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The Consumption-Investment Decision of a Prospect Theory Household: A Two-Period Model *

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Abstract

This study extends the literature on portfolio choice under prospect theory preferences by introducing a two-period life cycle model, where the household decides on optimal consumption and investment in a portfolio with one risk-free and one risky asset. The optimal solution depends primarily on the household's choice of the present value of the consumption reference levels relative to the present value of its endowment income. If the present value of the consumption reference levels is set below the present value of endowment income, then the household behaves in such a way to avoid relative losses in consumption in any present or future state of nature (good or bad). As a result the degree of loss aversion does not directly affect optimal consumption and risk taking activity. However, it must be sufficiently high in order to rule out outcomes with relative losses. On the other hand, if the present value of the consumption reference levels is set exactly equal to the present value of the endowment income, i.e., the household sets its reference levels such that they are in balance with its income, then the household's optimal consumption is the reference consumption in both periods and the household will not invest in the risky asset. Finally, if the present value of the household's consumption reference levels is set above the present value of its endowment income, then the household cannot avoid experiencing a relative loss in consumption, either now or in the future. As a result, loss aversion directly affects consumption and risky investment. Reference levels play a significant role in consumption and risk taking activity. In most cases the household will "follow the Joneses" if the reference levels are set equal to the consumption levels of the Joneses. Independent of how consumption reference levels are set, being more ambitious, i.e., increasing one's reference levels, will result in less happiness. The only case when this is not true is when reference levels increase with growing income (and the present value of reference levels is set below the present value of endowment income).

Keywords: prospect theory, loss aversion, consumption-savings decision, portfolio allocation, happiness

JEL classification: G02, G11, E20

1 Introduction

In this paper we explore the factors that influence a household's consumption and savings decision, based on behavioral economics preferences. Households make decisions on how much to consume today and how much to save for the future when, e.g., they retire. Savings are the means of transferring consumption into the future and of having income for retirement or the means of transferring future income to the present in order to be able to afford more consumption today. Moreover, households do not only decide how much to save but also how to allocate their savings into different types of assets. These decisions are made knowing that the future is risky and uncertain.

Traditionally, the expected utility (EUT) framework has been used to model such behavior.[1] This research will deviate from the EUT model and will explore a different type of preferences. In particular, we will assume prospect theory preferences that were introduced and developed by Kahneman and Tversky (1979) and Tversky and Kahneman (1992) and that take into account also psychological aspects of households' behavior. Prospect theory can be characterized by the following properties. Decision makers under risk evaluate gains and losses with respect to some reference level, rather than evaluating absolute values (of their wealth or consumption). Households exhibit loss aversion, which means that they are more sensitive to losses than to gains of the same magnitude. In addition, households display risk aversion in the domain of gains but show risk appetite in the domain of losses, which is described by an S-shaped value function that is concave in the domain of gains and convex in the domain of losses.[2] For a comprehensive overview on prospect theory see, e.g., Barberis (2013) and DellaVigna (2009).

We address a number of issues on the savings behavior under prospect theory preferences that have only partially been explored in the literature before. How do households decide on consumption and portfolio decisions when faced with prospect theory type of preferences? Do households have to be sufficiently loss averse to yield reasonable optimal solutions for consumption and investment decisions? Does loss aversion affect consumption and portfolio decisions? Do reference levels affect the households' consumption, savings and portfolio choice to transfer consumption into the future? If yes, how? Do households "follow the Joneses", (i.e., compare themselves to and follow neighbors or associates) when making consumption and savings decisions?

There are many different types of reference points that can be considered in exploring the savings behavior and portfolio choice. The first reference levels that were used are subsistence levels of consumption (see, e.g., Stone, 1954 and Geary, 1951). Under such preferences, households get utility from consumption in excess of a subsistence level. Individuals have to

[1] For original work in this area see Sandmo (1968, 1969) and Merton (1969, 1971).

[2] Another property, not included in this study, is that the probabilities assigned to the utility of the outcomes are not objective but subjective (so-called decision weights), as people seem to underestimate large probabilities and overestimate small probabilities.

consume a certain minimal level irrespective of its price or the person's income. The savings and portfolio choice with subsistence consumption has recently been explored by Achury et al. (2012). They use the Stone-Geary expected utility model to explain a number of observed empirical facts such as why the rich have a higher savings rate, higher holdings of risky assets relative to personal wealth and a higher consumption volatility than the poor.

Another commonly used reference dependent preference model is habit persistence. The habit persistence model assumes that households derive utility from consumption relative to a reference level that depends on past consumption levels. Habit persistence models have been used in many applications in macroeconomics and finance and can to some extent explain, for instance, the equity premium puzzle and the behavior of asset returns (Abel, 1990; Constantinides, 1990; Campbell and Cochrane, 1999), excess smoothness in consumption expenditures (Lettau and Uhlig, 2000) and business cycles characteristics (Boldrin et al., 2001; Christiano et al., 2005).

Reference levels can also be set by comparing one's consumption levels to others (Falk and Knell, 2004).[3] According to the psychology literature, people can be governed by self-enhancement and/or self-improvement motives. The former motive occurs when people want to make themselves feel better by setting their references at low levels, possibly reflecting the wealth of poorer people. However, people also place importance on the self-improvement motive. Here people compare themselves with others who are more successful and as a result set their reference levels high.

Many of the applications of prospect theory in portfolio selection assume that the reference level is the investor's return from investing all initial wealth into the risk-free asset (Barberis and Huang, 2001; Gomes, 2005; Barberis and Xiong, 2009; Bernard and Ghossoub, 2010; He and Zhou, 2011). Barberis et al. (2001), Berkelaar et al. (2004), Fortin and Hlouskova (2011, 2015) and Gomes (2005) use also a dynamic updating rule for the reference point. Future utility, in one-period models, is derived from the excess return of the risky asset holdings. One of the major findings of this literature is that investors may not invest in risky assets even if its expected return is higher than the risk-free rate.

Some work has been devoted to exploring the consequences of reference dependent preferences for inter-temporal two-period habit-persistence consumption decisions, when future income is uncertain and when households are loss averse, see, e.g., Bowman et al., 1999. They find that a household will resist reducing its consumption level when there is bad news about future income. Furthermore, the resistance to reducing consumption with bad news is greater than the resistance to increasing consumption in response to good news.

Koszegi and Rabin (2006) assume rational expectations in the formation of reference levels. Assuming agents are more affected by news about current consumption than by news about future consumption, they find that people would intend to overconsume today relative to

[3] For a literature review see Clark et al. (2008).

their optimal plans. They would increase consumption right away when good news regarding wealth arrives, but would postpone decreasing consumption when receiving bad news. Thus, higher wealth reduces the painful impact of bad news, and as a result people save more for precaution.

Van Bilsen et al. (2014) investigate optimal consumption and portfolio choice paths of a loss averse household but with an endogenous reference level. They find that households strive to protect themselves against consumption losses in order to avoid bad states of nature. They attribute this behavior to loss aversion. Due to the dynamic nature of their set-up they can investigate the effect of financial shocks and find that consumption choices adjust only slowly to financial shocks and that welfare losses are substantial with suboptimal consumption and portfolio selections.

Our research complements the work by Van Bilsen et al. (2014) in that it provides additional insights as discussed below. We provide a closed-form solution to the inter-temporal consumption and portfolio decision of a prospect theory household in a theoretical two-period model, where uncertainty arises from the risky asset. We assume that the asset's return follows a Bernoulli distribution, i.e., there are two states of nature realizing with certain probabilities, and that the household's consumption reference levels are set exogenously. These reference levels are compared with the household's consumption levels and the household derives its utility from the difference between its consumption and the reference level. Consuming above the reference level means that the household incurs relative gains while consuming below the reference level means that it incurs relative losses. It turns out that the consumption reference levels (in both periods) as well as the loss aversion parameter are crucial in the analysis. The solution depends on the household's choice of the consumption reference levels, more precisely, on the present value of the chosen reference levels relative to the present value of the endowment income. Hence we have three different types of households with reference levels below, equal to or above the income.

Our main results are the following. If the household sets its references levels such that the present value is *below* the present value of its endowment income, then it behaves in such a way that it avoids relative losses in any present or future state of nature (good or bad). So optimal consumption is always above the reference level. This implies that the degree of loss aversion does not directly affect optimal consumption and risk taking activity. However, loss aversion must be sufficiently high in order to prevent relative losses. Further, the household always invests in the risky asset. If, on the other hand, the household sets its references levels such that the present value is *equal* to the present value of its endowment income, i.e., the household completely balances reference levels and income, then the optimal consumption is equal to the reference consumption in both periods. Also the household does not invest in the risky asset in this case. Finally, if the household sets its reference levels such that the present value is *above* the present value of its endowment income, then it cannot avoid relative losses at all times. Either in the first or in the second period (good or bad state of nature) the household has to accept consuming below the reference level. This implies that loss aversion directly affects consumption and investment in the risky asset. Investment in the risky asset is again positive in this case. We look at various examples of how consumption reference levels are set, including the case when households set their reference levels according to the consumption of the "Joneses" (neighbors or associates) and examine what happens to the implied optimal consumption. Mostly prospect theory households "follow the Joneses" in the sense that their optimal consumption follows the Joneses' consumption. Another interesting result of the sensitivity analysis is that increasing one's reference level, i.e., increasing one's targets and thus being more ambitious leads to less happiness.[4] This is true for all three types of households, i.e., independent of whether households set their consumption reference levels below, equal to, or above the income.

The paper proceeds as follows. In the next section we describe the set-up of the model. In section 3 we investigate the case where the households sets its reference levels such that the present value is below the present value of its endowment income (low aspirations). Section 4 explores the case where the present value of consumption reference levels is exactly equal to the present value of the household's endowment income. Section 5 examines the case where the household sets its reference levels such that the present value is above the present value of its endowment income (high aspirations). Finally, we summarize and offer some concluding remarks and future extensions.

2 Problem set-up

We consider a household that lives for two periods. In the first period it receives a non-stochastic exogenous income (labor income, endowment income), $Y_1 > 0$, which it can allocate to current consumption, C_1, risk-free investment, m, and risky investment, α, where the sum of the risky and risk-free investment are savings S. Thus, in the first period

$$Y_1 = C_1 + m + \alpha = C_1 + S \tag{1}$$

We consider two assets, a risk-free asset with a net of the dollar return $r_f > 0$ and a risky asset with stochastic net of the dollar return r that yields r_g in the good state of nature, which occurs with probability p, and r_b in the bad state of nature, which occurs with probability $1-p$. We assume that $-1 < r_b < r_f < r_g$, $0 < p < 1$, and $\mathbb{E}(r) = p\, r_g + (1-p)r_b > r_f$. Thus, in the second period the household consumes

$$C_{2i} = Y_2 + (1 + r_f)m + (1 + r_i)\alpha$$

[4] The term "happiness" is used to denote the indirect (optimal) utility.

where $Y_2 \geq 0$ is the non-stochastic income of the household in the second period, which can also be thought of as an exogenous government pension income. There are no liquidity constraints that prevent the household from consuming any exogenous future income in the first period, but consumption is not allowed to be negative in either period, so that it can only partially borrow against uncertain future income. This means that risk-free savings, m, can be negative to a certain extent. The value $(1+r_f)m + (1+r_i)\alpha$ represents the wealth acquired from capital investment, $i \in \{b, g\}$. So, in the second period the household consumes C_{2b} in the bad state of nature and C_{2g} in the good state of nature. Based on this and (1) the consumption in the second period is

$$C_{2i} = Y_2 + (1+r_f)(Y_1 - C_1) + (r_i - r_f)\alpha \tag{2}$$

The household's preferences are described by the following reference based utility function

$$U(C_1, \alpha) = V(C_1 - \bar{C}_1) + \delta\, V(C_2 - \bar{C}_2) \tag{3}$$

where $\bar{C}_1$ and $\bar{C}_2$ are exogenous consumption reference or comparison levels, such that $0 \leq \bar{C}_1 < Y_1 + \frac{Y_2}{1+r_f}$ and $0 \leq \bar{C}_2 < (1+r_f)Y_1 + Y_2$, i.e., $Y_1 + \frac{Y_2}{1+r_f} > \max\left\{\bar{C}_1, \frac{\bar{C}_2}{1+r_f}\right\}$, δ is the discount factor, $0 < \delta < 1$, and $V(\cdot)$ is a prospect theory (S-shaped) value function defined as

$$V(C_i - \bar{C}_i) = \left\{ \begin{array}{cc} \frac{(C_i - \bar{C}_i)^{1-\gamma}}{1-\gamma}, & C_i \geq \bar{C}_i \\ \\ -\lambda \frac{(\bar{C}_i - C_i)^{1-\gamma}}{1-\gamma}, & C_i < \bar{C}_i \end{array} \right\} \tag{4}$$

for $i = 1, 2$. Parameter $\lambda > 1$ is the loss aversion parameter and $\gamma \in (0, 1)$ is the parameter determining the curvature of the utility function. If consumption is above the reference level we talk about (relative) gains, if it is below the reference level we talk about (relative) losses. The utility has a kink at the consumption reference level and it is steeper for losses than for gains, i.e., a decrease in consumption is more severely penalized in the domain of losses than in the domain of gains. Finally, the utility function is concave above the reference point and convex below it. The household is thus risk averse in the domain of gains (i.e., above the consumption reference level) and risk seeking in the domain of losses (i.e., below the consumption reference level), see Figure 1.

The household maximizes the following expected utility as given by (3) and (4)

$$\text{Max}_{(\text{C}_1, \alpha)}: \quad \mathbb{E}(U(C_1, \alpha)) = V(C_1 - \bar{C}_1) + \delta\, \mathbb{E}V(C_2 - \bar{C}_2)$$

$$\text{such that}: \quad C_1 \geq 0,\ C_{2b} \geq 0,\ C_{2g} \geq 0 \text{ and } \alpha \geq 0$$

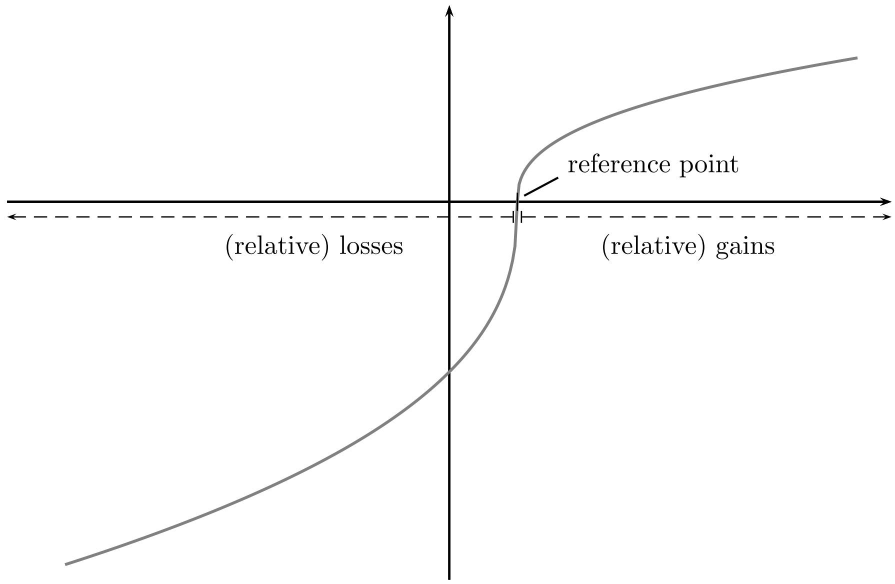

Figure 1: Loss aversion (S-shaped) utility

Based on this and (2) the household's maximization problem can be formulated as follows

$$\begin{aligned}
\mathrm{Max}_{(C_1,\alpha)}: \quad \mathbb{E}(U(C_1,\alpha)) \quad &= \quad V(C_1-\bar{C}_1)+\delta\,\mathbb{E}V\left((1+r_f)(Y_1-C_1)+(r_i-r_f)\alpha+Y_2-\bar{C}_2\right) \\
\text{such that}: \qquad\qquad 0 \le \quad C_1 \quad &\le Y_1+\tfrac{Y_2}{1+r_f}-\tfrac{r_f-r_b}{1+r_f}\alpha, \\
0 \le \quad \alpha \quad &\le \tfrac{(1+r_f)Y_1+Y_2}{r_f-r_b}
\end{aligned} \tag{5}$$

Note that the upper bound on C_1 follows from $C_{2b} \ge 0$ and the upper bound on α follows from the imposition of the upper bound on C_1 being non-negative, i.e. $Y_1 + \frac{Y_2}{1+r_f} - \frac{r_f-r_b}{1+r_f}\alpha \ge 0$.[5] The condition on α means that short sales are not allowed.[6]

[5]Imposing positive lower bounds on consumption in both periods (i.e., on C_1, C_{2b} and C_{2g}), so that the household does not "starve", would not substantially change our results. In occurrences when the optimal consumption hits zero now, it would hit the lower bound then. Thus, the behavioral implications of our findings related to the sensitivity analysis and thus comparisons to others would not change.

