even though she despises a bit the opinion of her neighbor to the left (with weight $-\alpha^2$). Solving the equation system for utility functions yields

$$u_i(a, T) = \frac{\delta^{T-1} \left(a_i + \alpha a_{(i+1) \bmod 3} \right)}{1 + \alpha^3}$$

for all $a \in \Delta$, all $T = 1, 2, \ldots$, and $i = 1, 2, 3$, i.e., players care about their own share and their neighbor's. Hence, utilities are continuous and Theorem 1 applies. An equilibrium—one of many, of course—is easily found: Every proposer i asks $1/(1 + \delta^2)$ for herself, offers 0 to $(i + 1) \bmod 3$, and $\delta^2/(1 + \delta^2)$ to $(i + 2) \bmod 3$, for all $i = 1, 2, 3$. Every responder j accepts any offer with $a_j \geq \delta^2/(1 + \delta^2)$ and rejects otherwise, for $j = 1, 2, 3$. It is not difficult to verify that this is an equilibrium, because the responder, who is offered a positive share, has to wait for two rounds before it is her turn to make the offer. Note that this equilibrium is independent of the parameter α that measures altruism.

5.4 Multi-Issue Bargaining

In Section 3 it was assumed that bargaining outcomes $a \in \Delta$ live in a simplex. This amounts to assuming that there is only a single issue that is negotiated. On the other hand, an important strand of the literature has studied *multi-issue* bargaining in the sense that several surpluses may be distributed (Inderst 2000, Busch and Horstmann 2002, In and Serrano 2003, 2004). The proof of Theorem 1 makes no use of the dimensionality of Δ, though. Since it relies exclusively on continuity and compactness, the simplex Δ could easily be replaced by a cube Δ^K that captures $K > 1$ distinct issues about which players bargain. The arguments establishing existence of equilibrium remain unchanged.

5.5 Procedural Preferences

In the main part of the paper we have taken a non-procedural stance, according to which bargaining partners care only about how the surplus is finally split and about when agreement is reached—but not about how this is brought about. Formally this is expressed by defining utility functions on the space Z of economic outcomes. If the procedure of bargaining in itself—how outrageous or how modest offers are, how stubbornly people behave, whether they enjoy or despise lengthy negotiations, etc.—influences well-being, then preferences need to be defined directly on plays. That is, the appropriate domain for "procedural" preferences is then the set W of plays, rather than the set Z. This would not pose a problem for the existence proof, though. For, the underlying theorem (Alós-Ferrer and Ritzberger 2016b, Theorem 1; 2016a, Theorem 7.4) holds for preferences defined on plays, even if those are purely ordinal, i.e., not necessarily representable by utility functions. Of course, the main hypothesis, continuity with respect to a (compact separated) topology on the set of plays, would then have to apply to these preferences defined on plays, but, with this modified hypothesis, Theorem 1 can be immediately extended to cover procedural preferences. The proof actually becomes simpler, since the part that establishes continuity of $\varphi : W \to Z$ can be dispensed with.

5.6 A Simple Non-Stationary Example Without Immediate Agreement

The following example illustrates a non-stationary, player-asymmetric environment where there is no immediate agreement.

Example 8. Let $n = 3$ and set $r^t = (1, 2, 3)$ for all t odd, $r^t = (2, 1, 3)$ for all t even. That is, players 1 and 2 make proposals alternatingly, with the other player voting first. Player 3 never gets to propose, and always votes second. For $t = 1, 2$, the voting rule is unanimity, i.e. $\psi^t(b) = 1$ if and only if $b = (1, 1)$. However, after $t = 3$ player 3 becomes a dummy player. For all $t \geq 3$, $\psi^t(b) = 1$ if and only if $b_2 = 1$, that is, the rule becomes unanimity among players 1 and 2.

Let $u_i(a, t) = \delta^{t-1} a_i$ for $i = 1, 2$. Since player 3 is effectively out of the game for all $t \geq 3$, it follows that in all subgames at or after $t = 3$ the only subgame perfect equilibrium is as in Example 5. That is, player 1 always proposes $\left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}, 0\right)$ and accepts if and only if offered a share of at least $\frac{\delta}{1+\delta}$, and symmetrically for player 2.

Suppose that player 3 has other-regarding, spiteful preferences of the form

$$u_3(a, t) = \delta^{t-1} a_3 + \min\{t - 1, 2\} \left(1 - \delta^{t-1}(1 - a_3)\right).$$

That is, player 3 values getting a large share for himself (and $u_3(a,t)$ is increasing in a_3 for any fixed t), but also values reducing the discounted joint payoffs of the other two players as much as possible, and the utility weight of this reduction increases every period until $t = 3$ (when the player becomes a dummy). Note that u_i is continuous in the sense introduced in Section 3 for all $i = 1, 2, 3$, with $u_1(\infty) = u_2(\infty) = 0$, $u_3(\infty) = 2$, and this example falls within the scope of Theorem 1. In any subgame perfect equilibrium, at $t = 2$, players know that if player 2's offer is rejected, the allocation $\left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}, 0\right)$ will be realized at $t = 3$. Hence, if an offer at $t = 2$ is to be preferred by both players 1 and 2 to waiting until $t = 3$, it must give player 1 at least $\frac{\delta}{1+\delta}$ and player 2 at least $\frac{\delta^2}{1+\delta}$. Hence, no offer at $t = 2$ can be made and accepted if it gives player 3 strictly more than $1 - \frac{\delta}{1+\delta} - \frac{\delta^2}{1+\delta} = 1 - \delta$. For, given such an allocation, either player 2 prefers to wait (and hence will not make the offer if it is to be accepted) or player 1 prefers to wait (and hence will vote against). However, a direct computation shows that if an offer at $t = 2$ gave player 3 $1 - \delta$, that player would vote against it, because he prefers to receive 0 one period later than $1 - \delta$ at $t = 2$. Indeed,

$$
\begin{aligned}
&u_3(1-\delta, 2) < u_3(0,3) \Leftrightarrow \delta(1-\delta) + (1 - \delta(\delta)) < 2\left(1-\delta^2\right) \\
&\Leftrightarrow \delta(1-\delta) < 1 - \delta^2 b \Leftrightarrow \delta < 1 + \delta
\end{aligned}
$$

and the last inequality holds immediately. It follows that player 3 would also reject any offer at $t = 2$ giving him strictly less than $1 - \delta$. We conclude that, in a subgame perfect equilibrium, the equilibrium offer of player 2 at $t = 2$ must be rejected.

