Elsevier Editorial System(tm) for Journal of Mathematical Economics

Manuscript Draft

Manuscript Number: JME-D-16-00246R2

Title: The Consumption-Investment Decision of a Prospect Theory Household: A Two-Period Model

Article Type: Research Paper

Keywords: prospect theory, loss aversion, consumption-savings decision, portfolio allocation, happiness

Corresponding Author: Dr. Panagiotis Tsigaris,

Corresponding Author's Institution: Thompson Rivers University

First Author: Jaroslava Hlouskova, Ph.D.

Order of Authors: Jaroslava Hlouskova, Ph.D.; Ines Fortin, Ph.D.; Panagiotis Tsigaris

Abstract: This study extends the literature on portfolio choice under prospect theory preferences by introducing a two-period life cycle model, where the sufficiently loss averse household decides on optimal consumption and investment in a portfolio with one risk-free and one risky asset. The optimal solution depends primarily on whether the household's present value of the consumption reference levels is below, equal to, or above the present value of its endowment income. Reference levels below the endowment income are associated with the self-enhancement motive. In this case, the household avoids relative losses in consumption in any present or future state of nature (good or bad). As a result the degree of loss aversion does not directly affect optimal consumption and risk taking activity. Reference levels equal to the endowment income are associated with the belonging motive. An example would be a household comparing to others that belong to the same social class. In this case the household's optimal consumption is the reference consumption and the household will not invest in the risky asset. Finally, reference levels above the endowment income are associated with the self-improvement motive (or high aspirations). For such high reference levels, households cannot avoid experiencing a relative loss in consumption, either now or in the future. As a result, loss aversion directly affects consumption and risky investment.

*Manuscript
Click here to view linked References

The Consumption-Investment Decision of a Prospect Theory Household: A Two-Period Model*†

Jaroslava Hlouskova
Financial Markets and Econometrics, Institute for Advanced Studies, Vienna, Austria
Department of Economics, Thompson Rivers University, Kamloops, BC, Canada

Ines Fortin
Financial Markets and Econometrics, Institute for Advanced Studies, Vienna, Austria

Panagiotis Tsigaris
Department of Economics, Thompson Rivers University, Kamloops, BC, Canada

*Jaroslava Hlouskova gratefully acknowledges financial support from the Austrian Science Fund FWF (project number V 438-N32) and Ines Fortin gratefully acknowledges financial support from Oesterreichische Nationalbank (Anniversary Fund, Grant No. 15992).

†The authors would like to thank two anonymous referees for helpful comments and suggestions.

Abstract

This study extends the literature on portfolio choice under prospect theory preferences by introducing a two-period life cycle model, where the sufficiently loss averse household decides on optimal consumption and investment in a portfolio with one risk-free and one risky asset. The optimal solution depends primarily on whether the household's present value of the consumption reference levels is below, equal to, or above the present value of its endowment income. Reference levels below the endowment income are associated with the self-enhancement motive. In this case, the household avoids relative losses in consumption in any present or future state of nature (good or bad). As a result the degree of loss aversion does not directly affect optimal consumption and risk taking activity. Reference levels equal to the endowment income are associated with the belonging motive. An example would be a household comparing to others that belong to the same social class. In this case the household's optimal consumption is the reference consumption and the household will not invest in the risky asset. Finally, reference levels above the endowment income are associated with the self-improvement motive (or high aspirations). For such high reference levels, households cannot avoid experiencing a relative loss in consumption, either now or in the future. As a result, loss aversion directly affects consumption and risky investment.

Keywords: prospect theory, loss aversion, consumption-savings decision, portfolio allocation, happiness

JEL classification: G02, G11, E20

1 Introduction

In this paper we explore how behavioral traits such as loss aversion and reference levels influence the household's consumption and savings decision, based on prospect theory preferences. Households make decisions on how much to consume today and how much to save for the future when, e.g., they retire. Savings are the means of transferring consumption into the future and of having income for retirement or the means of transferring future income to the present in order to be able to afford more consumption today. Moreover, households do not only decide how much to save but also how to allocate their savings into different types of assets. These decisions are made knowing that the future is risky and uncertain.

Traditionally, the expected utility (EUT) framework has been used to model such behavior.[1] This research will deviate from the EUT model and will explore a different type of preferences. In particular, we will assume prospect theory preferences that were introduced and developed by Kahneman and Tversky (1979) and Tversky and Kahneman (1992) and that take into account also psychological aspects of households' behavior. Prospect theory can be characterized by the following properties. Decision makers under risk evaluate gains and losses with respect to some reference level, rather than evaluating absolute values (of their wealth or consumption). Households exhibit loss aversion, which means that they are more sensitive to losses than to gains of the same magnitude. In addition, households display risk aversion in the domain of gains but show risk appetite in the domain of losses, which is described by an S-shaped value function that is concave in the domain of gains and convex in the domain of losses.[2] For a comprehensive overview on prospect theory see, e.g., Barberis (2013) and DellaVigna (2009).

We address a number of issues on the savings behavior under prospect theory preferences that have only partially been explored in the literature before. How do households decide on consumption and portfolio decisions when faced with prospect theory type of preferences? Do households have to be sufficiently loss averse to yield reasonable optimal solutions for consumption and investment decisions? Does loss aversion affect consumption and portfolio decisions? Do reference levels affect the households' consumption, savings and portfolio choice to transfer consumption into the future? If yes, how? Do households "follow the Joneses", (i.e., compare themselves to and follow neighbors or associates) when making consumption and savings decisions?

There are many different types of reference points that can be considered in exploring the savings behavior and portfolio choice. The first reference levels that were used are subsistence levels of consumption (see, e.g., Stone, 1954 and Geary, 1951). Under such preferences, households get utility from consumption in excess of a subsistence level. Individuals have to consume a certain minimal level irrespective of its price or the person's income. The savings and portfolio choice with subsistence consumption has recently been explored by Achury et al. (2012). They use the

[1] For original work in this area see Sandmo (1968, 1969) and Merton (1969, 1971).

[2] Another property, not included in this study, is that the probabilities assigned to the utility of the outcomes are not objective but subjective (so-called decision weights), as people seem to underestimate large probabilities and overestimate small probabilities.

Stone-Geary expected utility model to explain a number of observed empirical facts such as why the rich have a higher savings rate, higher holdings of risky assets relative to personal wealth and a higher consumption volatility than the poor.

Another commonly used reference dependent preference model is habit persistence. The habit persistence model assumes that households derive utility from consumption relative to a reference level that depends on past consumption levels. Habit persistence models have been used in many applications in macroeconomics and finance and can to some extent explain, for instance, the equity premium puzzle and the behavior of asset returns (Abel, 1990; Constantinides, 1990; Campbell and Cochrane, 1999), excess smoothness in consumption expenditures (Lettau and Uhlig, 2000) and business cycles characteristics (Boldrin et al., 2001; Christiano et al., 2005).

Many of the applications of prospect theory in portfolio selection assume that the reference level is the investor's return from investing all initial wealth into the risk-free asset (Barberis and Huang, 2001; Gomes, 2005; Barberis and Xiong, 2009; Bernard and Ghossoub, 2010; He and Zhou, 2011). Barberis et al. (2001), Berkelaar et al. (2004), Fortin and Hlouskova (2011, 2015) and Gomes (2005) use also a dynamic updating rule for the reference point. Future utility, in one-period models, is derived from the excess return of the risky asset holdings. One of the major findings of this literature is that investors may not invest in risky assets even if its expected return is higher than the risk-free rate.

Some work has been devoted to exploring the consequences of reference dependent preferences for inter-temporal two-period habit-persistence consumption decisions, when future income is uncertain and when households are loss averse, see, e.g., Bowman et al., 1999. They find that a household will resist reducing its consumption level when there is bad news about future income. Furthermore, the resistance to reducing consumption with bad news is greater than the resistance to increasing consumption in response to good news.

Koszegi and Rabin (2006) assume rational expectations in the formation of reference levels. Assuming agents are more affected by news about current consumption than by news about future consumption, they find that people would intend to overconsume today relative to their optimal plans. They would increase consumption right away when good news regarding wealth arrives, but would postpone decreasing consumption when receiving bad news. Thus, higher wealth reduces the painful impact of bad news, and as a result people save more for precaution.

Van Bilsen et al. (2014) investigate optimal consumption and portfolio choice paths of a loss averse household but with an endogenous reference level. They find that households strive to protect themselves against consumption losses in order to avoid bad states of nature. They attribute this behavior to loss aversion. Due to the dynamic nature of their set-up they can investigate the effect of financial shocks and find that consumption choices adjust only slowly to financial shocks and that welfare losses are substantial with suboptimal consumption and portfolio selections. Our research complements the work by Van Bilsen et al. (2014) in that it provides additional insights as discussed below.

We provide closed-form solutions in an inter-temporal two-period consumption and portfolio decision model of a prospect theory household, where uncertainty arises due to the risky asset. The asset's return is assumed to follow a Bernoulli distribution, i.e., there are two states of nature realizing with certain probabilities, and the household's consumption reference levels are assumed to be exogenous. These reference levels are compared with the household's consumption levels and the household derives its utility from the difference between its consumption and the reference level. Consuming above the reference level means that the household incurs relative gains while consuming below the reference level means that it incurs relative losses. It turns out that the consumption reference levels (in both periods) as well as the loss aversion parameter are crucial in the analysis. The solution depends on whether the consumption reference levels are below, equal to, or above the household's income (in present value terms).

We analyze these three different types of households as well as the impact of changes in the behavioral traits (i.e., changes in loss aversion and the reference levels) on consumption and investment. Households can be characterized by comparing their economic status to the status of others (as suggested by Falk and Knell, 2004) who are better off, worse off or have a similar economic status. Upward and downward social comparisons are commonly done by humans.[3] Related drivers of the human behavior are self-enhancement and self-improvement motives, see Banaji and Prentice (1994).[4]

The self-enhancement motive occurs when people want to make themselves feel better and maintain their self-esteem.[5] Households with such a motivation have the desire to create and maintain positive feelings about themselves (see Taylor and Brown, 1988; Taylor et al., 2003). In our setting this means that they feel better by having low references levels, where the low reference levels may be interpreted as the wealth of people with a lower economic status (the people they compare to).[6] Households with reference levels below their wealth may thus be seen as households governed by self-enhancement.

On the other hand, households may be governed by the self-improvement motive (see Wood, 1989) in comparing themselves to others who are more successful or better off. Their high reference levels may then be interpreted as the wealth of people with a higher economic status. These households want to improve their economic status and sometimes make risky choices in order to catch up or exceed their targets. In our setting households with reference levels above their wealth can be seen as households governed by the self-improvement motive or ones with high aspirations.

The main results for a sufficiently loss averse household are the following. If the household's references levels, in the present value, are *below* the present value of its endowment income (due to the self-enhancement motive) then the household behaves in such a way that it avoids relative losses in any present or future state of nature (good or bad). So optimal consumption is always above the reference level. This implies that the degree of loss aversion does not directly affect optimal consumption and risk taking activity. In addition, the household always invests in the risky asset.

[3]The social comparison theory is originated by Festinger (1954). For a literature review see Clark et al. (2008).

[4]Gaertner et al. (2012) provide cross evidence that both motives are present in many cultures.

[5]Sedikides and Gregg (2008) provide a comprehensive literature review on self-enhancement.

[6]Wealth in our model is the present value of endowment income.

If, on the other hand, the household's references levels, in present value, are *equal* to the present value of its endowment income (as with households whose reference levels are equal to others with the same economic status) then the optimal consumption is equal to the reference consumption in both periods. In addition, the household does not invest in the risky asset in this case (even if its expected return is greater than the risk-free rate). Finally, if the household's reference levels, in present value, are *above* the present value of its endowment income (due to self-improvement and high aspirations motives) then it cannot avoid relative losses at all times. Either in the first or in the second period (good or bad state of nature) the household has to accept consuming below the reference level. This implies that loss aversion directly affects consumption and investment in the risky asset. Investment in the risky asset is again positive in this case. Independent of whether consumption reference levels are high (due to the self-improvement motive and high aspirations) or low (due to the self-enhancement motive and low aspirations), increasing reference levels will result in less happiness. Thus, less ambitious households are happier than more ambitious households. Finally, households are also less happy when the degree of loss aversion is increased for households driven by the self-improvement motive, while it remains unaffected for households governed by the self-enhancement motive or for households that want to belong to the same social class as is their status.

The paper proceeds as follows. In the next section we describe the set-up of the model. In Section 3 we investigate the case when the household's reference levels are such that their present value is below the present value of its endowment income (due to the self-enhancement motive). Section 4 explores the special case when the present value of consumption reference levels is exactly equal to the present value of the household's endowment income (i.e., the household compares itself to households belonging to the same social class). Section 5 examines the case when the household's reference levels are such that the present value is above the present value of its endowment income (due to the self-improvement motive and/or high aspirations). Section 6 provides an example where the same households compare themselves to poorer and richer neighbors. In Section 7 we summarize the main research results and present a brief discussion on what would happen if households were not sufficiently loss averse. Finally, some concluding remarks and future extensions are suggested.

2 Problem set-up

We consider a household that lives for two periods. In the first period it receives a non-stochastic exogenous income (labor income, endowment income), $Y_1 > 0$, which it can allocate to current consumption, C_1, risk-free investment, m, and risky investment, α, where the sum of the risky and risk-free investment are savings S. Thus, in the first period

$$Y_1 = C_1 + m + \alpha = C_1 + S \tag{1}$$

We consider two assets, a risk-free asset with a net of the dollar return $r_f > 0$ and a risky asset with stochastic net of the dollar return r that yields r_g in the good state of nature, which occurs with probability p, and r_b in the bad state of nature, which occurs with probability $1-p$. We assume that $-1 < r_b < r_f < r_g$, $0 < p < 1$, and $\mathbb{E}(r) = p\,r_g + (1-p)r_b > r_f$. Thus, in the second period the household consumes

$$C_{2i} = Y_2 + (1+r_f)m + (1+r_i)\alpha$$

$i \in \{b, g\}$, where $Y_2 \geq 0$ is the non-stochastic income of the household in the second period, which can also be thought of as an exogenous government pension income. There are no liquidity constraints that prevent the household from consuming any exogenous future income in the first period, but consumption is not allowed to be negative in either period, so that it can only partially borrow against uncertain future income. This means that risk-free savings, m, can be negative to a certain extent. The value $(1+r_f)m + (1+r_i)\alpha$ represents the wealth acquired from capital investment, $i \in \{b, g\}$. So, in the second period the household consumes C_{2b} in the bad state of nature and C_{2g} in the good state of nature. Based on this and (1) the consumption in the second period is

$$C_{2i} = Y_2 + (1+r_f)(Y_1 - C_1) + (r_i - r_f)\alpha \tag{2}$$

The household's preferences are described by the following reference based utility function

$$U(C_1, \alpha) = V(C_1 - \bar{C}_1) + \delta\, V(C_2 - \bar{C}_2) \tag{3}$$

where $\bar{C}_1$ and $\bar{C}_2$ are exogenous consumption reference or comparison levels, such that $0 \leq \bar{C}_1 < Y_1 + \frac{Y_2}{1+r_f}$ and $0 \leq \bar{C}_2 < (1+r_f)Y_1 + Y_2$, i.e., $Y_1 + \frac{Y_2}{1+r_f} > \max\left\{\bar{C}_1, \frac{\bar{C}_2}{1+r_f}\right\}$, δ is the discount factor, $0 < \delta < 1$, and $V(\cdot)$ is a prospect theory (S-shaped) value function defined as

$$V(C_i - \bar{C}_i) = \left\{ \begin{array}{cc} \frac{(C_i - \bar{C}_i)^{1-\gamma}}{1-\gamma}, & C_i \geq \bar{C}_i \\ \\ -\lambda\frac{(\bar{C}_i - C_i)^{1-\gamma}}{1-\gamma}, & C_i < \bar{C}_i \end{array} \right\} \tag{4}$$

for $i = 1, 2$. Parameter $\lambda > 1$ is the loss aversion parameter and $\gamma \in (0,1)$ is the parameter determining the curvature of the utility function. If consumption is above the reference level we talk about (relative) gains, if it is below the reference level we talk about (relative) losses. The utility has a kink at the consumption reference level and it is steeper for losses than for gains, i.e., a decrease in consumption is more severely penalized in the domain of losses than in the domain of gains. Finally, the utility function is concave above the reference point and convex below it. The household is thus risk averse in the domain of gains (i.e., above the consumption reference level) and risk seeking in the domain of losses (i.e., below the consumption reference level), see Figure 1.

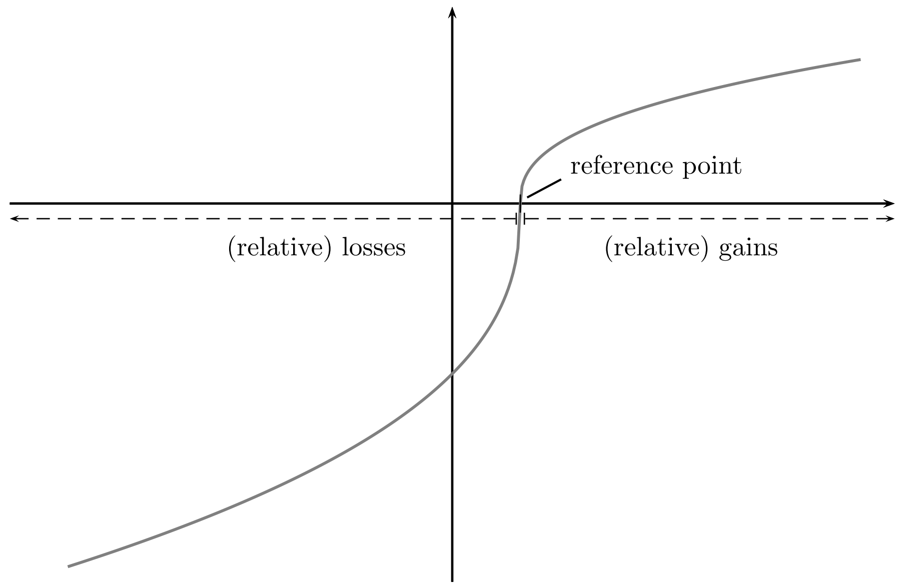

Figure 1: Loss aversion (S-shaped) utility

The household maximizes the following expected utility as given by (3) and (4)

$$\text{Max}_{(\mathrm{C}_1,\alpha)}: \quad \mathbb{E}(U(C_1,\alpha)) = V(C_1-\bar{C}_1) + \delta\,\mathbb{E}V(C_2-\bar{C}_2)$$

$$\text{such that}: \quad C_1 \geq 0,\ C_{2b} \geq 0,\ C_{2g} \geq 0 \text{ and } \alpha \geq 0$$

Based on this and (2) the household's maximization problem can be formulated as follows

$$\begin{aligned} \text{Max}_{(\mathrm{C}_1,\alpha)}: \quad \mathbb{E}(U(C_1,\alpha)) \quad &= \quad V(C_1-\bar{C}_1) + \delta\,\mathbb{E}V\left((1+r_f)(Y_1-C_1)+(r_i-r_f)\alpha+Y_2-\bar{C}_2\right) \\ \text{such that}: \quad 0 \leq \quad C_1 \quad &\leq Y_1 + \tfrac{Y_2}{1+r_f} - \tfrac{r_f-r_b}{1+r_f}\alpha, \\ 0 \leq \quad \alpha \quad &\leq \tfrac{(1+r_f)Y_1+Y_2}{r_f-r_b} \end{aligned} \tag{5}$$

Note that the upper bound on C_1 follows from $C_{2b} \geq 0$ and the upper bound on α follows from the imposition of the upper bound on C_1 being non-negative, i.e. $Y_1 + \frac{Y_2}{1+r_f} - \frac{r_f-r_b}{1+r_f}\alpha \geq 0$.[7] The condition on α means that short sales are not allowed.[8]

[7] Imposing positive lower bounds on consumption in both periods (i.e., on C_1, C_{2b} and C_{2g}), so that the household does not "starve", would not substantially change our results. In occurrences when the optimal consumption hits zero now, it would hit the lower bound then. Thus, the behavioral implications of our findings related to the sensitivity analysis and thus comparisons to others would not change.

[8] Fortin, Hlouskova and Tsigaris (2015) show that the assumption $p > \max\left\{\frac{r_f-r_b}{r_g-r_b}, \frac{(r_f-r_b)^{1-\gamma}}{(r_f-r_b)^{1-\gamma}+(r_g-r_f)^{1-\gamma}}\right\}$ rules out short-selling if there is no non-negativity restriction on α. As $\mathbb{E}(r) > r_f$ is equivalent to $p > \frac{r_f-r_b}{r_g-r_b}$ then $\mathbb{E}(r) > r_f$ is not sufficient to rule out short sales (except in Section 3).

Before proceeding further, we introduce the following notation

$$\begin{aligned}
\Omega &= (1+r_f)(Y_1-\bar{C}_1)+Y_2-\bar{C}_2 \\
&= (1+r_f)\left[\left(Y_1+\frac{Y_2}{1+r_f}\right)-\left(\bar{C}_1+\frac{\bar{C}_2}{1+r_f}\right)\right] \qquad (6)
\end{aligned}$$

$$K_\gamma = \frac{(1-p)(r_f-r_b)^{1-\gamma}}{p(r_g-r_f)^{1-\gamma}} \qquad (7)$$

$$M = \left(\delta(1+r_f)\,p\,\frac{r_g-r_b}{r_f-r_b}\right)^{\frac{1}{\gamma}}\frac{r_f-r_b+K_0^{\frac{1}{\gamma}}(r_g-r_f)}{r_g-r_b} \qquad (8)$$

$$k = \left[\delta(1+r_f)(1-p)\left(\frac{r_g-r_b}{r_g-r_f}\right)^{1-\gamma}\right]^{\frac{1}{\gamma}} \qquad (9)$$

$$k_2 = \left[\delta(1+r_f)p\left(\frac{r_g-r_b}{r_f-r_b}\right)^{1-\gamma}\right]^{\frac{1}{\gamma}} \qquad (10)$$

where Ω denotes the difference between the present value of total endowment income and the present value of the consumption reference levels multiplied by the gross return of a dollar investment in the risk-free rate.[9] The value $1/K_\gamma$ presents the attractiveness of investing in the risky asset and coincides with thresholds used in He and Zhou (2011), while M was introduced in order to simplify the conditions on the minimum degree of loss aversion in the following proposition. Finally, values k and k_2 were introduced to ease the exposition of the results. In addition notice that

$$K_0 = \frac{(1-p)(r_f-r_b)}{p\,(r_g-r_f)} = K_\gamma\left(\frac{r_f-r_b}{r_g-r_f}\right)^\gamma < 1$$

as $\mathbb{E}(r) > r_f$.

In the following analysis we consider three fundamentally different cases, which give rise to profoundly different types of optimal consumption behavior. These cases are characterized by how the household's consumption reference levels are related to its endowment income. Namely, whether the difference between the present value of total endowment income and the present value of the sum of the consumption reference levels is positive ($\Omega > 0$), zero ($\Omega = 0$), or negative ($\Omega < 0$). The case when Ω is positive is characteristic for households which are governed by the self-enhancement motivation (or low aspirations). The case when Ω is negative is typical for households governed by the self-improvement motive (i.e., those that have high aspirations). The former case relates to downward comparisons, while the latter relates to upward comparisons. The case when Ω is zero is a special case, where the present value of the household's total endowment income is exactly equal to the present value of its consumption reference levels. Here the household can be seen as comparing itself to other households that are in the same social class.

In the formal analysis we split the household's consumption decision problem (5) into eight

[9] Note that future income and consumption reference levels are discounted at the risk-free rate.

separate problems, (P1)–(P8), which differ in their respective domains, i.e., in their sets of feasible solutions. These domains are specified by whether first and second period (in the good and bad state of nature) consumption levels are above or below the respective reference levels. This yields a total of eight combinations, see Appendix A. Households with a positive Ω will operate on certain domains which differ from the domains on which households with a negative Ω operate. In all cases households need to be sufficiently loss averse.

3 Low reference levels ($\Omega > 0$)

We now consider the case when the household's consumption reference levels are such that their present value is below the present value of endowment income, i.e., when $\Omega > 0$. This can represent a household who is driven by the self-enhancement motive, i.e., the need to feel good and maintain self-esteem. Following this motive implies a "downward comparison": the household compares its own endowment income to the (smaller) wealth of others with a lower economic status, and this smaller wealth can be taken to be the household's reference level.

To proceed with the analysis let us introduce the following notation

$$\begin{aligned}\lambda^{\Omega\geq 0} &= \frac{\Omega^{1-\gamma}}{\delta(1-p)(1+r_f)\bar{C}_2^{1-\gamma}}\left[(1+r_f+k_2)^{\gamma}\left(1+\frac{r_g-r_f}{r_g-r_b}\frac{\bar{C}_2}{\Omega}\right)^{1-\gamma}-(1+r_f+M)^{\gamma}\right] \\ &= \left[\frac{1+r_f}{k}+\left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right]^{\gamma}\left(\frac{\Omega}{\bar{C}_2}\frac{r_g-r_b}{r_g-r_f}+1\right)^{1-\gamma} \\ &\quad -\left[\frac{1+r_f}{k}+\left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}+1\right]^{\gamma}\left(\frac{\Omega}{\bar{C}_2}\frac{r_g-r_b}{r_g-r_f}\right)^{1-\gamma} \qquad (11)\end{aligned}$$

in order to simplify the conditions on the minimum degree of loss aversion in the following proposition. We can now formulate the main result for the case when $\Omega > 0$.