[6]Fortin, Hlouskova and Tsigaris (2015) show that the assumption $p > \max\left\{\frac{r_f-r_b}{r_g-r_b}, \frac{(r_f-r_b)^{1-\gamma}}{(r_f-r_b)^{1-\gamma}+(r_g-r_f)^{1-\gamma}}\right\}$ rules out short-selling if there is no non-negativity restriction on α. Note that $\mathbb{E}(r) > r_f$ is equivalent to $p > \frac{r_f-r_b}{r_g-r_b}$, so only $\mathbb{E}(r) > r_f$ is not sufficient to rule out short sales (except in section 3).

Before proceeding further, we introduce the following notation

$$\begin{aligned}\Omega &= (1+r_f)(Y_1-\bar{C}_1)+Y_2-\bar{C}_2 \\ &= (1+r_f)\left[\left(Y_1+\frac{Y_2}{1+r_f}\right)-\left(\bar{C}_1+\frac{\bar{C}_2}{1+r_f}\right)\right] \qquad (6)\end{aligned}$$

$$K_\gamma = \frac{(1-p)(r_f-r_b)^{1-\gamma}}{p(r_g-r_f)^{1-\gamma}} \qquad (7)$$

$$M = \left(\delta(1+r_f)\,p\,\frac{r_g-r_b}{r_f-r_b}\right)^{\frac{1}{\gamma}} \frac{r_f-r_b+K_0^{\frac{1}{\gamma}}(r_g-r_f)}{r_g-r_b} \qquad (8)$$

In addition notice that

$$K_0 = \frac{(1-p)(r_f-r_b)}{p\,(r_g-r_f)} = K_\gamma\left(\frac{r_f-r_b}{r_g-r_f}\right)^\gamma < 1$$

In the following analysis we consider three fundamentally different situations, which give rise to profoundly different types of optimal consumption behavior. These situations are characterized by how the household sets its consumption reference levels in relation to its endowment income. Namely, whether the difference between the present value of total endowment income and the present value of the sum of the consumption reference levels is positive ($\Omega > 0$), zero ($\Omega = 0$), or negative ($\Omega < 0$).[7] The case when Ω is positive is characteristic for households with low aspirations, while the case when Ω is negative is typical for households with high aspirations. The case when Ω is zero is a special case, where the present value of the household's total endowment income is exactly equal to the present value of the consumption reference levels.

In the formal analysis we split the household's consumption decision problem (5) into eight separate problems, (P1)–(P8), which differ in their respective domains, i.e., in their sets of feasible solutions. These domains are specified by whether first and second period (in the good and bad state of nature) consumption levels are above or below the respective reference levels. This yields a total of eight combinations, see Appendix A. Households with a positive Ω will operate on certain domains which differ from the domains on which households with a negative Ω operate.

3 Low reference values relative to endowment income ($\Omega > 0$)

We now consider the case when the household sets its consumption reference levels such that the present value is below the present value of its endowment income, i.e., when $\Omega > 0$. This

[7]Note that Ω denotes the difference between the present value of total endowment income and the present value of the sum of the consumption reference levels multiplied by the gross return of a dollar investment in the risk-free rate, see equation (6). Note, in addition, that future income and consumption reference levels are discounted at the risk-free rate.

is done when the household has low aspirations. To proceed with the analysis let us introduce the following notation

$$\begin{aligned}\lambda^{\Omega\geq 0} &= \frac{\Omega^{1-\gamma}}{\delta(1-p)(1+r_f)\bar{C}_2^{1-\gamma}}\left[(1+r_f+k_2)^\gamma\left(1+\frac{r_g-r_f}{r_g-r_b}\frac{\bar{C}_2}{\Omega}\right)^{1-\gamma}-(1+r_f+M)^\gamma\right] \\ &= \left(\frac{1+r_f}{k}+\left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right)^\gamma\left(\frac{\Omega}{\bar{C}_2}\frac{r_g-r_b}{r_g-r_f}+1\right)^{1-\gamma}-\left(\frac{1+r_f}{k}+\left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}+1\right)^\gamma\left(\frac{\Omega}{\bar{C}_2}\frac{r_g-r_b}{r_g-r_f}\right)^{1-\gamma}\end{aligned} \tag{9}$$

We can now formulate the main result for the case when $\Omega > 0$.

Proposition 1 *Let $\Omega > 0$ and $\lambda > \max\left\{\frac{1}{K_\gamma}, \lambda^{\Omega\geq 0}, \left(\frac{M}{1+r_f}\right)^\gamma\right\}$. Then problem (5) obtains a unique maximum at (C_1^*, α^*) where*

$$\begin{aligned}C_1^* &= \bar{C}_1+\frac{\Omega}{1+r_f+M} \\ &= \bar{C}_1+\frac{1+r_f}{1+r_f+M}\left[\left(Y_1+\frac{Y_2}{1+r_f}\right)-\left(\bar{C}_1+\frac{\bar{C}_2}{1+r_f}\right)\right] > \bar{C}_1 \end{aligned} \tag{10}$$

$$\alpha^* = \frac{\left(1-K_0^{\frac{1}{\gamma}}\right)M}{r_f-r_b+K_0^{\frac{1}{\gamma}}(r_g-r_f)}(C_1^*-\bar{C}_1) > 0 \tag{11}$$

Proof. It follows directly from Lemma 1 in Appendix B. ■

When Ω is positive then aspirations are low, which should make it easier for a household to reach and exceed its consumption comparison levels than when aspirations are high. If, in addition, the utility is such that consumption below the reference levels is sufficiently penalized, i.e., the loss aversion parameter is large enough, then we expect optimal consumption to exceed its reference levels. This is indeed what we observe: optimal consumption levels in both periods are strictly larger than their corresponding reference levels provided that the household is sufficiently loss averse, i.e., $C_1^* > \bar{C}_1$ and $C_{2g}^* \geq C_{2b}^* > \bar{C}_2$, where

$$C_{2g}^* = \bar{C}_2+\frac{M\Omega}{(1+r_f+M)\left(1+K_\gamma^{\frac{1}{\gamma}}\right)}\frac{r_g-r_b}{r_f-r_b} \tag{12}$$

$$C_{2b}^* = \bar{C}_2+\frac{M\Omega}{(1+r_f+M)\left(1+K_\gamma^{\frac{1}{\gamma}}\right)}\frac{r_g-r_b}{r_f-r_b}K_0^{\frac{1}{\gamma}} \tag{13}$$

Thus the optimal behavior is characterized by avoiding any relative losses to happen or, in other words, the household's aspirations are fully attained. We note further that optimal investment in the risky asset is strictly positive, i.e., $\alpha^* > 0$, which implies that the household takes on risk in the financial market. Total savings, however, can be either positive or negative.[8]

Although the existence of the solution does depend on the loss aversion parameter λ, the solution itself, (C_1^*, α^*), does not directly depend on it. The reason for this is that the household's optimal solution is reached in problem (P1), where the solution is found in the domain given by $C_1 \geq \bar{C}_1$, $C_{2b} \geq \bar{C}_2$ and $C_{2g} \geq \bar{C}_2$, i.e., both periods' consumption levels are above their consumption reference levels and thus the utility does not depend on the loss aversion parameter λ (see Appendix A). However, for this to happen the household needs to be sufficiently loss averse, namely $\lambda > \max\left\{\frac{1}{K_\gamma}, \lambda^{\Omega\geq 0}, \left(\frac{M}{1+r_f}\right)^\gamma\right\}$. Hence, if the household is sufficiently loss averse it will make choices that avoid any relative losses from occurring. As the domains of all remaining problems (P2)–(P8) contain a relative loss, see Appendix A, a sufficiently loss averse household will never select solutions from these problems. This behavior is only possible, however, when the household does not set its goals (consumption reference levels) too high with respect to its income, thus, when Ω is positive.[9] Note, finally, that problem (P1) is known from the studies on habit formation, where the consumption habits are addictive and never fall below certain consumption targets (see, for example, Yu, 2015).

Table 1 summarizes the sensitivity results related to the solution presented in Proposition 1, so for a sufficiently loss averse household with low aspirations. In particular, we present the changes of the first and second period optimal consumption, of the optimal investment in the risky asset, of the first and second period consumption gap, of optimal savings[10] and of happiness (first row) with respect to changes in the loss aversion parameter and the first and second period consumption reference levels (first column). By "consumption gap" we mean the distance between the optimal consumption and its reference level, $|C_i^* - \bar{C}_i|$, $i = 1, 2$, and we use "happiness" to denote the household's indirect utility (i.e., its value at the optimum). We also use "relative consumption" to denote the difference between optimal consumption and the reference level, which is closely related to the previously defined consumption gap. The gap is always positive while relative consumption can be either positive or negative. Both definitions coincide if optimal consumption is above the reference level.

Since the solution does not explicitly depend on the loss aversion parameter, as discussed above, an exogenous increase in the loss aversion parameter, keeping everything else constant, does not change the solution or the utility at the solution (happiness).

An exogenous increase in the first period consumption reference level, keeping everything else constant, will increase the first period optimal consumption, decrease risky asset holdings and also decrease savings. As less income is transferred to the second period we would

[8] The assumption required for $S^* > 0$ is $M(Y_1 - \bar{C}_1) > (Y_2 - \bar{C}_2)$. This breaks down to simpler formulations in special cases.

[9] When Ω is negative, the household cannot totally avoid relative losses. It will have to face relative losses in the first or second period, or in the good or bad state of nature.

[10] The results for optimal savings follow from $\frac{dC_1^*}{d\bar{C}_1}$ and $Y_1 = C_1^* + S^*$.

	dC_1^*	dC_{2g}^*	dC_{2b}^*	$d\alpha^*$	$d(C_1^* - \bar{C}_1)$	$d(C_{2g}^* - \bar{C}_2)$	$d(C_{2b}^* - \bar{C}_2)$	dS^*	$d(\mathbb{E}(U(C_1^*, \alpha^*)))$
$d\lambda$	$= 0$	$= 0$	$= 0$	$= 0$	–	–	–	$= 0$	$= 0$
$d\bar{C}_1$	> 0	< 0	< 0	< 0	< 0	–	–	< 0	< 0
$d\bar{C}_2$	< 0	$\lesseqgtr 0$	> 0	< 0	–	< 0	< 0	> 0	< 0

Table 1: Sensitivity results when aspirations are low ($\Omega > 0$)

expect consumption to decrease in the second period. This is what we indeed observe: the second period consumption in either state of nature will fall with an increase in the first period consumption reference level. Even though optimal first period consumption increases in response to an exogenous increase in the first period consumption reference level, relative optimal consumption in the first period, i.e., the amount by which the reference level is exceeded, decreases. This means that the extent of the increase in the first period consumption reference level is not fully matched by the resulting increase in the first period optimal consumption. In summary, if the household increases the first period consumption reference level it will reduce the growth rate of consumption. Finally, an increase of the first period consumption reference level decreases the happiness level.

An increase in the second period consumption reference level, keeping everything else constant, will decrease the first period optimal consumption and risky asset holdings but increase both total savings and the risk-free investment. However, the increase of the risk-free investment is not sufficient to offset the reduction in risky assets in such a way that second period consumption will increase in both states of nature. Only in the bad state of nature optimal consumption in the second period will increase. In the good state of nature the response can be either an increase or a decrease of consumption. Probably the reduced risky investment – and hence the reduced potential to achieve high returns – is the reason why this is the case. Relative optimal consumption in the second period decreases if the second period consumption reference level is increased, which is in analogy to the situation when the first period consumption level is increased. The happiness level is negatively related to the second period consumption reference level, as it was to the first period consumption reference level. So if a household is "more ambitious" (i.e., if it increases its consumption reference level), in either the first or the second period, its happiness level will decrease.

The fact that consumption reference levels are exogenous gives us the opportunity to present some interesting examples. Consider, for instance, the case when the first period reference level is equal to the first period consumption level of other people that the household is associated with, i.e., the household *compares* itself to neighbors or peers. Then, if the first period consumption level of the other people increases, this household will respond by increasing its first period reference consumption level and because of this it will increase its first period optimal consumption level, reduce risk taking and reduce its future consumption in both states of nature. Hence, the household's behavior is one that "follows the Joneses" (i.e., the neighbors or peers the household wants to compare itself to).[11] In addition, the gap between the household's first period consumption and its consumption reference level narrows as the consumption level of the others increases. On the other hand, let the household's second period reference level be equal to the expected second period consumption level of other associates. Then, if the household expects the other people to have a higher expected future consumption, it will increase its second period consumption reference level, which will reduce its first period consumption, reduce risk taking but increase risk-free investment leading to an increase in consumption in the second period in the bad state of nature but not necessarily in the good state of nature. Here it is not clear that the household follows the Joneses in the second period, even when its first period consumption is reduced to achieve an increase in future consumption like the household's associates. However, the consumption gap in the second period declines in both states of nature when the second period consumption reference increases, bringing closer to the reference the consumption levels in the second period.

In what follows we will refer to *"following the Joneses" in the first period* when the increase (or decrease) of the first period consumption of the Joneses (a reference household) impacts this household such that its first period consumption will change in the same way as the one of the Joneses. I.e., it will increase, if the first period consumption of the Joneses increased and decrease if the first period consumption of the Joneses decreased. In our set-up this works in the way that the household adjusts its consumption reference level according to what the Joneses do. So it will increase its first period consumption reference level if the Joneses increase their first period consumption. In addition, we will refer to the *"following the Joneses" in the second period* when the increase (or decrease) of the expected second period consumption of the Joneses impacts the second period expected consumption (of the household under considerations) such that it will change in the same way as that of the Joneses. As before, the household will increase its second period consumption reference level if the Joneses increase their second period expected consumption. So the idea of "following the Joneses" is to introduce external preferences into the household's behavior. Based on this terminology we can say that a sufficiently loss averse household with low aspirations follows the Joneses in the first period but not necessarily in the second period.

If a household sets its reference levels according to the consumption of richer peers (the rich Joneses) who consume at higher levels and wants to catch up by increasing its reference levels, then this will decrease its happiness. If, on the other hand, a household compares itself to poorer peers (the poor Joneses) that consume at lower levels and wants to adapt to the others by decreasing its reference levels, then this will increase its happiness. So comparing yourself to richer people makes you less happy while comparing yourself to poorer people makes you happier.

[11] See Clark, Frijters and Shields, (2008).

Some examples of reference consumption levels for $\Omega > 0$

In the following we present some additional interesting examples of reference consumption levels $\bar{C}_1$ and $\bar{C}_2$.