Analogously, in any subgame perfect equilibrium, at $t = 1$ players know that if player 1's offer is rejected, the allocation $\left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}, 0\right)$ will be realized at $t = 3$. Hence, if an offer at $t = 1$ is to be preferred by both players 1 and 2 to waiting until $t = 3$, it must give player 1 at least $\frac{\delta^2}{1+\delta}$ and player 2 at least $\frac{\delta^3}{1+\delta}$. Hence, no offer at $t = 1$ can be made and accepted if it gives player 3 strictly more than $1 - \frac{\delta^2}{1+\delta} - \frac{\delta^3}{1+\delta} = 1 - \delta^2$. Again, a direct computation shows that player 3 will vote against any offer at $t = 1$ giving him $1 - \delta^2$, since

$$
u_3(1-\delta^2, 1) < u_3(0,3) \Leftrightarrow (1-\delta^2) < 2\left(1-\delta^2\right)
$$

and the last inequality is obviously true. That is, in all subgame perfect equilibria of this game players 1 and 2 purposefully make offers which will be rejected in the first two periods, and reach an agreement in period 3.

The structure of this example is minimalistic, but it can of course be generalized. The key of the example is the willingness of one player to wait in order to reduce other player's payoffs, a phenomenon allowed by other-regarding preferences, and the feasibility of shutting down this player by waiting until an appropriate environmental change, which is allowed by the non-stationarity of the setting. It also illustrates that the setting allows for the effective set of bargaining agents to be time-dependent.

6 Conclusions

This paper has studied a general class of multi-lateral bargaining models. No particular model of impatience is required, hence allowing for geometric or hyperbolic discounting, externalities and other-regarding preferences are allowed, and bargaining protocols can be arbitrary and vary over time—hence, no stationary structure is assumed. The main insight is that all games in that class have a subgame perfect equilibrium, provided the utility functions are continuous. Continuity here includes the possibility of perpetual disagreement, that is, it also holds "at infinity." This result hence eliminates the necessity to demonstrate existence of equilibrium for each bargaining problem separately, by providing a general existence proof for a wide class of problems.

A Topologies

A.1 The Space of Economic Outcomes

A bargaining outcome is an allocation and the time at which it is accepted, $(a,t) \in \Delta \times \{1,2,\ldots\}$, or perpetual disagreement, which is represented by the symbol ∞. Hence, the space of outcomes is

$$Z = (\Delta \times \{1,2,\ldots\}) \cup \{\infty\}.$$

To endow it with a natural topology, we will view it as a quotient set of the larger, instrumental space

$$\bar{Z} = \Delta \times \{1,2,\ldots,\infty\}.$$

This latter space has a product structure and is naturally well-behaved if the appropriate topologies are taken for the factors. Endow the space of allocations Δ with the Euclidean topology, which makes Δ compact and Hausdorff. The space $\{1,2,\ldots,\infty\}$ is also compact and Hausdorff if it is endowed with the one-point compactification of the discrete topology, that is, a subset of $\{1,2,\ldots,\infty\}$ is open if and only if its complement is either finite or contains ∞. This topology is just the discrete topology (all sets are open) if restricted to $\{1,2,\ldots\}$, but it requires open sets containing ∞ to contain some terminal segment $\{t,t+1,\ldots,\infty\}$. It will be shown below that this topology is closely related to notions of "continuity at infinity."[2]

The space $\bar{Z}$ is endowed with the product topology and is therefore compact and Hausdorff. Consider the projection $\pi : \bar{Z} \to Z$ given by $\pi(a,t) = (a,t)$ if $t \neq \infty$ and $\pi(a,\infty) = \infty$ for all $a \in \Delta$. This amounts to identifying all outcomes that are sufficiently far in the future to a single equivalence class. Thereby the space Z can be seen as the quotient space $\bar{Z}/(\Delta \times \{\infty\})$

[2] The topology on $\{1,2,\ldots,\infty\}$ is also known as the Fort topology with distinguished point ∞; see Steen and Seebach, Jr. (1978, p. 52).

and endowed with the identification topology (Steen and Seebach, Jr. 1978, p. 9), that is, a set $V \subseteq Z$ is open in Z if and only if $\pi^{-1}(V)$ is open in the product topology on $\bar{Z}$.

Lemma A.1. *The set Z endowed with the identification topology is compact and Hausdorff.*

Proof. Since $\bar{Z}$ is compact, the quotient set Z is also compact. To see that it is also Hausdorff, let $x, y \in \bar{Z}$ with $x \neq \infty \neq y$. Since $\bar{Z}$ is Hausdorff, there exist U_x, U_y open sets in $\bar{Z}$ such that $x \in U_x, y \in U_y$, and $U_x \cap U_y = \emptyset$. Since $\{1, 2, \ldots\}$ is open in $\{1, 2, \ldots, \infty\}$ and $x \neq \infty \neq y$, the sets $U'_x = U_x \cap \Delta \times \{1, 2, \ldots\}$ and $U'_y = U_y \cap \Delta \times \{1, 2, \ldots\}$ are also open sets in $\bar{Z}$ such that $x \in U'_x, y \in U'_y$, and $U'_x \cap U'_y = \emptyset$. Consider the (singleton) classes $[x]$ and $[y]$ in Z. Then $\pi^{-1}(U'_x)$ and $\pi^{-1}(U'_y)$ are open sets of Z with $[x] \in \pi^{-1}(U'_x)$, $[y] \in \pi^{-1}(U'_y)$, and $\pi^{-1}(U'_x) \cap \pi^{-1}(U'_y) = \emptyset$.