Proposition 1 *Let $\Omega > 0$ and $\lambda > \max\left\{\frac{1}{K_\gamma}, \lambda^{\Omega\geq 0}, \left(\frac{M}{1+r_f}\right)^{\gamma}\right\}$. Then problem (5) obtains a unique maximum at (C_1^*, α^*) where*

$$\begin{aligned}C_1^* &= \bar{C}_1+\frac{\Omega}{1+r_f+M} \\ &= \bar{C}_1+\frac{1+r_f}{1+r_f+M}\left[\left(Y_1+\frac{Y_2}{1+r_f}\right)-\left(\bar{C}_1+\frac{\bar{C}_2}{1+r_f}\right)\right] > \bar{C}_1 \qquad (12)\end{aligned}$$

$$\alpha^* = \frac{\left(1-K_0^{\frac{1}{\gamma}}\right)M}{r_f-r_b+K_0^{\frac{1}{\gamma}}(r_g-r_f)}(C_1^*-\bar{C}_1) > 0 \qquad (13)$$

Proof. It follows directly from Lemma 1 in Appendix B. ∎

When Ω is positive then reference levels are relatively low. This should make it easier for a household to reach and exceed its consumption comparison levels than when household reference levels are high as it is the case when the household compares with more successful households. If, in addition, the utility is such that consumption below the reference levels is sufficiently penalized, i.e., the loss aversion parameter is large enough, then we expect optimal consumption to exceed its reference levels. This is indeed what we observe: optimal consumption levels in both periods are strictly larger than their corresponding reference levels provided that the household is sufficiently loss averse, i.e., $C_1^* > \bar{C}_1$ and $C_{2g}^* \geq C_{2b}^* > \bar{C}_2$, where

$$C_{2g}^* = \bar{C}_2 + \frac{M\Omega}{(1+r_f+M)\left(1+K_\gamma^{\frac{1}{\gamma}}\right)} \frac{r_g - r_b}{r_f - r_b} \tag{14}$$

$$C_{2b}^* = \bar{C}_2 + \frac{M\Omega}{(1+r_f+M)\left(1+K_\gamma^{\frac{1}{\gamma}}\right)} \frac{r_g - r_b}{r_f - r_b} K_0^{\frac{1}{\gamma}} \tag{15}$$

Note that C_{2g}^* is the optimal consumption in the second period in the good state of nature while C_{2b}^* is the optimal consumption in the second period in the bad state of nature. Thus the optimal behavior is characterized by avoiding any relative losses to happen. In other words, the household adjusts its consumption to feel good and to maintain its self-esteem. We note further that optimal investment in the risky asset is strictly positive, i.e., $\alpha^* > 0$, which implies that the household takes on risk in the financial market. Total savings, however, can be either positive or negative.[10]

Although the existence of the solution does depend on the loss aversion parameter λ, the solution itself, (C_1^*, α^*), does not directly depend on it. The reason for this is that the household's optimal solution is reached in problem (P1), where the solution is found in the domain given by $C_1 \geq \bar{C}_1$, $C_{2b} \geq \bar{C}_2$ and $C_{2g} \geq \bar{C}_2$, i.e., both periods' consumption levels are above their consumption reference levels and thus the utility does not depend on the loss aversion parameter λ (see Appendix A). However, for this to happen the household needs to be sufficiently loss averse, namely $\lambda > \max\left\{\frac{1}{K_\gamma}, \lambda^{\Omega \geq 0}, \left(\frac{M}{1+r_f}\right)^\gamma\right\}$. Hence, if the household is sufficiently loss averse it will make choices that avoid any relative losses from occurring. The domains of all remaining problems (P2)–(P8) contain a relative loss (see Appendix A). A sufficiently loss averse household will never select solutions from these problems. This behavior is only possible, however, when the household does not have its aspirations (consumption reference levels) too high with respect to its income, thus, when Ω is positive.[11] Note, finally, that problem (P1) is known from the studies on habit formation, where the consumption habits are addictive and never fall below certain consumption targets (see, for example, Yu, 2015).

Table 1 summarizes the sensitivity results related to the solution presented in Proposition 1, so for a sufficiently loss averse household with low reference levels. In particular, we present the changes of the first and second period optimal consumption, of the optimal investment in the risky asset,

[10]The assumption required for $S^* > 0$ is $M(Y_1 - \bar{C}_1) > (Y_2 - \bar{C}_2)$ where M is defined by (8). This breaks down to simpler formulations in special cases.

[11]When Ω is negative, the household cannot totally avoid relative losses. It will have to face relative losses in the first or second period, or in the good or bad state of nature.

of the first and second period consumption gap, of optimal savings[12] and of happiness (first row) with respect to changes in the loss aversion parameter and the first and second period consumption reference levels (first column). By "consumption gap" we mean the distance between the optimal consumption and its reference level, $|C_i^* - \bar{C}_i|$, $i = 1, 2$, and we use "happiness" to denote the household's indirect utility (i.e., its value at the optimum). We also use "relative consumption" to denote the difference between optimal consumption and the reference level, which is closely related to the previously defined consumption gap. The gap is always positive while relative consumption can be either positive or negative. Both definitions coincide if optimal consumption is above the reference level.

	dC_1^*	dC_{2g}^*	dC_{2b}^*	$d\alpha^*$	$d(C_1^* - \bar{C}_1)$	$d(C_{2g}^* - \bar{C}_2)$	$d(C_{2b}^* - \bar{C}_2)$	dS^*	$d(\mathbb{E}(U(C_1^*, \alpha^*)))$
$d\lambda$	$= 0$	$= 0$	$= 0$	$= 0$	–	–	–	$= 0$	$= 0$
$d\bar{C}_1$	> 0	< 0	< 0	< 0	< 0	–	–	< 0	< 0
$d\bar{C}_2$	< 0	$\lesseqgtr 0$	> 0	< 0	–	< 0	< 0	> 0	< 0

Table 1: Sensitivity results when reference levels are low ($\Omega > 0$)

Since the solution does not explicitly depend on the loss aversion parameter, as discussed above, an exogenous increase in the loss aversion parameter, keeping everything else constant, does not change the solution or the utility at the solution (happiness).

An exogenous increase in the first period consumption reference level, keeping everything else constant, will increase the first period optimal consumption, decrease risky asset holdings and also decrease savings. As less income is transferred to the second period we would expect consumption to decrease in the second period. This is what we indeed observe: the second period consumption in either state of nature will fall with an increase in the first period consumption reference level. Even though optimal first period consumption increases in response to an exogenous increase in the first period consumption reference level, relative optimal consumption in the first period, i.e., the amount by which the reference level is exceeded, decreases. This means that the extent of the increase in the first period consumption reference level is not fully matched by the resulting increase in the first period optimal consumption. In summary, if the household increases the first period consumption reference level it will reduce the growth rate of consumption. Finally, an increase of the first period consumption reference level decreases the happiness level.

An increase in the second period consumption reference level, keeping everything else constant, will decrease the first period optimal consumption and risky asset holdings but increase both total savings and the risk-free investment. However, the increase of the risk-free investment is not sufficient to offset the reduction in risky assets in such a way that second period consumption will increase in both states of nature. Only in the bad state of nature optimal consumption in the second period will increase. In the good state of nature the response can be either an increase or a decrease of consumption. The reason why this is the case is the reduced risky investment

[12]The results for optimal savings follow from $\frac{dC_1^*}{d\bar{C}_1}$ and $Y_1 = C_1^* + S^*$.

– and hence the reduced potential to achieve high returns. Relative optimal consumption in the second period decreases if the second period consumption reference level is increased, which is in analogy to the situation when the first period consumption level is increased. The happiness level is negatively related to the second period consumption reference level, as it was to the first period consumption reference level. So if a household is "more ambitious" (indicated by an increase in the consumption reference level), in either the first or the second period, its happiness level will decrease.

The fact that consumption reference levels are exogenous gives us the opportunity to present some interesting examples. Consider, for instance, the case when the first period reference level is equal to the first period consumption level of other people that the household is associated with, i.e., the household *compares* itself to neighbors or peers. Then, if the first period consumption level of the other people increases, this household will respond by increasing its first period reference consumption level and because of this it will increase its first period optimal consumption level, reduce risk taking and reduce its future consumption in both states of nature. Hence, the household's behavior is one that "follows the Joneses" (i.e., the neighbors or peers the household wants to compare itself to).[13] In addition, the gap between the household's first period consumption and its consumption reference level narrows as the consumption level of the others increases. On the other hand, let the household's second period reference level be equal to the expected second period consumption level of other associates. Then, if the household expects the other people to have a higher expected future consumption, it will increase its second period consumption reference level, which will reduce its first period consumption, reduce risk taking but increase risk-free investment leading to an increase in consumption in the second period in the bad state of nature but not necessarily in the good state of nature. Here it is not clear whether the household follows the Joneses in the second period, even when its first period consumption is reduced to achieve an increase in future consumption like the household's associates. However, the consumption gap in the second period declines in both states of nature when the second period consumption reference increases, bringing closer to the reference the consumption levels in the second period.

We will refer to "following the Joneses" in the first period when the increase (or decrease) of the first period consumption of the Joneses (a reference household) impacts this household such that its first period consumption will change in the same way as the one of the Joneses. I.e., it will increase, if the first period consumption of the Joneses increases and vice versa. In our set-up this works through the household's increased or decreased (according to what the Joneses do) consumption reference levels. In addition, we will refer to the "following the Joneses" in the second period when the increase (or decrease) of the expected second period consumption of the Joneses impacts the second period expected consumption (of the household under considerations) such that it will change in the same way as that of the Joneses. So the idea of "following the Joneses" is to introduce external preferences into the household's behavior. Based on this terminology we can say that a sufficiently loss averse household with low aspirations follows the Joneses in the first period

[13] See Clark et al. (2008).

but not necessarily in the second period.

There is one particular example which deserves to be mentioned here, namely the traditional Merton type *expected utility* (EUT) with $\bar{C}_1 = \bar{C}_2 = 0$. Then, $V(C_i) \equiv \frac{C_i^{1-\gamma}}{1-\gamma}$ for $C_1, C_2 \geq 0$, which is a special case embedded in this behavioral study that does not include explicit reference levels. This means that the household does not compare its own wealth with anyone else's. The solution is given by problem (P1) which is then identical to (12) and (13) for the prospect theory utility with $\bar{C}_1 = \bar{C}_2 = 0$. Thus, the optimal consumption in the first period is

$$(C_1^*)^{EUT} = \frac{1+r_f}{1+r_f+M}\left(Y_1 + \frac{Y_2}{1+r_f}\right) = \frac{\Omega^{EUT}}{1+r_f+M} > 0$$

Note that in this case $\Omega^{EUT} > \Omega > 0$, where Ω is related to a prospect theory (PT) household and we assume that the PT household has at least one consumption reference level strictly positive, i.e., either $\bar{C}_1 > 0$ or $\bar{C}_2 > 0$, otherwise it boils down to the expected utility case. EUT optimal consumption is proportional to the present value of endowment income, where the factor of proportionality, representing the marginal propensity to consume out of the present value of total income, is less than unity. This marginal propensity to consume (out of the present value of total income) is the same as the one under PT preferences of a sufficiently loss averse household, assuming the curvature parameter γ remains unchanged.

In addition,

$$(\alpha^*)^{EUT} = \frac{\left(1 - K_0^{\frac{1}{\gamma}}\right) M}{r_f - r_b + K_0^{\frac{1}{\gamma}}(r_g - r_f)} (C_1^*)^{EUT} > 0$$

and thus the household's investment in the risky asset is also proportional to the present value of endowment income.[14] However, the EUT household will always invest in the risky asset, which is not necessarily the case for the PT household when $\Omega = 0$ and thus it will not invest in the risky financial market. The case when $\Omega = 0$ is discussed in Section 4. Note that for the EUT household the savings, $(S^*)^{EUT} = Y_1 - (C_1^*)^{EUT}$, are positive when $MY_1 > Y_2$, in which case the household transfers some of its first period income into the second period.

4 Belonging reference levels ($\Omega = 0$)

This special case occurs when the household is governed neither by self-enhancement motives (see the previous section) nor by self-improvement motives (see the following section). Instead, the

[14] In comparing optimal consumption and risky asset holdings between the two models one has to be careful and remember that the types of utility functions suggested by these models are different. For example, the EUT model implies a constant relative risk aversion, which is equal to γ, while the PT utility shows a decreasing relative risk aversion, which is equal to $\gamma C_1/(C_1 - \bar{C}_1)$ for $C_1 \neq \bar{C}_1$ and $\gamma C_2/(C_2 - \bar{C}_2)$ for $C_2 \neq \bar{C}_2$. In addition, the relative risk aversion for the EUT model is restricted to be below one (as a consequence from our restriction on γ, which states $0 < \gamma < 1$), while it has sometimes empirically been found to be larger than one (see Ahsan and Tsigaris, 2009, who provide some empirical examples).

household compares itself to other households who have exactly the same economic status. In this case, the household's reference levels are such that they are equal to its endowment income (in present value). In other words, the household's present value of endowment income matches exactly the discounted sum of its first and second period reference consumption levels, i.e., $Y_1 + \frac{Y_2}{1+r_f} = \bar{C}_1 + \frac{\bar{C}_2}{1+r_f}$. The household's reference levels (in present value) can be seen to be equal to the income of the households it compares to, i.e., its neighbors or associates. This type of the household desires to belong to the same social class as others that have a similar economic status. Hence we call the reference levels of such a household "belonging" reference levels.[15]

Looking at it from a purely technical point of view, the first period and second period reference levels are not independent from each other for the household to remain in the same social class. As the reference levels have to be equal to total income (in present value), it must always be true that $\bar{C}_1 = Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$ or, equivalently, $\bar{C}_2 = (1 + r_f)(Y_1 - \bar{C}_1) + Y_2$. The dependence between the two reference levels implies that if, for some reason, the first period reference level increases by some given amount then the second period reference level has to decrease by $(1 + r_f)$ times this amount, in order for the household to belong to the same wealth group as prior to the change of the first period reference level. A similar analysis can be applied if the second period reference level increases.

The following proposition presents the household's optimal choice

Proposition 2 *Let* $\Omega = 0$ *and* $\lambda > \max\left\{\left(\frac{1+r_f}{k} + \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right)^{\gamma}, \left(\frac{M}{1+r_f}\right)^{\gamma}\right\}$. *Then problem (5) obtains a unique maximum at* $(C_1^* = \bar{C}_1 = Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}, \alpha^* = 0)$.

Proof. It follows directly from Lemma 1 in Appendix B. ■

First note that the household has to be sufficiently loss averse in order to make its optimal choice. In fact the lower bounds on the loss aversion parameter are similar to the case when the reference levels are low ($\Omega > 0$), adjusted for the fact that $\Omega = 0$.[16]

If the household is sufficiently loss averse then the first period optimal consumption is exactly equal to the first period reference consumption, $C_1^* = \bar{C}_1$. In addition, the household does not invest in the risky asset, $\alpha^* = 0$, even though the expected return of the risky asset exceeds the risk-free return. This is a major difference with respect to the traditional expected utility model, where the household will always invest in the risky asset if the expected return of risky asset is greater than the risk-free asset. Note, in addition, that the second period optimal consumption is $C_2^* = Y_2 + (1 + r_f)(Y_1 - \bar{C}_1)$, see (2), but because the household's endowment wealth is equal to reference levels (in present value) it is also true that $C_{2g}^* = C_{2b}^* = \bar{C}_2$. The household may still transfer part of its income from the first period to the second period, or vice versa, in order to optimize its consumption path, but it will consume exactly at its reference level in both periods. This is a very particular situation.

[15] This is in line with the similarity/attraction theory stating that people like and are attracted to others who are similar to themselves, see International Encyclopedia of the Social Sciences (2016).

[16] Two of the previous lower bounds on the loss aversion parameter are now discarded as they are always smaller than other lower bounds included in the proposition.

In the light of this solution, the above restriction that both reference levels must be strictly positive makes also sense from an economic point of view: as a household lives for two periods it seems reasonable to require that its consumption is non-zero in both periods, which is guaranteed by $C_1^* = \bar{C}_1 > 0$ and $C_{2g}^* = C_{2g}^* = \bar{C}_2 > 0$. It can easily be seen that the savings, $S^* = m^* = Y_1 - \bar{C}_1$, are strictly positive if the consumption reference level in the first period is below the first period income, i.e., when $\bar{C}_1 < Y_1$, and thus the household wants to transfer some of its first period income into the second period. To do this the household will only invest in the risk-free asset and will consume in the second (e.g., retirement) period the amount of $(1+r_f)(Y_1 - \bar{C}_1)$ plus any exogenous future income Y_2. On the other hand, savings are negative if the consumption reference in the first period is above the first period income, i.e., when $\bar{C}_1 > Y_1$. In this case the household will transfer some part of its second period income, namely $Y_2 - \bar{C}_2$, into its first period and thus consume in the first period the amount $\frac{Y_2 - \bar{C}_2}{1+r_f} + Y_1$.

This household is at a very critical point and any changes in its reference levels, possibly triggered by changes of consumption (or income) of the households it associates with, can have significant implications in terms of inter-temporal decisions. For example, assume that the first period consumption of the Joneses increases (decreases) resulting in an increase (decrease) of this household's reference level. So suddenly the Joneses are not in the same social class as this household anymore (provided they do not decrease their second period consumption accordingly), and it is unclear now whether the household will continue to compare itself to the Joneses, who are now doing better (worse), or whether it will find other associates who belong to the same social class. The household's original motives might change and it might strive to self-improve to catch up with the suddenly richer associates (to self-enhance and feel good in following the suddenly poorer associates). If this happens then the household will solve a different problem which is presented in the next (previous) section, where $\Omega < 0$ ($\Omega > 0$). Moving across different types of Ω is also discussed in Section 6. Thus, the household in this category is at a crossroad with options to change motives and compare with others outside its own social class or to stay in the same class in terms of economic status and keep its belonging reference levels.

5 High reference levels ($\Omega < 0$)

Now we consider the case when the household's consumption reference levels are such that their present value is above the present value of endowment income. This can be viewed as a household with high aspirations or driven by the self-improvement motive. Such a household compares its own wealth to the (larger) wealth of other households with a higher economic status, and its reference levels can be determined by this larger wealth. Hence the household makes "upward comparisons". It could envy the rich neighbors and behave such as to catch up and exceed their status. But in a world of risk and uncertainty the household may still end up being in an worse economic status ex post.

In order to simplify the presentation of our main results in this section we introduce the following notation

$$M_1(\lambda) = k\left[\lambda^{\frac{1}{\gamma}} - \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right] \tag{16}$$

$$\tilde{c}^{P2} = \frac{(r_g - r_b)(-\Omega)}{(r_g - r_f)\bar{C}_2} \tag{17}$$

$$\bar{C}_2^{P2} = \frac{r_g - r_b}{r_f - r_b}\left((1+r_f)\left(Y_1 - \bar{C}_1\right) + Y_2\right) \tag{18}$$

$$\delta^+ = \frac{1}{1-p}\left[\frac{r_g - r_f}{(1+r_f)(r_g - r_b)}\right]^{1-\gamma} \tag{19}$$

Note that $M_1(\lambda)$ is an increasing function in λ and if $\lambda \geq \frac{1}{K_\gamma}$ then $M_1(\lambda) \geq 0$. A simple derivation shows that $\lambda > \frac{1+r_f}{k} + \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}$ is sufficient for $M_1(\lambda) > 1+r_f$. Note in addition that for $\bar{C}_2 < \bar{C}_2^{P2}$ is $\tilde{c}^{P2} < 1$. The threshold value $\bar{C}_2^{P2}$ is the upper bound for the second period consumption reference level and the main results in this section are derived under this condition.

We introduce the following additional notations

$$\hat{\lambda} = \left[\frac{k_2\left(1 + K_\gamma^{\frac{1}{\gamma}}\right)}{1+r_f}\right]^\gamma = \left[\frac{k\left(1 + \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right)}{1+r_f}\right]^\gamma \tag{20}$$

$$\lambda_1^{\Omega<0} = \left[\frac{\frac{1+r_f}{k} + \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}}{1-\tilde{c}^{P2}}\right]^\gamma \quad \text{if} \quad \bar{C}_2 < \bar{C}_2^{P2} \tag{21}$$

$$\lambda_2^{\Omega<0} = \hat{\lambda}\left[\frac{\bar{C}_1}{Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}}\right]^\gamma \tag{22}$$

$$\tilde{\lambda}^{\Omega<0} = \frac{1}{p}\left[\frac{1}{\delta}\left(\frac{Y_1 - \bar{C}_1 + \frac{Y_2}{1+r_f}}{\bar{C}_2}\right)^{1-\gamma} - (1-p)\left(1-\tilde{c}^{P2}\right)\lambda_1^{\Omega<0}\right] \quad \text{if} \quad \bar{C}_2 < \bar{C}_2^{P2} \tag{23}$$

to simplify the presentation of both the solution and the conditions on the minimum degree of loss aversion.

The following proposition presents the household's optimal choice and states the required assumptions for high reference values relative to endowment income.

Proposition 3 *Let* $\Omega < 0$, $\bar{C}_2 < \bar{C}_2^{P2}$ *and* $\lambda > \max\left\{\lambda_1^{\Omega<0}, \lambda_2^{\Omega<0}, \tilde{\lambda}^{\Omega<0}\right\}$. *Then the following holds*

$$C_1^* = \left\{ \begin{array}{ll} C_1^{P2} = \bar{C}_1 + \frac{-\Omega}{M_1(\lambda)-1-r_f} > \bar{C}_1 & \text{if} \quad \delta \leq \delta^+ \\ \\ 0 < C_1^{P5} = \frac{\left(Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}\right)\lambda^{\frac{1}{\gamma}} - \bar{C}_1 \hat{\lambda}^{\frac{1}{\gamma}}}{\lambda^{\frac{1}{\gamma}} - \hat{\lambda}^{\frac{1}{\gamma}}} < \bar{C}_1 & \text{if} \quad \delta > \delta^+ \end{array} \right\} \tag{24}$$

$$\alpha^* = \left\{ \begin{array}{ll} \alpha^{P2} = \frac{\left(\left(\frac{1}{K_0}\right)^{\frac{1}{\gamma}} + \lambda^{\frac{1}{\gamma}}\right)k}{r_g - r_f}(C_1^* - \bar{C}_1) > 0 & \text{if} \quad \delta \leq \delta^+ \\ \\ \alpha^{P5} = \frac{1-K_0^{\frac{1}{\gamma}}}{r_f - r_b + K_0^{\frac{1}{\gamma}}(r_g - r_f)} \frac{\hat{\lambda}^{\frac{1}{\gamma}}}{\lambda^{\frac{1}{\gamma}} - \hat{\lambda}^{\frac{1}{\gamma}}}(-\Omega) > 0 & \text{if} \quad \delta > \delta^+ \end{array} \right\} \tag{25}$$

Proof. See Appendix C. ■

Proposition 3 is derived for a relatively low[17] second period reference level, $\bar{C}_2 < \bar{C}_2^{P2}$, and for a sufficiently loss averse household, $\lambda \geq \max\left\{\lambda_1^{\Omega<0}, \lambda_2^{\Omega<0}, \tilde{\lambda}^{\Omega<0}\right\}$. The fact that $\Omega < 0$ implies that the household's aspirations are high, which should make it more difficult to exceed the consumption reference levels than when the household's reference levels are low, as in the previous section. So, even if the utility is such that consumption below the reference level is heavily penalized (large values of the loss aversion parameter λ) we cannot expect optimal consumption levels to exceed their reference levels at all times. This is indeed what we observe: in the first solution the second period optimal consumption in the bad state of nature is below its reference level, while in the second solution the first period optimal consumption is below its reference level. The first solution is denoted by superscript P2, which refers to problem (P2) where the solution was reached, and the second solution is denoted by superscript P5, which refers to problem (P5) where the solution was reached, see Appendix A. Thus, when the household's aspirations are above the present value of the endowment income, a relative loss (either in the first or second period) cannot be avoided.

Unlike households with low reference levels, we now have two different solutions, denoted $\left(C_1^{P2}, \alpha^{P2}\right)$ and $\left(C_1^{P5}, \alpha^{P5}\right)$,[18] and which one applies depends on the rate δ at which future utility is discounted. The first solution applies to households with lower discount factors ($\delta \leq \delta^+$), which put relatively more emphasis on the well-being in the present and near future and thus display a high time preference, while the second applies to households with higher discount factors ($\delta > \delta^+$), which care relatively more about the distant future and discount future utility at a lower rate and thus show a low time preference. The threshold value δ^+ separating the two types is a function of the rates r_f, r_g and r_b, of the probability of the good state of nature p and of the curvature

[17] Note that reference levels cannot be arbitrarily high for a given endowment income.

[18] Again, for the proof of the household's consumption decision we split problem (5) into eight separate problems, (P1)–(P8), which differ in the respective domains of feasible solutions. These domains are specified by whether (first and second period, good and bad state of nature) consumption is above or below the respective reference level. See Appendix A.

parameter γ, see equation (19). Note that it is increasing in p and γ while it is decreasing in r_f, all other things equal. The restriction $\delta > \delta^+$ only yields feasible candidates for the discount factor, which is bound to be below one, if $\delta^+ < 1$. This is the case when the probability of the good state is not too large.[19]

We will discuss the two solutions separately starting with $\left(C_1^{P2}, \alpha^{P2}\right)$. First note that the optimal consumption in the first period is strictly above the consumption reference level. As the solution $\left(C_1^{P2}, \alpha^{P2}\right)$ is reached in problem (P2) the optimal consumption in the second period is above the reference level $\bar{C}_2$ in the good state of nature, $C_{2g}^{P2} > \bar{C}_2$, and below $\bar{C}_2$ in the bad state of nature, $C_{2b}^{P2} < \bar{C}_2$. Thus, the household cannot avoid a relative loss if the bad state of nature materializes. Further, the optimal investment in the risky asset is strictly positive. Like in the case for households with low reference levels, savings can be either positive or negative in general.[20] If, however, the household's consumption reference level is equal to or above its income in the first period[21] then optimal savings are always negative, i.e., the household will transfer future income to the present period in order to satisfy its optimal consumption path. Given that optimal risky investment is positive, borrowing in the risk-free market has to be sufficiently large then, in order to produce negative savings. In fact, in this situation of negative savings the household will invest in the risky asset in order to (partially) fund the borrowing and thus the income transfer from the second to the first period. If in one period the household's income is below the consumption reference level and vice versa in the other period, then the answer to the question whether optimal savings are positive depends (also) on the loss aversion parameter. Keeping everything else constant, a larger loss aversion parameter will increase savings (see below). Note that the optimal savings of a household with low aspirations did not depend on the loss aversion parameter.