Example 1 (Merton type expected utility): $\bar{C}_1 = \bar{C}_2 = 0$

A special case embedded in this behavioral study is the traditional *expected utility* (EUT), where $\bar{C}_1 = \bar{C}_2 = 0$ and thus $V(C_i) \equiv \frac{C_i^{1-\gamma}}{1-\gamma}$ for $C_1, C_2 \geq 0$. In this case we solve problem (P1) and the solution is then identical to (10) and (11) for the prospect theory utility with $\bar{C}_1 = \bar{C}_2 = 0$. Thus, the optimal consumption in the first period is

$$(C_1^*)^{EUT} = \frac{1 + r_f}{1 + r_f + M}\left(Y_1 + \frac{Y_2}{1 + r_f}\right) > 0$$

Note that in this case $\Omega^{EUT} > \Omega > 0$, where Ω is related to a prospect theory (PT) household and we assume that the PT household has at least one consumption reference level strictly positive, i.e., either $\bar{C}_1 > 0$ or $\bar{C}_2 > 0$, otherwise it boils down to the expected utility case. EUT optimal consumption is proportional to the present value of endowment income, where the factor of proportionality, representing the marginal propensity to consume out of the present value of total income, is less than unity. This marginal propensity to consume (out of the present value of total income) is the same as the one under PT preferences, assuming the curvature parameter γ remains unchanged.

In addition,

$$(\alpha^*)^{EUT} = \frac{\left(1 - K_0^{\frac{1}{\gamma}}\right) M}{r_f - r_b + K_0^{\frac{1}{\gamma}}(r_g - r_f)}(C_1^*)^{EUT} > 0$$

and thus the household's investment in the risky asset is also proportional to the present value of endowment income.[12] However, the EUT household will always invest in the risky asset, which is not necessarily the case for the PT household when $\Omega = 0$ and thus it will not invest in the risky financial market. The case when $\Omega = 0$ is discussed in section 4. Note that for the EUT household the savings, $(S^*)^{EUT} = Y_1 - (C_1^*)^{EUT}$, are positive when $MY_1 > Y_2$, in which case the household transfers some of its first period income into the second period.

Example 2 (comparison to *poorer* peers): $\bar{C}_1 + \frac{\bar{C}_2}{1+r_f} = Y_1^P + \frac{Y_2^P}{1+r_f} < Y_1 + \frac{Y_2}{1+r_f}$

[12] In comparing optimal consumption and risky asset holdings between the two models one has to be careful and remember that the types of utility functions suggested by these models are different. For example, the EUT model implies a constant relative risk aversion, which is equal to γ, while the PT utility shows a decreasing relative risk aversion, which is equal to $\gamma C_1/(C_1 - \bar{C}_1)$ for $C_1 \neq \bar{C}_1$ and $\gamma C_2/(C_2 - \bar{C}_2)$ for $C_2 \neq \bar{C}_2$. In addition, the relative risk aversion for the EUT model is restricted to be below one (as a consequence from our restriction on γ, which states $0 < \gamma < 1$), while it has sometimes empirically been found to be larger than one (see Ahsan and Tsigaris, 2009, who provide some empirical examples).

In this situation the household with income levels Y_1 and Y_2 sets its first and second period consumption reference levels such that its present value of total reference consumption is equal to the present value of the endowment income stream of some poorer household. By a *poorer* household we mean a household whose total discounted endowment income is below the total discounted endowment income of this household. Namely, $Y_1^P + \frac{Y_2^P}{1+r_f} < Y_1 + \frac{Y_2}{1+r_f}$, where Y_1^P and Y_2^P are the first and the second period income levels of the poorer household such that $Y_1^P \geq 0$ and $Y_2^P \geq 0$. In this case Ω represents the household's wealth net of the wealth of the poorer household, i.e., $\Omega = (1+r_f)\left[Y_1 + \frac{Y_2}{1+r_f} - \left(Y_1^P + \frac{Y_2^P}{1+r_f}\right)\right] > 0$. In other words, less ambitious households place themselves into the *comfort zone* by comparing themselves to peers with a smaller wealth level. The impact of changes in the poorer household's income on this household's consumption and investment behavior were discussed previously as examples of changes in the reference levels. Note that the EUT model, example 1, is observationally equivalent to households who compare themselves to people that have no endowment income, i.e., $Y_1^P = Y_2^P = 0$.

Example 3 (consumption overreaction to income): $\bar{C}_1 = \bar{C} + cY_1, \bar{C}_2 = (1+r_f)\bar{C} + cY_2$ where $\bar{C} \in \left[-c\min\left\{Y_1, \frac{Y_2}{1+r_f}\right\}, \frac{1-c}{2}\left(Y_1 + \frac{Y_2}{1+r_f}\right)\right)$ and $c \in (0,1)$

In this case, the consumption reference levels are set such that one part, $\bar{C}$ or $(1+r_f)\bar{C}$, is independent of the household's current income and the remaining part is a fraction of its respective income.[13] In order to satisfy the $\Omega = (1-c)\left((1+r_f)Y_1 + Y_2\right) - 2(1+r_f)\bar{C} > 0$ assumption we require $\bar{C} < \frac{1-c}{2}\left(Y_1 + \frac{Y_2}{1+r_f}\right)$. In addition we assume $\bar{C} \geq -cY_1$ such that $\bar{C}_1 \geq 0$ and we assume $\bar{C} \geq -\frac{c}{1+r_f}Y_2$ such that $\bar{C}_2 \geq 0$. In this model the household increases reference consumption levels if its endowment income increases, so aspirations increase with growing income. The optimal first period consumption and investment in the risky asset are

$$
\begin{aligned}
C_1^* &= \frac{M-(1+r_f)}{1+r_f+M}\bar{C} + \frac{1+r_f+cM}{1+r_f+M}Y_1 + \frac{1-c}{1+r_f+M}Y_2 > \bar{C}_1 \ , \\
\alpha^* &= \frac{\left(1-K_0^{\frac{1}{\gamma}}\right)M}{r_f - r_b + K_0^{\frac{1}{\gamma}}(r_g - r_f)}\left(C_1^* - \bar{C}_1\right) > 0
\end{aligned}
$$

Note that now an increase in the first period income has a larger effect on optimal consumption than in the case when the first period reference level is independent of income. The reason why marginal propensity to consume is larger is that two effects are operating. First, the household increases consumption because of the increase in income and second, this effect is reinforced with an increase in the first period reference level. This situation can thus be seen as a *consumption overreaction* to current income changes. Note, in addition, that the

[13] All the statements made for example 3 would not change qualitatively if one used the same exogenous part of the reference level in both periods ($\bar{C}_1 = \bar{C} + cY_1, \bar{C}_2 = \bar{C} + cY_2$), different fractions of income in the two periods ($\bar{C}_1 = \bar{C} + c_1Y_1, \bar{C}_2 = (1+r_f)\bar{C} + c_2Y_2$) or the same exogenous part of the reference level and different fractions of income ($\bar{C}_1 = \bar{C} + c_1Y_1, \bar{C}_2 = \bar{C} + c_2Y_2$).

marginal propensity to consume (out of the first period income) is less than unity. Risky investment increases with an increase in the first period income but not as much as it would without this dependency. An increase in the second period income has similar effects. First, current period consumption increases but to a lower degree than in the independency case. Second, investment in the risky asset also increases, and again to a smaller extent than in the independency case.

If reference levels were set in this way households would be happier with an increase in the reference level if driven (only) by an increase in current period income. Households would be less happy, however, if the reference level was increased by a factor that is independent of the income. This situation is different from the case when reference levels are independent of income, where an increase in the reference level always decreases happiness. So now an increasing reference point can have either a positive or a negative effect on the household's happiness, depending on the source of the increase.

Example 4 (risky asset overreaction to income): $\bar{C}_1 = \bar{C} - cY_1$, $\bar{C}_2 = (1+r_f)\bar{C} - cY_2$ where $\bar{C} \in \left[c \max\left\{Y_1, \frac{Y_2}{1+r_f}\right\}, \frac{1+c}{2}\left(Y_1 + \frac{Y_2}{1+r_f}\right)\right)$ and $c \in (0,1)$

This is the opposite of the previous example with the household reducing the first (second) period consumption reference level as its first (second) period income increases. Hence the reference levels depend again partly on income and partly on a factor independent of income ($\bar{C}$ or $(1+r_f)\bar{C}$). For $\Omega > 0$ we assume that $\bar{C} < \frac{1+c}{2}\left(Y_1 + \frac{Y_2}{1+r_f}\right)$, and to satisfy the nonnegativity constraints on $\bar{C}_1$ and $\bar{C}_2$ we require $\bar{C} \geq cY_1$ and $\bar{C} \geq \frac{c}{1+r_f}Y_2$. The optimal solution is

$$
\begin{aligned}
C_1^* &= \frac{M-(1+r_f)}{1+r_f+M}\bar{C} + \frac{1+r_f-cM}{1+r_f+M}Y_1 + \frac{1+c}{1+r_f+M}Y_2 > \bar{C}_1 \ , \\
\alpha^* &= \frac{\left(1-K_0^{\frac{1}{\gamma}}\right)M}{r_f - r_b + K_0^{\frac{1}{\gamma}}(r_g - r_f)}\left(C_1^* - \bar{C}_1\right) > 0
\end{aligned}
$$

In this case the impact from a change in current income on optimal current consumption is smaller than in the case when the first period reference level is independent of income. This same impact is also smaller than in the previous example, because the increase in current income reduces the household's first period reference level. This indirect effect of current income on current consumption is negative and would have to be smaller in absolute value than the direct effect (which is positive) to make first period consumption a normal good.[14] Risky investment increases with an increase in current income, and it does so by a larger degree than when the reference level is independent of income. Hence, the household does not overreact with respect to current consumption, as in the previous example, but with

[14] This would be the case, i.e., $\frac{dC_1^*}{dY_1} > 0$, if $c < \frac{1+r_f}{M}$.

respect to investment in the risky asset. We call this a *risky asset overreaction* to current income changes.

Contrary to the previous example, and similar to cases when the reference levels are independent of income, the increase of reference levels driven (only) by a decrease of the income decreases the happiness level, which decreases also if the reference level is increased by a factor that is independent of the income. So now an increasing reference point will always have a negative effect on the household's happiness, independent of the source of the increase.

4 Balanced reference values relative to endowment income ($\Omega = 0$)

This case describes the situation when the household *adjusts* its consumption reference levels such that they are completely *in balance* with its total income. In other words, the household's present value of endowment income matches exactly the discounted sum of its first and second period reference consumption levels, i.e., $Y_1 + \frac{Y_2}{1+r_f} = \bar{C}_1 + \frac{\bar{C}_2}{1+r_f}$. This occurs when the household's goal is to achieve exactly what it can afford based on its endowment income stream. In this case the household cannot set its reference consumption levels independently in the first and second periods. It always has to balance the two targets in such a way that their sum will exactly match the total endowment (after discounting). This means that one reference level will be a function of the other, yielding

$$\begin{aligned} \bar{C}_1 &= Y_1 + \frac{Y_2 - \bar{C}_2}{1 + r_f} \quad \text{or} \\ \bar{C}_2 &= (1 + r_f)(Y_1 - \bar{C}_1) + Y_2 \end{aligned}$$

Depending on how one looks at it, the household either decides on its second period reference level and sets the first period reference level accordingly, or it sets the first period reference level and the second period reference level follows. In the latter case the household's second period consumption reference level will be equal to the sum of the second period income and the amount by which the first period income exceeds consuming at the reference level, transferred (through the risk-free asset) to the second period.

The dependence between the two reference levels implies that if the household, for some reason, *increases* its first period reference level by some given amount then it has to *decrease* the second period reference level by $(1 + r_f)$ times this amount, i.e., by the same amount transferred to second period value terms. If, on the other hand, the household *increases* the second period reference level by some amount then it has to *decrease* its first period reference level by $1/(1 + r_f)$ times this amount, i.e., by the same amount discounted to the first period. The above definitions of the two reference levels together with the upper limits on the reference levels given in the problem set-up imply that the reference levels are bound to be strictly positive, i.e., $\bar{C}_1 > 0$ and $\bar{C}_2 > 0$.

The following proposition presents the household's optimal choice and states the required assumptions when reference levels are balanced.

Proposition 2 *Let $\Omega = 0$ and $\lambda > \max\left\{\left(\frac{1+r_f}{k} + \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right)^\gamma, \left(\frac{M}{1+r_f}\right)^\gamma\right\}$. Then problem (5) obtains a unique maximum at $(C_1^* = \bar{C}_1, \alpha^* = 0)$.*

Proof. It follows directly from Lemma 1 in Appendix B. ■

First note that the household has to be sufficiently loss averse in order to make its optimal choice. In fact the lower bounds on the loss aversion parameter are similar to the case when the reference levels are low ($\Omega > 0$), adjusted for the fact that $\Omega = 0$.[15]

If the household is sufficiently loss averse then the first period optimal consumption is exactly equal to the first period reference consumption, $C_1^* = \bar{C}_1$. In addition, the household does not invest in the risky asset, $\alpha^* = 0$, even though the expected return of the risky asset exceeds the risk-free return. This is a major difference with respect to the traditional expected utility model, where the household will always invest in the risky asset if the expected return of risky asset is greater than the risk-free asset. Note, in addition (see equations (12) and (13)), that also the second period optimal consumption in both states corresponds to the second period reference consumption, namely, $C_{2g}^* = C_{2b}^* = \bar{C}_2$. The household may still transfer part of its income from the first period to the second period, or vice versa, in order to optimize its consumption path, but it will consume exactly at its reference level in both periods. This is a very particular situation.

In the light of this solution the above restriction that both reference levels must be strictly positive makes also sense from an economic point of view: assuming that a household lives for two periods it seems reasonable to require that its consumption is non-zero in both periods, which is guaranteed by $C_1^* = \bar{C}_1 > 0$ and $C_{2g}^* = C_{2g}^* = \bar{C}_2 > 0$. It can easily be seen that the savings, $S^* = m^* = Y_1 - \bar{C}_1$, are strictly positive if the consumption reference in the first period is below the first period income, i.e., when $\bar{C}_1 < Y_1$, and thus the household wants to transfer some of its first period income into the second period. To do this the household will only invest in the risk-free asset and will consume in the second (e.g., retirement) period the amount of $(1 + r_f)(Y_1 - \bar{C}_1)$ plus any exogenous future income Y_2. On the other hand, savings are negative if the consumption reference in the first period is above the first period income, i.e., when $\bar{C}_1 > Y_1$. In this case the household will transfer some part of its second period income, namely $Y_2 - \bar{C}_2$, into its first period and thus consume in the first period the amount $\frac{Y_2 - \bar{C}_2}{1+r_f} + Y_1$.

[15] Two of the previous lower bounds on the loss aversion parameter are now discarded as they are always smaller than other lower bounds included in the proposition.

We summarize the results on the sensitivity analysis related to the solution presented in Proposition 2 in Table 2. It presents the changes of the first and second period optimal consumption, of the optimal investment in the risky asset, of the consumption gap in the first and second period, of optimal savings and of happiness (first row) with respect to changes in the loss aversion parameter and the first and second period consumption reference levels (first column).

	dC_1^*	dC_{2g}^*	dC_{2b}^*	$d\alpha^*$	$d(C_1^* - \bar{C}_1)$	$d(C_{2g}^* - \bar{C}_2)$	$d(C_{2b}^* - \bar{C}_2)$	dS^*	$d(\mathbb{E}(U(C_1^*, \alpha^*)))$
$d\lambda$	0	0	0	0	0	0	0	0	0
$d\bar{C}_1$	1	$-(1+r_f)$	$-(1+r_f)$	0	0	–	–	-1	0
$d\bar{C}_2$	$-\frac{1}{1+r_f}$	1	1	0	–	0	0	$\frac{1}{1+r_f}$	0

Table 2: Sensitivity results when aspirations are balanced ($\Omega = 0$)

Similarly as for households with low reference levels an exogenous increase in the loss aversion parameter, keeping everything else constant, does not affect the solution. Note that now ($\Omega = 0$) the household cannot set its first and second period reference levels independently from each other. So if we want to analyze the effect of an exogenous increase in the first period consumption reference level, for instance, we also have to consider the resulting change (a decrease) in the second period consumption reference level. Taking this into consideration, an increase in the first period consumption reference will increase first period optimal consumption, will not affect risky asset holdings (they are always equal to zero) and will decrease (risk-free) savings. As less income is transferred to the second period, future consumption will in fact decrease in both states of nature. Following the same argument, an increase in the second period consumption reference level will increase the second period optimal consumption and reduce the first period consumption, since more (risk-free) savings have to be transferred to the second period. The consumption gap is equal to zero in both periods, so it is not affected by a change in either of the two consumption reference levels.