Hence all classes $[x]$, $[y]$ different from $[\infty]$ can be separated by open sets. To complete the proof, let $x = (a, t) \in \bar{Z}$. We have to show that the classes $[x]$ and $[\infty]$ can be separated by open sets in the quotient space. To see this, let $U_x = \Delta \times \{1, \ldots, t\}$ and $U_\infty = \Delta \times \{t+1, \ldots, \infty\}$. These sets are disjoint and open in $\bar{Z}$, and fulfill $x \in U_x$ and $\infty \in U_\infty$. Then $\pi^{-1}(U_x)$ and $\pi^{-1}(U_\infty)$ are open sets of Z with $[x] \in \pi^{-1}(U_x)$, $[\infty] \in \pi^{-1}(U_\infty)$, and $\pi^{-1}(U_x) \cap \pi^{-1}(U_\infty) = \emptyset$. ■

Abusing notation, denote the class $[x] = \{x\}$ in Z simply by x for each $x \neq \infty$, and the class $[\infty] = \Delta \times \{\infty\}$ by ∞. For convenience, if $V \subseteq Z$, the set $V \setminus \{\infty\}$ can be viewed as a subset of $\bar{Z}$ and hence cumbersome notation can be avoided. In particular, if $\infty \notin V$ the set V can be seen both as a subset of Z and as a subset of $\bar{Z}$. If $\infty \in V \subseteq Z$, the set $\pi^{-1}(V) \subseteq \bar{Z}$ corresponds to V with the addition of all pairs of the form (a, ∞). Formally,

$$\pi^{-1}(V) = \begin{cases} V & \text{if } \infty \notin V \\ (V \setminus \{\infty\}) \cup (\Delta \times \{\infty\}) & \text{if } \infty \in V. \end{cases}$$

That is, if an open set $V \subseteq Z$ contains ∞, the open set $\pi^{-1}(V) \subseteq \bar{Z}$ contains the whole "limit slice" $\Delta \times \{\infty\}$. The following lemma identifies an important property of such sets, which will be used in the proofs below.

Lemma A.2. *Let $U \subseteq \bar{Z}$ be an open set in the product topology on $\bar{Z}$ such that $\Delta \times \{\infty\} \subseteq U$. Then, there exists $T \geq 1$ such that $\Delta \times \{T, T+1, \ldots, \infty\} \subseteq U$.*

Proof. For each $a \in \Delta$, $(a, \infty) \in U$. Since U is open, there exist $U_1(a)$ open in Δ and $U_2(a)$ open in $\{1, 2, \ldots, \infty\}$ such that $(a, \infty) \in U_1(a) \times U_2(a) \subseteq U$.

The sets $\{U_1(a) \mid a \in \Delta\}$ form an open cover of Δ. Since Δ with the Euclidean topology is compact, there exists a finite subcover, i.e. $a_1, \ldots, a_m \in \Delta$ such that $U_1(a_1) \cup U_1(a_2) \cup \ldots \cup U_1(a_m) = \Delta$. For each $j = 1, \ldots, m$, the set $U_2(a_j)$ is open in $\{1, 2, \ldots, \infty\}$ but contains ∞. Open sets in the topology of this space are those whose complements are either finite or contain ∞. Since the former cannot happen, it follows that for each $j = 1, 2, \ldots, m$, there exists t_j such that $\{t_j, t_j+1, \ldots, \infty\} \subseteq U_2(a_j)$. Let $T = \max_{j=1,\ldots,m} t_j$. Then, $U_1(a_j) \times \{T, T+1, \ldots, \infty\} \subseteq U$ for all j and, since the sets $U_1(a_j)$ cover Δ, it follows that $\Delta \times \{T, T+1, \ldots, \infty\} \subseteq U$. ■

A.2 The Space of Plays

The space of plays is given by

$$W = \left(\bigcup_{T=1}^{\infty} W^T \right) \cup W^\infty$$

and the mapping assigning to each play its bargaining outcome is given by $\varphi : W \to Z$ with

$$\varphi(w) = \begin{cases} \left(a^T, T\right) & \text{if } w = \left(a^t, b^t\right)_{t=1}^T \in W^T,\ T < \infty \\ \infty & \text{if } w \in W^\infty \end{cases} \tag{1}$$

Viewed as an infinite union the space W does not lend itself to a natural well-behaved topology. Therefore, we will embed it into an instrumental space of sequences from an enlarged space of instantaneous actions. The latter can be endowed with a natural well-behaved product topology, but does contain many situations which cannot be interpreted in terms of the game. Thus, the actual space of plays will be a proper subset of the set of sequences. This subset will be shown to be closed in the product topology. In this way, the relative topology (inherited by the closed subset) will provide a well-behaved topology for the space of plays.

Expand the set of allocations Δ with a dummy action $*$, representing that an agreement was achieved in the past. Likewise, expand the set of voting profiles B with a dummy profile $**$, representing that no voting is called for because the game has already ended. Then take the set of instantaneous action profiles as

$$\Omega = (\Delta \cup \{*\}) \times (B \cup \{**\}).$$

Endow Δ with the Euclidean topology and $\Delta \cup \{*\}$ with a Euclidean-expanded topology where $V \subseteq \Delta \cup \{*\}$ is open if and only if $V \cap \Delta$ is open in Δ. That is, the topology on $\Delta \cup \{*\}$ simply "ignores" the additional point, and the singleton $\{*\}$ is declared open. This topology makes $\Delta \cup \{*\}$ compact and Hausdorff. The finite set $B \cup \{**\}$ is endowed with the discrete topology, which of course makes it also compact and Hausdorff. The set $\Omega = (\Delta \cup \{*\}) \times (B \cup \{**\})$ is endowed with the product topology and hence it is also compact and Hausdorff. Then consider the infinite product

$$\Omega^\infty = [(\Delta \cup \{*\}) \times (B \cup \{**\})]^\infty$$

endowed with the product topology. This space is compact and Hausdorff, but it has no natural correspondence with the space of plays. For instance, sequences where some $a \in \Delta$ appears strictly after an occurrence of $*$ cannot be a play. The following definition identifies the sequences in Ω^∞ that can be interpreted as plays.

Definition A.1. A sequence $\left(\bar{a}^t, \bar{b}^t\right)_{t=1}^\infty \in \Omega^\infty$ is a *bargaining sequence* if the following conditions hold.