For C_1^{P2} and α^{P2} given by (24) and (25) the following holds:

(i) $\lim_{\lambda\to\infty} C_1^{P2} = \bar{C}_1$

(ii) $\lim_{\lambda\to\infty} C_{2g}^{P2} = \bar{C}_2$

(iii) $\lim_{\lambda\to\infty} C_{2b}^{P2} = \frac{r_f - r_b}{r_g - r_f}\left(\bar{C}_2^{P2} - \bar{C}_2\right) < \bar{C}_2$

(iv) $\lim_{\lambda\to\infty} \alpha^{P2} = \frac{-\Omega}{r_g - r_f} \lesseqgtr 0$

The following table summarizes the results on the impact of the behavioural parameters on the decision variables

As opposed to households with low reference levels, the optimal consumption and optimal investment in the risky asset of households with high aspirations are sensitive with respect to the loss aversion parameter λ. An exogenous increase in the loss aversion parameter, keeping everything else constant, will decrease the first and the second period optimal consumption in the good state of

[19]More precisely, $p < 1 - \left(\frac{1}{1+r_f}\frac{r_g - r_f}{r_g - r_b}\right)^{1-\gamma}$ is needed in order to guarantee $\delta^+ < 1$. Note that there is always a solution for p as the term in the brackets is smaller than one.

[20]The assumption required for $S^* > 0$ is $M_1(\lambda)(Y_1 - \bar{C}_1) > -(Y_2 - \bar{C}_2)$.

[21]This condition is sufficient but not necessary.

	dC_1^{P2}	dC_{2g}^{P2}	dC_{2b}^{P2}	$d\alpha^{P2}$	$d(C_1^{P2}-\bar{C}_1)$	$d(C_{2g}^{P2}-\bar{C}_2)$	$d(\bar{C}_2-C_{2b}^{P2})$	dS^{P2}	$d\mathbb{E}(U(C_1^{P2},\alpha^{P2}))$
$d\lambda$	<0	<0	>0	<0	<0	<0	<0	>0	<0
$d\bar{C}_1$	>0	>0	<0	>0	>0	–	–	<0	<0
$d\bar{C}_2$	>0	>0	<0	>0	–	>0	>0	<0	<0

Table 2: Sensitivity results when aspirations are high ($\Omega < 0$): solution $\left(C_1^{P2}, \alpha^{P2}\right)$

nature, decrease the investment in the risky asset, increase savings and thus increase the investment in the risk-free asset. In addition, increased loss aversion decreases the consumption gap in both periods. Also the happiness level decreases with increasing loss aversion, i.e., more loss averse households are less happy than less loss averse households.

Continuing our previous *following the Joneses* example, an exogenous increase in the first period consumption reference level (following an increase in first period consumption of the successful person this household compares to), keeping everything else constant, will increase the first period optimal consumption as well as the second period optimal consumption in the good state (which is above the consumption reference level) and the investment in the risky asset, but will decrease the second period optimal consumption in the bad state (which is below the consumption reference level). At the same time the household will decrease optimal savings, which actually implies the observed decrease in optimal consumption in bad state of nature. Not only does optimal consumption rise in the first period and in the good state in the second period, but also the corresponding relative optimal consumption rises. Note that the household's happiness decreases with an increase in the consumption level of the already more successful person this household compares to (which can be seen as this household's reference level). An exogenous increase in the second period consumption reference level yields exactly the same sensitivities in terms of signs as an increase in the first period consumption reference level.

Thus the household follows the Joneses in the first period.[22] However, contrary to the case when $\Omega > 0$ the household does not reduce the consumption gap but widens this gap. This means that the household increases its consumption even more than the Joneses do. A household with high reference levels (high aspirations) reacts thus more intensely than one with low reference levels (driven by the feel good motive), even though both follow the Joneses. As in the case when $\Omega > 0$, it is not clear whether the household follows the Joneses in the second period, since its second period consumption in the good and the bad states of nature responds in opposite directions to the change in its second period reference level, and thus it is not clear whether the expected household's consumption will reflect an increase or a decrease of consumption. In the second period, in both the good and the bad states of nature, the consumption gap will widen in response to an increase in the second period reference level. This, however, implies a decrease of the optimal consumption when it is below the reference level, which is the case in the bad state of nature. Note, finally, that an infinitely loss averse household will optimally consume the (respective) consumption reference level

[22] Assuming the household increases its first period reference level as a response to an increase of the Joneses' first period consumption, it will also increase its optimal consumption in the first period.

in the first period (i.e., the consumption level of the successful household it compares to) and in the second period in the good state of nature, while it will have strictly positive optimal consumption below its reference level in the bad state of nature. In addition, it will invest the strictly positive amount of $-\Omega/(r_g - r_f)$ in the risky asset.

We now turn to the discussion of the second solution of Proposition 3, $\left(C_1^{P5}, \alpha^{P5}\right)$, which holds for households with a low time preference, i.e., for households with a high discount factor $\delta > \delta^+$. As the notation suggests, this solution is reached in problem (P5), where the first and second period consumption domains are given by $0 \le C_1 \le \bar{C}_1$ and $C_{2g} \ge C_{2b} \ge \bar{C}_2$. The optimal consumption in the first period is thus below the reference level and the optimal consumption in the second period is above the reference level $\bar{C}_2$ in both states of nature. Even though the household is rather loss averse (and so the penalty for consumption below the reference level is rather large) the optimal consumption in the first period is below the consumption reference level. With a sufficiently low time preference, i.e., a sufficiently high discount factor $\delta > \delta^+$, the household values future consumption so much that it prefers to consume above the reference level in both states of nature in the second period, accepting to consume below the reference level in the first period. Again the optimal investment in the risky asset is strictly positive.

As in the first solution, savings can be either positive or negative in general.[23] If, however, the household's first period consumption reference level is equal to or below its first period income[24] then optimal savings are always positive, i.e., the household transfers current income to the future period in order to satisfy its optimal consumption path. Note that a larger loss aversion parameter will *de*crease savings while before (in the first solution) it increased savings.

For the solution $\left(C_1^{P5}, \alpha^{P5}\right)$ the following holds:

(i) $\lim_{\lambda\to\infty} C_1^{P5} = Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f} < \bar{C}_1$

(ii) $\lim_{\lambda\to\infty} C_{2g}^{P5} = \bar{C}_2$

(iii) $\lim_{\lambda\to\infty} C_{2b}^{P5} = \bar{C}_2$

(iv) $\lim_{\lambda\to\infty} \alpha^{P5} = 0$

The following table summarizes the results of impacts of the behavioural traits on the decision variables:

	dC_1^{P5}	dC_{2g}^{P5}	dC_{2b}^{P5}	$d\alpha^{P5}$	$d(\bar{C}_1 - C_1^{P5})$	$d(C_{2g}^{P5} - \bar{C}_2)$	$d(C_{2b}^{P5} - \bar{C}_2)$	dS^{P5}	$d\mathbb{E}(U(C_1^{P5}, \alpha^{P5}))$
$d\lambda$	>0	<0	<0	<0	<0	<0	<0	<0	<0
$d\bar{C}_1$	<0	>0	>0	>0	>0	–	–	>0	<0
$d\bar{C}_2$	<0	>0	>0	>0	–	>0	>0	>0	<0

Table 3: Sensitivity results when aspirations are high ($\Omega < 0$): solution $\left(C_1^{P5}, \alpha^{P5}\right)$

[23] The assumption required for $S^* > 0$ is $\hat{\lambda}^{1/\gamma}(Y_1 - \bar{C}_1) < -\lambda^{1/\gamma}\frac{Y_2 - \bar{C}_2}{1+r_f}$.

[24] This condition is sufficient but not necessary.

Again, as for households with a higher time preference (i.e., a lower discount factor), the optimal consumption and optimal investment in the risky asset are sensitive with respect to the loss aversion parameter λ. An exogenous increase in the loss aversion parameter, keeping everything else constant, will increase first period optimal consumption (which is below the reference level), and decrease everything else, i.e., the second period optimal consumption in both states of nature (which is above the reference level), the investment in the risky asset, savings, the consumption gaps in both periods as well as the level of happiness.

Putting this into the context of social comparisons, an exogenous increase in the first period consumption reference level (following an increase in first period consumption of the already successful neighbor), keeping everything else constant, will decrease the first period optimal consumption and thus increase optimal savings, and, additionally, increase the consumption in the second period in both states of nature as well as investment in the risky asset. The level of happiness will again decrease with an increasing reference level. The sensitivity analysis with respect to the second period consumption reference level (of all variables under consideration) is – in terms of signs – the same as with respect to the first period consumption reference level.

In terms of *following the Joneses* we observe the following: the household does not follow the Joneses in the first period while it indeed follows the Joneses in the second period. In addition, in the first period the household's optimal consumption does not only move in the opposite direction with respect to that of the Joneses, but also the consumption gap increases. Note that – as in the first solution and contrary to the case when $\Omega > 0$ – the household increases its consumption even to a larger degree than the Joneses. This is true in the first period for households with a higher time preference and it is true in the second period for households with a lower time preference.

Note, finally, that an infinitely loss averse household will optimally consume the first period income plus the discounted relative consumption (i.e., the difference between the second period income and the second period consumption reference level). It will consume at the reference level in the second period in both states of nature and it will only invest in the risk-free asset.

6 Comparing across different social classes

In order to compare the consumption-investment behavior across different types of households we consider the following example: we look at two households with identical income who compare themselves to a poorer neighbor and to a richer neighbor.[25] Thus, the first household is governed by the self-enhancement or feel good motives, making downward comparisons, while the second household is governed by the self-improvement motive or high aspirations, making upward comparisons. The households' reference levels are equal to the income of their respective neighbors. More precisely, the reference levels of the household that compares itself to the poorer (richer) neighbor are such that their present value is equal to the present value of the income of the poorer (richer) neighbor (see Table 4). By a poorer (richer) household we mean a household whose total discounted

[25] We would like to thank one of the referees for suggesting that we explore this example.

income is below (above) the total discounted income of the household under consideration. Our assumptions imply low reference levels with respect to income ($\Omega^{Pn} > 0$) for the household that compares itself to the poorer neighbor and high reference levels with respect to income ($\Omega^{Rn} < 0$) for the household that compares itself to the richer neighbor.[26] Except for the reference levels all parameters for the two households are assumed to be the same. Table 4 summarizes the optimal behavior for the two households.[27] Our assumptions imply that the household in the poor neighborhood has lower reference levels than the household in the rich neighborhood, when summing and discounting over the two periods.[28]

Comparing the optimal solutions of the two different households yields a complex picture (see Table 5). It is not true, for example, that the first period consumption of the household in the rich neighborhood is always larger than that of the household in the poor neighborhood. This depends on the relation between the first period reference level of the household with the richer neighbor and the optimal consumption of the household with the poorer neighbor, and on the degree of loss aversion. Also investment in the risky asset of the household in the rich neighborhood can be either smaller or larger than that of the household in the poor neighborhood, depending on the size of Ω of the household in the rich neighborhood and, in most cases, also on the degree of loss aversion.

For illustration purposes and to get a better feeling for the consumption-investment behavior of a prospect theory household in general, let us consider a concrete example with the following assumptions about the two households (see Table 6). The first and second period income are assumed to be 100 and 80, and the loss aversion parameter is equal to 5.[29] The curvature parameter and the discount rate are assumed to be 0.6 and 0.8. For the risk-free interest rate, the return in the good state and the return in the bad state, we take rates of 1.5, 15, and 0.5, respectively.[30] The probability of the good state is assumed to be 0.5, which implies an expected return of the risky asset of 7.75.[31] Finally, the first and second period reference levels of the household with the poorer neighbor are assumed to be 95 and 70, and those of the household with the richer neighbor are 105 and 90. This implies $\Omega = 22.5$ for the household governed by the self-enhancement motive, and $\Omega = -22.5$ for the household governed by the self-improvement motive. With the given parameters the conditions in Propositions 1 and 3 hold, and for the latter household the first solution of

[26] The superscript Pn (Rn) indicates the household that compares itself to the poorer (richer) neighbor, the superscript P (R) indicates the poorer (richer) neighbor.

[27] The general solutions are given in Propositions 1 and 3, and we assume that the required conditions on the loss aversion parameter and the second period reference point are satisfied. Note that in the case when the household compares itself to the richer neighbor the region where the solution occurs depends on the household's time preference.

[28] I.e., $\bar{C}_1^{Pn} + \frac{\bar{C}_2^{Pn}}{1+r_f} < \bar{C}_1^{Rn} + \frac{\bar{C}_2^{Rn}}{1+r_f}$ as $(1+r_f)\bar{C}_1^{Pn} + \bar{C}_2^{Pn} = (1+r_f)Y_1^P + Y_2^P < (1+r_f)Y_1 + Y_2 < (1+r_f)Y_1^R + Y_2^R = (1+r_f)\bar{C}_1^{Rn} + \bar{C}_2^{Rn}$.

[29] Note the minimum loss aversion to obtain an optimal solution for the person who compares with poorer people is 2.83 while the minimum value of the same household who compares with rich is 4.22.

[30] Assuming that one period covers, e.g., 40 years these rates correspond to annual returns of 2.3%, 7.9%, and 1.0%, respectively. These annual rates seem rather realistic, even though one might be surprised at the initial (large) numbers.

[31] This implies an annual expected return of the risky asset of 5.6% and hence an annual premium of the risky asset of 3.3%.

	Comparison to poorer neighbor self-enhancement motive	Comparison to richer neighbor self-improvement motive
endow.	$(1+r_f)Y_1+Y_2>(1+r_f)Y_1^P+Y_2^P$	$(1+r_f)Y_1+Y_2<(1+r_f)Y_1^R+Y_2^R$
ref. level	$(1+r_f)\bar{C}_1^{Pn}+\bar{C}_2^{Pn}=(1+r_f)Y_1^P+Y_2^P$	$(1+r_f)\bar{C}_1^{Rn}+\bar{C}_2^{Rn}=(1+r_f)Y_1^R+Y_2^R$
Ω	$\Omega^{Pn}=(1+r_f)(Y_1-Y_1^P)+Y_2-Y_2^P>0$	$\Omega^{Rn}=(1+r_f)(Y_1-Y_1^R)+Y_2-Y_2^R<0$
C_1^*	$C_1^{*Pn}=\bar{C}_1^{Pn}+\frac{\Omega^{Pn}}{1+r_f+M}>\bar{C}_1^{Pn}$	$C_1^{*Rn,1}=\bar{C}_1^{Rn}+\frac{\left(-\Omega^{Rn}\right)}{M_1(\lambda)-1-r_f}>\bar{C}_1^{Rn}$ if $\delta\le\delta^+$
		$0<C_1^{*Rn,2}=\frac{\left(Y_1+\frac{Y_2-\bar{C}_2^{Rn}}{1+r_f}\right)\lambda^{\frac{1}{\gamma}}-\bar{C}_1^{Rn}\hat{\lambda}^{\frac{1}{\gamma}}}{\lambda^{\frac{1}{\gamma}}-\hat{\lambda}^{\frac{1}{\gamma}}}<\bar{C}_1^{Rn}$ if $\delta>\delta^+$
α^*	$\alpha^{*Pn}=\frac{\left(1-K_0^{1/\gamma}\right)M}{r_f-r_b+K_0^{1/\gamma}(r_g-r_f)}\frac{\Omega^{Pn}}{1+r_f+M}>0$	$\alpha^{*Rn,1}=\frac{\left(\left(\frac{1}{K_0}\right)^{\frac{1}{\gamma}}+\lambda^{\frac{1}{\gamma}}\right)k}{r_g-r_f}\frac{\left(-\Omega^{Rn}\right)}{M_1(\lambda)-1-r_f}>0$ if $\delta\le\delta^+$
		$\alpha^{*Rn,2}=\frac{1-K_0^{\frac{1}{\gamma}}}{r_f-r_b+K_0^{\frac{1}{\gamma}}(r_g-r_f)}\frac{\hat{\lambda}^{\frac{1}{\gamma}}}{\lambda^{\frac{1}{\gamma}}-\hat{\lambda}^{\frac{1}{\gamma}}}(-\Omega^{Rn})>0$ if $\delta>\delta^+$

Table 4: Households comparing themselves to poorer and richer neighbors: endowment, reference levels and solutions.
The example looks at two households with identical income who compare themselves to a poorer neighbor and to a richer neighbor, and set their reference levels equal to the income of their neighbors. Except for the reference levels all parameters are identical. We assume that the required conditions on λ and $\bar{C}_2$ of Propositions 1 and 3 hold. The superscript Pn (Rn) indicates the household that compares itself to the poorer (richer) neighbor, the superscript P (R) indicates the poorer (richer) neighbor.

Proposition 3, where $\delta \le \delta^+$, applies.[32]

The results show that the consumption of the household governed by the self-enhancement or feel good motive ($\Omega^{Pn} > 0$) is smaller in the first period as well as in the second period in the good state of nature than the consumption of the household driven by the self-improvement motive or high aspirations ($\Omega^{Rn} < 0$). However, in the bad state of nature, it is the household in the poor neighborhood, motivated by self-enhancement, that consumes considerably more than the household in the rich neighborhood, motivated by high aspirations. Both households invest in the risky asset and borrow at the risk-free interest rate, but at different proportions, such that total savings are positive for the self-enhancement motivated household in the poor neighborhood and negative for the self-improvement motivated household in the rich neighborhood. So, the household in the poor neighborhood, which is driven by the feel good motive, consumes less than its income in the first period and transfers part of it to the second period while the opposite is true for the household in the rich neighborhood, which is driven by high aspirations. Note that the second period consumption in the bad state of the household with the rich neighborhood (36.2) is considerably below its reference level (90). This indicates that the aggressive risk taking activity

[32] Note that the income and reference values are normalized numbers and can be multiplied with any constant without changing the characteristics of the solutions. All monetary values (consumption, savings, etc.) have to be multiplied by the same constant, all other values, e.g., the required minimum degree of loss aversion, are not affected by the multiplication (normalization).

Comparison of solutions
Lower discount factor: $\delta \leq \delta^{+}$
(i) $C_1^{*Rn} > C_1^{*Pn}$ if $\bar{C}_1^{Rn} > C_1^{*Pn}$ or if $(\bar{C}_1^{Rn} < C_1^{*Pn}$ and $\lambda < \lambda_1)$
(ii) $C_1^{*Rn} < C_1^{*Pn}$ if $\bar{C}_1^{Rn} < C_1^{*Pn}$ and $\lambda > \lambda_1$
(iii) $C_1^{*Rn} = C_1^{*Pn}$ if $\bar{C}_1^{Rn} < C_1^{*Pn}$ and $\lambda = \lambda_1$
(iv) $\alpha^{*Rn} > \alpha^{*Pn}$ if $-\Omega^{Rn} \geq \Omega_1$ or if $(-\Omega^{Rn} < \Omega_1$ and $\lambda < \lambda_2)$
(v) $\alpha^{*Rn} < \alpha^{*Pn}$ if $-\Omega^{Rn} < \Omega_1$ and $\lambda > \lambda_2$
(vi) $\alpha^{*Rn} = \alpha^{*Pn}$ if $-\Omega^{Rn} < \Omega_1$ and $\lambda = \lambda_2$
Higher discount factor: $\delta > \delta^{+}$
(i) $C_1^{*Rn} > C_1^{*Pn}$ if $\Omega^{Pn} > \Omega_2$ and $\lambda^{1/\gamma} > \lambda_3$ or if $(\Omega^{Pn} < \Omega_2$ and $\lambda^{1/\gamma} < \lambda_3)$
(ii) $C_1^{*Rn} < C_1^{*Pn}$ if $\bar{C}_1^{Rn} < C_1^{*Pn}$ or if $(\Omega^{Pn} > \Omega_2$ and $\lambda^{1/\gamma} < \lambda_3)$ or if $(\Omega^{Pn} < \Omega_2$ and $\lambda^{1/\gamma} > \lambda_3)$
(iii) $C_1^{*Rn} = C_1^{*Pn}$ if $\Omega^{Pn} \neq \Omega_2$, $\lambda^{1/\gamma} = \lambda_3$
(iv) $\alpha^{*Rn} > \alpha^{*Pn}$ if $\lambda < \lambda_4$
(v) $\alpha^{*Rn} < \alpha^{*Pn}$ if $\lambda > \lambda_4$
(vi) $\alpha^{*Rn} = \alpha^{*Pn}$ if $\lambda = \lambda_4$
where
$\Omega_1 = \frac{M}{1+r_f+M} \frac{1-K_0^{1/\gamma}}{\frac{r_f-r_b}{r_g-r_f}+K_0^{1/\gamma}} \Omega^{Pn}, \ \Omega_2 = \left(1+\frac{1+r_f}{M}\right)\left(\bar{C}_2^{Rn}-\bar{C}_2^{Pn}\right)$
$\lambda_1 = \left[\frac{1}{k}\left[1+r_f+\frac{-\Omega^{Rn}}{C_1^{*Pn}-\bar{C}_1^{Rn}}\right]+\left(\frac{1}{K_\gamma}\right)^{1/\gamma}\right]^{\gamma}, \ \lambda_2 = \left[\frac{\left(\left(\frac{1}{K_\gamma}\right)^{1/\gamma}\frac{1+r_f}{k}\right)(r_g-r_f)\alpha^{*Pn}-\left(\frac{1}{K_0}\right)^{1/\gamma}\Omega^{Rn}}{\Omega^{Rn}+\frac{M}{1+r_f+M}\frac{1-K_0^{1/\gamma}}{\frac{r_f-r_b}{r_g-r_f}+K_0^{1/\gamma}}\Omega^{Pn}}\right]^{\gamma}$
$\lambda_3 = (1+r_f)\frac{M(\bar{C}_1^{Rn}-\bar{C}_1^{Pn})-\Omega^{Rn}+\bar{C}_2^{Pn}-\bar{C}_2^{Rn}}{M(\Omega^{Pn}-\tilde{\Omega}_1)}\hat{\lambda}^{1/\gamma}, \ \lambda_4 = \left[1+\left(1+\frac{1+r_f}{M}\right)\frac{-\Omega^{Rn}}{\Omega^{Pn}}\right]^{\gamma}\hat{\lambda}$

Table 5: Households comparing themselves to poorer and richer neighbors: comparison of solutions. The example looks at two households with identical income who compare themselves to a poorer neighbor and to a richer neighbor, and set their reference levels equal to the income of their neighbors. Except for the reference levels all parameters are identical. We assume that the required conditions on λ and $\bar{C}_2$ of Propositions 1 and 3 hold. The superscript Pn (Rn) indicates the household that compares itself to the poorer (richer) neighbor, the superscript P (R) indicates the poorer (richer) neighbor.

(in the sense of accepting a considerably high variation of second period consumption), in order to self-improve, can worsen the household's economic status (in the bad state) but at the same time it is optimal for the household to engage in such risk taking activity. The extent to which consumption in the bad state falls short of the reference level should decrease with an increasing degree of loss aversion, as the penalty for consuming below the reference level will be larger with a higher degree of loss aversion.

The expected value of second period consumption is much larger for the household with the richer neighbor. This is achieved through the household's willingness to take the risk as stated above: it accepts a very low consumption in the bad state of nature (36.2) in order to be able to consume at a very high level in the good state of nature (371.7). This also means a very high variation (standard deviation) of second period consumption. If we control for risk and look at expected second period consumption per unit of risk, then it is the self-enhancement motivated household with the poorer neighbor that performs better.

Let us increase now the loss aversion parameter from 5 to 7, all other things equal then the self-improvement motivated household in the rich neighborhood will reshuffle its optimal consumption in the following way. The second period consumption in the bad state (which is below the reference point) will increase substantially from 36.2 to 54.8, thereby reducing the distance to the reference value (and thus also the penalty for consuming below the reference value) while the second period consumption in the good state as well as the first period consumption (which are both above the reference level) will decrease – the former considerably from 371.7 to 195.3, and the latter only slightly from 108.3 to 106.2. So, for the higher degree of loss aversion, both second period consumption in the good and in the bad state of nature are now smaller for the self-improvement motivated household with the richer neighbor than for the self-enhancement motivated household with the poorer neighbor. In addition, a higher loss aversion will result in less happiness for the self-improvement motivated household.[33]

Solutions across different values of Ω

In order to provide some intuition on what happens when the household moves across different reference values relative to income (i.e., different values of Ω), we present the optimal consumption levels and investment as functions of Ω for households with identical income ($Y_1 = 100$ and $Y_2 = 80$) but with different first and second period reference levels. Both reference levels are decreasing (linearly) in such a way that the values at $\Omega = -22.5$ ($\bar{C}_1 = 105$, $\bar{C}_2 = 90$) and at $\Omega = 22.5$ ($\bar{C}_1 = 95$, $\bar{C}_2 = 70$) correspond to our numerical example. This implies $\bar{C}_1 = 100$ and $\bar{C}_2 = 80$ for $\Omega = 0$. All other parameters describing the consumption-investment decision are identical. We take the same parameter values as in our previous numerical example, i.e., $r_f = 1.5$, $r_g = 15$, $r_b = 0.5$, $p = 0.5$, $\gamma = 0.6$, $\delta = 0.8$, and $\lambda = 5$.

The top graph of Figure 2 shows the optimal consumption in the first period (solid line) and in the second period, in the good (dashed line) and in the bad (dotted line) state of nature. The bottom graph shows the optimal investment in the risky (solid line) and in the risk-free (dashed line) asset and optimal savings (dotted line). For $\Omega < 0$ and the given parameters, the first solution of Proposition 3 applies, where $\delta \leq \delta^+$. This implies that the solution is reached in problem (P2), where the first period consumption and the second period consumption in the good state are above their respective reference levels and the second period consumption in the bad state is below its reference level (see Proposition 3). For $\Omega > 0$ all consumption levels are above their corresponding reference levels (see Proposition 1). These theoretical properties are also reflected in the graph.