It can easily be seen that the indirect utility is not affected by either of the two consumption reference levels nor is it affected by the degree of loss aversion (as long as the household is sufficiently loss averse). I.e., the household's level of happiness will be insensitive to an increase of the (first or second period) consumption reference level as well as to any changes of the degree of loss aversion.

Continuing our *following the Joneses* example (where the household sets its reference consumption level according to its neighbor's consumption), the household actually mimics the Joneses behavior in the first or the second period, but not necessarily in both periods. If this household sets its first period reference level according to the Joneses then $C_1^* = \bar{C}_1 = C_{1,Joneses}^*$, so the optimal first period consumption levels of this household and of the Joneses will be identical, and this household will follow the Joneses in the first period.

Since the two reference levels are tied together, however, this also means that $C_2^* = \bar{C}_2 = (1 + r_f)(Y_1 - C^*_{1,Joneses}) + Y_2$, so this household's reaction in the second period will be to decrease (increase) its optimal consumption provided the Joneses increased (decreased) their consumption in the first period. If, on the other hand, this household sets its second period reference level according to the Joneses, it mimics the Joneses in the second period in the following sense: it consumes exactly at the expected optimal consumption of the Joneses, namely, $C_2^* = \bar{C}_2 = \mathbb{E}(C^*_{2,Joneses})$, so this household will follow the Joneses in the second period. Again, since the reference levels are not independent, this means at the same time that $\bar{C}_1 = Y_1 + (Y_2 - \mathbb{E}(C^*_{2,Joneses}))/(1 + r_f)$ and hence the household's reaction in the first period would be to decrease (increase) its optimal consumption provided the Joneses increase (decrease) their expected optimal consumption in the second period. In summary, our results imply that the balanced household follows the Joneses either in the first or in the second period but not in both, provided that the Joneses change their first and second period consumption in the same direction, i.e, increase – or alternatively decrease – their (expected) optimal consumption in both periods.

Some examples of reference consumption levels for $\Omega = 0$

In the following we present some examples worth mentioning for the balanced household.

Example 5: $\bar{C}_1 = Y_1$ and $\bar{C}_2 = Y_2$

This situation occurs when the household sets its consumption reference levels equal to its respective incomes in both periods. In this case the household will invest neither in the risky asset nor in the risk-free asset, so no income is transferred to enable a larger future or current consumption. The household will totally consume its first period income in the first period and its second period income in the second period. Thus, households that belong to this category do not save or borrow anything and rely exclusively on their exogenous income to consume.

Example 6 (status quo): $\bar{C}_2 = (1 + r_f)(Y_1 - \bar{C}_1)$ and $Y_2 = 0$

In this case, the second period consumption reference level is set equal to the gross return of investing the first period endowment income net of the first period consumption reference level which can be considered as the counterpart to reference levels in one-period models, which are equal to the gross return from investing all initial wealth into the risk-free asset.[16] In our case the initial wealth would correspond to $Y_1 - \bar{C}_1$. In addition, $Y_2 = 0$, i.e., the household does not receive any exogenous second period income (as in one-period models).

[16]See Hlouskova and Tsigaris (2012).

5 High reference values relative to endowment income ($\Omega < 0$)

Now we consider the case when the household sets its consumption reference levels such that the present value is above the present value of its endowment income. This is done when the household has high aspirations. Let us first introduce the following notation

$$
\begin{aligned}
M_1(\lambda) &= k\left[\lambda^{\frac{1}{\gamma}} - \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right] && (14)\\
\tilde{c}^{P2} &= \frac{(r_g - r_b)(-\Omega)}{(r_g - r_f)\bar{C}_2} && (15)\\
\bar{C}_2^{P2} &= \frac{r_g - r_b}{r_f - r_b}\left((1+r_f)\left(Y_1 - \bar{C}_1\right) + Y_2\right) && (16)\\
\delta^{+} &= \frac{1}{1-p}\left[\frac{r_g - r_f}{(1+r_f)(r_g - r_b)}\right]^{1-\gamma} && (17)\\
k &= \left[\delta(1+r_f)(1-p)\left(\frac{r_g - r_b}{r_g - r_f}\right)^{1-\gamma}\right]^{\frac{1}{\gamma}} && (18)\\
k_2 &= \left[\delta(1+r_f)p\left(\frac{r_g - r_b}{r_f - r_b}\right)^{1-\gamma}\right]^{\frac{1}{\gamma}} && (19)
\end{aligned}
$$

Note that $M_1(\lambda)$ is an increasing function in λ and if $\lambda \geq \frac{1}{K_\gamma}$ then $M_1(\lambda) \geq 0$. A simple derivation shows that $\lambda > \frac{1+r_f}{k} + \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}$ is sufficient for $M_1(\lambda) > 1 + r_f$. Note in addition that for $\bar{C}_2 < \bar{C}_2^{P2}$ is $\tilde{c}^{P2} < 1$.

We introduce an additional notation

$$
\begin{aligned}
\hat{\lambda} &= \left[\frac{k_2\left(1 + K_\gamma^{\frac{1}{\gamma}}\right)}{1+r_f}\right]^{\gamma} = \left[\frac{k\left(1 + \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right)}{1+r_f}\right]^{\gamma} && (20)\\
\lambda_1^{\Omega<0} &= \left[\frac{\frac{1+r_f}{k} + \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}}{1 - \tilde{c}^{P2}}\right]^{\gamma} \quad \text{if} \quad \bar{C}_2 < \bar{C}_2^{P2} && (21)\\
\lambda_2^{\Omega<0} &= \hat{\lambda}\left[\frac{\bar{C}_1}{Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}}\right]^{\gamma} && (22)\\
\tilde{\lambda}^{\Omega<0} &= \frac{1}{p}\left[\frac{1}{\delta}\left(\frac{Y_1 - \bar{C}_1 + \frac{Y_2}{1+r_f}}{\bar{C}_2}\right)^{1-\gamma} - (1-p)\left(1 - \tilde{c}^{P2}\right)\lambda_1^{\Omega<0}\right] \quad \text{if} \quad \bar{C}_2 < \bar{C}_2^{P2} && (23)
\end{aligned}
$$

The following proposition presents the household's optimal choice and states the required assumptions for high reference values relative to endowment income.

Proposition 3 *Let* $\Omega < 0$, $\bar{C}_2 < \bar{C}_2^{P2}$ *and* $\lambda > \max\left\{\lambda_1^{\Omega<0}, \lambda_2^{\Omega<0}, \tilde{\lambda}^{\Omega<0}\right\}$. *Then the following holds*

$$C_1^* = \left\{ \begin{array}{ll} C_1^{P2} = \bar{C}_1 + \frac{-\Omega}{M_1(\lambda)-1-r_f} > \bar{C}_1 & \text{if} \quad \delta \leq \delta^+ \\ \\ 0 < C_1^{P5} = \frac{\left(Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}\right)\lambda^{\frac{1}{\gamma}} - \bar{C}_1 \hat{\lambda}^{\frac{1}{\gamma}}}{\lambda^{\frac{1}{\gamma}} - \hat{\lambda}^{\frac{1}{\gamma}}} < \bar{C}_1 & \text{if} \quad \delta > \delta^+ \end{array} \right\} \tag{24}$$

$$\alpha^* = \left\{ \begin{array}{ll} \alpha^{P2} = \frac{\left(\left(\frac{1}{K_0}\right)^{\frac{1}{\gamma}} + \lambda^{\frac{1}{\gamma}}\right)k}{r_g - r_f}(C_1^* - \bar{C}_1) > 0 & \text{if} \quad \delta \leq \delta^+ \\ \\ \alpha^{P5} = \frac{1 - K_0^{\frac{1}{\gamma}}}{r_f - r_b + K_0^{\frac{1}{\gamma}}(r_g - r_f)} \frac{\hat{\lambda}^{\frac{1}{\gamma}}}{\lambda^{\frac{1}{\gamma}} - \hat{\lambda}^{\frac{1}{\gamma}}}(-\Omega) > 0 & \text{if} \quad \delta > \delta^+ \end{array} \right\} \tag{25}$$

Proof. See Appendix C. ■

Proposition 3 is derived for a relatively low[17] second period reference level, $\bar{C}_2 < \bar{C}_2^{P2}$, and for a sufficiently loss averse household, $\lambda \geq \max\left\{\lambda_1^{\Omega<0}, \lambda_2^{\Omega<0}, \tilde{\lambda}^{\Omega<0}\right\}$. The fact that $\Omega < 0$ implies that the household's aspirations are high, which should make it more difficult to exceed the consumption reference levels than when the household's aspirations are low. So, even if the utility is such that consumption below the reference level is heavily penalized (large values of the loss aversion parameter λ) we cannot expect optimal consumption levels to exceed their reference levels at all times. This is indeed what we observe: in the first solution the second period optimal consumption in the bad state of nature is below its reference level, while in the second solution the first period optimal consumption is below its reference level. The first solution is denoted by superscript P2, which refers to problem (P2) where the solution was reached, and the second solution is denoted by superscript P5, which refers to problem (P5) where the solution was reached, see Appendix A. Thus, when the household's aspirations are set above the present value of the endowment income, a relative loss (either in the first or second period) cannot be avoided.

Unlike households with low aspirations, we now have two different solutions, denoted $\left(C_1^{P2}, \alpha^{P2}\right)$ and $\left(C_1^{P5}, \alpha^{P5}\right)$,[18] and which one applies depends on the rate δ at which future utility is discounted. The first solution applies to households with lower discount factors

[17] Note that households cannot set arbitrarily high reference levels for a given endowment income.

[18] Again, for the proof of the household's consumption decision we split problem (5) into eight separate problems, (P1)–(P8), which differ in the respective domains of feasible solutions. These domains are specified by whether (first and second period, good and bad state of nature) consumption is above or below the respective reference level. See Appendix A.

($\delta \leq \delta^+$), which put relatively more emphasis on the well-being in the present and near future and thus display a high time preference, while the second applies to households with higher discount factors ($\delta > \delta^+$), which care relatively more about the distant future and discount future utility at a lower rate and thus show a low time preference. The threshold value δ^+ separating the two types is a function of the rates r_f, r_g and r_b, of the probability of the good state of nature p and of the curvature parameter γ, see equation (17). Note that it is increasing in p and γ while it is decreasing in r_f, all other things equal. The restriction $\delta > \delta^+$ only yields feasible candidates for the discount factor, which is bound to be below one, if $\delta^+ < 1$. This is the case when the probability of the good state is not too large.[19]

We will discuss the two solutions separately starting with $\left(C_1^{P2}, \alpha^{P2}\right)$. First note that the optimal consumption in the first period is strictly above the consumption reference level. As the solution $\left(C_1^{P2}, \alpha^{P2}\right)$ is reached in problem (P2) the optimal consumption in the second period is above the reference level $\bar{C}_2$ in the good state of nature, $C_{2g}^{P2} > \bar{C}_2$, and below $\bar{C}_2$ in the bad state of nature, $C_{2b}^{P2} < \bar{C}_2$. Thus, the household cannot avoid a relative loss if the bad state of nature materializes. Further, the optimal investment in the risky asset is strictly positive. Like in the case for households with low aspirations, savings can be either positive or negative in general.[20] If, however, the household's consumption reference level is equal to or above its income in the first period[21] then optimal savings are always negative, i.e., the household will transfer future income to the present period in order to satisfy its optimal consumption path. Given that optimal risky investment is positive, borrowing in the risk-free market has to be sufficiently large then, in order to produce negative savings. In fact, in this situation of negative savings the household will invest in the risky asset in order to (partially) fund the borrowing and thus the income transfer from the second to the first period. If in one period the household's income is below the consumption reference level and vice versa in the other period, then the answer to the question whether optimal savings are positive depends (also) on the loss aversion parameter. Keeping everything else constant, a larger loss aversion parameter will increase savings (see below). Note that the optimal savings of a household with low aspirations did not depend on the loss aversion parameter.

For C_1^{P2} and α^{P2} given by (24) and (25) the following holds:

(i) $\lim_{\lambda \to \infty} C_1^{P2} = \bar{C}_1$

(ii) $\lim_{\lambda \to \infty} C_{2g}^{P2} = \bar{C}_2$

(iii) $\lim_{\lambda \to \infty} C_{2b}^{P2} = \frac{r_f - r_b}{r_g - r_f}\left(\bar{C}_2^{P2} - \bar{C}_2\right) < \bar{C}_2$

[19] More precisely, $p < 1 - \left(\frac{1}{1+r_f}\frac{r_g - r_f}{r_g - r_b}\right)^{1-\gamma}$ is needed in order to guarantee $\delta^+ < 1$. Note that there is always a solution for p as the term in the brackets is smaller than one.

[20] The assumption required for $S^* > 0$ is $M_1(\lambda)(Y_1 - \bar{C}_1) > -(Y_2 - \bar{C}_2)$.

[21] This condition is sufficient but not necessary.

(iv) $\lim_{\lambda\to\infty} \alpha^{P2} = \frac{-\Omega}{r_g - r_f} \leq \alpha^{P2}$

The following table summarizes the results on the sensitivity analysis of optimal consumption in both periods, optimal risky investment, the consumption gap in both periods, optimal savings and happiness (first row) with respect to the loss aversion parameter λ and the first and second period consumption reference levels $\bar{C}_1$ and $\bar{C}_2$ (first column).

	dC_1^{P2}	dC_{2g}^{P2}	dC_{2b}^{P2}	$d\alpha^{P2}$	$d(C_1^{P2} - \bar{C}_1)$	$d(C_{2g}^{P2} - \bar{C}_2)$	$d(\bar{C}_2 - C_{2b}^{P2})$	dS^{P2}	$d\mathbb{E}(U(C_1^{P2}, \alpha^{P2}))$
$d\lambda$	< 0	< 0	> 0	< 0	< 0	< 0	< 0	> 0	< 0
$d\bar{C}_1$	> 0	> 0	< 0	> 0	> 0	–	–	< 0	< 0
$d\bar{C}_2$	> 0	> 0	< 0	> 0	–	> 0	> 0	< 0	< 0

Table 3: Sensitivity results when aspirations are high ($\Omega < 0$): solution $\left(C_1^{P2}, \alpha^{P2}\right)$

As opposed to households with low aspirations the optimal consumption and optimal investment in the risky asset of households with high aspirations are sensitive with respect to the loss aversion parameter λ. An exogenous increase in the loss aversion parameter, keeping everything else constant, will decrease the first and the second period optimal consumption in the good state of nature, decrease the investment in the risky asset, increase savings and thus increase the investment in the risk-free asset. In addition, increased loss aversion decreases the consumption gap in both periods. Also the happiness level decreases with increasing loss aversion, i.e., more loss averse households are less happy than less loss averse households.