(S.1) $\bar{a}^1 \in \Delta$;

(S.2) $\bar{a}^t = * \Leftrightarrow \bar{b}^t = **$, for all $t = 1, 2 \ldots$;

(S.3) $\left(\bar{a}^t, \bar{b}^t\right) \in \Delta \times B \Leftrightarrow \psi^s\left(\bar{b}^s\right) = 0\, \forall s = 1, \ldots, t-1$ for all $t = 1, 2, \ldots$

Condition (S.1) states that bargaining begins with an offer. (S.2) demands that dummy options always occur in both coordinates of the instantaneous action profile. (S.3) stipulates that after rejections bargaining continues and after an acceptance it ends. Let $\bar{W} \subset \Omega^\infty$ denote the set of all bargaining sequences.

Lemma A.3. *The set $\bar{W}$ is closed in Ω^∞.*

Proof. It will be proved that the complement $\Omega^\infty \setminus \bar{W}$ of $\bar{W}$ is open (in the product topology on Ω^∞) by showing that it contains an open neighborhood for all its points. More precisely, for any sequence $\bar{w} \in \Omega^\infty \setminus \bar{W}$ and every possible violation of (S.1-3) we will exhibit a basic open set U that contains $\bar{w}$ and is contained in $\Omega^\infty \setminus \bar{W}$.[3] Let $\bar{w} = \left(\bar{a}^t, \bar{b}^t\right)_{t=1}^\infty \in \Omega^\infty \setminus \bar{W}$. Then $\bar{w}$ violates at least one of the conditions from Definition A.1.

If $\bar{w}$ fails (S.1), i.e. $\bar{a}^1 = *$, take $U = (\{*\} \times (B \cup \{**\})) \times \Omega^\infty$. If $\bar{w}$ violates (S.2), then there is $t \geq 2$ such that either $\bar{a}^t = *$ and $\bar{b}^t \in B$ or $\bar{a}^t \in \Delta$ and $\bar{b}^t = **$. In the first case let $U = \Omega^{t-1} \times (\{*\} \times B) \times \Omega^\infty$, in the second case let $U = \Omega^{t-1} \times (\Delta \times \{**\}) \times \Omega^\infty$.

If $\bar{w}$ fails the "if"-part of (S.3) with $\bar{a}^t \in \Delta$ or its "only if"-part, then there is t such that either $\psi^s(\bar{b}^s) = 0$ for all $s = 1, \ldots, t-1$ but $\bar{b}^t = **$ or $\left(\bar{a}^t, \bar{b}^t\right) \in \Delta \times B$ but $\psi^s(\bar{b}^s) = 1$ for some $s = 1, \ldots, t-1$. For these cases let $U = \left(\times_{s=1}^t \left((\Delta \cup \{*\}) \times \{\bar{b}^s\}\right)\right) \times \Omega^\infty$. Finally, if $\bar{w}$ violates the "if"-part of (S.3) with $\bar{a}^t = *$, let

$$U = \left(\times_{s=1}^{t-1} \left((\Delta \cup \{*\}) \times \{\bar{b}^s\}\right)\right) \times (\{*\} \times (B \cup \{**\})) \times \Omega^\infty.$$

Since it is not difficult to check that all the sets U are basic open neighborhoods of $\bar{w}$ contained in $\Omega \setminus \bar{W}$, the proof is complete. ■

Since $\bar{W}$ is closed in Ω^∞ and the latter space is compact and Hausdorff with the product topology, the set $\bar{W}$ together with the relative topology inherited from Ω^∞ (as a subspace) is compact and Hausdorff. All that remains is to show that there is a natural bijection between $\bar{W}$

[3] A basis for the product topology consists of the product sets for which all but finitely many of the factors are unrestricted. An open set from this basis is called a *basic open set*.

and the space of plays W. Let $\Lambda : W \to \bar{W}$ be given by

$$\Lambda(w) = \begin{cases} \left(a^1, b^1, \dots, a^T, b^T, *, **, *, **, \dots\right) & \text{if } w = \left(a^t, b^t\right)_{t=1}^T \in W^T,\ T < \infty \\ w & \text{if } w \in W^\infty. \end{cases}$$

This mapping is obviously injective (one to one). To show that is surjective (onto), let $\bar{w} \in \bar{W}$. If $\bar{w} \in (\Delta \times B)^\infty$, then $\bar{w} \in W^\infty$ and $\Lambda(\bar{w}) = \bar{w}$. If $\bar{w} = \left(\bar{a}^t, \bar{b}^t\right)_{t=1}^\infty \notin (\Delta \times B)^\infty$, by (S.3) there is some $t \geq 1$ such that $\bar{a}^t = *$. Let $\bar{t} = \min\left\{t \mid \bar{a}^t = *\right\}$. By (S.3) again, $\bar{b}^s \neq **$, for all $s < \bar{t}$. By (S.2) and (S.3), $\psi^s(\bar{b}^s) = 0$ for all $s < \bar{t} - 1$. By (S.3) again, $\psi^s(\bar{b}^{\bar{t}-1}) = 1$. It follows that $\left(\bar{a}^t, \bar{b}^t\right)_{t=1}^{\bar{t}-1} \in W^{\bar{t}-1}$. Further, by (S.2) $\bar{a}^s = *$ and $\bar{b}^s = **$ for all $s \geq \bar{t}$, hence $\Lambda\left(\left(\bar{a}^t, \bar{b}^t\right)_{t=1}^{\bar{t}-1}\right) = \bar{w}$.

Hence, Λ is a bijection between W and $\bar{W}$. Formally, the topology on $\bar{W}$ defines a compact Hausdorff topology on W by declaring $U \subseteq W$ to be open if and only if $\Lambda(U)$ is open in $\bar{W}$. In practice, we can simply identify the spaces W and $\bar{W}$ and rely on the topology on $\bar{W}$. To economize on notation, in the sequel we will follow this approach and identify W and $\bar{W}$, meaning that a play $w \in W$ can also be seen as the corresponding bargaining sequence in the sense of Definition A.1.

A.3 The Outcome Mapping

Having endowed the set W of plays and the set Z of outcomes with compact Hausdorff topologies, the next step is to show that the mapping φ from (1), as a mapping between these topological spaces, is continuous.