The first period consumption is decreasing with Ω over the whole region, i.e., for both $\Omega < 0$ and $\Omega > 0$. On the other hand the second period consumption in the good state is decreasing for $\Omega < 0$

[33] Note that the solution of the household with the poorer neighbor does not change with a higher degree of loss aversion, as its total income is large enough to allow consumption above the reference levels at all times ($\Omega^{Pn} > 0$).

	Comparison to poorer neighbor self-enhancement motive	Comparison to richer neighbor self-improvement motive
Parameters		
endowment	$Y_1 = 100,\ Y_2 = 80$	$Y_1 = 100,\ Y_2 = 80$
reference level	$\bar{C}_1^{Pn} = 95,\ \bar{C}_2^{Pn} = 70$	$\bar{C}_1^{Rn} = 105,\ \bar{C}_2^{Rn} = 90$
γ, δ	$\gamma = 0.6, \delta = 0.8$	$\gamma = 0.6, \delta = 0.8$
λ, λ_{min}	$\lambda = 5,\ \lambda_{min}^{Pn} = 2.83$	$\lambda = 5,\ \lambda_{min}^{Rn} = 4.22$
returns	$r_f = 1.5, r_g = 15, r_b = 0.5$	$r_f = 1.5, r_g = 15, r_b = 0.5$
probability	$p = 0.5$	$p = 0.5$
Ω	22.5	-22.5
Solutions		
C_1^*	97.4 $<$	108.3
α^*	13.9 $<$	23.1
m^*	-11.3 $>$	-31.4
S^*	2.6 $>$	-8.3
C_{2g}^*	274.3 $<$	371.7
C_{2b}^*	72.7 $>$	36.2
$E(U^*)$	13.4 $>$	-11.1
Properties of C_2 implied by the solutions		
$\mathbb{E}(C_2)$	173.5 $<$	203.9
$\sigma(C_2)$	100.8 $<$	167.7
$\mathbb{E}(C_2)$ per unit of risk	1.72 $>$	1.22

Table 6: Numerical example: households comparing themselves to poorer and richer neighbors. The example considers two households with identical income who compare themselves to a poorer neighbor and to a richer neighbor, and set their reference levels equal to the income of their neighbors. For the given parameters the conditions in Propositions 1 and 3 hold and the first solution of Proposition 3 applies (where $\delta \leq \delta^+$). λ_{min}^{Pn} and λ_{min}^{Rn} are the required minimum values of loss aversion for Propositions 1 and 3 to hold.

and increasing for $\Omega > 0$. For the second period consumption in the bad state the opposite is true. This means that consumption in the good state of nature is increasing when the household moves away from the $\Omega = 0$ level (in both directions), where it compares itself to households of similar economic status, while consumption in the bad state of nature is decreasing when the household moves away from the $\Omega = 0$ level (in both directions).[34]

The investment in the risky asset is zero for a household that has reference levels that mimic others of similar economic status ($\Omega = 0$). If the household moves to higher or lower Ω values then the investment in the risky asset is positive and increasing. On the other hand the investment in the risk-free asset is always negative for self-enhancement or self-improvement motivated households and it is decreasing for higher, or lower, Ω values. Finally, total savings are zero for the household that compares itself to households of similar economic status ($\Omega = 0$), negative for households governed by the self-improvement motive ($\Omega < 0$), and positive for households driven by the self-enhancement motive ($\Omega > 0$).[35]

7 Summary of main results

Table 7 summarizes the main results for the three different types of households, i.e., for households with low, equal and high reference values relative to endowment income. Let us re-emphasize that the household's behavior depends crucially on how the household's discounted value of the reference levels compares to its discounted value of endowment, yielding three different types of households. If the household's reference levels relative to income are low (i.e., governed by the self-enhancement motive, $\Omega > 0$) and the household is sufficiently loss averse then it can afford to consume above the reference level in both periods (see Table 7). If, on the other hand, the reference levels of a sufficiently loss averse household are high relative to income (i.e., the household is governed by the self-improvement motive, $\Omega < 0$) then the household will consume below the reference level in either the first or the second period (see Table 7). Whether it consumes below the reference level in the first or in the second period depends on its time preference. With a relatively high time preference (i.e., a relatively small discount factor, $\delta \leq \delta^+$) the household values the first period (sufficiently) more than the second one and hence chooses to consume above the reference level in the first period (see Table 7). Note, in addition, that in this case the happiness of the household

[34]These statements are true, in general, for $\Omega < 0$, as in this region the effects of changing the first or the second period reference point (and this is what changes Ω) work in the same directions. For $\Omega > 0$, however, the above statements may not hold, as the slopes of the consumption levels depend on which of the two (opposite) effects at work dominate, the one due to the change of the first period or the one due to the change of the second period reference level. Note that "generally true" for $\Omega < 0$ applies to the first solution of Proposition 3, which is found in problem (P2).

[35]The negative (positive) slope of the optimal investment in the risky asset for $\Omega < 0$ ($\Omega > 0$) holds, in general, as the effects of changing the first or the second period reference level (and this is what changes Ω) work in the same directions for both $\Omega < 0$ and $\Omega > 0$. For optimal savings, however, the slope (shown in our graph) is generally true only for $\Omega < 0$. For $\Omega > 0$, it depends on which of the two (opposite) effects at work dominate, the one due to the change of the first period or the one due to the change of the second period reference level. The statement that investment in the risky asset is positive for $\Omega \neq 0$ is always true. However, the above statements about investment in the risk-free asset and savings being positive or negative do not hold generally. Note that "generally true" for $\Omega < 0$ applies to the first solution of Proposition 3, which is found in problem (P2).

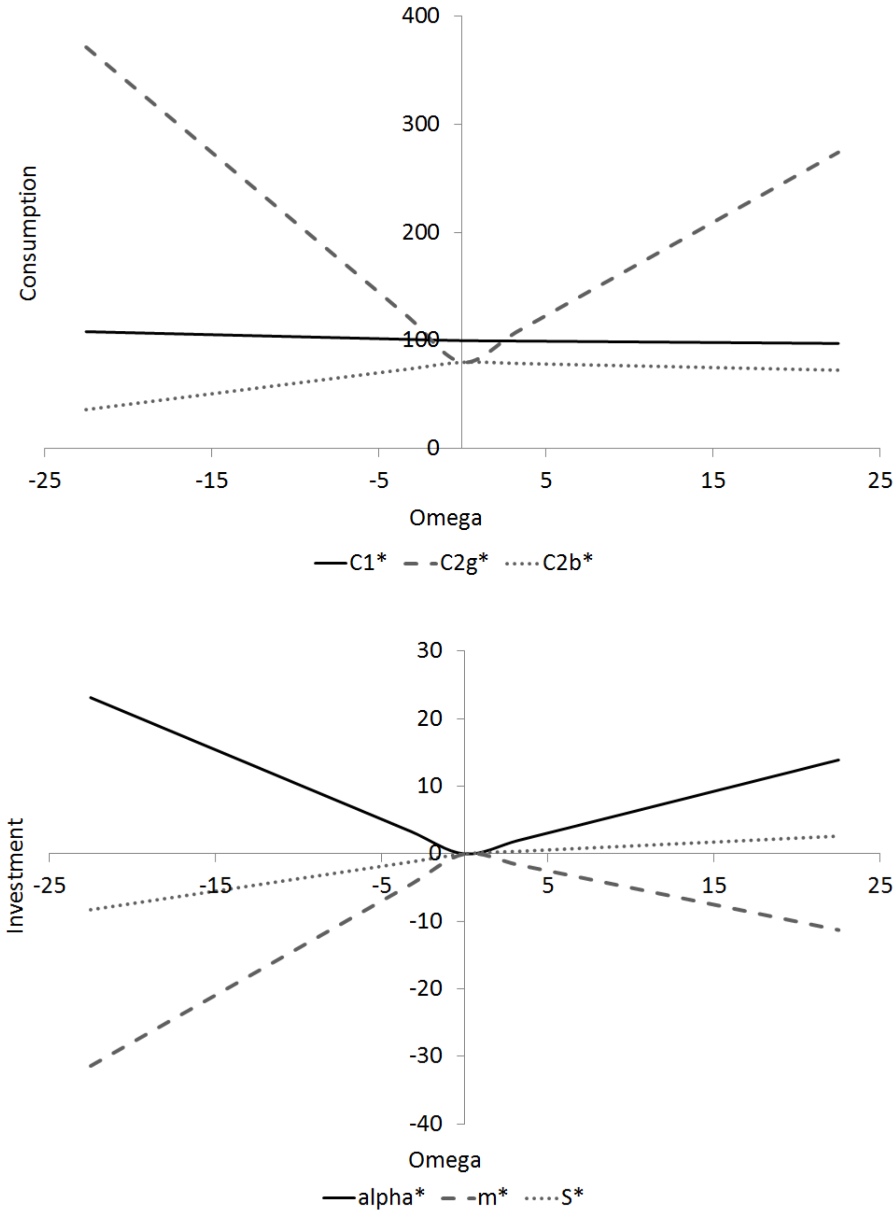

Figure 2: Optimal consumption levels and investment for different types of households.
The graphs show the optimal first and second period (good state, bad state) consumption (top graph), and the optimal investment in the risky and risk-free asset and optimal savings (bottom graph) as functions of Ω. Different values of Ω are due to different values of the reference points. Both reference points are decreasing (linearly) in such a way that the values at $\Omega = -22.5$ ($\bar{C}_1 = 105$, $\bar{C}_2 = 90$) and $\Omega = 22.5$ ($\bar{C}_1 = 95$, $\bar{C}_2 = 70$) correspond to our numerical example. This implies $\bar{C}_1 = 100$ and $\bar{C}_2 = 80$ for $\Omega = 0$. We assume $Y_1 = 100$, $Y_2 = 80$, $\lambda = 5$, $\gamma = 0.6$, $\delta = 0.8$, $r_f = 1.5$, $r_g = 15$, $r_b = 0.5$, and $p = 0.5$, like in our numerical example.

increases with a decreasing degree of loss aversion.[36] The case of $\Omega = 0$ is a special case, when the household's present value of endowment income matches exactly the discounted sum of its first and second period reference levels. Here the household compares itself to associates that are in the same social class and have the same endowment income (in present value). The sufficiently loss averse household consumes its consumption reference levels in both periods.

Even though the dependence of the household's behavior on the comparison of reference levels and endowment income may not be obvious at first sight, it can be explained by the particular feature of the loss aversion utility. On the one hand, the household is driven to consume at or above the reference level in order to avoid the penalty for consuming below the reference level. On the other hand, the endowment is limited and thus will not necessarily be large enough to enable consumption above the reference level at all times (i.e., in both periods and in both states of the nature). Intuitively, it is clear that with a large enough income it will be easy to always consume above the reference level while with a sufficiently small income the household will be forced to consume below the reference level at some point (in the first or second period, in the good or bad state of nature) and then also the degree of loss aversion will matter.[37] Consequently it should not be surprising that there are different mechanisms at work determining the household's behavior, which depend on exactly this difference in the relation of income and reference levels.[38]

	Low ref. levels self-enhancement $\Omega > 0$	Break point ref. levels same social class $\Omega = 0$	High ref. levels self-improvement $\Omega < 0$	
			$\delta \leq \delta^+$ high time preference	$\delta > \delta^+$ low time preference
Solution	(P1)	border	(P2)	(P5)
Restrictions	min. λ	min. λ	min. λ, max. $\bar{C}_2$	min. λ, max. $\bar{C}_2$
C_1^*	$C_1^* > \bar{C}_1$	$C_1^* = \bar{C}_1$	$C_1^* > \bar{C}_1$	$C_1^* < \bar{C}_1$
C_{2g}^*, C_{2b}^*	$C_{2g}^* > C_{2b}^* > \bar{C}_2$	$C_{2g}^* = C_{2b}^* = \bar{C}_2$	$C_{2g}^* > \bar{C}_2 > C_{2b}^*$	$C_{2g}^* > C_{2b}^* > \bar{C}_2$
α^*	> 0	$= 0$	> 0	> 0
$d(\mathbb{E}(U^*)/d\lambda$	$= 0$	$= 0$	< 0	< 0
$d(\mathbb{E}(U^*)/d\bar{C}_1$	< 0	$= 0$	< 0	< 0
$d(\mathbb{E}(U^*)/d\bar{C}_2$	< 0	$= 0$	< 0	< 0

Table 7: Summary of results for different reference levels relative to income.
$C_1^*, C_{2g}^*, C_{2b}^*$ and α^* are the optimal consumption levels and investment of the respective problems. The border solution is feasible for all problems (P1), ..., (P8).

Although there are many differences concerning the results for the different types of households (see Sections 3 to 5), there is an interesting observation relating to the household's happiness,

[36] We call $\mathbb{E}(U^*) = \mathbb{E}(U(C_1^*, \alpha^*))$ the household's happiness.

[37] For example, the problem the household will face is different if suddenly the household moves from the category belonging to a social class, where $\Omega = 0$, to a category where it compares itself to another social class ($\Omega \neq 0$).

[38] Also Gomes (2005) considers three distinct cases, when he investigates the optimal portfolio allocation under loss aversion. He calls these cases negative surplus wealth, zero surplus wealth, and positive surplus wealth. In addition, we have provided support for these cases from the psychology literature.

which is equally true for self-enhanced and self-improved motivated households: an increase in the economic status of a neighbor (implying an increase in this household's reference level) will make the household less happy (see Table 7). The only way the household can avoid this "unhappiness trap" is to compare to households who belong to the same social class (i.e., $\Omega = 0$). Note that households driven by the self-improvement motive are happier with a *decreasing* level of loss aversion.

Households not being sufficiently loss averse

The focus of our paper is the theoretical analysis of sufficiently (or more) loss averse households and thus Propositions 1 to 3 include assumptions on the minimum values of the loss aversion parameter, for which the statements hold. These conditions imply that solutions of the respective problems are reached in specific domains.[39] For households with low reference levels ($\Omega > 0$), for example, the minimum condition on the loss aversion parameter implies that the solution is found in problem (P1), i.e., the optimal consumption in both the first and the second period (in the good and the bad state of nature) is above its respective reference level (see Proposition 1). If the loss aversion parameter is smaller than this required minimum value but still larger than some other value, then the optimal solution will be found in a different domain, where consumption in the first or in the second period is decreased below its corresponding reference level. For an even smaller degree of loss aversion, the optimal solution will be found in still another domain (with a different "constellation" of the first and the second period consumption with respect to their reference levels). This continues until the loss aversion parameter reaches its lower bound (unity). Unfortunately, there is no given sequence of domains that will be hit consecutively. The order and also the number of different domains that will be hit depend on the given (combination of) parameters. For example, with lower discount factors it is comparatively less important (than with higher discount factors) what happens in the second period, and thus consumption below the reference point in the second period might be more plausible. Consequently, the regions where consumption is below the reference point in the second period (in the good or in the bad state, or in both states) may be hit sooner (with larger loss aversion parameters) than for higher discount factors. When reference levels are equal to or higher than income ($\Omega = 0$ or $\Omega < 0$) then the situation is similar: loss aversion parameters which are below the required minimum values stated in Proposition 2 or Proposition 3[40] will lead to solutions in different domains, where, again, the sequence and the number of domains hit for decreasing degrees of loss aversion are not determined.

8 Concluding remarks and future extensions

We can conclude from this study that reference levels and loss aversion play a very important role in determining not only optimal portfolio decisions, as has been found in the literature until

[39] These domains are specified by whether the first and the second period consumption levels are above or below the respective reference levels.

[40] The solutions described in Proposition 3 are reached in problems (P2) or (P5), depending on whether the discount factor is smaller or greater than a given number, which is a function of r_f, r_g, r_b, p, and γ.

now,[41] but also in determining inter-temporal decisions on current and future consumption levels, which depend on the total savings transferred and the risky investment activity undertaken. One-period models, investigated to date, impose the assumption that current consumption is fixed at a certain level and hence the household invests the exogenous initial wealth to the safe and the risky assets. In this model current consumption and savings are not fixed but optimally selected by the household, which generalizes the one-period model to a two-period model that can be thought of as a life cycle model. The optimal solution depends on how the household's exogenous consumption reference levels relate to the endowment income (in present value terms) and on loss aversion being sufficiently large. The relation between the consumption reference levels and the endowment income can be explained by psychological factors such as the self-enhancement or "feel good" motive when comparing to poorer households, the self-improvement motive or having high aspirations when comparing to richer households, or the "belonging motive" when the household compares itself to associates that belong to the same social class.

We find that a sufficiently loss averse and self-enhancement motivated household avoids any relative losses, both in the current period and in any future states of nature (good or bad). As a consequence the degree of loss aversion does not directly affect optimal consumption and risk taking activity. In both periods, the gap between the household's consumption and reference level shrinks as the reference level increases. In addition, the investment in the risky asset, which is always positive, decreases as the reference level increases. Following others who consume more (i.e., an increase in the reference levels) will actually hurt the household and make it less happy.

For a sufficiently loss averse household that compares to others who are in the same social class the optimal consumption is equal to the reference consumption in both periods. In addition, the investment in the risky asset is zero even if its expected return is greater than the risk-free rate. However, changes in the reference levels, driven by an increase or a decrease of income (or consumption) of the associates, can have profound effects on the household's inter-temporal decisions. From a purely technical point of view the household can not change its first period reference level without appropriately changing its second period reference level (in the opposite direction), if it wants to remain in its own social class. Hence when associates change to another social class this household has to reconsider its options available. These are: continue to compare to households of the same economic status or compare to the (higher or lower) economic status of the class the associates changed to.

A sufficiently loss averse and self-improvement motivated household cannot avoid experiencing a relative loss in the first or in the second period. As a result, loss aversion directly affects consumption and risky investment. In both periods, the gap between the household's consumption and reference level widens as the reference level increases (which is the opposite of what happens for the self-enhanced motivated household), while the gap shrinks as the loss aversion parameter increases. Moreover the investment in the risky asset, which is always positive, increases with an increasing reference level (which, again, is the opposite of what happens for the self-enhanced motivated

[41] See Hlouskova and Tsigaris (2012) for a literature review.

household), while it decreases with a growing degree of loss aversion. Note that the household's happiness increases with a decreasing level of loss aversion. Another statement about happiness, which was already made earlier, is still valid: following others in wanting to consume more (i.e., an increase in the reference levels) will make the household less happy. So this statement is equally true for self-enhancement and self-improvement motivated households.

The policy implications are clear: people need to realize that comparing their economic status to that of others will lead to increased unhappiness. This could in turn cause aggressive behavior and deteriorating mental, social and health conditions. It is not a new finding that comparing to others causes less happiness. Already Theodore Roosevelt, statesman and president of the U.S. once said "Comparison is the thief of joy." Why people still continue to compare themselves to others, when this leads to less happiness, is left for future research.

There are a number of extensions of this research that might be worth undertaking in the future. One could be to introduce uncertainty in the second period exogenous income instead of uncertainty in the returns of the risky asset. Another extension could be to consider an endogenous second period consumption reference level instead of considering it exogenous. For example, the second period consumption reference level could be a weighted average of the first period reference level and the first period consumption (habit persistence). Still another extension could be to develop a model where the utility includes consumption reference levels directly (and not only through relative consumption) in order to give households not only disutility but also pleasure from having them to compare. Hence happiness could be directly derived by following self-enhancement or self-improvement motives. Finally, one could explore the impact of taxation on the decisions to take risk and to consume today. This could be either a tax on the exogenous endowment income or a tax on capital income or a tax on both. Taxes on equity are often designed to close the gap between social classes, but if the rich and the poor classes were taxed differently this could lead to dramatic changes in the household's behavior in our set-up, depending on self-improvement and self-enhancement motivating factors.

References

[1] Abel, A., 1990. Asset prices under habit formation and catching up with the Joneses, *American Economic Review,* 80, 38–42.

[2] Achury, C., S. Hubar and C. Koulovatianos, 2012. Saving rates and portfolio choice with subsistence consumption, *Review of Economic Dynamics,* 15, 108–126.

[3] Banaji, M.R. and D.A. Prentice, 1994. The self in social contexts, *Annual review of psychology*, 45, 297–332.

[4] Barberis, N., 2013. Thirty years of prospect theory in economics: A review and assessment, *Journal of Economic Perspectives,* 27, 173–196.

[5] Barberis, N. and M. Huang, 2001. Mental accounting, loss aversion and individual stock returns, *Journal of Finance,* 56, 1247–1292.

[6] Barberis, N. and W. Xiong, 2009. What drives the disposition effect? An analysis of a long-standing preference-based explanation, *Quarterly Journal of Economics,* 116, 1–53.

[7] Berkelaar, A., R., Kouwenberg and T., Post, 2004. Optimal portfolio choice under loss aversion, *The Review of Economics and Statistics,* 86, 973–987.

[8] Bernard, C. and M. Ghossoub, 2010. Static portfolio choice under cumulative prospect theory, *Mathematics and Financial Economics*, 2, 277–306.

[9] van Bilsen, S., R.J.A. Laeven and T.E. Nijman, 2014. Consumption and portfolio choice under loss aversion and endogenous updating of the reference level, Netspar discussion papers, DP 11/2014-048.

[10] Boldrin, M., L. Christiano and J. Fisher, 2001. Habit persistence, asset returns, and the business cycle, *American Economic Review*, 91, 149–166.

[11] Bowman, D., D. Minehart and M. Rabin, 1999. Loss aversion in a consumption-savings model, *Journal of Economic Behavior and Organization*, 38, 155–178.

[12] Campbell, J. and J.H. Cochrane, 1999. By force of habit: A consumption-based explanation of aggregate stock market behavior, *Journal of Political Economy*, 107, 205–251.

[13] Christiano, L., M. Eichenbaum and C. Evans, 2005. Nominal rigidities and the dynamic effects of a shock to monetary policy, *Journal of Political Economy*, 113, 1–45.

[14] Clark, A.E., P. Frijters and M.A. Shields, 2008. Relative income, happiness, and utility: An explanation for the Easterlin paradox and other puzzles, *Journal of Economic Literature*, 46, 95–144.

[15] Constantinides, G.M., 1990. Habit formation: A resolution of the equity premium puzzle, *Journal of Political Economy,* 98, 531–552.

[16] DellaVigna, S., 2009. Psychology and economics: Evidence from the field, *Journal of Economic Literature*, 47, 315–372.

[17] Falk A. and M. Knell, 2004. Choosing the Joneses: Endogenous goals and reference standards, *Scandinavian Journal of Economics*, 106, 417–435.

[18] Festinger, L., 1954. A theory of social comparison processes, *Human relations,* 7, 117–140.

[19] Gaertner, L., Sedikides, C. and H. Cai, 2012. Wanting to be great and better but not average on the pancultural desire for self-enhancing and self-improving feedback, *Journal of Cross-Cultural Psychology*, 43, 521–526.

[20] Fortin I. and J. Hlouskova, 2011. Optimal asset allocation under linear loss aversion, *Journal of Banking and Finance*, 35, 2974–2990.

[21] Fortin I. and J. Hlouskova, 2015. Downside loss aversion: Winner or loser?, *Mathematical Methods of Operations Research*, 81, 181–233.

[22] Fortin I., J. Hlouskova and P. Tsigaris, 2015. Two-period model for consumption-investment decision with a prospect theory household, mimeo.

[23] Geary R.C., 1951. A constant utility index of the cost of living, *Review of Economic Studies*, 18, 65–66.

[24] Gomes F.J., 2005. Portfolio choice and trading volume with loss-averse investors, *Journal of Business*, 78, 675–706.

[25] He, X.D. and X.Y. Zhou, 2011. Portfolio choice under cumulative prospect theory: An analytical treatment, *Management Science*, 57, 315–331.

[26] Hlouskova, J. and P. Tsigaris, 2012. Capital income taxation and risk taking under prospect theory, *International Tax and Public Finance*, 19, 554–573.

[27] International Encyclopedia of the Social Sciences, 2016. Similarity/attraction theory, retrieved 2 November 2016 from Encyclopedia.com: http://www.encyclopedia.com/social-sciences/applied-and-social-sciences-magazines/similarityattraction-theory.

[28] Kahneman, D. and A. Tversky, 1979. Prospect theory: An analysis of decision under risk, *Econometrica*, 47, 363–391.

[29] Koszegi, B. and M. Rabin, 2006. A model of reference-dependent preferences, *Quarterly Journal of Economics*, 121, 1133–1165.

[30] Lettau, M. and H. Uhlig, 2000. Can habit formation be reconciled with business cycle facts?, *Review of Economic Dynamics*, 3, 79–99.

[31] Merton, R.C., 1969. Lifetime portfolio selection under uncertainty: The continuous time case, *Review of Economics and Statistics*, 50, 247–257.

[32] Merton, R.C., 1971. Optimum consumptio and portfolio rules in a continuous time model, *Journal of Economic Theory*, 3, 373–413.

[33] Sandmo, A., 1968. Portfolio choice in a theory of saving, *Swedish Journal of Economics*, 70, 106–122.

[34] Sandmo, A., 1969. Capital risk, consumption, and portfolio choice, *Econometrica*, 37, 586–599.

[35] Sedikides, C. and A.P. Gregg, 2008. Self-enhancement: Food for thought, *Perspectives on Psychological Science*, 3, 102–116.

[36] Stone, R., 1954. Linear expenditure systems and demand analysis: An application to the pattern of British demand, *Economic Journal*, 64, 511–527.

[37] Taylor, S. and J. Brown, 1988. Illusion and well-being: A social psychological perspective on mental health, *Psychological Bulletin*, 103, 193–210.

[38] Taylor, S., J. Lerner, D. Sherman, R. Sage and N. McDowell, 2003. Are self-enhancing cognitions associated with healthy or unhealthy biological profiles?, *Journal of Personality and Social Psychology*, 85, 605–615.

[39] Tversky, A. and D. Kahneman, 1992. Advances in prospect theory: Cumulative representation of uncertainty, *Journal of Risk and Uncertainty,* 5, 297–323.

[40] Wood, J.V., 1989. Theory and research concerning social comparisons of personal attributes, *Psychological bulletin*, 106, 231–248.