An exogenous increase in the first period consumption reference level, keeping everything else constant, will increase the first period optimal consumption as well as the second period optimal consumption in the good state (which is above the consumption reference level) and the investment in the risky asset, but will decrease the second period optimal consumption in the bad state (which is below the consumption reference level). At the same time the household will decrease optimal savings, which actually implies the observed decrease in optimal consumption in bad state of nature. Not only does optimal consumption rise in the first period and in the good state in the second period, but also the corresponding relative optimal consumption rises. Note that the household's happiness decreases with an increasing consumption reference level, i.e., more ambitious households are less happy than less ambitious ones. An exogenous increase in the second period consumption reference level yields exactly the same sensitivities in terms of signs as an increase in the first period consumption reference level.

Continuing our previous example, where the household compares its optimal consumption to its neighbor's consumption level we see that the household again follows the Joneses in the first period.[22] However, contrary to the case when $\Omega > 0$ the household does not reduce the consumption gap but widens this gap. This means that the household increases its

[22] Assuming the household increases its first period reference level as a response to an increase of the Joneses' first period consumption, it will also increase its optimal consumption in the first period.

consumption even more than the Joneses do. A household with high aspirations reacts thus more intensely than one with low aspirations, even though both follow the Joneses.

As in the case when $\Omega > 0$, it is not clear whether the household follows the Joneses in the second period, since its second period consumption in the good and the bad states of nature responds in opposite directions to the change in its second period reference level, and thus it is not clear whether the expected household's consumption will reflect an increase or a decrease of consumption. In addition in the second period, in both the good and the bad states of nature, the consumption gap will widen in response to an increase in the second period reference level. This, however, implies a decrease of the optimal consumption when it is below the reference level, which is the case in the bad state of nature.

Note, finally, that an infinitely loss averse household will optimally consume the consumption reference level in the first period and in the second period in the good state of nature, it will have strictly positive optimal consumption below its reference level in the bad state of nature and will invest the strictly positive amount of $-\Omega/(r_g - r_f)$ in the risky asset.

We now turn to the discussion of the second solution of Proposition 3, $\left(C_1^{P5}, \alpha^{P5}\right)$, which holds for households with a low time preference, i.e., for households with a high discount factor $\delta > \delta^+$. As the notation suggests, this solution is reached in problem (P5), where the first and second period consumption domains are given by $0 \leq C_1 \leq \bar{C}_1$ and $C_{2g} \geq C_{2b} \geq \bar{C}_2$. The optimal consumption in the first period is thus below the reference level and the optimal consumption in the second period is above the reference level $\bar{C}_2$ in both states of nature. Even though the household is rather loss averse (and so the penalty for consumption below the reference level is rather large) the optimal consumption in the first period is below the consumption reference level. With a sufficiently low time preference, i.e., a sufficiently high discount factor $\delta > \delta^+$, the household values future consumption so much that it prefers to consume above the reference level in both states of nature in the second period, accepting to consume below the reference level in the first period. Again the optimal investment in the risky asset is strictly positive.

As in the first solution, savings can be either positive or negative in general.[23] If, however, the household sets its consumption reference level equal to or below its income in the first period[24] then optimal savings are always positive, i.e., the household transfers current income to the future period in order to satisfy its optimal consumption path. Note that a larger loss aversion parameter will *de*crease savings while before (in the first solution) it increased savings.

For the solution $\left(C_1^{P5}, \alpha^{P5}\right)$ the following holds:

(i) $\lim_{\lambda\to\infty} C_1^{P5} = Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f} < \bar{C}_1$

[23]The assumption required for $S^* > 0$ is $\hat{\lambda}^{1/\gamma}(Y_1 - \bar{C}_1) < -\lambda^{1/\gamma}\frac{Y_2 - \bar{C}_2}{1+r_f}$.

[24]This condition is sufficient but not necessary.

(ii) $\lim_{\lambda\to\infty} C_{2g}^{P5} = \bar{C}_2$

(iii) $\lim_{\lambda\to\infty} C_{2b}^{P5} = \bar{C}_2$

(iv) $\lim_{\lambda\to\infty} \alpha^{P5} = 0$

The following table summarizes the results on the sensitivity analysis of optimal consumption in both periods, optimal risky investment, the consumption gap in both periods, optimal savings and happiness (first row) with respect to the loss aversion parameter λ and the first and second period consumption reference levels $\bar{C}_1$ and $\bar{C}_2$ (first column).

	dC_1^{P5}	dC_{2g}^{P5}	dC_{2b}^{P5}	$d\alpha^{P5}$	$d(\bar{C}_1 - C_1^{P5})$	$d(C_{2g}^{P5} - \bar{C}_2)$	$d(C_{2b}^{P5} - \bar{C}_2)$	dS^{P5}	$d\mathbb{E}(U(C_1^{P5}, \alpha^{P5}))$
$d\lambda$	> 0	< 0	< 0	< 0	< 0	< 0	< 0	< 0	< 0
$d\bar{C}_1$	< 0	> 0	> 0	> 0	> 0	–	–	> 0	< 0
$d\bar{C}_2$	< 0	> 0	> 0	> 0	–	> 0	> 0	> 0	< 0

Table 4: Sensitivity results when aspirations are high ($\Omega < 0$): solution $\left(C_1^{P5}, \alpha^{P5}\right)$

Again, as for households with a higher time preference (i.e., a lower discount factor), the optimal consumption and optimal investment in the risky asset are sensitive with respect to the loss aversion parameter λ. An exogenous increase in the loss aversion parameter, keeping everything else constant, will increase first period optimal consumption (which is below the reference level), and decrease everything else, i.e., the second period optimal consumption in both states of nature (which is above the reference level), the investment in the risky asset, savings, the consumption gaps in both periods as well as the level of happiness.

An exogenous increase in the first period consumption reference level, keeping everything else constant, will decrease the first period optimal consumption and thus increase optimal savings, and, additionally, increase the consumption in the second period in both states of nature as well as investment in the risky asset. The level of happiness will again decrease with an increasing reference level and thus more ambitious households are less happy than the less ambitious ones. The sensitivity analysis with respect to the second period consumption reference level (of all variables under consideration) is – in terms of signs – the same as with respect to the first period consumption reference level.

Putting this into the context of *following the Joneses* we observe the following: the household does not follow the Joneses in the first period while it indeed follows the Joneses in the second period. In addition, in the first period the household's optimal consumption does not only move in the opposite direction with respect to that of the Joneses also the consumption gap increases. Note that – as in the first solution and contrary to the case when $\Omega > 0$ – the household increases its consumption even to a larger degree than the Joneses, provided it does follow the Joneses. So households react stronger when they have high aspirations than when they have low aspirations: in following the Joneses, households increase their optimal consumption even more than the Joneses do (i.e., they widen their consumption gaps). This is true in the first period for households with a higher time preference and it is true in the second period for households with a lower time preference.

Note, finally, that an infinitely loss averse household will optimally consume the first period income plus the discounted relative consumption (i.e., the difference between the second period income and the second period consumption reference level). It will consume at the reference level in the second period in both states of nature and it will only invest in the risk-free asset.

Comparison to *richer* peers: $Y_1 + \frac{Y_2}{1+r_f} < Y_1^R + \frac{Y_2^R}{1+r_f} = \bar{C}_1 + \frac{\bar{C}_2}{1+r_f}$

In this situation the household with income levels Y_1 and Y_2 sets its first and second period consumption references such that the total reference consumption is equal to the total income of some richer household (in the first or second period's value terms). By a *richer* household we mean a household whose total income is larger than the total income of this household; namely, $Y_1^R + \frac{Y_2^R}{1+r_f} > Y_1 + \frac{Y_2}{1+r_f}$, where Y_1^R and Y_2^R are the first and the second period income levels of the richer household such that $Y_1^R > 0$ and $Y_2^R \geq 0$. In this case Ω represents the household's total income relative to the total income of the richer household, i.e., $\Omega = (1+r_f)Y_1 + Y_2 - \big((1+r_f)Y_1^R + Y_2^R\big) < 0$. In other words, more ambitious households place themselves into the *discomfort zone* by comparing themselves to peers with a higher total income. The impact of changes in the rich person's consumption on this household's consumption and investment were discussed previously in the sensitivity analysis.

6 Concluding remarks and future extensions

We can conclude from this study that reference levels and loss aversion play a very important role in determining not only optimal portfolio decisions, as has been found in the literature until now,[25] but also in determining inter-temporal decisions on current and future consumption levels, which depend on the total savings transferred and the risky investment activity undertaken. One-period models, investigated to date, impose the assumption that current consumption is fixed at a certain level and hence the household invests the exogenous initial wealth to the safe and the risky assets. In this model current consumption and savings are not fixed but optimally selected by the household, which generalizes the one-period model to a two-period model that can be thought of as a life cycle model. The optimal solutions depend on the household's choice of the present value of the reference levels relative to the present value of its endowment incomes.

If the present value of the consumption reference levels is lower than the present value of the endowment income ($\Omega > 0$) then the household behaves in such a way to avoid any relative losses, both in the current period and in any future states of nature (good or bad).

[25] See Hlouskova and Tsigaris (2012) for a literature review.

As a consequence the degree of loss aversion does not directly affect optimal consumption and risk taking activity. But loss aversion is needed to be sufficiently high to prevent relative losses. On the other hand, also reference consumption levels play a significant role in affecting consumption and risk taking activity. People often compare their own income or consumption levels to that of others and hence use reference levels determined by other people's income, wealth or consumption. We find that following others in wanting to consume more may actually hurt households and make them less happy. However, if the reference level depends on endowment income this is not always true, in particular when reference levels increase with growing income. In the first period we observe that the prospect theory household follows what the Joneses are doing if the reference level is set equal to the consumption level of the Joneses. In the second period, however, the household does not (necessarily) follow the Joneses. In both periods, the gap between the household's consumption and reference level shrinks as the reference level increases. In addition, if the consumption reference levels are increased then the investment in the risky asset is reduced.

On the other hand, if the discounted present value of the consumption reference levels coincides with the present value of the endowment income, i.e., the household sets its reference levels such that they are in balance with its total income ($\Omega = 0$) then the household's optimal consumption is the reference consumption in both periods, and the investment in the risky asset is zero. In addition, the household follows the Joneses only and exactly in one period, either in the first one (and not in the second) or in the second one (and not in the first). Also being more ambitious will not make the household more happy, e.g., when it compares itself to richer households. In fact, just the opposite is true: households that are more ambitious are less happy.

Finally, if the present value of the consumption reference levels is higher than the present value of the endowment income ($\Omega < 0$) then the household cannot avoid experiencing a relative loss, either today or in the future. As a result, loss aversion directly affects consumption and risky investment. Here, too, the reference levels play an important role in affecting the household's behavior. For example, in half of the cases the household will follow the Joneses if the reference levels are equal to the consumption levels of the Joneses. However, in this case the gap between the household's optimal consumption and its reference level widens as the reference level increases.

If a prospect theory household is more ambitious, i.e., if it increases its consumption reference level, this will decrease its happiness. And this is equally true for all three types of households, those with low, balanced and high aspirations. On the other hand, households with low and high aspirations differ completely in their following the Joneses behavior, provided they do follow the Joneses. While households with low aspirations follow the Joneses only under-proportionately (i.e., they increase their optimal consumption to a lower degree than the Joneses) households with high aspirations follow the Joneses over-proportionately (i.e., they increase their optimal consumption even more than the Joneses do).

There are a number of extensions that might be worth undertaking in the future. One could be to introduce uncertainty in the second period exogenous income instead of uncertainty in the returns of the risky asset. Another extension could be to consider an endogenous second period consumption reference instead of considering it exogenous, as we did in this study.[26] For example, the second period consumption reference level could be a weighted average of the first period reference level and the first period consumption (habit persistence). Still another extension could be to develop a model where the utility includes consumption reference levels directly (and not only through relative consumption) in order to give households not only disutility but also pleasure from having them to compare. Households would then select the reference levels endogenously by setting the marginal benefit equal to the marginal cost. Finally, one could explore the impact of taxation on the decisions to take risk and to consume today. This could be either a tax on the exogenous endowment income or a tax on capital income or a tax on both.

[26]This feature, however, allowed us to investigate certain types of reference levels which could not have been done otherwise.

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Appendix A: Optimization problems

There are eight cases to consider in proofs of lemmas 1 and 2 when $\alpha \geq 0$:

(P1) $C_1 \geq \bar{C}_1$, $C_{2b} \geq \bar{C}_2$ and $C_{2g} \geq \bar{C}_2$

(P2) $C_1 \geq \bar{C}_1$, $C_{2b} \leq \bar{C}_2$ and $C_{2g} \geq \bar{C}_2$

(P3) $C_1 \geq \bar{C}_1$, $C_{2b} \geq \bar{C}_2$ and $C_{2g} \leq \bar{C}_2$

(P4) $C_1 \geq \bar{C}_1$, $C_{2b} \leq \bar{C}_2$ and $C_{2g} \leq \bar{C}_2$

(P5) $C_1 \leq \bar{C}_1$, $C_{2b} \geq \bar{C}_2$ and $C_{2g} \geq \bar{C}_2$

(P6) $C_1 \leq \bar{C}_1$, $C_{2b} \leq \bar{C}_2$ and $C_{2g} \geq \bar{C}_2$

(P7) $C_1 \leq \bar{C}_1$, $C_{2b} \geq \bar{C}_2$ and $C_{2g} \leq \bar{C}_2$

(P8) $C_1 \leq \bar{C}_1$, $C_{2b} \leq \bar{C}_2$ and $C_{2g} \leq \bar{C}_2$

The corresponding problems are

$$\left.\begin{array}{ll}\mathrm{Max}_{(\mathrm{C}_1,\alpha)}: & \mathbb{E}(U(C_1,\alpha)) = \\ & \frac{(C_1-\bar{C}_1)^{1-\gamma}}{1-\gamma} + \delta p \frac{((1+r_f)(Y_1-C_1)+(r_g-r_f)\alpha+Y_2-\bar{C}_2)^{1-\gamma}}{1-\gamma} \\ & +\delta(1-p)\frac{((1+r_f)(Y_1-C_1)-(r_f-r_b)\alpha+Y_2-\bar{C}_2)^{1-\gamma}}{1-\gamma} \\ \text{such that}: & \bar{C}_1 \leq C_1 \leq Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f} - \frac{r_f-r_b}{1+r_f}\alpha \\ & \max\left\{0, -\frac{\Omega}{r_g-r_f}\right\} \leq \alpha \leq \frac{\Omega}{r_f-r_b}\end{array}\right\} \quad \text{(P1)}$$

$$\left.\begin{array}{ll}\mathrm{Max}_{(\mathrm{C}_1,\alpha)}: & \mathbb{E}(U(C_1,\alpha)) = \\ & \frac{(C_1-\bar{C}_1)^{1-\gamma}}{1-\gamma} + \delta p \frac{((1+r_f)(Y_1-C_1)+(r_g-r_f)\alpha+Y_2-\bar{C}_2)^{1-\gamma}}{1-\gamma} \\ & -\lambda\delta(1-p)\frac{(\bar{C}_2-Y_2-(1+r_f)(Y_1-C_1)+(r_f-r_b)\alpha)^{1-\gamma}}{1-\gamma} \\ \text{such that}: & Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f} - \frac{r_f-r_b}{1+r_f}\alpha \leq C_1 \leq Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f} + \frac{r_g-r_f}{1+r_f}\alpha \\ & \bar{C}_1 \leq C_1 \leq Y_1 + \frac{Y_2}{1+r_f} - \frac{r_f-r_b}{1+r_f}\alpha \\ & \max\left\{0, \frac{-\Omega}{r_g-r_f}\right\} \leq \alpha \leq \frac{(1+r_f)(Y_1-\bar{C}_1)+Y_2}{r_f-r_b} = \frac{\Omega+\bar{C}_2}{r_f-r_b}\end{array}\right\} \quad \text{(P2)}$$