Lemma A.4. *The outcome mapping $\varphi : W \to Z$ is continuous.*

Proof. Let $V \subseteq Z$ be an open set. By construction of the topology on Z, as a quotient topology derived from $\bar{Z} = \Delta \times \{1, 2, \dots, \infty\}$, the subset $\pi^{-1}(V)$ of $\bar{Z}$ is open in the product topology on $\bar{Z}$. To prove that $\varphi^{-1}(V)$ is open in W, it will be shown that $\varphi^{-1}(V)$ contains an open neighborhood for each of its elements. Let $w \in \varphi^{-1}(V)$ and let $v = \varphi(w) \in V$. Two cases need to be distinguished.

First, suppose that $v \neq \infty$. In this case $\pi^{-1}(v) = \{v\}$. Then, $w \in W^T$ and $v = \left(a^T, T\right)$ for some finite $T \geq 1$ and $a^T \in \Delta$. Denoting $w = \left(\tilde{a}^t, \tilde{b}^t\right)_{t=1}^T$, it also follows that $\tilde{a}^T = a^T$ and $\psi^T(\tilde{b}^T) = 1$. Since V is open in $\bar{Z}$ and $v \in \pi^{-1}(V)$, there are U_1 open in Δ and U_2 open in $\{1, 2, \dots, \infty\}$ such that $v = \left(a^T, T\right) \in U_1 \times U_2 \subseteq \pi^{-1}(V)$. Then the set

$$U = (\Delta \times B)^{T-1} \times \left(U_1 \times \{\tilde{b}^T\}\right) \times \Omega^\infty.$$

is an open basic set of the product topology on Ω^∞. Hence, $U \cap \bar{W}$ is an open set of $\bar{W}$ and, through the identification of this set with the space of plays, of W. For every $w' \in U \cap \bar{W}$, since

$\psi^T(\tilde{b}^T) = 1$, it follows that $\pi^{-1}(\varphi(w')) = \{\varphi(w')\} \subseteq U_1 \times \{T\} \subseteq U_1 \times U_2 \subseteq \pi^{-1}(V)$. Therefore, $w \in U \cap \bar{W} \subseteq \varphi^{-1}(V)$.

Suppose now that $v = \infty$, that is, w is a perpetual disagreement path. In this case, $\pi^{-1}(v) = \Delta \times \{\infty\} \subseteq \pi^{-1}(V)$. By Lemma A.2, there is $T \geq 1$ such that $\Delta \times \{T, T+1, \ldots, \infty\} \subseteq \pi^{-1}(V)$. Then the set $U = (\Delta \times B)^T \times (\Omega)^\infty$ is an open basic set of the product topology on Ω^∞, $U \cap \bar{W}$ is an open set of $\bar{W}$, and—by the identification of this set with the space of plays—of W. For every $w' \in U \cap \bar{W}$, either agreement occurs at a finite time larger than or equal to T, in which case $\pi^{-1}(\varphi(w')) = \{\varphi(w')\}$, or there is perpetual disagreement, $\varphi(w') = \infty$. In both cases, $\pi^{-1}(\varphi(w')) \subseteq \Delta \times \{T, T+1, \ldots, \infty\} \subseteq \pi^{-1}(V)$. Hence $w \in U \cap \bar{W} \subseteq \varphi^{-1}(V)$. ■

B The Bargaining Tree

Following the original approach of von Neumann and Morgenstern (1944), a node in an extensive form game is simply a collection of outcomes. Specifically, a node can be seen as the set of outcomes which are still available when a player decides at that node. A node precedes another node if and only if the latter node is properly contained in the former. Intuitively, decisions discard possible outcomes and hence reduce the size of the nodes. This approach has been developed for arbitrarily large extensive form games (see Alós-Ferrer and Ritzberger 2016a), showing e.g. the relation with popular "graphical approaches" where trees are viewed as graphs on abstract nodes. For the perfect information case treated here, the collection of nodes (the "tree") and the specification of which player is active when fully specifies the game, because the choices of players are simply to pick an immediate successor of each given node.

The nodes in the tree of the bargaining game are as follows. At each round $s = 1, 2, \ldots$ there are n "slices," $(s, 1)$ to (s, n), where the proposer and the second to last responder act, respectively. At slice $(s, 1)$, the player selected as proposer in that round (which might vary depending on previous votes) makes an offer.

The nodes at this slice are of the form

$$x_i^s\left(\left(\bar{a}^t, \bar{b}^t\right)_{t=1}^{s-1}\right) = \bigcup_{T \in \{s, \ldots, \infty\}} \left\{\left(a^t, b^t\right)_{t=1}^T \in W^T \;\middle|\; a^t = \bar{a}^t,\ b^t = \bar{b}^t\ \forall t \leq s-1\right\}$$

where $i = r_1^s(\bar{b}^1, \ldots, \bar{b}^{s-1})$ is the proposer (who acts at this node), for each $\left(\bar{a}^t, \bar{b}^t\right)_{t=1}^{s-1}$ describing a previous history of rejected offers, that is, fulfilling $\psi\left(\bar{b}^t\right) = 0$ for all $t = 1, \ldots, s-1$. (The subindex in the union describing the node x_i^s runs from s to ∞ and, in particular, $T = \infty$ is allowed for notational convenience, to capture plays with perpetual disagreement.) Hence, the set of non-terminal nodes at slice $(s,1)$ is given by

$$X_1^s = \left\{ x_{i,1}^s \left(\left(\bar{a}^t, \bar{b}^t \right)_{t=1}^{s-1} \right) \;\middle|\; \begin{array}{l} \left(\bar{a}^t, \bar{b}^t \right) \in \Delta \times B \text{ and} \\ \psi\left(\bar{b}^t \right) = 0 \;\forall t \leq s-1 \\ i = r_1^s(\bar{b}^1, \ldots, \bar{b}^{s-1}) \end{array} \right\}.$$

In particular, at slice $(1,1)$ the set X_1^1 consists of only one node which contains all plays of the game and is, as a set, identical with W—the root of the game.