[41] Yu, X., 2015. Utility maximization with addictive consumption habit formation in incomplete smimartingale markets, *Annals of Applied Probability*, 25, 1383–1419.

Appendix A: Optimization problems

There are eight cases to consider in proofs of lemmas 1 and 2 when $\alpha \geq 0$:

(P1) $C_1 \geq \bar{C}_1$, $C_{2b} \geq \bar{C}_2$ and $C_{2g} \geq \bar{C}_2$

(P2) $C_1 \geq \bar{C}_1$, $C_{2b} \leq \bar{C}_2$ and $C_{2g} \geq \bar{C}_2$

(P3) $C_1 \geq \bar{C}_1$, $C_{2b} \geq \bar{C}_2$ and $C_{2g} \leq \bar{C}_2$

(P4) $C_1 \geq \bar{C}_1$, $C_{2b} \leq \bar{C}_2$ and $C_{2g} \leq \bar{C}_2$

(P5) $C_1 \leq \bar{C}_1$, $C_{2b} \geq \bar{C}_2$ and $C_{2g} \geq \bar{C}_2$

(P6) $C_1 \leq \bar{C}_1$, $C_{2b} \leq \bar{C}_2$ and $C_{2g} \geq \bar{C}_2$

(P7) $C_1 \leq \bar{C}_1$, $C_{2b} \geq \bar{C}_2$ and $C_{2g} \leq \bar{C}_2$

(P8) $C_1 \leq \bar{C}_1$, $C_{2b} \leq \bar{C}_2$ and $C_{2g} \leq \bar{C}_2$

The corresponding problems are

$$\left.\begin{array}{ll}
\text{Max}_{(\mathrm{C}_1,\alpha)}: & \mathbb{E}(U(C_1,\alpha)) = \\
& \frac{(C_1-\bar{C}_1)^{1-\gamma}}{1-\gamma} + \delta p \frac{((1+r_f)(Y_1-C_1)+(r_g-r_f)\alpha+Y_2-\bar{C}_2)^{1-\gamma}}{1-\gamma} \\
& +\delta(1-p)\frac{((1+r_f)(Y_1-C_1)-(r_f-r_b)\alpha+Y_2-\bar{C}_2)^{1-\gamma}}{1-\gamma} \\
\text{such that}: & \bar{C}_1 \leq C_1 \leq Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f} - \frac{r_f-r_b}{1+r_f}\alpha \\
& 0 \leq \alpha \leq \frac{\Omega}{r_f-r_b}
\end{array}\right\} \quad \text{(P1)}$$

$$\left.\begin{array}{ll}
\text{Max}_{(\mathrm{C}_1,\alpha)}: & \mathbb{E}(U(C_1,\alpha)) = \\
& \frac{(C_1-\bar{C}_1)^{1-\gamma}}{1-\gamma} + \delta p \frac{((1+r_f)(Y_1-C_1)+(r_g-r_f)\alpha+Y_2-\bar{C}_2)^{1-\gamma}}{1-\gamma} \\
& -\lambda\delta(1-p)\frac{(\bar{C}_2-Y_2-(1+r_f)(Y_1-C_1)+(r_f-r_b)\alpha)^{1-\gamma}}{1-\gamma} \\
\text{such that}: & Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f} - \frac{r_f-r_b}{1+r_f}\alpha \leq C_1 \leq Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f} + \frac{r_g-r_f}{1+r_f}\alpha \\
& \bar{C}_1 \leq C_1 \leq Y_1 + \frac{Y_2}{1+r_f} - \frac{r_f-r_b}{1+r_f}\alpha \\
& \max\left\{0, \frac{-\Omega}{r_g-r_f}\right\} \leq \alpha \leq \frac{(1+r_f)(Y_1-\bar{C}_1)+Y_2}{r_f-r_b} = \frac{\Omega+\bar{C}_2}{r_f-r_b}
\end{array}\right\} \quad \text{(P2)}$$

To guarantee that $\frac{-\Omega}{r_g-r_f} \leq \frac{\Omega+\bar{C}_2}{r_f-r_b}$ the condition $\bar{C}_2 \leq \bar{C}_2^{P2} \equiv \frac{r_g-r_b}{r_f-r_b}\big((1+r_f)(Y_1-\bar{C}_1)+Y_2\big)$ needs to be satisfied.[42]

[42]Note that the condition on $\bar{C}_2$ follows from the constraint $C_{2b} \geq 0$ given by $C_1 \leq Y_1 + \frac{Y_2}{1+r_f} - \frac{r_f-r_b}{1+r_f}\alpha$.

$$\left.\begin{array}{ll}\text{Max}_{(\text{C}_1,\alpha)}: & \mathbb{E}(U(C_1,\alpha)) = \\ & \frac{(C_1-\bar{C}_1)^{1-\gamma}}{1-\gamma} - \lambda\delta p\frac{(\bar{C}_2-Y_2-(1+r_f)(Y_1-C_1)-(r_g-r_f)\alpha)^{1-\gamma}}{1-\gamma} \\ & +\delta(1-p)\frac{((1+r_f)(Y_1-C_1)-(r_f-r_b)\alpha+Y_2-\bar{C}_2)^{1-\gamma}}{1-\gamma} \\ \text{such that}: & Y_1+\frac{Y_2-\bar{C}_2}{1+r_f}+\frac{r_g-r_f}{1+r_f}\alpha \leq C_1 \leq Y_1+\frac{Y_2-\bar{C}_2}{1+r_f}-\frac{r_f-r_b}{1+r_f}\alpha \\ & \bar{C}_1 \leq C_1 \leq Y_1+\frac{Y_2}{1+r_f}+\frac{r_g-r_f}{1+r_f}\alpha \\ & 0 \leq \alpha \leq \min\left\{0, \frac{\Omega}{r_f-r_b}\right\}\end{array}\right\} \quad \text{(P3)}$$

Note that the only feasible solution is $\left(C_1 = Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}, \alpha = 0\right)$ when $\Omega > 0$ and $\left(C_1 = \bar{C}_1, \alpha = 0\right)$ when $\Omega = 0$. There is no feasible solution when $\Omega < 0$.

$$\left.\begin{array}{ll}\text{Max}_{(\text{C}_1,\alpha)}: & \mathbb{E}(U(C_1,\alpha)) = \\ & \frac{(C_1-\bar{C}_1)^{1-\gamma}}{1-\gamma} - \lambda\delta p\frac{(\bar{C}_2-Y_2-(1+r_f)(Y_1-C_1)-(r_g-r_f)\alpha)^{1-\gamma}}{1-\gamma} \\ & -\lambda\delta(1-p)\frac{(\bar{C}_2-Y_2-(1+r_f)(Y_1-C_1)+(r_f-r_b)\alpha)^{1-\gamma}}{1-\gamma} \\ \text{such that}: & Y_1+\frac{Y_2-\bar{C}_2}{1+r_f}+\frac{r_g-r_f}{1+r_f}\alpha \leq C_1 \\ & \bar{C}_1 \leq C_1 \leq Y_1+\frac{Y_2}{1+r_f}-\frac{r_f-r_b}{1+r_f}\alpha \\ & 0 \leq \alpha \leq \frac{\bar{C}_2}{r_g-r_b}\end{array}\right\} \quad \text{(P4)}$$

$$\left.\begin{array}{ll}\text{Max}_{(\text{C}_1,\alpha)}: & \mathbb{E}(U(C_1,\alpha)) = \\ & -\lambda\frac{(\bar{C}_1-C_1)^{1-\gamma}}{1-\gamma} + \delta p\frac{((1+r_f)(Y_1-C_1)+(r_g-r_f)\alpha+Y_2-\bar{C}_2)^{1-\gamma}}{1-\gamma} \\ & +\delta(1-p)\frac{((1+r_f)(Y_1-C_1)-(r_f-r_b)\alpha+Y_2-\bar{C}_2)^{1-\gamma}}{1-\gamma} \\ \text{such that}: & 0 \leq C_1 \leq \min\left\{Y_1+\frac{Y_2-\bar{C}_2}{1+r_f}-\frac{r_f-r_b}{1+r_f}\alpha, \bar{C}_1\right\} \\ & 0 \leq \alpha \leq \frac{(1+r_f)Y_1+Y_2-\bar{C}_2}{r_f-r_b} = \frac{\Omega+(1+r_f)\bar{C}_1}{r_f-r_b}\end{array}\right\} \quad \text{(P5)}$$

$$
\left.
\begin{aligned}
\text{Max}_{(\mathrm{C}_1,\alpha)}: \quad & \mathbb{E}(U(C_1,\alpha)) = \\
& -\lambda\tfrac{(\bar{C}_1-C_1)^{1-\gamma}}{1-\gamma} + \delta p\tfrac{((1+r_f)(Y_1-C_1)+(r_g-r_f)\alpha+Y_2-\bar{C}_2)^{1-\gamma}}{1-\gamma} \\
& -\lambda\delta(1-p)\tfrac{(\bar{C}_2-Y_2-(1+r_f)(Y_1-C_1)+(r_f-r_b)\alpha)^{1-\gamma}}{1-\gamma} \\
\text{such that}: \quad & Y_1+\tfrac{Y_2-\bar{C}_2}{1+r_f}-\tfrac{r_f-r_b}{1+r_f}\alpha \le C_1 \le Y_1+\tfrac{Y_2-\bar{C}_2}{1+r_f}+\tfrac{r_g-r_f}{1+r_f}\alpha \\
& 0 \le C_1 \le \min\left\{Y_1+\tfrac{Y_2}{1+r_f}-\tfrac{r_f-r_b}{1+r_f}\alpha, \bar{C}_1\right\} \\
& \max\left\{0, \tfrac{\Omega}{r_f-r_b}\right\} \le \alpha \le \tfrac{(1+r_f)Y_1+Y_2}{r_f-r_b}
\end{aligned}
\right\} \tag{P6}
$$

$$
\left.
\begin{aligned}
\text{Max}_{(\mathrm{C}_1,\alpha)}: \quad & \mathbb{E}(U(C_1,\alpha)) = \\
& -\lambda\tfrac{(\bar{C}_1-C_1)^{1-\gamma}}{1-\gamma} - \lambda\delta p\tfrac{(\bar{C}_2-Y_2-(1+r_f)(Y_1-C_1)-(r_g-r_f)\alpha)^{1-\gamma}}{1-\gamma} \\
& +\delta(1-p)\tfrac{((1+r_f)(Y_1-C_1)-(r_f-r_b)\alpha+Y_2-\bar{C}_2)^{1-\gamma}}{1-\gamma} \\
\text{such that}: \quad & Y_1+\tfrac{Y_2-\bar{C}_2}{1+r_f}+\tfrac{r_g-r_f}{1+r_f}\alpha \le C_1 \le \min\left\{Y_1+\tfrac{Y_2-\bar{C}_2}{1+r_f}-\tfrac{r_f-r_b}{1+r_f}\alpha, \bar{C}_1\right\} \\
& 0 \le \alpha \le \min\left\{0, -\tfrac{\Omega}{r_g-r_f}\right\}
\end{aligned}
\right\} \tag{P7}
$$

Note that the only feasible solution is $\left(C_1 = Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}, \alpha = 0\right)$ when $\Omega < 0$ and $\left(C_1 = \bar{C}_1, \alpha = 0\right)$ when $\Omega = 0$. There is no feasible solution when $\Omega > 0$.

$$
\left.
\begin{aligned}
\text{Max}_{(\mathrm{C}_1,\alpha)}: \quad & \mathbb{E}(U(C_1,\alpha)) = \\
& -\lambda\tfrac{(\bar{C}_1-C_1)^{1-\gamma}}{1-\gamma} - \lambda\delta p\tfrac{(\bar{C}_2-Y_2-(1+r_f)(Y_1-C_1)-(r_g-r_f)\alpha)^{1-\gamma}}{1-\gamma} \\
& -\lambda\delta(1-p)\tfrac{(\bar{C}_2-Y_2-(1+r_f)(Y_1-C_1)+(r_f-r_b)\alpha)^{1-\gamma}}{1-\gamma} \\
\text{such that}: \quad & Y_1+\tfrac{Y_2-\bar{C}_2}{1+r_f}+\tfrac{r_g-r_f}{1+r_f}\alpha \le C_1 \le \min\left\{Y_1+\tfrac{Y_2}{1+r_f}-\tfrac{r_f-r_b}{1+r_f}\alpha, \bar{C}_1\right\} \\
& 0 \le \alpha \le -\tfrac{\Omega}{r_g-r_f}
\end{aligned}
\right\} \tag{P8}
$$

Appendix B: $\Omega \ge 0$

Before proceeding further, we introduce the following notation

$$
M \quad = \quad k\left[1+\left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right] = k_2\left[1+K_\gamma^{\frac{1}{\gamma}}\right] = k + k_2
$$

In addition notice that

$$\begin{aligned} k^{\gamma} &= K_{\gamma} k_2^{\gamma} \\ K_0^{\frac{1}{\gamma}} + K_{\gamma}^{\frac{1}{\gamma}} &= \frac{r_g - r_b}{r - r_b} K_0^{\frac{1}{\gamma}} = \frac{r_g - r_b}{r_g - r_f} K_{\gamma}^{\frac{1}{\gamma}} \\ \left(\frac{1}{K_0}\right)^{\frac{1}{\gamma}} + \left(\frac{1}{K_{\gamma}}\right)^{\frac{1}{\gamma}} &= \left(\frac{1}{K_0}\right)^{\frac{1}{\gamma}} \frac{r_g - r_b}{r_g - r_f} \end{aligned} \tag{26}$$

where k and k_2 are defined by (9) and (10).

Lemma 1 *Let $\bar{C}_1 \leq Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$, i.e., $\Omega \geq 0$ and $\lambda > \max\left\{\frac{1}{K_{\gamma}}, \lambda^{\Omega \geq 0}, \left(\frac{M}{1+r_f}\right)^{\gamma}\right\}$. Then problem (5) obtains a unique maximum at (C_1^*, α^*) where*

$$\begin{aligned} C_1^* &= \bar{C}_1 + \frac{\Omega}{1 + r_f + M} \\ &= \bar{C}_1 + \frac{1 + r_f}{1 + r_f + M}\left[\left(Y_1 + \frac{Y_2}{1 + r_f}\right) - \left(\bar{C}_1 + \frac{\bar{C}_2}{1 + r_f}\right)\right] \geq \bar{C}_1 \end{aligned} \tag{27}$$

$$\alpha^* = \frac{\left(1 - K_0^{\frac{1}{\gamma}}\right) M}{r_f - r_b + K_0^{\frac{1}{\gamma}}(r_g - r_f)}(C_1^* - \bar{C}_1) \geq 0 \tag{28}$$

Proof.

<u>**Problem (P1).**</u> Note at first that there is no feasible solution for (P1) if $C_1 > Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$. If $C_1 = Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$ then the only feasible solution is $\left(C_1 = Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}, \alpha = 0\right)$. Let $C_1 < Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$. At first we solve the concave programming problem (P1) as an unconstrained problem, i.e., we solve two equations in two unknown variables C_1 and α, namely $\frac{d\mathbb{E}(U)}{dC_1} = 0$ and $\frac{d\mathbb{E}(U)}{d\alpha} = 0$ ($\nabla \mathbb{E}(U) = 0$), obtain the optimum solution (C_1^*, α^*) and finally verify that $C_{2b}^* \geq \bar{C}_2$ and $C_{2g}^* \geq \bar{C}_2$, $C_1^* \geq \bar{C}_1$ and $-\frac{\Omega}{r_g - r_f} \leq \alpha^* \leq \frac{\Omega}{r_f - r_b}$, i.e. that the solution is also feasible.

The first order conditions are

$$\left.\begin{aligned} \frac{d\mathbb{E}(U)}{dC_1} = \; & (C_1 - \bar{C}_1)^{-\gamma} - \delta p \left[(1 + r_f)(Y_1 - C_1) + (r_g - r_f)\alpha + Y_2 - \bar{C}_2\right]^{-\gamma}(1 + r_f) \\ & - \delta(1 - p)\left[(1 + r_f)(Y_1 - C_1) - (r_f - r_b)\alpha + Y_2 - \bar{C}_2\right]^{-\gamma}(1 + r_f) = 0 \\ \frac{d\mathbb{E}(U)}{d\alpha} = \; & \delta p \left[(1 + r_f)(Y_1 - C_1) + (r_g - r_f)\alpha + Y_2 - \bar{C}_2\right]^{-\gamma}(r_g - r_f) \\ & - \delta(1 - p)\left[(1 + r_f)(Y_1 - C_1) - (r_f - r_b)\alpha + Y_2 - \bar{C}_2\right]^{-\gamma}(r_f - r_b) = 0 \end{aligned}\right\} \tag{29}$$

$\frac{d\mathbb{E}(U)}{d\alpha} = 0$ from (29) implies the following

$$\begin{aligned} & p\left[(1 + r_f)(Y_1 - C_1) - (r_f - r_b)\alpha + Y_2 - \bar{C}_2\right]^{\gamma}(r_g - r_f) \\ & = (1 - p)\left[(1 + r_f)(Y_1 - C_1) + (r_g - r_f)\alpha + Y_2 - \bar{C}_2\right]^{\gamma}(r_f - r_b) \end{aligned}$$

which after using the definition of K_γ as given by (7) gives

$$K_0^{-\frac{1}{\gamma}}\left[(1+r_f)(Y_1-C_1)-(r_f-r_b)\alpha+Y_2-\bar{C}_2\right]=(1+r_f)(Y_1-C_1)+(r_g-r_f)\alpha+Y_2-\bar{C}_2$$

This implies that

$$\alpha=\frac{1-K_0^{\frac{1}{\gamma}}}{r_f-r_b+K_0^{\frac{1}{\gamma}}(r_g-r_f)}((1+r_f)(Y_1-C_1)+Y_2-\bar{C}_2) \tag{30}$$

If we plug the last expression for α into the C_1 part of the FOC in (29) we obtain

$$\begin{aligned}
\frac{(C_1-\bar{C}_1)^{-\gamma}}{\delta(1+r_f)} &= p\left[\Omega-(1+r_f)(C_1-\bar{C}_1)+\frac{(1-K_0^{\frac{1}{\gamma}})(r_g-r_f)}{r_f-r_b+K_0^{\frac{1}{\gamma}}(r_g-r_f)}\left(\Omega-(1+r_f)(C_1-\bar{C}_1)\right)\right]^{-\gamma}\\
&+ (1-p)\left[\Omega-(1+r_f)(C_1-\bar{C}_1)-\frac{(1-K_0^{\frac{1}{\gamma}})(r_f-r_b)}{r_f-r_b+K_0^{\frac{1}{\gamma}}(r_g-r_f)}\left(\Omega-(1+r_f)(C_1-\bar{C}_1)\right)\right]^{-\gamma}\\
&= \left[\frac{r_f-r_b+K_0^{\frac{1}{\gamma}}(r_g-r_f)}{\left(\Omega-(1+r_f)(C_1-\bar{C}_1)\right)(r_g-r_b)}\right]^{\gamma}\left[p\left(1-K_0^{-1}\right)+K_0^{-1}\right] \qquad (31)
\end{aligned}$$

with assuming that $\Omega-(1+r_f)(C_1-\bar{C}_1)>0$ which is equivalent to $C_1<Y_1+\frac{Y_2-\bar{C}_2}{1+r_f}$.

After some simplifications we obtain

$$\begin{aligned}
(1+r_f)(Y_1-C_1)+Y_2-\bar{C}_2 &= (C_1-\bar{C}_1)\frac{r_f-r_b+K_0^{\frac{1}{\gamma}}(r_g-r_f)}{r_g-r_b}\left[\delta(1+r_f)p\frac{r_g-r_b}{r_f-r_b}\right]^{\frac{1}{\gamma}}\\
&= (C_1-\bar{C}_1)M \qquad (32)
\end{aligned}$$

which gives $C_1=C_1^*\geq\bar{C}_1$. Note that (27) and the assumption $\Omega\geq 0$ imply that $C_1^*\geq\bar{C}_1$. In addition, after plugging C_1^* into (30) we obtain α^* as given in (28). Note that $C_1^*<Y_1+\frac{Y_2-\bar{C}_2}{1+r_f}$ and $\alpha^*\geq 0$ as $K_0<1$ (which follows from $\mathbb{E}(r)>r_f$).

Using (29), it is easy to verify that $\frac{d^2\mathbb{E}(U)}{dC_1^2}<0$, $\frac{d^2\mathbb{E}(U)}{d\alpha^2}<0$, and $\nabla^2\mathbb{E}(U(C_1,C_2))=\frac{d^2\mathbb{E}(U)}{dC_1^2}\frac{d^2\mathbb{E}(U)}{d\alpha^2}-\left(\frac{d^2\mathbb{E}(U)}{dC_1d\alpha}\right)^2>0$ and thus problem (P1) is a concave programming problem and (C_1^*,α^*) is its unique global maximum.

Finally, C_{2g}^* and C_{2b}^* can be written as

$$\begin{aligned}
C_{2g}^* &= \bar{C}_2+\frac{M\Omega}{(1+r_f+M)\left(1+K_\gamma^{\frac{1}{\gamma}}\right)}\frac{r_g-r_b}{r_f-r_b}\\
C_{2b}^* &= \bar{C}_2+\frac{M\Omega}{(1+r_f+M)\left(1+K_\gamma^{\frac{1}{\gamma}}\right)}\frac{r_g-r_b}{r_f-r_b}K_0^{\frac{1}{\gamma}}
\end{aligned}$$

and thus they both are such that $C_{2g}^* \geq \bar{C}_2$ and $C_{2b}^* \geq \bar{C}_2$ as $K_0 \geq 0$ and $r_b < r_f < r_g$.

It can be shown that

$$(1-\gamma)\mathbb{E}(U(C_1^*, \alpha^*)) = \left(\frac{\Omega}{1+r_f}\right)^{1-\gamma}\left(1+\frac{M}{1+r_f}\right)^{\gamma} = \frac{\Omega^{1-\gamma}}{1+r_f}(1+r_f+M)^{\gamma} \tag{33}$$

As we have already mentioned the only feasible solution for $C_1 = Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}$ is $\left(C_1 = Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}, \alpha = 0\right)$ with

$$(1-\gamma)\mathbb{E}\left(U\left(Y_1+\frac{Y_2-\bar{C}_2}{1+r_f}, 0\right)\right) = \left(\frac{\Omega}{1+r_f}\right)^{1-\gamma} \tag{34}$$

which is below the value of the expected utility function at (C_1^*, α_1^*), see (33), as $M > 0$. Thus, the maximum of (P1) is reached at (C_1^*, α^*).

If $\Omega = 0$ then definition of problem (P1) implies that the only feasible solution is $(C_1, \alpha) = (\bar{C}_1, 0)$ which is then also the maximum. I.e., $(C_1^*, \alpha^*) = (\bar{C}_1, 0)$ and $C_{2b}^* = C_{2g}^* = \bar{C}_2 \geq 0$. Note in addition that $\mathbb{E}(U(C_1^*, \alpha^*)) = 0$.

Problem (P2). From the proof of Proposition 3 (problem (P2)) it can be seen that its stationary point (which is the same as in this case) happens to be (C_1^{P2}, α^{P2}), see (24). For $\Omega > 0$ is this stationary point infeasible for (P2) as $C_1^{P2} < \bar{C}_1$ and thus the maximum will occur at the border. The feasible solutions at the border for (P2) that come into consideration are given by: (i) $C_{2g} = \bar{C}_2$, (ii) $C_{2b} = \bar{C}_2$, (iii) $C_{2b} = 0$ and (iv) $C_1 = \bar{C}_1$.

Case (i). $C_{2g} = \bar{C}_2$ when $C_1 = Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f} + \frac{r_g-r_f}{1+r_f}\alpha$ and $0 \leq \alpha \leq \frac{\bar{C}_2}{r_g-r_b}$. Note that point $\left(Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}, 0\right)$ is feasible for (P1) as $\Omega \geq 0$ and $C_{2b} = C_{2g} = \bar{C}_2$.

It can be seen that

$$\begin{aligned}(1-\gamma)\mathbb{E}&\left(U\left(Y_1+\frac{Y_2-\bar{C}_2}{1+r_f}+\frac{r_g-r_f}{1+r_f}\,\alpha, \alpha\right)\right)\\ &= \left(\frac{\Omega+(r_g-r_f)\alpha}{1+r_f}\right)^{1-\gamma} - \lambda\delta(1-p)(r_g-r_b)^{1-\gamma}\alpha^{1-\gamma}\end{aligned} \tag{35}$$

and thus the potential maximum occurs either at $\alpha = 0$ or $\alpha = \frac{\bar{C}_2}{r_g-r_b}$ or at the stationary point of function given by (35) which can be easily derived and has the value $\alpha = \bar{\alpha} \equiv \frac{\lambda^{\frac{1}{\gamma}}k}{1+r_f-\lambda^{\frac{1}{\gamma}}k}\frac{\Omega}{r_g-r_f}$ when $\lambda < \left(\frac{1+r_f}{k}\right)^{\gamma}$ and is infeasible for $\lambda > \left(\frac{1+r_f}{k}\right)^{\gamma}$. Thus, for $\lambda < \left(\frac{1+r_f}{k}\right)^{\gamma}$

$$\begin{aligned}(1-\gamma)\mathbb{E}\left(U\left(C_1 = Y_1+\frac{Y_2-\bar{C}_2}{1+r_f}+\frac{r_g-r_f}{1+r_f}\,\bar{\alpha}, \bar{\alpha}\right)\right) &= \frac{\Omega^{1-\gamma}}{1+r_f}\left(1+r_f-\lambda^{\frac{1}{\gamma}}k\right)^{\gamma}\\ \leq \left(\frac{\Omega}{1+r_f}\right)^{1-\gamma} &= (1-\gamma)\mathbb{E}\left(U\left(Y_1+\frac{Y_2-\bar{C}_2}{1+r_f}, 0\right)\right)\end{aligned}$$

therefore point $\left(C_1 = Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f} + \frac{r_g - r_f}{1+r_f}\,\bar{\alpha}, \bar{\alpha}\right)$ can not be a maximum as its utility function is below the utility function of $\left(Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}, 0\right)$ which is feasible for (P1).