To guarantee that $\frac{-\Omega}{r_g-r_f} \leq \frac{\Omega+\bar{C}_2}{r_f-r_b}$ the condition $\bar{C}_2 \leq \bar{C}_2^{P2} \equiv \frac{r_g-r_b}{r_f-r_b}\big((1+r_f)(Y_1-\bar{C}_1)+Y_2\big)$ needs to be satisfied.[27]

$$
\left.
\begin{array}{rl}
\text{Max}_{(\mathrm{C}_1,\alpha)}: & \mathbb{E}(U(C_1,\alpha)) = \\
& \frac{(C_1-\bar{C}_1)^{1-\gamma}}{1-\gamma} - \lambda\delta p\frac{(\bar{C}_2-Y_2-(1+r_f)(Y_1-C_1)-(r_g-r_f)\alpha)^{1-\gamma}}{1-\gamma} \\
& +\delta(1-p)\frac{((1+r_f)(Y_1-C_1)-(r_f-r_b)\alpha+Y_2-\bar{C}_2)^{1-\gamma}}{1-\gamma} \\
\text{such that}: & Y_1+\frac{Y_2-\bar{C}_2}{1+r_f}+\frac{r_g-r_f}{1+r_f}\alpha \leq C_1 \leq Y_1+\frac{Y_2-\bar{C}_2}{1+r_f}-\frac{r_f-r_b}{1+r_f}\alpha \\
& \bar{C}_1 \leq C_1 \leq Y_1+\frac{Y_2}{1+r_f}+\frac{r_g-r_f}{1+r_f}\alpha \\
& 0 \leq \alpha \leq \min\left\{0, \frac{\Omega}{r_f-r_b}\right\}
\end{array}
\right\} \quad \text{(P3)}
$$

Note that the only feasible solution is $\left(C_1 = Y_1+\frac{Y_2-\bar{C}_2}{1+r_f}, \alpha = 0\right)$ when $\Omega > 0$ and $\left(C_1 = \bar{C}_1, \alpha = 0\right)$ when $\Omega = 0$. There is no feasible solution when $\Omega < 0$.

$$
\left.
\begin{array}{rl}
\text{Max}_{(\mathrm{C}_1,\alpha)}: & \mathbb{E}(U(C_1,\alpha)) = \\
& \frac{(C_1-\bar{C}_1)^{1-\gamma}}{1-\gamma} - \lambda\delta p\frac{(\bar{C}_2-Y_2-(1+r_f)(Y_1-C_1)-(r_g-r_f)\alpha)^{1-\gamma}}{1-\gamma} \\
& -\lambda\delta(1-p)\frac{(\bar{C}_2-Y_2-(1+r_f)(Y_1-C_1)+(r_f-r_b)\alpha)^{1-\gamma}}{1-\gamma} \\
\text{such that}: & Y_1+\frac{Y_2-\bar{C}_2}{1+r_f}+\frac{r_g-r_f}{1+r_f}\alpha \leq C_1 \\
& \bar{C}_1 \leq C_1 \leq Y_1+\frac{Y_2}{1+r_f}-\frac{r_f-r_b}{1+r_f}\alpha \\
& 0 \leq \alpha \leq \frac{\bar{C}_2}{r_g-r_b}
\end{array}
\right\} \quad \text{(P4)}
$$

$$
\left.
\begin{array}{rl}
\text{Max}_{(\mathrm{C}_1,\alpha)}: & \mathbb{E}(U(C_1,\alpha)) = \\
& -\lambda\frac{(\bar{C}_1-C_1)^{1-\gamma}}{1-\gamma} + \delta p\frac{((1+r_f)(Y_1-C_1)+(r_g-r_f)\alpha+Y_2-\bar{C}_2)^{1-\gamma}}{1-\gamma} \\
& +\delta(1-p)\frac{((1+r_f)(Y_1-C_1)-(r_f-r_b)\alpha+Y_2-\bar{C}_2)^{1-\gamma}}{1-\gamma} \\
\text{such that}: & 0 \leq C_1 \leq \min\left\{Y_1+\frac{Y_2-\bar{C}_2}{1+r_f}-\frac{r_f-r_b}{1+r_f}\alpha, \bar{C}_1\right\} \\
& 0 \leq \alpha \leq \frac{(1+r_f)Y_1+Y_2-\bar{C}_2}{r_f-r_b} = \frac{\Omega+(1+r_f)C_1}{r_f-r_b}
\end{array}
\right\} \quad \text{(P5)}
$$

[27]Note that the condition on $\bar{C}_2$ follows from the constraint $C_{2b} \geq 0$ given by $C_1 \leq Y_1 + \frac{Y_2}{1+r_f} - \frac{r_f-r_b}{1+r_f}\alpha$.

$$\left.\begin{array}{ll}
\text{Max}_{(\mathrm{C}_1,\alpha)}: & \mathbb{E}(U(C_1,\alpha)) = \\
& -\lambda\frac{(\bar{C}_1-C_1)^{1-\gamma}}{1-\gamma} + \delta p\frac{((1+r_f)(Y_1-C_1)+(r_g-r_f)\alpha+Y_2-\bar{C}_2)^{1-\gamma}}{1-\gamma} \\
& -\lambda\delta(1-p)\frac{(\bar{C}_2-Y_2-(1+r_f)(Y_1-C_1)+(r_f-r_b)\alpha)^{1-\gamma}}{1-\gamma} \\
\text{such that}: & Y_1+\frac{Y_2-\bar{C}_2}{1+r_f}-\frac{r_f-r_b}{1+r_f}\alpha \leq C_1 \leq Y_1+\frac{Y_2-\bar{C}_2}{1+r_f}+\frac{r_g-r_f}{1+r_f}\alpha \\
& 0 \leq C_1 \leq \min\left\{Y_1+\frac{Y_2}{1+r_f}-\frac{r_f-r_b}{1+r_f}\alpha, \bar{C}_1\right\} \\
& \max\left\{0, \frac{\Omega}{r_f-r_b}\right\} \leq \alpha \leq \frac{(1+r_f)Y_1+Y_2}{r_f-r_b}
\end{array}\right\} \quad \text{(P6)}$$

$$\left.\begin{array}{ll}
\text{Max}_{(\mathrm{C}_1,\alpha)}: & \mathbb{E}(U(C_1,\alpha)) = \\
& -\lambda\frac{(\bar{C}_1-C_1)^{1-\gamma}}{1-\gamma} - \lambda\delta p\frac{(\bar{C}_2-Y_2-(1+r_f)(Y_1-C_1)-(r_g-r_f)\alpha)^{1-\gamma}}{1-\gamma} \\
& +\delta(1-p)\frac{((1+r_f)(Y_1-C_1)-(r_f-r_b)\alpha+Y_2-\bar{C}_2)^{1-\gamma}}{1-\gamma} \\
\text{such that}: & Y_1+\frac{Y_2-\bar{C}_2}{1+r_f}+\frac{r_g-r_f}{1+r_f}\alpha \leq C_1 \leq \min\left\{Y_1+\frac{Y_2-\bar{C}_2}{1+r_f}-\frac{r_f-r_b}{1+r_f}\alpha, \bar{C}_1\right\} \\
& 0 \leq \alpha \leq \min\left\{0, -\frac{\Omega}{r_g-r_f}\right\}
\end{array}\right\} \quad \text{(P7)}$$

Note that the only feasible solution is $\left(C_1 = Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}, \alpha = 0\right)$ when $\Omega < 0$ and $\left(C_1 = \bar{C}_1, \alpha = 0\right)$ when $\Omega = 0$. There is no feasible solution when $\Omega > 0$.

$$\left.\begin{array}{ll}
\text{Max}_{(\mathrm{C}_1,\alpha)}: & \mathbb{E}(U(C_1,\alpha)) = \\
& -\lambda\frac{(\bar{C}_1-C_1)^{1-\gamma}}{1-\gamma} - \lambda\delta p\frac{(\bar{C}_2-Y_2-(1+r_f)(Y_1-C_1)-(r_g-r_f)\alpha)^{1-\gamma}}{1-\gamma} \\
& -\lambda\delta(1-p)\frac{(\bar{C}_2-Y_2-(1+r_f)(Y_1-C_1)+(r_f-r_b)\alpha)^{1-\gamma}}{1-\gamma} \\
\text{such that}: & Y_1+\frac{Y_2-\bar{C}_2}{1+r_f}+\frac{r_g-r_f}{1+r_f}\alpha \leq C_1 \leq \min\left\{Y_1+\frac{Y_2}{1+r_f}-\frac{r_f-r_b}{1+r_f}\alpha, \bar{C}_1\right\} \\
& 0 \leq \alpha \leq -\frac{\Omega}{r_g-r_f}
\end{array}\right\} \quad \text{(P8)}$$

Appendix B: $\Omega \geq 0$

Before proceeding further, we introduce the following notation

$$M = k\left[1+\left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right] = k_2\left[1+K_\gamma^{\frac{1}{\gamma}}\right] = k+k_2$$

In addition notice that

$$
\begin{aligned}
k^{\gamma} &= K_{\gamma} k_2^{\gamma} \\
K_0^{\frac{1}{\gamma}} + K_{\gamma}^{\frac{1}{\gamma}} &= \frac{r_g - r_b}{r - r_b} K_0^{\frac{1}{\gamma}} = \frac{r_g - r_b}{r_g - r_f} K_{\gamma}^{\frac{1}{\gamma}} \\
\left(\frac{1}{K_0}\right)^{\frac{1}{\gamma}} + \left(\frac{1}{K_{\gamma}}\right)^{\frac{1}{\gamma}} &= \left(\frac{1}{K_0}\right)^{\frac{1}{\gamma}} \frac{r_g - r_b}{r_g - r_f}
\end{aligned} \tag{26}
$$

where k and k_2 are defined by (18) and (19).

Lemma 1 *Let $\bar{C}_1 \leq Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$, i.e., $\Omega \geq 0$ and $\lambda > \max\left\{\frac{1}{K_{\gamma}}, \lambda^{\Omega \geq 0}, \left(\frac{M}{1+r_f}\right)^{\gamma}\right\}$. Then problem (5) obtains a unique maximum at (C_1^*, α^*) where*

$$
\begin{aligned}
C_1^* &= \bar{C}_1 + \frac{\Omega}{1 + r_f + M} \\
&= \bar{C}_1 + \frac{1 + r_f}{1 + r_f + M} \left[\left(Y_1 + \frac{Y_2}{1 + r_f}\right) - \left(\bar{C}_1 + \frac{\bar{C}_2}{1 + r_f}\right) \right] \geq \bar{C}_1 && (27) \\
\alpha^* &= \frac{\left(1 - K_0^{\frac{1}{\gamma}}\right) M}{r_f - r_b + K_0^{\frac{1}{\gamma}} (r_g - r_f)} (C_1^* - \bar{C}_1) \geq 0 && (28)
\end{aligned}
$$

Proof. We proceed in two steps. At first we assume that $C_1 \leq Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$ and then $C_1 > Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$. Note that as we assume that $\bar{C}_1 \leq Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$, i.e., $\Omega \geq 0$, then in case $C_1 > Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$ only problems (P1)–(P4) are feasible.

<u>**Problem (P1).**</u> Note at first that there is no feasible solution for (P1) if $C_1 > Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$. If $C_1 = Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$ then the only feasible solution is $\left(C_1 = Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}, \alpha = 0\right)$. Let $C_1 < Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$. At first we solve the concave programming problem (P1) as an unconstrained problem, i.e., we solve two equations in two unknown variables C_1 and α, namely $\frac{d\mathbb{E}(U)}{dC_1} = 0$ and $\frac{d\mathbb{E}(U)}{d\alpha} = 0$ ($\nabla \mathbb{E}(U) = 0$), obtain the optimum solution (C_1^*, α^*) and finally verify that $C_{2b}^* \geq \bar{C}_2$ and $C_{2g}^* \geq \bar{C}_2$, $C_1^* \geq \bar{C}_1$ and $-\frac{\Omega}{r_g - r_f} \leq \alpha^* \leq \frac{\Omega}{r_f - r_b}$, i.e. that the solution is also feasible.

The first order conditions are

$$
\left.
\begin{aligned}
\frac{d\mathbb{E}(U)}{dC_1} = \ & (C_1 - \bar{C}_1)^{-\gamma} - \delta p \left[(1 + r_f)(Y_1 - C_1) + (r_g - r_f)\alpha + Y_2 - \bar{C}_2\right]^{-\gamma} (1 + r_f) \\
& - \delta(1 - p) \left[(1 + r_f)(Y_1 - C_1) - (r_f - r_b)\alpha + Y_2 - \bar{C}_2\right]^{-\gamma} (1 + r_f) = 0 \\
\frac{d\mathbb{E}(U)}{d\alpha} = \ & \delta p \left[(1 + r_f)(Y_1 - C_1) + (r_g - r_f)\alpha + Y_2 - \bar{C}_2\right]^{-\gamma} (r_g - r_f) \\
& - \delta(1 - p) \left[(1 + r_f)(Y_1 - C_1) - (r_f - r_b)\alpha + Y_2 - \bar{C}_2\right]^{-\gamma} (r_f - r_b) = 0
\end{aligned}
\right\} \tag{29}
$$

$\frac{d\mathbb{E}(U)}{d\alpha} = 0$ from (29) implies the following

$$\begin{aligned} & p\left[(1+r_f)(Y_1 - C_1) - (r_f - r_b)\alpha + Y_2 - \bar{C}_2\right]^\gamma (r_g - r_f) \\ & = (1-p)\left[(1+r_f)(Y_1 - C_1) + (r_g - r_f)\alpha + Y_2 - \bar{C}_2\right]^\gamma (r_f - r_b) \end{aligned}$$

which after using the definition of K_γ as given by (7) gives

$$K_0^{-\frac{1}{\gamma}}\left[(1+r_f)(Y_1 - C_1) - (r_f - r_b)\alpha + Y_2 - \bar{C}_2\right] = (1+r_f)(Y_1 - C_1) + (r_g - r_f)\alpha + Y_2 - \bar{C}_2$$

This implies that

$$\alpha = \frac{1 - K_0^{\frac{1}{\gamma}}}{r_f - r_b + K_0^{\frac{1}{\gamma}}(r_g - r_f)}((1+r_f)(Y_1 - C_1) + Y_2 - \bar{C}_2) \tag{30}$$

If we plug the last expression for α into the C_1 part of the FOC in (29) we obtain

$$\begin{aligned} \frac{(C_1 - \bar{C}_1)^{-\gamma}}{\delta(1+r_f)} &= p\left[\Omega - (1+r_f)(C_1 - \bar{C}_1) + \frac{(1 - K_0^{\frac{1}{\gamma}})(r_g - r_f)}{r_f - r_b + K_0^{\frac{1}{\gamma}}(r_g - r_f)}\left(\Omega - (1+r_f)(C_1 - \bar{C}_1)\right)\right]^{-\gamma} \\ &+ (1-p)\left[\Omega - (1+r_f)(C_1 - \bar{C}_1) - \frac{(1 - K_0^{\frac{1}{\gamma}})(r_f - r_b)}{r_f - r_b + K_0^{\frac{1}{\gamma}}(r_g - r_f)}\left(\Omega - (1+r_f)(C_1 - \bar{C}_1)\right)\right]^{-\gamma} \\ &= \left[\frac{r_f - r_b + K_0^{\frac{1}{\gamma}}(r_g - r_f)}{\left(\Omega - (1+r_f)(C_1 - \bar{C}_1)\right)(r_g - r_b)}\right]^\gamma \left[p\left(1 - K_0^{-1}\right) + K_0^{-1}\right] \end{aligned} \tag{31}$$

with assuming that $\Omega - (1+r_f)(C_1 - \bar{C}_1) > 0$ which is equivalent to $C_1 < Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$.