For every $k = 2, \ldots, N$, the slice (s,k) consists of nodes of the form

$$x_{j,k}^s \left(\left(\bar{a}^t, \bar{b}^t \right)_{t=1}^{s-1}, \bar{a}^s, \left(\bar{b}_\ell^s \right)_{\ell=2}^{k-1} \right) = \bigcup_{T \in \{s,\ldots,\infty\}} \left\{ \left(a^t, b^t \right)_{t=1}^{T} \in W^T \;\middle|\; \begin{array}{l} a^t = \bar{a}^t \;\forall t \leq s, \; b^t = \bar{b}^t \;\forall t \leq s-1, \\ b_\ell^s = \bar{b}_\ell^s \;\forall \ell = 2, \ldots, k-1 \end{array} \right\}$$

following a sequence of rejected offers, where $j = r_k^s(\bar{b}^1, \ldots, \bar{b}^{s-1})$ is the $(k-1)$-th responder, who plays at this node and whose identity depends on previous voting rounds. That is, the set of nodes at slice (s,k) with $k \geq 2$ is given by

$$X_k^s = \left\{ x_{j,k}^s \left(\left(\bar{a}^t, \bar{b}^t \right)_{t=1}^{s-1}, \bar{a}^s, \left(\bar{b}_\ell^s \right)_{\ell=2}^{k-1} \right) \;\middle|\; \begin{array}{l} \bar{a}^t \in \Delta \;\forall t \leq s, \\ \bar{b}^t \in B \;\forall t \leq s-1, \\ \bar{b}_\ell^s \in \{0,1\} \;\forall \ell = 2, \ldots, k-1, \\ \psi\left(\bar{b}^t \right) = 0 \;\forall t = 1, \ldots, s-1 \\ j = r_k^s(\bar{b}^1, \ldots, \bar{b}^{s-1}) \end{array} \right\}.$$

Finally, the set of terminal nodes arising after acceptance of offers at round s, which formally belong to slice $(s+1,1)$, is given by the singletons of the corresponding finite plays, $E^{s+1} = \{\{w\} \mid w \in W^s\}$ for each $s = 1, 2, \ldots$

Following the notation in Alós-Ferrer and Ritzberger (2016a; 2016b), the slice (s,k), viewed as a collection of nodes, is denoted by $Y_{s,k}$. Thus, $Y_{1,1} = \{W\}$ and $Y_{s,1} = X_1^s \cup E^s$ for each $s \geq 1$. Since terminal nodes only occur after all votes have been cast, also $Y_{s,k} = X_k^s$ for each $k \geq 2$.

B.1 Properties of the Bargaining Tree

In this section it is shown that the bargaining tree defined above satisfies the hypotheses for the existence theorem of Alós-Ferrer and Ritzberger (2016b, Theorem 1; 2016a, Theorem 7.4).

B.2 Well-Behaved

The first condition stipulates that the game is "well-behaved." This condition has two parts. The first is that the preferences of all players are continuous with respect to the topology on the space of plays. However, here the payoff functions are defined on bargaining outcomes, $u_i : Z \to \mathbb{R}$. In order to apply the existence theorem, we need to consider payoffs defined on the full set of plays. Plays are mapped into outcomes by the function $\varphi : W \to Z$, and hence payoffs from plays are given by the compositions $u_i \circ \varphi : W \to \mathbb{R}$, for each $i \in I$. Since φ is continuous by Lemma A.4, it follows that the composition $u_i \circ \varphi$ is also continuous, provided that $u_i : Z \to \mathbb{R}$ is continuous. The latter amounts to continuity with respect the topology on $Z = (\Delta \times \{1, 2, \ldots\}) \cup \{\infty\}$ derived as the identification topology of $\Delta \times \{1, 2, \ldots, \infty\}$ with respect to $\Delta \times \{\infty\}$. The following result shows that our continuity assumptions on u_i are exactly equivalent to continuity with respect to this topology. Note that (ii) below amounts to "continuity at infinity."

Proposition B.1. *The function $u_i : Z \to \mathbb{R}$ is continuous if and only if the following two conditions hold.*

(i) For each $t = 1, 2, \ldots$, the function $u_i^t : \Delta \to \mathbb{R}$ given by $u_i^t(a) = u_i(a, t)$ for each $a \in \Delta$ is continuous (with respect to the Euclidean topologies on Δ and $\mathbb{R}$).

(ii) For each $\varepsilon > 0$ there exists $T \in \{1, 2, \ldots\}$ such that $|u_i(a, t) - u_i(\infty)| < \varepsilon$ for all $a \in \Delta$ and all $t \geq T$.

Proof. "if:" Suppose that (i) and (ii) hold. To see that u_i is continuous, let $V \subseteq \mathbb{R}$ be an open set of real numbers. To prove that $u_i^{-1}(V)$ is open in Z, we will show that $u_i^{-1}(V)$ contains an open neighborhood for each for its elements. To see this, let $z \in u_i^{-1}(V)$ and distinguish two cases.

First, suppose $z \neq \infty$. Then, $z = (a, t)$ for some $a \in \Delta$ and $t \in \{1, 2, \ldots\}$. Therefore, $u_i^t(a) = u_i(a, z) \in V$ and, by continuity of u_i^t, $\left(u_i^t\right)^{-1}(V)$ is open in Z. By construction of the topology on Z, $\left(u_i^t\right)^{-1}(V) \times \{t\}$ is an open subset of Z such that $z \in \left(u_i^t\right)^{-1}(V) \times \{t\} \subseteq u_i^{-1}(V)$. Second, suppose $z = \infty$. Since V is open and $u_i(\infty) \in V$, there exists $\varepsilon > 0$ such that $(u_i(\infty) - \varepsilon, u_i(\infty) + \varepsilon) \subseteq V$. By condition (ii), there exists $T \in \{1, 2, \ldots\}$ such that for each $a \in \Delta$ and each $t \geq T$, $|u_i(a, t) - u_i(\infty)| < \varepsilon$. It follows that $\{\infty\} \cup (\Delta \times \{1, 2, \ldots\}) \subseteq u_i^{-1}\left((u_i(\infty) - \varepsilon, u_i(\infty) + \varepsilon)\right) \subseteq u_i^{-1}(V)$. Since $\{\infty\} \cup (\Delta \times \{1, 2, \ldots\}) = \pi(\Delta \times \{1, 2, \ldots, \infty\})$, this set is an open neighborhood of ∞ in Z.