The end-point $\alpha = \frac{\bar{C}_2}{r_g - r_b}$, which gives $C_{2b} = 0$ is dealt with in case (iii).

Case (ii). Any feasible solution of this case is also the feasible solution of (P1).

Case (iii). $C_{2b} = 0$ when $C_1 = Y_1 + \frac{Y_2}{1+r_f} - \frac{r_f - r_b}{1+r_f}\alpha$ and $\frac{\bar{C}_2}{r_g - r_b} \leq \alpha \leq \frac{(1+r_f)(Y_1 - \bar{C}_1) + Y_2}{r_f - r_b}$. It can be seen that

$$
\begin{aligned}
(1-\gamma)\mathbb{E}\left(U\left(Y_1 + \frac{Y_2}{1+r_f} - \frac{r_f - r_b}{1+r_f}\,\alpha, \alpha\right)\right) &= \left(Y_1 - \bar{C}_1 + \frac{Y_2}{1+r_f} - \frac{r_f - r_b}{1+r_f}\alpha\right)^{1-\gamma} \\
&+ \delta p\left((r_g - r_b)\alpha - \bar{C}_2\right)^{1-\gamma} - \lambda\delta(1-p)\bar{C}_2^{1-\gamma}
\end{aligned} \tag{36}
$$

As function (36) is concave in α the maximum is reached at the stationary point $\bar{\alpha}$, i.e. $\frac{d\mathbb{E}\left(U\left(Y_1 + \frac{Y_2}{1+r_f} - \frac{r_f - r_b}{1+r_f}\alpha, \alpha\right)\right)}{d\alpha}\Big|_{\alpha=\bar{\alpha}} = 0$ which is given as follows

$$
\begin{aligned}
\bar{\alpha} &= \frac{k_2((1+r_f)(Y_1 - \bar{C}_1) + Y_2) + \frac{r_f - r_b}{r_g - r_b}(1+r_f)\bar{C}_2}{(1+r_f+k_2)(r_f - r_b)} \\
&= \frac{1+r_f}{1+r_f+k_2}\frac{\bar{C}_2}{r_g - r_b} + \frac{k_2}{1+r_f+k_2}\frac{(1+r_f)(Y_1 - \bar{C}_1) + Y_2}{r_f - r_b}
\end{aligned}
$$

The expected utility at this point is given by

$$
\begin{aligned}
&(1-\gamma)\mathbb{E}\left(U\left(Y_1 + \frac{Y_2}{1+r_f} - \frac{r_f - r_b}{1+r_f}\,\bar{\alpha}, \bar{\alpha}\right)\right) \\
&= \frac{(1+r_f+k_2)^\gamma}{1+r_f}\left((1+r_f)(Y_1 - \bar{C}_1) + Y_2 - \frac{r_f - r_b}{r_g - r_b}\bar{C}_2\right)^{1-\gamma} - \lambda\delta(1-p)\bar{C}_2^{1-\gamma}
\end{aligned} \tag{37}
$$

If $\lambda > \lambda^{\Omega>0}$, see (11), then the utility given by (37) will be below the utility of (P1) at its maximum,[43] which is given by $\frac{\Omega^{1-\gamma}}{1+r_f}(1+r_f+M)^\gamma$, see (33).

Case (iv). $C_1 = \bar{C}_1$ for $\frac{\Omega}{r_f - r_b} \leq \alpha \leq \frac{(1+r_f)(Y_1 - \bar{C}_1) + Y - 2}{r_f - r_b}$ where $\alpha = \frac{\Omega}{r_f - r_b}$ is implied by $C_{2b} = \bar{C}_2$ and thus the point is feasible for (P1) and $\alpha = \frac{(1+r_f)(Y_1 - \bar{C}_1) + Y - 2}{r_f - r_b}$ is given by $C_{2b} = 0$ which is an example dealt with in case (iii). Thus, what remains to dealt with is the stationary point that can be easily derived, namely

$$
\alpha = \frac{1 + (\lambda K_0)^{1/\gamma}}{1 - (\lambda K_\gamma)^{1/\gamma}}\frac{\Omega}{r_f - r_b}
$$

which is feasible only for $\lambda < \frac{1}{K_\gamma}$ which contradicts our assumption.

It follows from the proof above that for $\Omega = 0$ the maximum, for $\lambda > \left(\frac{1+r_f}{k} + \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right)^\gamma$, is achieved at $(C_1 = \bar{C}_1, \alpha = 0)$ where the value of the expected utility is zero.

[43] And thus, for $\lambda < \lambda^{\Omega>0}$ the utility given by (37) will exceed the utility of (P1) at its maximum.

Problem (P3). The only feasible solution is $\left(Y_1+\frac{Y_2-\bar{C}_2}{1+r_f},0\right)$ which is feasible also for (P1).

Problem (P4). As $\frac{d^2\mathbb{E}(U)}{d\alpha^2}>0$ then there is no local interior maximum which implies that the maximum will occur at the border. The cases under consideration are: (i) $C_{2g}=\bar{C}_2$, (ii) $C_{2b}=\bar{C}_2$, (iii) $C_{2g}=0$, (iv) $C_{2b}=0$ and (v) $C_1=\bar{C}_1$. Note that case (i) coincides with case (i) when proving problem (P2) and that the only feasible solution in case (ii) and case (iii) is $\left(C_1=Y_1+\frac{Y_2-\bar{C}_2}{1+r_f},\alpha=0\right)$ which is feasible for (P1).

Case (iv). $C_{2b}=0$ when $C_1=Y_1+\frac{Y_2}{1+r_f}-\frac{r_f-r_b}{1+r_f}\alpha$ and $0\le\alpha\le\frac{\bar{C}_2}{r_g-r_b}$ and thus

$$(1-\gamma)\mathbb{E}\left(U\left(C_1=Y_1+\frac{Y_2}{1+r_f}-\frac{r_f-r_b}{1+r_f}\alpha,\alpha\right)\right)=\left(Y_1-\bar{C}_1+\frac{Y_2}{1+r_f}-\frac{r_f-r_b}{1+r_f}\alpha\right)^{1-\gamma} -\lambda\delta p\left(\bar{C}_2-(r_g-r_b)\alpha\right)^{1-\gamma}-\lambda\delta(1-p)\,\bar{C}_2^{1-\gamma} \tag{38}$$

The potential candidates for maximum are $\alpha=0$, $\alpha=\frac{\bar{C}_2}{r_g-r_b}$ and $\alpha=\bar{\alpha}$ where $\bar{\alpha}$ is a unique stationary point such that $\left.\frac{d\mathbb{E}(U)}{d\alpha}\right|_{\alpha=\bar{\alpha}}=0$ where

$$\bar{\alpha}=\frac{(\lambda\delta p(r_g-r_b))^{\frac{1}{\gamma}}\left(Y_1-\bar{C}_1+\frac{Y_2}{1+r_f}\right)-\left(\frac{r_f-r_b}{1+r_f}\right)^{\frac{1}{\gamma}}\bar{C}_2}{(\lambda\delta p(r_g-r_b))^{\frac{1}{\gamma}}\frac{r_f-r_b}{1+r_f}-\left(\frac{r_f-r_b}{1+r_f}\right)^{\frac{1}{\gamma}}(r_g-r_b)}>\frac{\bar{C}_2}{r_g-r_b}$$

and thus infeasible for $\Omega\ge0$ and $\lambda>\left(\frac{1+r_f}{k_2}\right)^{\gamma}$.

For $\alpha=\frac{\bar{C}_2}{r_g-r_b}$ is $C_{2g}=\bar{C}_2$ and thus the point $\left(C_1=Y_1+\frac{Y_2}{1+r_f}-\frac{r_f-r_b}{1+r_f}\frac{\bar{C}_2}{r_g-r_b},\alpha=\frac{\bar{C}_2}{r_g-r_b}\right)$ is feasible for (P2).

Finally, we show that the utility function at $\alpha=0$ is below the utility function at (C_1^*,α^*). Namely

$$\left(Y_1-\bar{C}_1+\frac{Y_2}{1+r_f}\right)^{1-\gamma}-\lambda\delta\bar{C}_2^{1-\gamma}=\left(\frac{\Omega+\bar{C}_2}{1+r_f}\right)^{1-\gamma}-\lambda\delta\bar{C}_2^{1-\gamma}\le\left(\frac{\Omega}{1+r_f}\right)^{1-\gamma}\left(1+\frac{M}{1+r_f}\right)^{\gamma}$$

which holds if

$$\lambda>\frac{\Omega^{1-\gamma}}{\delta(1+r_f)\bar{C}_2^{1-\gamma}}\left[\left(1+\frac{\bar{C}_2}{\Omega}\right)^{1-\gamma}(1+r_f)^{\gamma}-(1+r_f+M)^{\gamma}\right]$$

The last inequality holds as $\lambda>\lambda^{\Omega\ge0}$ and

$$\lambda^{\Omega\ge0}>\frac{\Omega^{1-\gamma}}{\delta(1+r_f)\bar{C}_2^{1-\gamma}}\left[\left(1+\frac{\bar{C}_2}{\Omega}\right)^{1-\gamma}(1+r_f)^{\gamma}-(1+r_f+M)^{\gamma}\right]$$

Case (v). There is no feasible solution for $C_1=\bar{C}_1$ for $\Omega>0$ and for $\Omega=0$ the only feasible solution is $(\bar{C}_1,0)$.

For $\Omega = 0$ the conclusions obtained in (P2) apply, i.e., if
$\lambda > \left(\frac{1+r_f}{k} + \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right)^\gamma$ then maximum is reached at $(C_1 = \bar{C}_1, \alpha = 0)$.

Regarding **problem (P5)** we show that for $C_1 \leq Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$ and $\Omega \geq 0$ is its maximum reached at the point that is feasible also for (P1), namely at $(\bar{C}_1, \bar{\alpha})$ (with $\bar{\alpha}$ being defined later), and as utility functions of (P1) and (P5) coincide at this point then the utility function of (P1) at (C_1^*, α^*) exceeds the one at $(\bar{C}_1, \bar{\alpha})$.[44]

In more detail, as $\frac{d\mathbb{E}(U)}{d\alpha}$ is the same for both (P1) and (P5) then the second equation in (29) implies that for any fixed C_1 is the expected utility of (P5) concave and thus its maximum is achieved at (30). I.e., if $\tilde{C}_1$ is such that $0 \leq \tilde{C}_1 \leq \bar{C}_1 \leq Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$ then for this fixed $\tilde{C}_1$ is maximum of (P5) reached at $(\tilde{C}_1, \tilde{\alpha})$ where $\tilde{\alpha} = \frac{1 - K_0^{\frac{1}{\gamma}}}{r_f - r_b + K_0^{\frac{1}{\gamma}}(r_g - r_f)}((1+r_f)(Y_1 - \tilde{C}_1) + Y_2 - \bar{C}_2)$. Note that $(\tilde{C}_1, \tilde{\alpha})$ is feasible for (P5). Thus, the only candidates for the maximum for (P5) are $(\tilde{C}_1, \tilde{\alpha})$ with $0 \leq \tilde{C}_1 \leq \bar{C}_1$. By plugging this point into the expected utility of (P5) we obtain

$$
\begin{aligned}
& (1-\gamma)\mathbb{E}(U) = -\lambda(\bar{C}_1 - \tilde{C}_1)^{1-\gamma} \\
+ \quad & \delta p \left[(1+r_f)(Y_1 - \tilde{C}_1) + Y_2 - \bar{C}_2 + \frac{(r_g - r_f)(1 - K_0^{1/\gamma})}{r_f - r_b + K_0^{1/\gamma}(r_g - r_f)} \Big((1+r_f)(Y_1 - \tilde{C}_1) + Y_2 - \bar{C}_2\Big)\right]^{1-\gamma} \\
+ \quad & \delta(1-p) \left[(1+r_f)(Y_1 - \tilde{C}_1) + Y_2 - \bar{C}_2 - \frac{(r_f - r_b)(1 - K_0^{1/\gamma})}{r_f - r_b + K_0^{1/\gamma}(r_g - r_f)} \Big((1+r_f)(Y_1 - \tilde{C}_1) + Y_2 - \bar{C}_2\Big)\right]^{1-\gamma}
\end{aligned}
$$

which after some derivations gives

$$
\begin{aligned}
(1-\gamma)\mathbb{E}(U) &= -\lambda(\bar{C}_1 - \tilde{C}_1)^{1-\gamma} + \delta \Big((1+r_f)(Y_1 - \tilde{C}_1) + Y_2 - \bar{C}_2\Big)^{1-\gamma} \left(\frac{r_g - r_b}{r_f - r_b + K_0^{1/\gamma}(r_g - r_f)}\right)^{1-\gamma} \\
&\quad \times \left(p + (1-p)K_0^{\frac{1-\gamma}{\gamma}}\right) \\
&= -\lambda(\bar{C}_1 - \tilde{C}_1)^{1-\gamma} + \left(\frac{k_2\left(1 + K_\gamma^{\frac{1}{\gamma}}\right)}{1+r_f}\right)^\gamma \left(Y_1 - \tilde{C}_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}\right)^{1-\gamma}
\end{aligned}
$$

[44] Note that there is no feasible solution in (P5) for $C_1 > Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$.

If the expected utility of (P5) is increasing function in $\tilde{C}_1$, i.e., $\frac{d\mathbb{E}(U)}{d\tilde{C}_1} > 0$ then the maximum will be reached at $(\bar{C}_1, \bar{\alpha})$, where $\bar{\alpha} = \frac{1-K_0^{\frac{1}{\gamma}}}{r_f - r_b + K_0^{\frac{1}{\gamma}}(r_g - r_f)}\Omega$. In more detail, the inequality below

$$\frac{d\mathbb{E}(U)}{d\tilde{C}_1} = \lambda(\bar{C}_1 - \tilde{C}_1)^{-\gamma} - \left(\frac{k_2\left(1+K_\gamma^{\frac{1}{\gamma}}\right)}{1+r_f}\right)^{\gamma}\left(Y_1 - \tilde{C}_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}\right)^{-\gamma} > 0$$

holds if

$$\lambda > \left(\frac{k_2\left(1+K_\gamma^{\frac{1}{\gamma}}\right)}{1+r_f}\right)^{\gamma}\left(\frac{\bar{C}_1 - \tilde{C}_1}{\frac{\Omega}{1+r_f} + \bar{C}_1 - \tilde{C}_1}\right)^{\gamma} \tag{39}$$

If $\tilde{C}_1 < \bar{C}_1$ then the right hand side of the inequality (39) is below $\left(\frac{k_2\left(1+K_\gamma^{\frac{1}{\gamma}}\right)}{1+r_f}\right)^{\gamma} = \left(\frac{M}{1+r_f}\right)^{\gamma}$, where M is defined by (8), and as this is exceeded by λ, i.e., $\lambda > \left(\frac{M}{1+r_f}\right)^{\gamma}$, see assumptions of Proposition 1, then $\mathbb{E}(U)$ is increasing in $\tilde{C}_1$ and the maximum is reached at $(\bar{C}_1, \bar{\alpha})$, what we wanted to show.

It can be derived in the same way that for $\Omega = 0$ the utility of (P5)

$$(1-\gamma)\mathbb{E}(U) = (\bar{C}_1 - \tilde{C}_1)^{1-\gamma}\left[-\lambda + \left(\frac{k_2\left(1+K_\gamma^{\frac{1}{\gamma}}\right)}{1+r_f}\right)^{\gamma}\right]$$

is an increasing function in $\tilde{C}_1$ for $\lambda > \left(\frac{M}{1+r_f}\right)^{\gamma}$ and thus the maximum will be reached for $(C_1 = \bar{C}_1, \alpha = 0)$.

Problem (P6) can be dealt with in a very similar way as problem (P6) for $\Omega < 0$ (see the proof of Proposition 3), where there is no feasible stationary point and thus all possible border solutions are examined. Note that there is no feasible solution of **problem (P7)** when $\Omega > 0$ and the only feasible solution for $\Omega = 0$ is $(C_1 = \bar{C}_1, \alpha = 0)$. Finally, **problem (P8)** has no feasible solution. ■

Appendix C: $\Omega < 0$

Proof of Proposition 3.

We proceed as follows: At first we assume that $C_1 \leq Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$ and then $C_1 > Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$.

Note that for $C_1 \leq Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f} < \bar{C}_1$ (as $\Omega < 0$) it follows that only cases $C_1 < \bar{C}_1$ could be considered and thus only problems (P5)–(P8) need to be solved.

Problem (P1). For $\Omega < 0$ there is no feasible solution for (P1).

Problem (P2). There is no feasible solution for $C_1 \leq Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}$. Let $C_1 > Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}$ and in addition we assume $\lambda > \lambda_1^{\Omega<0}$ and $\bar{C}_2 < \frac{r_g-r_b}{r_f-r_b}\left((1+r_f)\left(Y_1-\bar{C}_1\right)+Y_2\right) = \bar{C}_2^{P2}$. We proceed in the following way: At first we solve the problem (P2) as an unconstrained problem i.e., we solve $\nabla\mathbb{E}(U) = 0$, so that the FOC are satisfied, obtain the unique solution (C_1^{P2}, α^{P2}), verify that the objective function of (P2) is concave at (C_1^{P2}, α^{P2}) and that the solution is also feasible. As the utility function is differentiable at the domain under consideration, (C_1^{P2}, α^{P2}) is the only local extrema (namely local maximum) and if the objective function at the border of (P2) does not exceed its value at (C_1^{P2}, α^{P2}), then this point is also a global maximum of (P2) when $\lambda > \lambda_1^{\Omega<0}$.

The first order conditions are

$$\left.\begin{aligned}
\frac{d\mathbb{E}(U)}{dC_1} = \quad & (C_1-\bar{C}_1)^{-\gamma} \quad -\delta p\left[(1+r_f)(Y_1-C_1)+Y_2-\bar{C}_2+(r_g-r_f)\alpha\right]^{-\gamma}(1+r_f) \\
& \qquad -\lambda\delta(1-p)\left[\bar{C}_2-Y_2-(1+r_f)(Y_1-C_1)+(r_f-r_b)\alpha\right]^{-\gamma}(1+r_f) = 0 \\
\frac{d\mathbb{E}(U)}{d\alpha} = \quad & \qquad \delta p\left[(1+r_f)(Y_1-C_1)+Y_2-\bar{C}_2+(r_g-r_f)\alpha\right]^{-\gamma}(r_g-r_f) \\
& \qquad -\lambda\delta(1-p)\left[\bar{C}_2-Y_2-(1+r_f)(Y_1-C_1)+(r_f-r_b)\alpha\right]^{-\gamma}(r_f-r_b) = 0
\end{aligned}\right\} \quad (40)$$

$\frac{d\mathbb{E}(U)}{d\alpha} = 0$ from (40) implies the following

$$\begin{aligned}
& p\left[\bar{C}_2-Y_2-(1+r_f)(Y_1-C_1)+(r_f-r_b)\alpha\right]^{\gamma}(r_g-r_f) \\
& = \lambda(1-p)\left[(1+r_f)(Y_1-C_1)+Y_2-\bar{C}_2+(r_g-r_f)\alpha\right]^{\gamma}(r_f-r_b)
\end{aligned}$$

which gives

$$\left(\frac{1}{K_0}\right)^{\frac{1}{\gamma}}\left[\bar{C}_2-Y_2-(1+r_f)(Y_1-C_1)+(r_f-r_b)\alpha\right] = \lambda^{\frac{1}{\gamma}}\left[(1+r_f)(Y_1-C_1)+Y_2-\bar{C}_2+(r_g-r_f)\alpha\right]$$

This implies that

$$\begin{aligned}
\alpha &= \frac{\lambda^{\frac{1}{\gamma}}+\left(\frac{1}{K_0}\right)^{\frac{1}{\gamma}}}{\lambda^{\frac{1}{\gamma}}(r_g-r_f)-\left(\frac{1}{K_0}\right)^{\frac{1}{\gamma}}(r_f-r_b)}(\bar{C}_2-Y_2-(1+r_f)(Y_1-C_1)) \\
&= \frac{\lambda^{\frac{1}{\gamma}}+\left(\frac{1}{K_0}\right)^{\frac{1}{\gamma}}}{\left(\lambda^{\frac{1}{\gamma}}-\left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right)(r_g-r_f)}(\bar{C}_2-Y_2-(1+r_f)(Y_1-C_1)) \qquad (41)
\end{aligned}$$

If we plug the last expression for α into the C_1 part of the FOC in (40) we obtain

$$\begin{aligned}\frac{(C_1-\bar{C}_1)^{-\gamma}}{\delta(1+r_f)} &= \left[\bar{C}_2 - Y_2 - (1+r_f)(Y_1-C_1)\right]^{-\gamma} \\ &\times \left[p\left(\frac{\left(\frac{1}{K_0}\right)^{\frac{1}{\gamma}}+\left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}}{\lambda^{\frac{1}{\gamma}}-\left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}}\right)^{-\gamma} + \lambda(1-p)\left(1+\frac{\lambda^{\frac{1}{\gamma}}+\left(\frac{1}{K_0}\right)^{\frac{1}{\gamma}}}{\lambda^{\frac{1}{\gamma}}-\left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}}\frac{r_f-r_b}{r_g-r_f}\right)^{-\gamma}\right]\end{aligned}$$

After some simplifications we obtain

$$\begin{aligned}\bar{C}_2 - Y_2 - (1+r_f)(Y_1-C_1) &= (C_1-\bar{C}_1)\frac{r_g-r_f}{r_g-r_b}\left[\delta(1+r_f)(1-p)\frac{r_g-r_b}{r_g-r_f}\right]^{\frac{1}{\gamma}}\left[\lambda^{\frac{1}{\gamma}}-\left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right] \\ &= (C_1-\bar{C}_1)M_1(\lambda) \qquad (42)\end{aligned}$$

which gives (24). In addition, after plugging C_1^{P2} into (41) we obtain α^{P2} as given in (25). Note that

$$\bar{C}_2 - Y_2 - (1+r_f)(Y_1 - C_1^{P2}) = \frac{M_1(\lambda)(-\Omega)}{M_1(\lambda)-1-r_f}$$

and thus assumption $C_1 > Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}$ is satisfied for $C_1 = C_1^{P2}$ only if $\Omega < 0$ which happens to be our assumption.

Note that (24), the assumptions $\Omega < 0$ and $\lambda > \lambda_1^{\Omega<0}$ (which gives $M_1(\lambda) > 1 + r_f$) imply that $C_1^{P2} > \bar{C}_1$.

What remains to be shown is when is the expected utility function strictly concave at (C_1^{P2}, α^{P2}). For this to hold it is sufficient to show that the following holds at (C_1^{P2}, α^{P2}): $\frac{d^2\mathbb{E}(U)}{d\alpha^2} < 0$ and $D \equiv \nabla^2\mathbb{E}(U(C_1,C_2)) = \frac{d^2\mathbb{E}(U)}{dC_1^2}\frac{d^2\mathbb{E}(U)}{d\alpha^2} - \left(\frac{d^2\mathbb{E}(U)}{dC_1 d\alpha}\right)^2 > 0$. Note that

$$\begin{aligned}C_{2g}^{P2} - \bar{C}_2 &= \frac{\left(\frac{1}{K_0}\right)^{\frac{1}{\gamma}}(r_g-r_b)}{(r_g-r_f)\left(\lambda^{\frac{1}{\gamma}}-\left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right)}\frac{M_1(\lambda)}{M_1(\lambda)-1-r_f}\left(\bar{C}_2 - Y_2 - (1+r_f)(Y_1-\bar{C}_1)\right) \\ &= k\frac{-\Omega}{M_1(\lambda)-1-r_f}\frac{r_g-r_b}{r_g-r_f}\left(\frac{1}{K_0}\right)^{\frac{1}{\gamma}} \qquad (43)\end{aligned}$$

$$\begin{aligned}\bar{C}_2 - C_{2b}^{P2} &= \frac{\lambda^{\frac{1}{\gamma}}(r_g-r_b)}{(r_g-r_f)\left(\lambda^{\frac{1}{\gamma}}-\left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right)}\frac{M_1(\lambda)}{M_1(\lambda)-1-r_f}\left(\bar{C}_2 - Y_2 - (1+r_f)(Y_1-\bar{C}_1)\right) \\ &= k\frac{-\Omega}{M_1(\lambda)-1-r_f}\frac{r_g-r_b}{r_g-r_f}\lambda^{\frac{1}{\gamma}} \qquad (44)\end{aligned}$$

and thus $\bar{C}_2 - C_{2b}^{P2} = (K_0\lambda)^{\frac{1}{\gamma}}(C_{2g}^{P2} - \bar{C}_2)$. Using (40), (43) and (44) we obtain the following

$$\frac{1}{\gamma}\frac{d^2\mathbb{E}(U)}{dC_1^2}|_{(C_1,\alpha)=(C_1^{P2},\alpha^{P2})} = \left[\frac{-\Omega}{M_1(\lambda)-1-r_f}\right]^{-1-\gamma} \times\left[-1+\frac{1+r_f}{k}\left(\frac{r_g-r_f}{r_g-r_b}\right)^2\left(\lambda^{-\frac{1}{\gamma}} - K_0^{\frac{1}{\gamma}}\frac{r_f-r_b}{r_g-r_f}\right)\right] \tag{45}$$

$$\frac{1}{\gamma}\frac{d^2\mathbb{E}(U)}{d\alpha^2}|_{(C_1,\alpha)=(C_1^{P2},\alpha^{P2})} = \frac{(r_f-r_b)^2}{k(1+r_f)}\left(\frac{r_g-r_f}{r_g-r_b}\right)^2\left[\frac{-\Omega}{M_1(\lambda)-1-r_f}\right]^{-1-\gamma}\left(\lambda^{-\frac{1}{\gamma}} - K_\gamma^{\frac{1}{\gamma}}\right) \tag{46}$$

$$\frac{1}{\gamma}\frac{d^2\mathbb{E}(U)}{dC_1d\alpha}|_{(C_1,\alpha)=(C_1^{P2},\alpha^{P2})} = \frac{r_f-r_b}{k}\left(\frac{r_g-r_f}{r_g-r_b}\right)^2\left[\frac{-\Omega}{M_1(\lambda)-1-r_f}\right]^{-1-\gamma}\left(\lambda^{-\frac{1}{\gamma}} + K_0^{\frac{1}{\gamma}}\right)$$

Note that (46) and $\lambda > \frac{1}{K_\gamma}$ imply that $\frac{d^2\mathbb{E}(U)}{d\alpha^2}|_{(C_1,\alpha)=(C_1^{P2},\alpha^{P2})} < 0$. In addition,

$$\frac{1}{\gamma^2}\left[\frac{-\Omega}{M_1(\lambda)-1-r_f}\right]^{2(1+\gamma)} D = \left[-1+\frac{1+r_f}{k}\left(\frac{r_g-r_f}{r_g-r_b}\right)^2\left(\lambda^{-\frac{1}{\gamma}} - \frac{r_f-r_b}{r_g-r_f}K_0^{\frac{1}{\gamma}}\right)\right] \times\frac{r_f-r_b}{k(1+r_f)}\left(\frac{r_g-r_f}{r_g-r_b}\right)^2\left[(r_f-r_b)\lambda^{-\frac{1}{\gamma}} - (r_g-r_f)K_0^{\frac{1}{\gamma}}\right] - \left(\frac{r_f-r_b}{k}\right)^2\left(\frac{r_g-r_f}{r_g-r_b}\right)^4\left(\lambda^{-\frac{1}{\gamma}} + K_0^{\frac{1}{\gamma}}\right)^2$$

and thus

$$\frac{1}{\gamma^2}\left[\frac{-\Omega}{M_1(\lambda)-1-r_f}\right]^{2(1+\gamma)}\left(\frac{r_g-r_b}{r_g-r_f}\right)^2\frac{k}{r_f-r_b}D = \frac{1}{1+r_f}\left[(r_g-r_f)K_0^{\frac{1}{\gamma}} - (r_f-r_b)\lambda^{-\frac{1}{\gamma}}\right]$$

$$+ \frac{1}{k}\left(\frac{r_g-r_f}{r_g-r_b}\right)^2\left(\lambda^{-\frac{1}{\gamma}} - \frac{r_f-r_b}{r_g-r_f}K_0^{\frac{1}{\gamma}}\right)\left[(r_f-r_b)\lambda^{-\frac{1}{\gamma}} - (r_g-r_f)K_0^{\frac{1}{\gamma}}\right] - \frac{r_f-r_b}{k}\left(\frac{r_g-r_f}{r_g-r_b}\right)^2\left(\lambda^{-\frac{1}{\gamma}} + K_0^{\frac{1}{\gamma}}\right)^2$$

After some derivations we obtain

$$\frac{1}{\gamma^2}\left[\frac{-\Omega}{M_1(\lambda)-1-r_f}\right]^{2(1+\gamma)}\left(\frac{r_g-r_b}{r_g-r_f}\right)^2\frac{k}{r_f-r_b}D = \frac{1}{1+r_f}\left[(r_g-r_f)K_0^{\frac{1}{\gamma}} - (r_f-r_b)\lambda^{-\frac{1}{\gamma}}\right] - \frac{r_g-r_f}{k}\lambda^{-\frac{1}{\gamma}}K_0^{\frac{1}{\gamma}}$$

Now it can be easily shown that if $\lambda > \left[\frac{1+r_f}{k} + \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right]^\gamma$, which follows from assumption $\lambda > \lambda_1^{\Omega<0}$ and (21), then $D > 0$.