After some simplifications we obtain

$$\begin{aligned} (1+r_f)(Y_1 - C_1) + Y_2 - \bar{C}_2 &= (C_1 - \bar{C}_1)\frac{r_f - r_b + K_0^{\frac{1}{\gamma}}(r_g - r_f)}{r_g - r_b}\left[\delta(1+r_f)p\frac{r_g - r_b}{r_f - r_b}\right]^{\frac{1}{\gamma}} \\ &= (C_1 - \bar{C}_1)M \end{aligned} \tag{32}$$

which gives $C_1 = C_1^* \geq \bar{C}_1$. Note that (27) and the assumption $\Omega \geq 0$ imply that $C_1^* \geq \bar{C}_1$. In addition, after plugging C_1^* into (30) we obtain α^* as given in (28). Note that $C_1^* < Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$ and $\alpha^* \geq 0$ as $K_0 < 1$ (which follows from $\mathbb{E}(r) > r_f$).

Using (29), it is easy to verify that $\frac{d^2\mathbb{E}(U)}{dC_1^2} < 0$, $\frac{d^2\mathbb{E}(U)}{d\alpha^2} < 0$, and $\nabla^2\mathbb{E}(U(C_1, C_2)) = \frac{d^2\mathbb{E}(U)}{dC_1^2}\frac{d^2\mathbb{E}(U)}{d\alpha^2} - \left(\frac{d^2\mathbb{E}(U)}{dC_1 d\alpha}\right)^2 > 0$ and thus problem (P1) is a concave programming problem and (C_1^*, α^*) is its unique global maximum.

Finally, C_{2g}^* and C_{2b}^* can be written as

$$
\begin{aligned}
C_{2g}^* &= \bar{C}_2 + \frac{M\Omega}{(1+r_f+M)\left(1+K_\gamma^{\frac{1}{\gamma}}\right)}\frac{r_g-r_b}{r_f-r_b} \\
C_{2b}^* &= \bar{C}_2 + \frac{M\Omega}{(1+r_f+M)\left(1+K_\gamma^{\frac{1}{\gamma}}\right)}\frac{r_g-r_b}{r_f-r_b}K_0^{\frac{1}{\gamma}}
\end{aligned}
$$

and thus they both are such that $C_{2g}^* \geq \bar{C}_2$ and $C_{2b}^* \geq \bar{C}_2$ as $K_0 \geq 0$ and $r_b < r_f < r_g$.

It can be shown that

$$
(1-\gamma)\mathbb{E}(U(C_1^*,\alpha^*)) = \left(\frac{\Omega}{1+r_f}\right)^{1-\gamma}\left(1+\frac{M}{1+r_f}\right)^{\gamma} = \frac{\Omega^{1-\gamma}}{1+r_f}(1+r_f+M)^{\gamma} \tag{33}
$$

As we have already mentioned the only feasible solution for $C_1 = Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}$ is $\left(C_1 = Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}, \alpha = 0\right)$ with

$$
(1-\gamma)\mathbb{E}\left(U\left(Y_1+\frac{Y_2-\bar{C}_2}{1+r_f},0\right)\right) = \left(\frac{\Omega}{1+r_f}\right)^{1-\gamma} \tag{34}
$$

which is below the value of the expected utility function at (C_1^*, α_1^*), see (33), as $M > 0$. Thus, the maximum of (P1) is reached at (C_1^*, α^*).

If $\Omega = 0$ then definition of problem (P1) implies that the only feasible solution is $(C_1, \alpha) = (\bar{C}_1, 0)$ which is then also the maximum. I.e., $(C_1^*, \alpha^*) = (\bar{C}_1, 0)$ and $C_{2b}^* = C_{2g}^* = \bar{C}_2 \geq 0$. Note in addition that $\mathbb{E}(U(C_1^*, \alpha^*)) = 0$.

Next we show that for $\Omega \geq 0$ and $C_1 \leq Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}$ all possible candidates for maximum in problems (P2), (P3) and (P4) are also feasible solutions of (P1) and as the expected utilities of problems (P2), (P3) and (P4) in these points coincide with the expected utility of (P1) then utility of (P1) at (C_1^*, α^*) exceeds utility functions of problems (P2), (P3) and (P4) at their feasible solutions.

Problem (P2). Let $C_1 \leq Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}$. From the proof of Proposition 3 (problem (P2)) it can be seen that its stationary point exists only for $C_1 > Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}$, so there are no stationary points when $C_1 > Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}$. Note that the utility of (P2) is a decreasing function in α for any fixed C_1

$$
\begin{aligned}
\frac{d\mathbb{E}(U)}{d\alpha} &= \delta p\left[(1+r_f)(Y_1-C_1)+Y_2-\bar{C}_2+(r_g-r_f)\alpha\right]^{-\gamma}(r_g-r_f) \\
&\quad -\lambda\delta(1-p)\left[(1+r_f)(C_1-Y_1)+\bar{C}_2-Y_2+(r_f-r_b)\alpha\right]^{-\gamma}(r_f-r_b) < 0
\end{aligned}
$$

if

$$\lambda > \frac{p(r_g - r_f)}{(1-p)(r_f - r_b)} \left[\frac{(1+r_f)(C_1 - Y_1) + \bar{C}_2 - Y_2 + (r_f - r_b)\alpha}{(1+r_f)(Y_1 - C_1) + Y_2 - \bar{C}_2 + (r_g - r_f)\alpha} \right]^\gamma$$

The latter is achieved if

$$\frac{1}{K_\gamma} \geq \frac{p(r_g - r_f)}{(1-p)(r_f - r_b)} \left[\frac{(1+r_f)(C_1 - Y_1) + \bar{C}_2 - Y_2 + (r_f - r_b)\alpha}{(1+r_f)(Y_1 - C_1) + Y_2 - \bar{C}_2 + (r_g - r_f)\alpha} \right]^\gamma \tag{35}$$

as it is assumed that $\lambda > \frac{1}{K_\gamma}$ where K_γ is given by (7). It can be shown that (35) is satisfied if $C_1 \leq Y_1 + \frac{Y_2 - \bar{C}_2}{1+r}$ which is our assumption. In addition, the set of feasible solutions for (P2) can be written as

$$\left.\begin{array}{rrll} & \bar{C}_1 & \leq C_1 & \leq Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f} \\ \left(C_{2b} \leq \bar{C}_2 \Rightarrow\right) & Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f} - \frac{r_f-r_b}{1+r_f}\alpha & \leq C_1 & \leq Y_1 + \frac{Y_2}{1+r_f} - \frac{r_f-r_b}{1+r_f}\alpha \quad (\Leftarrow C_{2b} \geq 0) \\ & 0 & \leq \alpha & \leq \frac{(1+r_f)(Y_1-\bar{C}_1)+Y_2}{r_f-r_b} \end{array}\right\} \tag{36}$$

Let $\tilde{C}_1$ be fixed and such that $\bar{C}_1 \leq \tilde{C}_1 \leq Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}$. Based on the first inequality in the second row of (36) and the fact that the utility of (P2) is decreasing in α for given $C_1 = \tilde{C}_1$, the smallest possible $\alpha = \tilde{\alpha}$ such that $(\tilde{C}_1, \tilde{\alpha})$ remains feasible is given by

$$Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f} - \frac{r_f - r_b}{1+r_f}\tilde{\alpha} = \tilde{C}_1$$

and thus $\tilde{C}_{2b} = \bar{C}_2$ and

$$\tilde{\alpha} = \frac{(1+r_f)(Y_1 - \tilde{C}_1) + Y_2 - \bar{C}_2}{r_f - r_b} \in \left[0, \frac{\Omega}{r_f - r_b}\right]$$

Note that $(\tilde{C}_1, \tilde{\alpha})$ completes also the second inequality in (36), namely: $\tilde{C}_1 \leq Y_1 + \frac{Y_2}{1+r_f} - Y_1 + \tilde{C}_1 - \frac{Y_2-\bar{C}_2}{1+r_f} = \tilde{C}_1 + \frac{\bar{C}_2}{1+r_f}$ i.e., $\tilde{C}_{2b} = \bar{C}_2 \geq 0$, as $\bar{C}_2 \geq 0$. Thus, for any given $\tilde{C}_1$ that satisfies $\bar{C}_1 \leq \tilde{C}_1 \leq Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}$ is the point $\left(\tilde{C}_1, \frac{(1+r_f)(Y_1-\tilde{C}_1)+Y_2-\bar{C}_2}{r_f-r_b}\right)$ where the utility of (P2) achieves its maxima. As point $(\tilde{C}_1, \tilde{\alpha})$ is feasible also for (P1) and as utilities of (P1) and (P2) coincide at this point then the utility function of (P1) at (C_1^*, α^*) is bigger or equal to the utility function of (P2) at any point $(\tilde{C}_1, \tilde{\alpha})$.

Note that for $\tilde{C}_1 = Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}$ is $\tilde{\alpha} = 0$ and point $\left(Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}, 0\right)$ is feasible also for (P1) and for $\Omega = 0$ is the maximum reached at $(\bar{C}_1, 0)$.

Let $C_1 \geq Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f} \geq \bar{C}_1$. From the proof of Proposition 3 (problem (P2)) it can be seen that its stationary point (which is the same as in this case) happens to be (C_1^{P2}, α^{P2}), see (24). For $\Omega > 0$ is this stationary point infeasible for (P2) as $C_1^{P2} < \bar{C}_1$ and thus the maximum will occur at the border. The feasible solutions at the border for (P2) that come into consideration are given by: (i) $C_{2g} = \bar{C}_2$, (ii) $C_{2b} = \bar{C}_2$, (iii) $C_{2b} = 0$ and (iv) $C_1 = Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$. On the other hand, for $\Omega = 0$ the stationary point is $\bar{C}_1, 0$ and the utility function at this point can also be compared to the value of the utility function at the feasible solutions at the border.

Case (i). $C_{2g} = \bar{C}_2$ when $C_1 = Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f} + \frac{r_g - r_f}{1+r_f}\alpha$ and $0 \le \alpha \le \frac{\bar{C}_2}{r_g - r_b}$. It can be seen that

$$\begin{aligned}
&(1-\gamma)\mathbb{E}\left(U\left(Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f} + \frac{r_g - r_f}{1+r_f}\,\alpha, \alpha\right)\right) \\
&= \left(\frac{\Omega + (r_g - r_f)\alpha}{1+r_f}\right)^{1-\gamma} - \lambda\delta(1-p)(r_g - r_b)^{1-\gamma}\alpha^{1-\gamma}
\end{aligned} \tag{37}$$

and thus the potential maximum occurs either at $\alpha = 0$ or $\alpha = \frac{\bar{C}_2}{r_g - r_b}$ or at the stationary point of function given by (37) which can be easily derived and has the value $\alpha = \bar{\alpha} \equiv \frac{\lambda^{\frac{1}{\gamma}} k}{1+r_f - \lambda^{\frac{1}{\gamma}} k} \frac{\Omega}{r_g - r_f}$ when $\lambda < \left(\frac{1+r_f}{k}\right)^{\gamma}$. For $\lambda \ge \left(\frac{1+r_f}{k}\right)^{\gamma}$ is function (37) decreasing in α and thus the maximum occurs at $\alpha = 0$ where point $\left(Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}, 0\right)$ is feasible also for (P1). Note in addition that

$$\begin{aligned}
(1-\gamma)\mathbb{E}\left(U\left(C_1 = Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f} + \frac{r_g - r_f}{1+r_f}\,\bar{\alpha}, \bar{\alpha}\right)\right) &= \frac{\Omega^{1-\gamma}}{1+r_f}\left(1 + r_f - \lambda^{\frac{1}{\gamma}} k\right)^{\gamma} \\
\le \left(\frac{\Omega}{1+r_f}\right)^{1-\gamma} &= (1-\gamma)\mathbb{E}\left(U\left(Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}, 0\right)\right)
\end{aligned}$$

when $\lambda < \left(\frac{1+r_f}{k}\right)^{\gamma}$ and thus point $\left(C_1 = Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f} + \frac{r_g - r_f}{1+r_f}\,\bar{\alpha}, \bar{\alpha}\right)$ can not be a maximum. Regarding the end-point $\alpha = \frac{\bar{C}_2}{r_g - r_b}$, one can see that

$$\begin{aligned}
&(1-\gamma)\mathbb{E}\left(U\left(C_1 = Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f} + \frac{r_g - r_f}{r_g - r_b}\frac{\bar{C}_2}{1+r_f}, \alpha = \frac{\bar{C}_2}{r_g - r_b}\right)\right) \\
&= \left(\frac{\Omega + \frac{r_g - r_f}{r_g - r_b}\bar{C}_2}{1+r_f}\right)^{1-\gamma} - \lambda\delta(1-p)\,\bar{C}_2^{1-\gamma} \\
&\le \frac{\Omega^{1-\gamma}}{1+r_f}\left(1 + r_f + M\right)^{\gamma} = (1-\gamma)\mathbb{E}\left(U\left(C_1^*, \alpha^*\right)\right)
\end{aligned}$$

if

$$\lambda \ge \tilde{\lambda} \equiv \frac{\Omega^{1-\gamma}}{\delta(1-p)(1+r_f)\bar{C}_2^{1-\gamma}}\left[(1+r_f)^{\gamma}\left(1 + \frac{r_g - r_f}{r_g - r_b}\frac{\bar{C}_2}{\Omega}\right)^{1-\gamma} - (1 + r_f + M)^{\gamma}\right] \tag{38}$$

As we assume that $\lambda > \lambda^{\Omega>0}$, see (9), then (38) holds as $\lambda^{\Omega>0} \ge \tilde{\lambda}$.

There is no feasible solution for case (ii), i.e., when $C_{2b} = \bar{C}_2$.

Case (iii). $C_{2b}=0$ when $C_1 = Y_1 + \frac{Y_2}{1+r_f} - \frac{r_f-r_b}{1+r_f}\alpha$ and $\frac{\bar{C}_2}{r_g-r_b} \le \alpha \le \frac{\bar{C}_2}{r_f-r_b}$. It can be seen that

$$\begin{aligned}(1-\gamma)\mathbb{E}\left(U\left(Y_1+\frac{Y_2}{1+r_f}-\frac{r_f-r_b}{1+r_f}\,\alpha,\alpha\right)\right) &= \left(Y_1-\bar{C}_1+\frac{Y_2}{1+r_f}-\frac{r_f-r_b}{1+r_f}\alpha\right)^{1-\gamma} \\ &+ \delta p\left((r_g-r_b)\alpha-\bar{C}_2\right)^{1-\gamma}-\lambda\delta(1-p)\bar{C}_2^{1-\gamma}\end{aligned} \tag{39}$$

As function (39) is concave in α the maximum is reached at the stationary point $\bar{\alpha}$, i.e. $\left.\frac{d\mathbb{E}\left(U\left(Y_1+\frac{Y_2}{1+r_f}-\frac{r_f-r_b}{1+r_f}\alpha,\alpha\right)\right)}{d\alpha}\right|_{\alpha=\bar{\alpha}}=0$ which is given as follows

$$\begin{aligned}\bar{\alpha} &= \frac{k_2((1+r_f)(Y_1-\bar{C}_1)+Y_2)+\frac{r_f-r_b}{r_g-r_b}(1+r_f)\bar{C}_2}{(1+r_f+k_2)(r_f-r_b)} \\ &= \frac{1+r_f}{1+r_f+k_2}\frac{\bar{C}_2}{r_g-r_b}+\frac{k_2}{1+r_f+k_2}\frac{(1+r_f)(Y_1-\bar{C}_1)+Y_2}{r_f-r_b}\end{aligned}$$

The expected utility is at this point is given by

$$\begin{aligned}&(1-\gamma)\mathbb{E}\left(U\left(Y_1+\frac{Y_2}{1+r_f}-\frac{r_f-r_b}{1+r_f}\,\bar{\alpha},\bar{\alpha}\right)\right) \\ &= \frac{(1+r_f+k_2)^{\gamma}}{1+r_f}\left((1+r_f)(Y_1-\bar{C}_1)+Y_2-\frac{r_f-r_b}{r_g-r_b}\bar{C}_2\right)^{1-\gamma}-\lambda\delta(1-p)\bar{C}_2^{1-\gamma}\end{aligned} \tag{40}$$

If $\lambda > \lambda^{\Omega>0}$, see (9), then the utility given by (40) will be below the utility of (P1) at its maximum, which is given by $\frac{\Omega^{1-\gamma}}{1+r_f}(1+r_f+M)^{\gamma}$, see (33).