"only if:" Let $u_i : Z \to \mathbb{R}$ be continuous. To see (i), fix $t \in \{1, 2, \ldots\}$ and let $V \subseteq \mathbb{R}$ be an open set of real numbers. Then,

$$\left(u_i^t\right)^{-1}(V) = \{a \in \Delta \mid u_i(a, t) \in V\} = u_i^{-1}(V) \cap (\Delta \times \{t\}) .$$

The set $u_i^{-1}(V)$ is open in Z because u_i is continuous, and the set $\Delta \times \{t\}$ is open by construction of the topology on Z, since $\pi^{-1}(\Delta \times \{t\}) = \Delta \times \{t\}$. Hence, $\left(u_i^t\right)^{-1}(V)$ is open in Z. Since V was arbitrary, u_i^t is continuous.

To see (ii), let $\varepsilon > 0$. Since $\infty \in Z$, $u_i(\infty) \in \mathbb{R}$ and the interval of real numbers $(u_i(\infty) - \varepsilon, u_i(\infty) + \varepsilon)$ is open in $\mathbb{R}$. By continuity of u_i, $u_i^{-1}\left((u_i(\infty) - \varepsilon, u_i(\infty) + \varepsilon)\right)$ is an open set of Z containing

∞. By Lemma A.2, there exists $T \geq 1$ such that

$$\Delta \times \{T, T+1, \ldots, \infty\} \subseteq \pi^{-1}\left(u_i^{-1}\left((u_i(\infty) - \varepsilon, u_i(\infty) + \varepsilon)\right)\right).$$

Hence, for all $t \geq T$ and all $a \in \Delta$, $(a, t) = \pi^{-1}(a, t)$ fulfills $u_i(a, t) \in (u_i(\infty) - \varepsilon, u_i(\infty) + \varepsilon)$, establishing (ii). ■

The second part of well-behavedness demands that the non-terminal nodes at each slice $Y_{s,k}$ are partitioned into finitely many cells, each of which consists of decision points of a single player, such that the set of plays passing through each cell is closed relative to the plays passing through the slice (the set $W(Y_{s,k})$ of plays).

Begin with slice $Y_{s,1}$ for some $s \geq 1$. In fact, for $s = 1$ the condition is trivially true, because at the root a single player moves. For $s > 1$ and a given player $i \in I$ who may make an offer at slice $(s, 1)$, choose a play $\bar{w} = \left(\bar{a}^t, \bar{b}^t\right)_{t=1}^T \in W\left(Y_{s,1}\right)$ (where T may be ∞), which does *not* pass through any decision point of i at $Y_{s,1}$. Then the set $U = \times_{t=1}^{s-1}\left(\Delta \times \{\bar{b}^t\}\right) \times \Omega^\infty$ is a basic open set in Ω^∞, hence open in $\bar{W}$ and, by the identification of plays with bargaining sequences, open in W. Clearly, U contains $\bar{w}$. Further, no play in U can pass through a decision point of i at $Y_{s,1}$, because if it would, last round's voting profile $\bar{b}^{s-1}$ would have given her the move also along $\bar{w}$. Therefore, $U \cap W\left(Y_{s,1}\right)$ is relatively open in $W\left(Y_{s,1}\right)$, contains $\bar{w}$, and is contained in the complement of i's decision points at $Y_{s,1}$. Since $\bar{w} \in W\left(Y_{s,1}\right)$ was arbitrary, i's cell at $Y_{s,1}$ is closed as a set of plays, as desired.

Next, for $s \geq 1$ and $k > 1$ at slice $Y_{s,k}$ and a given player $i \in I$, who has a move at $Y_{s,k}$, choose a play $\bar{w} = \left(\bar{a}^t, \bar{b}^t\right)_{t=1}^T \in W(Y_{s,k})$ (where again T may be ∞), which does *not* pass through any decision point of i at $Y_{s,k}$. Then the set $U = \times_{t=1}^{s}\left(\Delta \times \{\bar{b}^t\}\right) \times \Omega^\infty$ is a basic open set in Ω^∞, hence open in $\bar{W}$ and, by the identification of plays and bargaining sequences, open in W. Since U contains $\bar{w}$ but i does not move at $Y_{s,k}$ along any member of U, the relatively open set $U \cap W(Y_{s,k})$ is contained in the complement of i's decision points at $Y_{s,k}$ and forms a relative neighborhood of $\bar{w}$. Therefore, i's cell at $Y_{s,k}$ is closed.

Putting the facts above together completes the verification that the first necessary condition holds. That is, the bargaining game is indeed well-behaved.

B.3 Closed Nodes

The second condition is that all nodes, viewed as sets of plays, are closed sets in the topology on the set of plays W.

Proposition B.2. *All nodes of the bargaining tree are closed sets in the topology on plays.*

Proof. Since the space of plays is Hausdorff, all singleton sets (which are precisely the terminal nodes) are closed. Consider the nodes of a proposer i in an arbitrary slice s, that is, the nodes in the set X_1^s where player i makes the proposal. Let $x = x_i^s\left(\left(\bar{a}^t, \bar{b}^t\right)_{t=1}^{s-1}\right)$ be one such node. It will be shown that $W \setminus x$ is open. To see this, let $w = \left(a^t, b^t\right)_{t=1}^{\infty} \in W \setminus x$ (viewed as a bargaining sequence). There exists some $\hat{t} \in \{1, \ldots, s-1\}$ such that either $a^{\hat{t}} \neq \bar{a}^{\hat{t}}$ or $b^{\hat{t}} \neq \bar{b}^{\hat{t}}$. Suppose the first holds (the proof for the second case is analogous). If $a^{\hat{t}} \neq *$, since the space of allocations Δ is Hausdorff, there exists an open set V of Δ such that $a^{\hat{t}} \in V$ but $\bar{a}^{\hat{t}} \notin V$. If $a^{\hat{t}} = *$, let $V = \{*\}$. The set $U = (\Omega)^{\hat{t}-1} \times (V \times B) \times (\Omega)^{\infty}$ is an open basic set of Ω^{∞}, hence $U \cap \bar{W}$ is open in $\bar{W}$ (and W). Obviously, $w \in U \cap \bar{W} \subseteq W \setminus x$. The proof for responders' nodes at a given slice is analogous. ■

B.4 Open Predecessors

The last condition, called the "open predecessors condition," requires that the predecessor mapping is open. Specifically, the condition is that, for every slice $Y_{s,k}$ and every set of nodes $V \subseteq Y_{s,k}$, if the set of plays $W(V)$ is open in the relative topology on $W(Y_{s,k})$, then the set of plays passing through the predecessors of nodes in V, $W(p(V))$, is open in the relative topology of the *previous* slice, that is, on the set $W(Y_{s,k-1})$ if $k \geq 2$ or on the set $W(Y_{s-1,N})$ if $k = 1$.