Regarding the feasibility, note that (43) and (44) imply that $C_{2g}^{P2} > \bar{C}_2$ and $C_{2b}^{P2} < \bar{C}_2$. In addition, (44), $\bar{C}_2 \leq \bar{C}_2^{P2}$ and $\lambda > \lambda_1^{\Omega<0}$ imply that $C_{2b}^{P2} \geq 0$.[45]

Note that

$$\begin{aligned}
(1-\gamma)\mathbb{E}(U(C_1^{P2}, \alpha^{P2})) &= \left(\frac{-\Omega}{M_1(\lambda)-1-r_f}\right)^{1-\gamma}\left[1+\frac{k}{1+r_f}\left(\left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}} - \lambda^{\frac{1}{\gamma}}\right)\right] \\
&= -\frac{(-\Omega)^{1-\gamma}}{1+r_f}\left[k\left(\lambda^{\frac{1}{\gamma}} - \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right) - 1 - r_f\right]^{\gamma} \\
&= -\frac{(-\Omega)^{1-\gamma}}{1+r_f}k^{\gamma}\left[\lambda^{\frac{1}{\gamma}} - (1-\tilde{c}^{P2})\left(\lambda_1^{\Omega<0}\right)^{\frac{1}{\gamma}}\right]^{\gamma} \\
&= -\frac{(-\Omega)^{1-\gamma}}{1+r_f}\left(M_1(\lambda)-1-r_f\right)^{\gamma} \qquad (47)
\end{aligned}$$

What remains to show is that feasible solutions at the border do not exceed the expected utility at (C_1^{P2}, α^{P2}), where (P2) obtains its local maximum. The feasible solution at the border that come into consideration are: (i) $C_{2g} = \bar{C}_2$, (ii) $C_{2b} = \bar{C}_2$, (iii) $C_{2b} = 0$ and (iv) $C_1 = \bar{C}_1$.

Case (i). $C_{2g} = \bar{C}_2$ when $C_1 = Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f} + \frac{r_g - r_f}{1+r_f}\alpha$ for $\frac{-\Omega}{r_g - r_f} \leq \alpha \leq \frac{\bar{C}_2}{r_g - r_b}$ and thus

$$\begin{aligned}
(1-\gamma)\mathbb{E}\left(U\left(Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f} + \frac{r_g - r_f}{1+r_f}\alpha,\, \alpha\right)\right) &= \left(\frac{\Omega + (r_g - r_f)\alpha}{1+r_f}\right)^{1-\gamma} \\
&\quad -\lambda\delta(1-p)(r_g - r_b)^{1-\gamma}\alpha^{1-\gamma}
\end{aligned}$$

The following can be easily shown

$$\lim_{\alpha \to^+ \frac{-\Omega}{r_g - r_f}} \frac{d\mathbb{E}\left(U\left(Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f} + \frac{r_g-r_f}{1+r_f}\,\alpha,\, \alpha\right)\right)}{d\alpha} = +\infty$$

and

$$\left.\frac{d\mathbb{E}\left(U\left(Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f} + \frac{r_g-r_f}{1+r_f}\,\alpha,\, \alpha\right)\right)}{d\alpha}\right|_{\alpha=\alpha_1} = 0$$

where

$$\alpha_1 = \frac{k\lambda^{\frac{1}{\gamma}}(-\Omega)}{\left(k\lambda^{\frac{1}{\gamma}} - 1 - r_f\right)(r_g - r_f)}$$

Note in addition that $\alpha_1 \leq \frac{\bar{C}_2}{r_g - r_b}$ for $\lambda > \left[\frac{1+r_f}{k(1-\tilde{c}^{P2})}\right]^{\gamma}$, where $\tilde{c}^{P2} = \frac{(r_g - r_b)(-\Omega)}{(r_g - r_f)\bar{C}_2}$ and that

$$\left.\frac{d^2\mathbb{E}\left(U\left(Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f} + \frac{r_g-r_f}{1+r_f}\,\alpha,\, \alpha\right)\right)}{d\alpha^2}\right|_{\alpha=\alpha_1} < 0$$

[45] Note that if we would not want to guarantee $C_{2b} \geq 0$ then it would be sufficient to have $\lambda_1^{\Omega<0} = \left[\frac{1+r_f}{k} + \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right]^{\gamma}$. The more complicated expression of $\lambda_1^{\Omega<0}$ defined by (21) follows from the constraint $C_{2b} \geq 0$. Also condition $\bar{C}_2 < \bar{C}_2^{P2}$ is implied by the constraint $C_{2b} \geq 0$.

Thus, for $\lambda > \left[\frac{1+r_f}{k(1-\tilde{c}^{P2})}\right]^\gamma$ is the maximum reached at α_1. As

$$(1-\gamma)\mathbb{E}\left(U\left(Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f} + \frac{r_g - r_f}{1+r_f}\,\alpha_1,\, \alpha_1\right)\right) = -\frac{(-\Omega)^{1-\gamma}}{1+r_f}\left(k\lambda^{\frac{1}{\gamma}} - 1 - r_f\right)^\gamma \tag{48}$$

then based on this it can be shown that for $\lambda > \lambda_1^{\Omega<0}$

$$(1-\gamma)\mathbb{E}\left(U\left(Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f} + \frac{r_g - r_f}{1+r_f}\,\alpha_1,\, \alpha_1\right)\right) < (1-\gamma)\mathbb{E}\left(U\left(C_1^{P2},\, \alpha^{P2}\right)\right)$$

where $(1-\gamma)\mathbb{E}\left(U\left(C_1^{P2},\, \alpha^{P2}\right)\right) = -\frac{(-\Omega)^{1-\gamma}}{1+r_f}\left[k\left(\lambda^{\frac{1}{\gamma}} - \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right) - 1 - r_f\right]^\gamma$, see (47). In summary, there are only two possible candidates for the maximum: (1) $\alpha = \frac{\bar{C}_2}{r_g - r_b}$, which is tackled in case (iii) and (2) $\alpha = \alpha_1$ for $\lambda > \left[\frac{1+r_f}{k(1-\tilde{c}^{P2})}\right]^\gamma$ where we have shown that expected utility at $\left(C_1 = Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f} + \frac{r_g - r_f}{1+r_f}\,\alpha_1,\, \alpha_1\right)$ is smaller than the expected utility at $\left(C_1^{P2},\, \alpha^{P2}\right)$.

Case (ii). $C_{2b} = \bar{C}_2$ when $C_1 = Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f} - \frac{r_f - r_b}{1+r_f}\alpha$. This case has no feasible solution as C_1 can not exceed $Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$.

Case (iii). $C_{2b} = 0$ when $C_1 = Y_1 + \frac{Y_2}{1+r_f} - \frac{r_f - r_b}{1+r_f}\alpha$ for $\frac{\bar{C}_2}{r_g - r_b} \le \alpha \le \frac{(1+r_f)(Y_1 - \bar{C}_1) + Y_2}{r_f - r_b}$ and thus

$$\begin{aligned}(1-\gamma)\mathbb{E}\left(U\left(Y_1 + \frac{Y_2}{1+r_f} - \frac{r_f - r_b}{1+r_f}\,\alpha,\, \alpha\right)\right) \quad &= \quad \left(\frac{(1+r_f)(Y_1 - \bar{C}_1) + Y_2 - (r_f - r_b)\alpha}{1+r_f}\right)^{1-\gamma} \\ &\quad + \delta p\, \left((r_g - r_b)\alpha - \bar{C}_2\right)^{1-\gamma} - \lambda\delta(1-p)\bar{C}_2^{1-\gamma}\end{aligned}$$

It can be shown that this expected utility function is concave and thus its maximum is reached either at α^{P20} where

$$\left.\frac{d\mathbb{E}\left(U\left(Y_1 + \frac{Y_2}{1+r_f} - \frac{r_f - r_b}{1+r_f}\alpha,\, \alpha\right)\right)}{d\alpha}\right|_{\alpha=\alpha^{P20}} = 0$$

or at one of the end-points. After some derivations we obtain

$$\alpha^{P20} = \frac{1+r_f}{k_2 + 1 + r_f}\frac{\bar{C}_2}{r_g - r_b} + \frac{k_2}{k_2 + 1 + r_f}\frac{(1+r_f)(Y_1 - \bar{C}_1) + Y_2}{r_f - r_b} \tag{49}$$

which is a convex combination of the end-points and thus the maximum is reached at $\alpha = \alpha^{P20}$ such that $\frac{\bar{C}_2}{r_g - r_b} < \alpha^{P20} < \frac{(1+r_f)(Y_1 - \bar{C}_1) + Y_2}{r_f - r_b}$. The last inequalities imply that optimal C_1 is strictly above $\bar{C}_1$ and optimal C_{2g} is strictly above $\bar{C}_2$. Further derivations give

$$(1-\gamma)\mathbb{E}\left(U\left(C_1^{P20} = Y_1 + \frac{Y_2}{1+r_f} - \frac{r_f - r_b}{1+r_f}\,\alpha^{P20},\, \alpha^{P20}\right)\right) =$$
$$= \frac{k^\gamma}{1+r_f}\left(\frac{r_g - r_f}{r_g - r_b}\,\bar{C}_2\right)^{1-\gamma}\left[\left(\left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}} + \frac{1+r_f}{k}\right)^\gamma (1-\tilde{c}^{P2})^{1-\gamma} - \lambda\right]$$
$$= \delta(1-p)\,\bar{C}_2^{1-\gamma}\left[\lambda_1^{\Omega<0}(1-\tilde{c}^{P2}) - \lambda\right] \tag{50}$$

Finally, based on this and (47) it can be shown that for $\lambda > \lambda_1^{\Omega<0}$

$$(1-\gamma)\mathbb{E}\left(U\left(C_1^{P20},\, \alpha^{P20}\right)\right) < (1-\gamma)\mathbb{E}\left(U\left(C_1^{P2},\, \alpha^{P2}\right)\right) \tag{51}$$

if

$$\left(\lambda^{\frac{1}{\gamma}} - \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}} - \frac{1+r_f}{k}\right)^\gamma \left(\tilde{c}^{P2}\right)^{1-\gamma} < \lambda - \left(\left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}} + \frac{1+r_f}{k}\right)^\gamma \left(1 - \left(\tilde{c}^{P2}\right)\right)^{1-\gamma}$$

or if

$$F(\lambda) \equiv \lambda - \left(\left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}} + \frac{1+r_f}{k}\right)^\gamma \left(1 - \left(\tilde{c}^{P2}\right)\right)^{1-\gamma} - \left(\lambda^{\frac{1}{\gamma}} - \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}} - \frac{1+r_f}{k}\right)^\gamma \left(\tilde{c}^{P2}\right)^{1-\gamma} > 0$$

The last inequality holds as $F(\lambda)$ is a convex function with the minimum being reached at $\lambda = \lambda_1^{\Omega<0}$, see (21), where $F(\lambda_1^{\Omega<0}) = 0$.

Case (iv). $C_1 = \bar{C}_1$ for $\frac{-\Omega}{r_g - r_f} \leq \alpha \leq \frac{(1+r_f)(Y_1 - \bar{C}_1) + Y_2}{r_f - r_b}$ and thus

$$(1-\gamma)\mathbb{E}\left(U\left(\bar{C}_1,\, \alpha\right)\right) \;=\; \delta p\left(\Omega + (r_g - r_f)\alpha\right)^{1-\gamma} - \lambda\delta(1-p)\left(-\Omega + (r_f - r_b)\alpha\right)^{1-\gamma} \tag{52}$$

The following can be easily shown

$$\lim_{\alpha \to^+ \frac{-\Omega}{r_g - r_f}} \frac{d\mathbb{E}\left(U\left(\bar{C}_1,\, \alpha\right)\right)}{d\alpha} = +\infty \tag{53}$$

and

$$\left.\frac{d\mathbb{E}\left(U\left(\bar{C}_1,\, \alpha\right)\right)}{d\alpha}\right|_{\alpha = \alpha_3} = 0$$

for

$$\alpha_3 = \frac{\lambda^{\frac{1}{\gamma}} + \left(\frac{1}{K_0}\right)^{\frac{1}{\gamma}}}{\lambda^{\frac{1}{\gamma}} - \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}}\frac{(-\Omega)}{r_g - r_f}$$

and $\lambda > \frac{1}{K_\gamma}$. As

$$(1-\gamma)\mathbb{E}\left(U\left(\bar{C}_1,\, \alpha_3\right)\right) = -\frac{k^\gamma(-\Omega)^{1-\gamma}}{1+r_f}\left(\lambda^{\frac{1}{\gamma}} - \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right)^\gamma \tag{54}$$

then based on this and (47) it can be shown that for $\lambda > \lambda_1^{\Omega<0}$

$$(1-\gamma)\mathbb{E}\left(U\left(\bar{C}_1,\, \alpha_3\right)\right) < (1-\gamma)\mathbb{E}\left(U\left(C_1^{P2},\, \alpha^{P2}\right)\right)$$

Note that the maximum can not be reached at $\frac{-\Omega}{r_g-r_f}$, see (53), and another end-point, $\alpha = \frac{(1+r_f)(Y_1-\bar{C}_1)+Y_2}{r_f-r_b}$, is tackled in case (iii).

There is no feasible solution for **problem (P3)** when $\Omega < 0$.

Problem (P4). There is no feasible solution for $C_1 \leq Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}$. Let $C_1 \geq Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}$. At first we show that no local extreme can be a local maximum. This implies that, as the function is continuous, a maximum will occur at the border of the set of feasible solutions.

In more detail

$$\begin{aligned}\frac{d\mathbb{E}(U)}{d\alpha} &= \lambda\delta p\left[\bar{C}_2 - Y_2 - (1+r_f)(Y_1-C_1) - (r_g-r_f)\alpha\right]^{-\gamma}(r_g-r_f) \\ &\quad -\lambda\delta(1-p)\left[\bar{C}_2 - Y_2 - (1+r_f)(Y_1-C_1) + (r_f-r_b)\alpha\right]^{-\gamma}(r_f-r_b)\end{aligned}$$

and thus

$$\begin{aligned}\frac{d^2\mathbb{E}(U)}{d\alpha^2} &= \lambda\gamma\delta p\left[\bar{C}_2 - Y_2 - (1+r_f)(Y_1-C_1) - (r_g-r_f)\alpha\right]^{-\gamma-1}(r_g-r_f)^2 \\ &\quad +\lambda\gamma\delta(1-p)\left[\bar{C}_2 - Y_2 - (1+r_f)(Y_1-C_1) + (r_f-r_b)\alpha\right]^{-\gamma-1}(r_f-r_b)^2 > 0\end{aligned}$$

This excludes the possibility of the objective function of (P4) to obtain its maximum in the interior and thus it would occur at the border of the feasible solutions of problem (P4). Now we will consider feasible solutions at the border, namely: (i) $C_{2g} = \bar{C}_2$, (ii) $C_{2b} = \bar{C}_2$, (iii) $C_{2g} = 0$, (iv) $C_{2b} = 0$ and (v) $C_1 = \bar{C}_1$. Note that case (i) coincides with case (i) when proving (P2) and there is only one feasible solution in cases (ii) and (iii), namely $\left(C_1 = Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}, \alpha = 0\right)$, which is feasible also for (P2).

Case (iii). The only feasible solution for $C_{2g} = 0$ is $(C_1 = Y_1 + \frac{Y_2}{1+r_f}, \alpha = 0)$ with the utility function being

$$(1-\gamma)\mathbb{E}\left(U\left(Y_1 + \frac{Y_2}{1+r_f}, 0\right)\right) = \left(Y_1 - \bar{C}_1 + \frac{Y_2}{1+r_f}\right)^{1-\gamma} - \lambda\delta\bar{C}_2^{1-\gamma}$$

which is dealt in case (iv) below.

$$(1-\gamma)\mathbb{E}(U(C_1,\alpha)) = \left(\frac{(1+r_f)(Y_1-\bar{C}_1)+Y_2-(r_f-r_b)\alpha}{1+r_f}\right)^{1-\gamma} - \lambda\delta p(\bar{C}_2-(r_g-r_b)\alpha)^{1-\gamma} - \lambda\delta(1-p)\bar{C}_2^{1-\gamma} \tag{55}$$

<u>Case (iv).</u> $C_{2b}=0$ when $C_1 = Y_1 + \frac{Y_2}{1+r_f} - \frac{r_f-r_b}{1+r_f}\alpha$ and $0 \le \alpha \le \frac{\bar{C}_2}{r_g-r_b}$ and the utility function is given by (38). The potential candidates for maximum are $\alpha = 0$, $\alpha = \frac{\bar{C}_2}{r_g-r_b}$ and $\alpha = \bar{\alpha}$ where $\bar{\alpha}$ is a unique stationary point such that $\left.\frac{d\mathbb{E}(U)}{d\alpha}\right|_{\alpha=\bar{\alpha}} = 0$ where

$$\bar{\alpha} = \frac{(\lambda\delta p(r_g-r_b))^{\frac{1}{\gamma}}\left(Y_1-\bar{C}_1+\frac{Y_2}{1+r_f}\right) - \left(\frac{r_f-r_b}{1+r_f}\right)^{\frac{1}{\gamma}}\bar{C}_2}{(\lambda\delta p(r_g-r_b))^{\frac{1}{\gamma}}\frac{r_f-r_b}{1+r_f} - \left(\frac{r_f-r_b}{1+r_f}\right)^{\frac{1}{\gamma}}(r_g-r_b)}$$

Note that for $\lambda > \left(\frac{1+r_f}{k_2}\right)^{\gamma}$ and $\bar{C}_2 < \bar{C}_2^{P2}$ is $\bar{\alpha}$ infeasible and for $\bar{C}_2 = \bar{C}_2^{P2}$ is $\bar{\alpha} = \frac{\bar{C}_2}{r_g-r_b}$. For $\alpha = \frac{\bar{C}_2}{r_g-r_b}$ is the point $\left(C_1 = Y_1 + \frac{Y_2}{1+r_f} - \frac{r_f-r_b}{1+r_f}\frac{\bar{C}_2}{r_g-r_b}, \alpha = \frac{\bar{C}_2}{r_g-r_b}\right)$ feasible for (P2). Finally, we show that the utility function at $\alpha = 0$ is below the utility function at $\left(C_1^{P2}, \alpha^{P2}\right)$; i.e., that

$$\mathbb{E}\left(U\left(Y_1 + \frac{Y_2}{1+r_f}, 0\right)\right) \le \mathbb{E}\left(U\left(C_1^{P2}, \alpha^{P2}\right)\right) \tag{56}$$

We proceed in two steps. If $\tilde{\lambda}^{\Omega<0} \ge \lambda_1^{\Omega<0}$ then we show that

$$\mathbb{E}\left(U\left(Y_1 + \frac{Y_2}{1+r_f}, 0\right)\right) \le \mathbb{E}\left(U\left(C_1^{P20}, \alpha^{P20}\right)\right) \tag{57}$$

where $\left(C_1^{P20}, \alpha^{P20}\right)$ and $\mathbb{E}\left(U\left(C_1^{P20}, \alpha^{P20}\right)\right)$ are given by (49) and (50). Inequality (57) holds for $\lambda \ge \tilde{\lambda}^{\Omega<0}$ which implies that also (56) holds as for $\lambda > \lambda_1^{\Omega<0}$ is $\mathbb{E}\left(U\left(C_1^{P20}, \alpha^{P20}\right)\right) < \mathbb{E}\left(U\left(C_1^{P2}, \alpha^{P2}\right)\right)$. On the other hand, if $\tilde{\lambda}^{\Omega<0} < \lambda_1^{\Omega<0}$ then (56) can be shown directly.

Let $\tilde{\lambda}^{\Omega<0} \ge \lambda_1^{\Omega<0}$. Then (57) holds if

$$\left(Y_1 - \bar{C}_1 + \frac{Y_2}{1+r_f}\right)^{1-\gamma} - \lambda\delta\,\bar{C}_2^{1-\gamma} \le \delta(1-p)\bar{C}_2^{1-\gamma}\left[\left(1-\tilde{c}^{P2}\right)\lambda_1^{\Omega<0} - \lambda\right]$$

which holds if

$$\lambda \ge \tilde{\lambda}^{\Omega<0} \equiv \frac{1}{p}\left[\frac{1}{\delta}\left(\frac{Y_1-\bar{C}_1+\frac{Y_2}{1+r_f}}{\bar{C}_2}\right)^{1-\gamma} - (1-p)\left(1-\tilde{c}^{P2}\right)\lambda_1^{\Omega<0}\right]$$

Let $\tilde{\lambda}^{\Omega<0} > \lambda_1^{\Omega<0}$. Then for $\lambda > \lambda_1^{\Omega<0}$ (56) holds if

$$\frac{(-\Omega)^{1-\gamma}}{1+r_f} k^\gamma \left(\lambda^{\frac{1}{\gamma}} - \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}} - \frac{1+r_f}{k} \right)^\gamma \leq \lambda\delta\, \bar{C}_2^{1-\gamma} - \left(Y_1 - \bar{C}_1 + \frac{Y_2}{1+r_f} \right)^{1-\gamma}$$

Let

$$G(\lambda) \equiv \frac{(-\Omega)^{1-\gamma}}{1+r_f} k^\gamma \left(\lambda^{\frac{1}{\gamma}} - \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}} - \frac{1+r_f}{k} \right)^\gamma - \lambda\delta\, \bar{C}_2^{1-\gamma} + \left(Y_1 - \bar{C}_1 + \frac{Y_2}{1+r_f} \right)^{1-\gamma} \tag{58}$$

As $G(\lambda)$ is a continuous decreasing function in λ[46] and as $G(\tilde{\lambda}^{\Omega<0}) \leq 0 \leq G(\lambda_1^{\Omega<0})$ then there exists $\lambda_0^{P2} \in \left[\lambda_1^{\Omega<0}, \tilde{\lambda}^{\Omega<0}\right]$ such that $G(\lambda_0^{P2}) = 0$. Thus, is obtained at $\left(C_1 = C_1^{P2},\, \alpha = \alpha^{P2}\right)$ if $\lambda > \lambda_0^{P2}$.

Thus, (56) holds for $\lambda > \max\left\{\tilde{\lambda}^{\Omega<0}, \lambda_1^{\Omega<0}\right\}$.

Case (v). The utility function of (P4) with $C_1 = \bar{C}_1$ is

$$\begin{aligned} \mathbb{E}(U(C_1,\alpha)) &= -\lambda\delta p \frac{(-\Omega - (r_g - r_f)\alpha)^{1-\gamma}}{1-\gamma} - \lambda\delta(1-p)\frac{(-\Omega + (r_f - r_b)\alpha)^{1-\gamma}}{1-\gamma} \\ & \quad \text{for } \; 0 \leq \alpha \leq \frac{-\Omega}{r_g - r_f} \end{aligned} \tag{59}$$

when $\bar{C}_2 \leq \bar{C}_2^{P2}$. It can be easily shown that utility function (59) is convex and thus its maximum is reached at either $\alpha = 0$, for which is $C_1 = \bar{C}_1, \alpha = 0$ infeasible, or $\alpha = \frac{-\Omega}{r_g - r_f}$, for which is $\left(\bar{C}_1, \frac{-\Omega}{r_g - r_f}\right)$ feasible for (P2).