Proposition B.3. *The bargaining tree fulfills the open predecessors condition.*

Proof. Let $V \subseteq Y_{s,k}$ be such that the set of plays $W(V)$ is open in the relative topology on $W(Y_{s,k})$. There are three cases to be distinguished.
Case 1: From $Y_{s,k}$ to $Y_{s,k-1}$ $(k \geq 3)$.

In this case $W(Y_{s,k}) = W(Y_{s,k-1})$. Let $w = \left(a^t, b^t\right)_{t=1}^{\infty}$ be a play (viewed as a bargaining sequence) with $w \in W(p(V)) \subseteq W(Y_{s,k-1})$. There exists a node $x \in V$ such that $w \in p(x) = x_{i,k-1}^s\left(\left(a^t, b^t\right)_{t=1}^{s-1}, a^s, (b_\ell^s)_{\ell=2}^{k-1}\right)$, where i is the player active at $p(x)$. In fact, we then have $x = x_{j,k}^s\left(\left(a^t, b^t\right)_{t=1}^{s-1}, a^s, (b_\ell^s)_{\ell=2}^{k}\right)$ for some player j. Let $\bar{w} \in x \subseteq W(V)$. Then the first coordinates of $\bar{w}$ agree with $\left(\left(a^t, b^t\right)_{t=1}^{s-1}, a^s, (b_\ell^s)_{\ell=2}^{k}\right)$. Since $W(V)$ is open, it follows that there are open sets of Ω, $U^t \times C^t$, $t = 1, \ldots, s$, such that

$$\bar{w} \in \left[\times_{t=1}^{s-1}\left(U^t \times C^t\right) \times \left(U^s \times \left\{\bar{b} \in B \;\middle|\; \bar{b}_\ell^s = b_\ell^s \;\forall \ell = 2, \ldots, k\right\}\right) \times \Omega^{\infty}\right] \cap \bar{W}$$

and the latter set is contained in $W(V)$. Note that both w and $\bar{w}$ agree with $\left(\left(a^t, b^t\right)_{t=1}^{s-1}, a^s, (b_\ell^s)_{\ell=2}^{k-1}\right)$. Consider the open basic set of Ω^{∞} given by

$$U = \left[\times_{t=1}^{s-1}\left(U^t \times C^t\right) \times \left(U^s \times \left\{\bar{b} \in B \;\middle|\; \bar{b}_\ell^s = b_\ell^s \;\forall \ell = 2, \ldots, k-1\right\}\right) \times (\Omega)^{\infty}\right]$$

Then $U \cap \bar{W}$ is open in $\bar{W}$, hence in W, with $w \in U \cap \bar{W} \subseteq W(p(V))$, demonstrating that the latter set is open.
Case 2: From $Y_{s,2}$ to $Y_{s,1}$ $(k = 2)$.

In this case $W(Y_{s,2}) = W(Y_{s,1})$. Let $w = \left(a^t, b^t\right)_{t=1}^{\infty}$ be a play (viewed as a bargaining sequence) with $w \in W(p(V)) \subseteq W(Y_{s,1})$. There exists a node $x \in V$ such that $w \in p(x) = x_{i,1}^s\left(\left(a^t, b^t\right)_{t=1}^{s-1}\right)$, where i is the proposer active at $p(x)$. Actually, we then have that $x = x_{j,2}^s\left(\left(a^t, b^t\right)_{t=1}^{s-1}, a^s\right)$, where j is the first responder after the node x.

Let $\bar{w} \in x \subseteq W(V)$. It follows that the first coordinates of $\bar{w}$ agree with $\left(\left(a^t, b^t\right)_{t=1}^{s-1}, a^s\right)$. Since $W(V)$ is open, there are open sets of Ω, $U^t \times C^t$, $t = 1, \ldots, s$, such that

$$\bar{w} \in \left[\times_{t=1}^{s-1}\left(U^t \times C^t\right) \times (U^s \times B) \times (\Omega)^{\infty}\right] \cap \bar{W} \subseteq W(V).$$

Both w and $\bar{w}$ agree with $\left(\left(a^t, b^t\right)_{t=1}^{s-1}\right)$. Consider the open basic set of Ω^{∞} given by $U = \left[\times_{t=1}^{s-1}\left(U^t \times C^t\right) \times (\Omega)^{\infty}\right]$. It follows that $U \cap \bar{W}$ is open in $\bar{W}$, hence in W, with $w \in U \cap \bar{W} \subseteq W(p(V))$, proving that the latter set is open.

Case 3: From $Y_{s,1}$ to $Y_{s-1,N}$ $(k = 1)$.

In this case, $W(Y_{s,1}) = W(Y_{s-1,N}) \cup E^{s-1}$. Let $w \in W(p(V))$. There exists $x \in V$ such that $w \in p(x)$. If x is a terminal node, then there exists a play $\bar{w}$ such that $x = \{\bar{w}\}$ with $\bar{w} \in W^{s-1}$. If x is not terminal, choose any $\bar{w} \in x$. Note that w and $\bar{w}$ coincide up to bargaining round $s-1$.

Since $W(V)$ is open, there are open sets of Ω, $U^t \times C^t$, $t = 1, \ldots, s$, such that

$$\bar{w} \in \left[\times_{t=1}^{s}\left(U^t \times C^t\right) \times (\Omega)^{\infty}\right] \cap \bar{W} \subseteq W(V).$$

Consider the open basic set of Ω^{∞} given by $U = \left[\times_{t=1}^{s-1}\left(U^t \times C^t\right) \times (\Omega)^{\infty}\right]$. It follows that $U \cap \bar{W}$ is open in $\bar{W}$, hence in W, with $w \in U \cap \bar{W} \subseteq W(p(V))$, demonstrating that the latter set is open. ■

This completes the verification that the bargaining games satisfy the hypotheses of the existence theorem in Alós-Ferrer and Ritzberger (2016b, Theorem 1; 2016a, Theorem 7.4).

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