Problem (P5). Note that for $C_1 \geq Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$ there is only one feasible solution, namely $\left(C_1 = Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}, \alpha = 0\right)$, which is thus feasible for case when $C_1 \leq Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$ (see below).

When $C_1 \leq Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$ it is easy to show that for any fixed $\tilde{C}_1$ such that $0 \leq \tilde{C}_1 \leq Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$ is the expected utility of (P5) concave and thus its maximum is achieved at

$$\tilde{\alpha} = \frac{1 - K_0^{\frac{1}{\gamma}}}{r_f - r_b + K_0^{\frac{1}{\gamma}}(r_g - r_f)} ((1+r_f)(Y_1 - \tilde{C}_1) + Y_2 - \bar{C}_2) \geq 0 \tag{60}$$

Note also that $(\tilde{C}_1, \tilde{\alpha})$ is feasible for (P5). Thus, the candidates for the maximum for (P5) are $(\tilde{C}_1, \tilde{\alpha})$ with $0 \leq \tilde{C}_1 \leq Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$ and $\tilde{\alpha}$ given by (60). By plugging this point into the expected utility of (P5) we obtain (after some derivations)

$$(1-\gamma)\mathbb{E}(U) = -\lambda(\bar{C}_1 - \tilde{C}_1)^{1-\gamma} + \hat{\lambda}\left(Y_1 - \tilde{C}_1 + \frac{Y_2 - \bar{C}_2}{1+r_f} \right)^{1-\gamma} \tag{61}$$

[46] This follows from $\lambda \geq \lambda_1^{\Omega<0} \geq \left(\frac{\left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}} + \frac{1+r_f}{k}}{1-(1-p)^{\frac{1}{1-p}}\, \tilde{c}^{P2}} \right)^\gamma$ where the latter inequality follows from $\lambda_1^{\Omega<0} \leq \tilde{\lambda}^{\Omega<0}$.

where

$$\hat{\lambda} = \left[\frac{k_2\left(1+K^{\frac{1}{\gamma}}_{\gamma}\right)}{1+r_f}\right]^{\gamma} = \left[\frac{k\left(1+\left(\frac{1}{K_{\gamma}}\right)^{\frac{1}{\gamma}}\right)}{1+r_f}\right]^{\gamma}$$

As this expected utility is not monotone or concave – in $\tilde{C}_1$ such that $0 \leq \tilde{C}_1 \leq Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}$ – the maximum of (61) can be reached at either end points (see cases (i) and (ii) below) or at the point (see case (iii) below) where $\left.\frac{d\mathbb{E}(U)}{d\tilde{C}_1}\right|_{\tilde{C}_1=C_1^{P5}} = 0$ with $\mathbb{E}(U)$ being given by (61). Thus, the cases under consideration are

(i) $\tilde{C}_1 = 0$ where

$$\mathcal{U}^{P5i} \equiv (1-\gamma)\mathbb{E}(U) = -\lambda\bar{C}_1^{1-\gamma} + \hat{\lambda}\left(Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}\right)^{1-\gamma}$$

(ii) $\tilde{C}_1 = Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}$ where

$$\mathcal{U}^{P5ii} \equiv (1-\gamma)\mathbb{E}(U) = -\lambda\left(\frac{-\Omega}{1+r_f}\right)^{1-\gamma}$$

(iii) $\tilde{C}_1 = C_1^{P5} \equiv \frac{\left(Y_1+\frac{Y_2-\bar{C}_2}{1+r_f}\right)\lambda^{\frac{1}{\gamma}} - \bar{C}_1\hat{\lambda}^{\frac{1}{\gamma}}}{\lambda^{\frac{1}{\gamma}}-\hat{\lambda}^{\frac{1}{\gamma}}}$ for $\lambda \geq \lambda_2^{\Omega<0}$ where

$$\lambda_2^{\Omega<0} = \hat{\lambda}\left(\frac{\bar{C}_1}{Y_1+\frac{Y_2-\bar{C}_2}{1+r_f}}\right)^{\gamma}$$

to guarantee that $0 \leq \tilde{C}_1 \leq Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}$. Note in addition that C_1^{P5} is the only stationary point of $\mathbb{E}(U)$ given by (61). Thus,

$$\begin{aligned}\mathcal{U}^{P5} \equiv (1-\gamma)\mathbb{E}(U) &= -\left(\frac{-\Omega}{1+r_f}\right)^{1-\gamma}\left(\lambda^{\frac{1}{\gamma}} - \frac{k_2\left(1+K^{\frac{1}{\gamma}}_{\gamma}\right)}{1+r_f}\right)^{\gamma} \qquad (62)\\ &= -\left(\frac{-\Omega}{1+r_f}\right)^{1-\gamma}\left(\lambda^{\frac{1}{\gamma}}-\hat{\lambda}^{\frac{1}{\gamma}}\right)^{\gamma}\end{aligned}$$

It can be shown that $\mathbb{E}(U)$ given by (61) is concave at $C_1 = C_1^{P5}$ given by case (iii) as C_1^{P5} is the only stationary point there. Thus, the maximum for (P5) with $\lambda \geq \lambda_2^{\Omega<0}$ is reached in case (iii), i.e., at point

$$\left(0 < C_1^{P5} = \frac{\left(Y_1+\frac{Y_2-\bar{C}_2}{1+r_f}\right)\lambda^{\frac{1}{\gamma}} - \bar{C}_1\hat{\lambda}^{\frac{1}{\gamma}}}{\lambda^{\frac{1}{\gamma}}-\hat{\lambda}^{\frac{1}{\gamma}}} < \bar{C}_1,\ \alpha^{P5} = \frac{1-K_0^{\frac{1}{\gamma}}}{r_f - r_b + K_0^{\frac{1}{\gamma}}(r_g-r_f)}\frac{\hat{\lambda}^{\frac{1}{\gamma}}}{\lambda^{\frac{1}{\gamma}}-\hat{\lambda}^{\frac{1}{\gamma}}}(-\Omega) > 0\right)$$

Let

$$\mathcal{U}^{P2} \equiv (1-\gamma)\mathbb{E}(U(C_1^{P2}, \alpha^{P2})) = -\frac{(-\Omega)^{1-\gamma}}{1+r_f}\left[k\left(\lambda^{\frac{1}{\gamma}} - \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right) - 1 - r_f\right]^\gamma$$

Note that (62) can be written also as

$$\mathcal{U}^{P5} = (1-\gamma)\mathbb{E}\left(U\left(C_1^{P5}, \alpha^{P5}\right)\right) = -\left(\frac{-\Omega}{1+r_f}\right)^{1-\gamma}\left(\lambda^{\frac{1}{\gamma}} - \frac{k\left(1+\left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right)}{1+r_f}\right)^\gamma$$

Then for $\lambda \geq \lambda_2^{\Omega<0}$ the utility function of problem (P2) at its maxima is related to the utility function of (P5) at its maximum as follows: $\mathcal{U}^{P2} > \mathcal{U}^{P5}$ for $k < 1+r_f$, $\mathcal{U}^{P2} = \mathcal{U}^{P5}$ for $k = 1+r_f$ and $\mathcal{U}^{P2} < \mathcal{U}^{P5}$ for $k > 1+r_f$. Note in addition that condition $k \leq 1+r_f$ is equivalent to $\delta \leq \delta^+$ and condition $k_2 \leq 1+r_f$ is equivalent to $\delta \leq \delta^-$.

Problem (P6). We show at first that there is no interior local maximum or minimum for (P6) which implies that the maximum will occur at the border of the set of feasible solutions for (P6). Then we check all potential feasible solutions at the border.

Let $C_1 \geq Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$. The first order conditions are

$$\left.\begin{aligned}
\frac{d\mathbb{E}(U)}{dC_1} = {} & \lambda(\bar{C}_1 - C_1)^{-\gamma} - \delta p\left[(1+r_f)(Y_1 - C_1) + Y_2 - \bar{C}_2 + (r_g - r_f)\alpha\right]^{-\gamma}(1+r_f) \\
& -\lambda\delta(1-p)\left[\bar{C}_2 - Y_2 - (1+r_f)(Y_1 - C_1) + (r_f - r_b)\alpha\right]^{-\gamma}(1+r_f) = 0 \\
\frac{d\mathbb{E}(U)}{d\alpha} = {} & \delta p\left[(1+r_f)(Y_1 - C_1) + Y_2 - \bar{C}_2 + (r_g - r_f)\alpha\right]^{-\gamma}(r_g - r_f) \\
& -\lambda\delta(1-p)\left[\bar{C}_2 - Y_2 - (1+r_f)(Y_1 - C_1) + (r_f - r_b)\alpha\right]^{-\gamma}(r_f - r_b) = 0
\end{aligned}\right\} \quad (63)$$

$\frac{d\mathbb{E}(U)}{d\alpha} = 0$ from (63) implies the expression for α given by (41) and if we plug it into the C_1 part of the FOC in (63) we obtain after some simplifications

$$\begin{aligned}
\lambda^{\frac{1}{\gamma}}(\bar{C}_2 - Y_2 - (1+r_f)(Y_1 - C_1)) &= (\bar{C}_1 - C_1)\frac{r_g - r_f}{r_g - r_b}\left[\delta(1+r_f)(1-p)\frac{r_g - r_b}{r_g - r_f}\right]^{\frac{1}{\gamma}}\left[\lambda^{\frac{1}{\gamma}} - \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right] \\
&= (\bar{C}_1 - C_1)M_1(\lambda)
\end{aligned}$$

which gives

$$C_1^+ = \frac{\bar{C}_1 M_1(\lambda) + \lambda^{\frac{1}{\gamma}}[(1+r_f)Y_1 + Y_2 - \bar{C}_2]}{M_1(\lambda) + \lambda^{\frac{1}{\gamma}}(1+r_f)} = \bar{C}_1 + \frac{\Omega}{\frac{M_1(\lambda)}{\lambda^{\frac{1}{\gamma}}} + 1 + r_f} \quad (64)$$

In addition, after plugging C_1^+ from (64) into (41) we obtain

$$\alpha^+ = \frac{k}{r_g - r_f}\left[\left(\frac{1}{K_0}\right)^{\frac{1}{\gamma}} + \lambda^{\frac{1}{\gamma}}\right]\frac{-\Omega}{M_1(\lambda) + \lambda^{\frac{1}{\gamma}}(1+r_f)}$$

Next we show that the expected utility function is indifferent at (C_1^+, α^+), namely, we show that at (C_1^+, α^+) are $\frac{d^2\mathbb{E}(U)}{d\alpha^2} < 0$, and $D_3 \equiv \nabla^2\mathbb{E}(U(C_1, C_2)) = \frac{d^2\mathbb{E}(U)}{dC_1^2}\frac{d^2\mathbb{E}(U)}{d\alpha^2} - \left(\frac{d^2\mathbb{E}(U)}{dC_1 d\alpha}\right)^2 < 0$. Note that

$$C_{2g}^+ - \bar{C}_2 = k\frac{-\Omega}{M_1(\lambda) + \lambda^{\frac{1}{\gamma}}(1+r_f)}\frac{r_g - r_b}{r_g - r_f}\left(\frac{1}{K_0}\right)^{\frac{1}{\gamma}} \quad (65)$$

$$\bar{C}_2 - C_{2b}^+ = k\frac{-\Omega}{M_1(\lambda) + \lambda^{\frac{1}{\gamma}}(1+r_f)}\frac{r_g - r_b}{r_g - r_f}\lambda^{\frac{1}{\gamma}} \quad (66)$$

and thus $\bar{C}_2 - (C_{2b}^+) = (K_0\lambda)^{\frac{1}{\gamma}}((C_{2g}^+) - \bar{C}_2)$. Using (63), (65) and (66) we obtain the following

$$\frac{1}{\gamma}\frac{d^2\mathbb{E}(U)}{dC_1^2}|_{(C_1^+,\alpha^+)} = \left[\frac{-\Omega}{M_1(\lambda) + \lambda^{\frac{1}{\gamma}}(1+r_f)}\right]^{-1-\gamma} \times \left[\frac{1}{\lambda^{\frac{1}{\gamma}}} + \frac{1+r_f}{k}\left(\frac{r_g - r_f}{r_g - r_b}\right)^2\left(\lambda^{-\frac{1}{\gamma}} - K_0^{\frac{1}{\gamma}}\frac{r_f - r_b}{r_g - r_f}\right)\right] \quad (67)$$

$$\frac{1}{\gamma}\frac{d^2\mathbb{E}(U)}{d\alpha^2}|_{(C_1^+,\alpha^+)} = \frac{(r_f - r_b)^2}{k(1+r_f)}\left(\frac{r_g - r_f}{r_g - r_b}\right)^2\left[\frac{-\Omega}{M_1(\lambda) + \lambda^{\frac{1}{\gamma}}(1+r_f)}\right]^{-1-\gamma}\left(\lambda^{-\frac{1}{\gamma}} - K_\gamma^{\frac{1}{\gamma}}\right) \quad (68)$$

$$\frac{1}{\gamma}\frac{d^2\mathbb{E}(U)}{dC_1 d\alpha}|_{(C_1^+,\alpha^+)} = \frac{r_f - r_b}{k}\left(\frac{r_g - r_f}{r_g - r_b}\right)^2\left[\frac{-\Omega}{M_1(\lambda) + \lambda^{\frac{1}{\gamma}}(1+r_f)}\right]^{-1-\gamma}\left(\lambda^{-\frac{1}{\gamma}} + K_0^{\frac{1}{\gamma}}\right) \quad (69)$$

Note that (68) and $\lambda > \frac{1}{K_\gamma}$ implies that $\frac{d^2\mathbb{E}(U)}{d\alpha^2}|_{(C_1^+,\alpha^+)} < 0$. In addition,

$$\frac{1}{\gamma^2}\left[\frac{-\Omega}{M_1(\lambda) + \lambda^{\frac{1}{\gamma}}(1+r_f)}\right]^{2(1+\gamma)} D = \left[\frac{1}{\lambda^{\frac{1}{\gamma}}} + \frac{1+r_f}{k}\left(\frac{r_g - r_f}{r_g - r_b}\right)^2\left(\lambda^{-\frac{1}{\gamma}} - \frac{r_f - r_b}{r_g - r_f}K_0^{\frac{1}{\gamma}}\right)\right] \times \frac{(r_f - r_b)^2}{k(1+r_f)}\left(\frac{r_g - r_f}{r_g - r_b}\right)^2\left[\lambda^{-\frac{1}{\gamma}} - K_\gamma^{\frac{1}{\gamma}}\right] - \left(\frac{r_f - r_b}{k}\right)^2\left(\frac{r_g - r_f}{r_g - r_b}\right)^4\left(\lambda^{-\frac{1}{\gamma}} + K_0^{\frac{1}{\gamma}}\right)^2$$

where $D = \nabla^2 \mathbb{E}(U(C_1, C_2))_{(C_1^+, \alpha^+)} = \frac{d^2\mathbb{E}(U)}{dC_1^2}\frac{d^2\mathbb{E}(U)}{d\alpha^2} - \left(\frac{d^2\mathbb{E}(U)}{dC_1 d\alpha}\right)^2 |_{(C_1^+, \alpha^+)}$. Thus,

$$\frac{1}{\gamma^2}\left[\frac{-\Omega}{M_1(\lambda) + \lambda^{\frac{1}{\gamma}}(1+r_f)}\right]^{2(1+\gamma)} \left(\frac{r_g - r_b}{r_g - r_f}\right)^2 \frac{k}{r_f - r_b} D = \frac{r_f - r_b}{1+r_f}\left[\lambda^{-\frac{1}{\gamma}} - K_\gamma^{\frac{1}{\gamma}}\right] - \frac{r_g - r_f}{k}\lambda^{-\frac{1}{\gamma}} K_0^{\frac{1}{\gamma}} < 0$$

for $\lambda > \frac{1}{K_\gamma}$ which gives that $D = \nabla^2 \mathbb{E}(U(C_1, C_2)) = \frac{d^2\mathbb{E}(U)}{dC_1^2}\frac{d^2\mathbb{E}(U)}{d\alpha^2} - \left(\frac{d^2\mathbb{E}(U)}{dC_1 d\alpha}\right)^2 < 0$. Thus, the expected utility is indifferent at (C_1^+, α_1), also for $\lambda \leq \frac{1}{K_\gamma}$, which is the only point satisfying the FOC and thus the maximum will occur at the border.

The feasible solutions at the border for (P6) that come into consideration are given by: (i) $C_{2g} = \bar{C}_2$, (ii) $C_{2b} = \bar{C}_2$, (iii) $C_{2b} = 0$, (iv) $C_1 = \bar{C}_1$ and (v) $C_1 = 0$.

Case (i): $C_{2g} = \bar{C}_2$ when $C_1 = Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f} + \frac{r_g - r_f}{1+r_f}\alpha$ and $0 \leq \alpha \leq \frac{-\Omega}{r_g - r_f}$. It can be seen that

$$(1-\gamma)\mathbb{E}\left(U\left(Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f} + \frac{r_g - r_f}{1+r_f}\alpha, \alpha\right)\right) = -\lambda\left(\frac{-\Omega - (r_g - r_f)\alpha}{1+r_f}\right)^{1-\gamma} - \lambda\delta(1-p)(r_g - r_b)^{1-\gamma}\alpha^{1-\gamma}$$

is a convex function in α and thus its maximum is reached either for $\alpha = 0$ or $\alpha = \frac{-\Omega}{r_g - r_f}$. Thus, the potential candidates for maximum in this case are $\left(C_1 = Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}, \alpha = 0\right)$ or $\left(C_1 = \bar{C}_1, \alpha = \frac{-\Omega}{r_g - r_f}\right)$. Note that point $\left(C_1 = Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}, \alpha = 0\right)$ is feasible also for (P5) and is also the only feasible solution for case (ii). On the other hand, point $\left(C_1 = \bar{C}_1, \alpha = \frac{-\Omega}{r_g - r_f}\right)$ is feasible solution for (P2).

Case (iii). $C_{2b} = 0$ when $C_1 = Y_1 + \frac{Y_2}{1+r_f} - \frac{r_f - r_b}{1+r_f}\alpha$ which is feasible for $\frac{(1+r_f)(Y_1 - \bar{C}_1) + Y_2}{r_f - r_b} \leq \alpha \leq \frac{\bar{C}_2}{r_f - r_b}$. It can be seen that

$$(1-\gamma)\mathbb{E}\left(U\left(Y_1 + \frac{Y_2}{1+r_f} - \frac{r_f - r_b}{1+r_f}\alpha, \alpha\right)\right) = -\lambda\left(\bar{C}_1 - Y_1 - \frac{Y_2}{1+r_f} + \frac{r_f - r_b}{1+r_f}\alpha\right)^{1-\gamma} + \delta p((r_g - r_b)\alpha - \bar{C}_2)^{1-\gamma} - \lambda\delta(1-p)\bar{C}_2^{1-\gamma} \tag{70}$$

The potential maximum of (70) thus can be reached either at the endpoints $\alpha = \frac{(1+r_f)(Y_1 - \bar{C}_1) + Y_2}{r_f - r_b}$ and $\alpha = \frac{\bar{C}_2}{r_f - r_b}$ or at the point α^{P6} such that $\frac{d\mathbb{E}\left(U\left(Y_1 + \frac{Y_2}{1+r_f} - \frac{r_f - r_b}{1+r_f}\alpha, \alpha\right)\right)}{d\alpha}\Big|_{\alpha^{P6}} = 0$. Simple derivation gives

$$\alpha^{P6} \equiv \frac{\lambda^{1/\gamma} - \frac{k_2}{1+r_f}\frac{\bar{C}_2^{P2}}{\bar{C}_2}}{\lambda^{1/\gamma} - \frac{k_2}{1+r_f}} \frac{\bar{C}_2}{r_g - r_b} \tag{71}$$

which for $\bar{C}_2 < \bar{C}_2^{P2}$ and $\lambda > \left(\frac{k_2}{1+r_f}\right)^\gamma$ is below $\frac{(1+r_f)(Y_1 - \bar{C}_1) + Y_2}{r_f - r_b}$ and thus infeasible. This implies then that for $\bar{C}_2 < \bar{C}_2^{P2}$ the maximum of (70) can be reached only at the end points, namely $\left(C_1 = \bar{C}_1, \alpha = \frac{(1+r_f)(Y_1-\bar{C}_1)+Y_2}{r_f-r_b}\right)$ or $\left(C_1 = Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}, \alpha = \frac{\bar{C}_2}{r_f-r_b}\right)$ where the former is feasible also for (P2) and the latter for (P6) which will be dealt with later (in case when $C_1 \le Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}$).

In case (iv) any feasible solution is also feasible for (P2). There is no feasible solution in case (v).

Let $C_1 \le Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}$. Note that the utility of (P6) is a decreasing function in α for any fixed C_1

$$\begin{aligned}\frac{d\mathbb{E}(U)}{d\alpha} &= \delta p\left[(1+r_f)(Y_1-C_1)+Y_2-\bar{C}_2+(r_g-r_f)\alpha\right]^{-\gamma}(r_g-r_f)\\ &\quad -\lambda\delta(1-p)\left[(1+r_f)(C_1-Y_1)+\bar{C}_2-Y_2+(r_f-r_b)\alpha\right]^{-\gamma}(r_f-r_b) < 0\end{aligned}$$

if

$$\lambda > \frac{p(r_g-r_f)}{(1-p)(r_f-r_b)}\left[\frac{(1+r_f)(C_1-Y_1)+\bar{C}_2-Y_2+(r_f-r_b)\alpha}{(1+r_f)(Y_1-C_1)+Y_2-\bar{C}_2+(r_g-r_f)\alpha}\right]^{\gamma}$$

The latter is achieved if

$$\frac{1}{K_\gamma} \ge \frac{p(r_g-r_f)}{(1-p)(r_f-r_b)}\left[\frac{(1+r_f)(C_1-Y_1)+\bar{C}_2-Y_2+(r_f-r_b)\alpha}{(1+r_f)(Y_1-C_1)+Y_2-\bar{C}_2+(r_g-r_f)\alpha}\right]^{\gamma}$$

as it is assumed that $\lambda > \frac{1}{K_\gamma}$ where K_γ is given by (7). It can be shown that the above inequality holds if $C_1 \le Y_1 + \frac{Y_2-\bar{C}_2}{1+r}$ which is our assumption. In more detail, the set of feasible solutions for (P6) can be written as

$$\begin{array}{rcl} 0 & \le C_1 & \le Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}\\ Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f} - \frac{r_f-r_b}{1+r_f}\,\alpha & \le C_1 & \le Y_1 + \frac{Y_2}{1+r_f} - \frac{r_f-r_b}{1+r_f}\,\alpha\\ 0 & \le \alpha & \le \frac{(1+r_f)Y_1+Y_2}{r_f-r_b}\end{array}$$

Let $\tilde{C}_1$ be fixed and such that $0 \le \tilde{C}_1 \le Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}$. Based on the first inequality in the second row of the inequalities above and the fact that the utility of (P6) is decreasing in α it follows that the smallest possible $\tilde{\alpha}$ such that the feasible set is satisfied for $C_1 = \tilde{C}_1$ is given by

$$Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f} - \frac{r_f-r_b}{1+r_f}\,\tilde{\alpha} = \tilde{C}_1$$

and thus

$$\tilde{\alpha} = \frac{(1+r_f)(Y_1-\tilde{C}_1)+Y_2-\bar{C}_2}{r_f-r_b} \in \left[0, \frac{(1+r_f)Y_1+Y_2}{r_f-r_b}\right]$$

Note that $(\tilde{C}_1, \tilde{\alpha})$ completes $\tilde{C}_1 \le Y_1 + \frac{Y_2}{1+r_f} - \frac{r_f-r_b}{1+r_f}\tilde{\alpha} = \tilde{C}_1 + \frac{\bar{C}_2}{1+r_f}$ as $\bar{C}_2 \ge 0$. Thus, for any given $\tilde{C}_1$ that satisfies $0 \le \tilde{C}_1 \le Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}$ is the point $\left(\tilde{C}_1, \frac{(1+r_f)(Y_1-\tilde{C}_1)+Y_2-\bar{C}_2}{r_f-r_b}\right)$ where the utility of (P6) achieves its maxima. As point $(\tilde{C}_1, \tilde{\alpha})$ is feasible also for (P5) and as utilities of (P5) and (P6) coincide at this point then the utility function of (P5) at its maximum is bigger or equal to the utility function of (P6) at any point $(\tilde{C}_1, \tilde{\alpha})$.

Problem (P7). The only feasible solution is $\left(Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}, 0\right)$ which is feasible also for (P5).

Problem (P8). Note that the only feasible solution for case when $C_1 \leq Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$ is $\left(C_1 = Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}, \alpha = 0\right)$. Let $C_1 \geq Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$. As $\frac{d^2 \mathbb{E}(U(C_1, \alpha))}{d\alpha^2} > 0$ then no local extreme can be a local maximum. Thus, an maximum will occur at the border of the set of feasible solutions. The feasible solutions at the border that come into considerations are: (i) $C_{2g} = \bar{C}_2$, (ii) $C_{2b} = \bar{C}_2$, (iii) $C_{2g} = 0$, (iv) $C_{2b} = 0$, (v) $C_1 = \bar{C}_1$ and (vi) $C_1 = 0$. Note that case (i) was already dealt with in case (i) of problem (P6) and the only feasible solution in case (ii) is $\left(Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}, 0\right)$ which is feasible also for (P5). In addition, there are no feasible solutions in cases (iii) and (vi) and neither in case (iv) for $\bar{C}_2 < \bar{C}_2^{P2}$. Finally, if $C_1 = \bar{C}$, case (v), then any feasible solution will be feasible also for (P4). ■