Case (iv). $C_1 = Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}$ for $0 \le \alpha \le \frac{\bar{C}_2}{r_f-r_b}$ with the utility function being

$$\begin{aligned}(1-\gamma)\mathbb{E}\left(U\left(Y_1+\frac{Y_2-\bar{C}_2}{1+r_f}\,\alpha,\alpha\right)\right) &= \left(\frac{\Omega}{1+r_f}\right)^{1-\gamma}+\delta p(r_g-r_f)^{1-\gamma}\alpha^{1-\gamma} \\ &-\lambda\delta(1-p)(r_f-r_b)^{1-\gamma}\alpha^{1-\gamma}\end{aligned}$$

which is decreasing in α for $\lambda > \frac{1}{K_\gamma}$ and thus the maximum is reached at $\left(Y_1+\frac{Y_2-\bar{C}_2}{1+r_f},0\right)$ which is feasible also for (P1).

It follows from the proof above that for $\Omega=0$ the maximum, for $\lambda > \left(\frac{1+r_f}{k}+\left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right)^{\gamma}$, is achieved at $(C_1=\bar{C}_1,\alpha=0)$ where the value of the expected utility is zero.

Problem (P3). The only feasible solution is $\left(Y_1+\frac{Y_2-\bar{C}_2}{1+r_f},0\right)$ which is feasible also for (P1).

The set of feasible solutions for **problem (P4)**

$$Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f} + \frac{1}{1+r_f}(r_g - r_f)\alpha \leq C_1$$

implies $C_1 \geq Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$. The other set of feasible solutions

$$C_1 \leq Y_1 + \frac{Y_2}{1+r_f} - \frac{r_f - r_b}{1+r_f}\alpha$$

gives $C_1 \leq Y_1 + \frac{Y_2}{1+r_f}$, which implies that $Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f} \leq C_1 \leq Y_1 + \frac{Y_2}{1+r_f}$.

The only feasible solution for $C_1 = Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}$ is $(C_1 = Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}, \alpha = 0)$ at which the utility function is below the utility function at (C_1^*, α^*).

Let $C_1 > Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}$. As $\frac{d^2\mathbb{E}(U)}{d\alpha^2} > 0$ then there is no local interior maximum which implies that the maximum will occur at the border. The cases when this could happen are: (i) $C_{2g} = \bar{C}_2$, (ii) $C_{2b} = \bar{C}_2$, (iii) $C_{2g} = 0$ and (iv) $C_{2b} = 0$. Note that case (i) coincides with case (i) when proving (P2) and the only feasible solution in case (ii) is $(C_1 = Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}, \alpha = 0)$ which is feasible for (P1).

Case (iii). The only feasible solution for $C_{2g} = 0$ is $(C_1 = Y_1 + \frac{Y_2}{1+r_f}, \alpha = 0)$ with the utility function being

$$(1-\gamma)\mathbb{E}\left(U\left(Y_1 + \frac{Y_2}{1+r_f}, 0\right)\right) = \left(Y_1 - \bar{C}_1 + \frac{Y_2}{1+r_f}\right)^{1-\gamma} - \lambda\delta\bar{C}_2^{1-\gamma}$$

which is dealt in case (iv) below.

Case (iv). $C_{2b} = 0$ when $C_1 = Y_1 + \frac{Y_2}{1+r_f} - \frac{r_f - r_b}{1+r_f}\alpha$ and $0 \leq \alpha \leq \frac{\bar{C}_2}{r_g - r_b}$ and thus

$$(1-\gamma)\mathbb{E}\left(U\left(C_1 = Y_1 + \frac{Y_2}{1+r_f} - \frac{r_f - r_b}{1+r_f}\alpha, \alpha\right)\right) = \left(Y_1 - \bar{C}_1 + \frac{Y_2}{1+r_f} - \frac{r_f - r_b}{1+r_f}\alpha\right)^{1-\gamma} - \lambda\delta p\left(\bar{C}_2 - (r_g - r_b)\alpha\right)^{1-\gamma} - \lambda\delta(1-p)\,\bar{C}_2^{1-\gamma} \quad (41)$$

The potential candidates for maximum are $\alpha = 0$, $\alpha = \frac{\bar{C}_2}{r_g - r_b}$ and $\alpha = \bar{\alpha}$ where $\bar{\alpha}$ is a unique stationary point such that $\left.\frac{d\mathbb{E}(U)}{d\alpha}\right|_{\alpha=\bar{\alpha}} = 0$ where

$$\bar{\alpha} = \frac{(\lambda\delta p(r_g - r_b))^{\frac{1}{\gamma}}\left(Y_1 - \bar{C}_1 + \frac{Y_2}{1+r_f}\right) - \left(\frac{r_f - r_b}{1+r_f}\right)^{\frac{1}{\gamma}}\bar{C}_2}{(\lambda\delta p(r_g - r_b))^{\frac{1}{\gamma}}\frac{r_f - r_b}{1+r_f} - \left(\frac{r_f - r_b}{1+r_f}\right)^{\frac{1}{\gamma}}(r_g - r_b)}$$

Note that $\Omega \geq 0$ and $r_f > 0$ imply that $\bar{\alpha} > \frac{\bar{C}_2}{r_g - r_b}$ and thus infeasible. For $\alpha = \frac{\bar{C}_2}{r_g - r_b}$ is $C_{2g} = \bar{C}_2$ and thus the point $\left(C_1 = Y_1 + \frac{Y_2}{1+r_f} - \frac{r_f - r_b}{1+r_f}\frac{\bar{C}_2}{r_g - r_b}, \alpha = \frac{\bar{C}_2}{r_g - r_b}\right)$ is feasible for (P2).

Finally, we show that the utility function at $\alpha = 0$ is below the utility function at (C_1^*, α^*). Namely

$$\left(Y_1 - \bar{C}_1 + \frac{Y_2}{1+r_f}\right)^{1-\gamma} - \lambda\delta\bar{C}_2^{1-\gamma} = \left(\frac{\Omega + \bar{C}_2}{1+r_f}\right)^{1-\gamma} - \lambda\delta\bar{C}_2^{1-\gamma} \leq \left(\frac{\Omega}{1+r_f}\right)^{1-\gamma}\left(1 + \frac{M}{1+r_f}\right)^{\gamma}$$

which holds if

$$\lambda > \frac{\Omega^{1-\gamma}}{\delta(1+r_f)\bar{C}_2^{1-\gamma}}\left[\left(1 + \frac{\bar{C}_2}{\Omega}\right)^{1-\gamma}(1+r_f)^{\gamma} - (1+r_f+M)^{\gamma}\right]$$

The last inequality holds as $\lambda > \lambda^{\Omega \geq 0}$ and

$$\lambda^{\Omega \geq 0} > \frac{\Omega^{1-\gamma}}{\delta(1+r_f)\bar{C}_2^{1-\gamma}}\left[\left(1 + \frac{\bar{C}_2}{\Omega}\right)^{1-\gamma}(1+r_f)^{\gamma} - (1+r_f+M)^{\gamma}\right]$$

For $\Omega = 0$ the conclusions obtained in (P2) apply, i.e., if $\lambda > \left(\frac{1+r_f}{k} + \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right)^{\gamma}$ then maximum is reached at $(C_1 = \bar{C}_1, \alpha = 0)$.

Regarding **problem (P5)** we show that for $C_1 \leq Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$ and $\Omega \geq 0$ is its maximum reached at the point that is feasible also for (P1), namely at $(\bar{C}_1, \bar{\alpha})$ (with $\bar{\alpha}$ being defined later), and as utility functions of (P1) and (P5) coincide at this point then the utility function of (P1) at (C_1^*, α^*) exceeds the one at $(\bar{C}_1, \bar{\alpha})$.

In more detail, as $\frac{d\mathbb{E}(U)}{d\alpha}$ is the same for both (P1) and (P5) then the second equation in (29) implies that for any fixed C_1 is the expected utility of (P5) concave and thus its maximum is achieved at (30). I.e., if $\tilde{C}_1$ is such that $0 \leq \tilde{C}_1 \leq \bar{C}_1 \leq Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$ then for this fixed $\tilde{C}_1$ is maximum of (P5) reached at $(\tilde{C}_1, \tilde{\alpha})$ where $\tilde{\alpha} = \frac{1 - K_0^{\frac{1}{\gamma}}}{r_f - r_b + K_0^{\frac{1}{\gamma}}(r_g - r_f)}((1+r_f)(Y_1 - \tilde{C}_1) + Y_2 - \bar{C}_2)$. Note that $(\tilde{C}_1, \tilde{\alpha})$ is feasible for (P5). Thus, the only candidates for the maximum for (P5) are $(\tilde{C}_1, \tilde{\alpha})$ with $0 \leq \tilde{C}_1 \leq \bar{C}_1$. By plugging this point into the expected utility of (P5) we obtain

$$\begin{aligned}
&(1-\gamma)\mathbb{E}(U) = -\lambda(\bar{C}_1 - \tilde{C}_1)^{1-\gamma} \\
+ \quad & \delta p\left[(1+r_f)(Y_1 - \tilde{C}_1) + Y_2 - \bar{C}_2 + \frac{(r_g - r_f)(1 - K_0^{1/\gamma})}{r_f - r_b + K_0^{1/\gamma}(r_g - r_f)}\Big((1+r_f)(Y_1 - \tilde{C}_1) + Y_2 - \bar{C}_2\Big)\right]^{1-\gamma} \\
+ \quad & \delta(1-p)\left[(1+r_f)(Y_1 - \tilde{C}_1) + Y_2 - \bar{C}_2 - \frac{(r_f - r_b)(1 - K_0^{1/\gamma})}{r_f - r_b + K_0^{1/\gamma}(r_g - r_f)}\Big((1+r_f)(Y_1 - \tilde{C}_1) + Y_2 - \bar{C}_2\Big)\right]^{1-\gamma}
\end{aligned}$$

which after some derivations gives

$$
\begin{aligned}
(1-\gamma)\mathbb{E}(U) &= -\lambda(\bar{C}_1-\tilde{C}_1)^{1-\gamma}+\delta\left((1+r_f)(Y_1-\tilde{C}_1)+Y_2-\bar{C}_2\right)^{1-\gamma}\left(\frac{r_g-r_b}{r_f-r_b+K_0^{1/\gamma}(r_g-r_f)}\right)^{1-\gamma}\\
&\quad\times\left(p+(1-p)K_0^{\frac{1-\gamma}{\gamma}}\right)\\
&= -\lambda(\bar{C}_1-\tilde{C}_1)^{1-\gamma}+\left(\frac{k_2\left(1+K_\gamma^{\frac{1}{\gamma}}\right)}{1+r_f}\right)^{\gamma}\left(Y_1-\tilde{C}_1+\frac{Y_2-\bar{C}_2}{1+r_f}\right)^{1-\gamma}
\end{aligned}
$$

If the expected utility of (P5) is increasing function in $\tilde{C}_1$, i.e., $\frac{d\mathbb{E}(U)}{d\tilde{C}_1}>0$ then the maximum will be reached at $(\bar{C}_1,\bar{\alpha})$, where $\bar{\alpha}=\frac{1-K_0^{\frac{1}{\gamma}}}{r_f-r_b+K_0^{\frac{1}{\gamma}}(r_g-r_f)}\Omega$. In more detail, the inequality below

$$
\frac{d\mathbb{E}(U)}{d\tilde{C}_1} = \lambda(\bar{C}_1-\tilde{C}_1)^{-\gamma}-\left(\frac{k_2\left(1+K_\gamma^{\frac{1}{\gamma}}\right)}{1+r_f}\right)^{\gamma}\left(Y_1-\tilde{C}_1+\frac{Y_2-\bar{C}_2}{1+r_f}\right)^{-\gamma}>0
$$

holds if

$$
\lambda>\left(\frac{k_2\left(1+K_\gamma^{\frac{1}{\gamma}}\right)}{1+r_f}\right)^{\gamma}\left(\frac{\bar{C}_1-\tilde{C}_1}{\frac{\Omega}{1+r_f}+\bar{C}_1-\tilde{C}_1}\right)^{\gamma} \tag{42}
$$

If $\tilde{C}_1<\bar{C}_1$ then the right hand side of the inequality (42) is below $\left(\frac{k_2\left(1+K_\gamma^{\frac{1}{\gamma}}\right)}{1+r_f}\right)^{\gamma}=\left(\frac{M}{1+r_f}\right)^{\gamma}$, where M is defined by (8), and as this is exceeded by λ, i.e., $\lambda>\left(\frac{M}{1+r_f}\right)^{\gamma}$, see assumptions of Proposition 1, then $\mathbb{E}(U)$ is increasing in $\tilde{C}_1$ and the maximum is reached at $(\bar{C}_1,\bar{\alpha})$, what we wanted to show. Finally, there is no feasible solution for the case when $\bar{C}_1\geq C_1>Y_1+\frac{Y_2-\bar{C}_2}{1+r_f}$ as it implies that $\Omega<0$ which is in contradiction with our assumption.

It can be derived in the same way that for $\Omega=0$ the utility of (P5)

$$
(1-\gamma)\mathbb{E}(U)=(\bar{C}_1-\tilde{C}_1)^{1-\gamma}\left[-\lambda+\left(\frac{k_2\left(1+K_\gamma^{\frac{1}{\gamma}}\right)}{1+r_f}\right)^{\gamma}\right]
$$

is an increasing function in $\tilde{C}_1$ for $\lambda>\left(\frac{M}{1+r_f}\right)^{\gamma}$ and thus the maximum will be reached for $(C_1=\bar{C}_1,\alpha=0)$.

Finally, in the identical way as in problem (P2) it can be shown for $\Omega \geq 0$ that all possible candidates for maximum in **problem (P6)** are also feasible solutions of (P5) and as the expected utility function of problem (P6) in these points coincide with the expected utility function of (P5) then utility of (P5) in its maximum exceeds the utility function of problem (P6) at its feasible solutions. Note that there is no feasible solution of **problem (P7)**when $\Omega > 0$ and the only feasible solution for $\Omega = 0$ is $(C_1 = \bar{C}_1, \alpha = 0)$.

Problem (P8) has no feasible solution. ■

Appendix C: $\Omega < 0$

Proof of Proposition 3.

As in the proof of Lemma 1 we proceed in two steps. At first we assume that $C_1 \leq Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$ and then $C_1 > Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$. Note that for $C_1 \leq Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f} < \bar{C}_1$ (as $\Omega < 0$) it follows that only cases $C_1 < \bar{C}_1$ could be considered and thus only problems (P5)–(P8) need to be solved.

Problem (P1). For $\Omega < 0$ there is no feasible solution for (P1).

Problem (P2). There is no feasible solution for $C_1 \leq Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$. Let $C_1 > Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$ and in addition we assume $\lambda > \lambda_1^{\Omega<0}$ and $\bar{C}_2 < \frac{r_g - r_b}{r_f - r_b}\left((1+r_f)\left(Y_1 - \bar{C}_1\right) + Y_2\right) = \bar{C}_2^{P2}$. We proceed in the following way: At first we solve the problem (P2) as an unconstrained problem i.e., we solve $\nabla \mathbb{E}(U) = 0$, so that the FOC are satisfied, obtain the unique solution (C_1^{P2}, α^{P2}), verify that the objective function of (P2) is concave at (C_1^{P2}, α^{P2}) and that the solution is also feasible. As the utility function is differentiable at the domain under consideration, (C_1^{P2}, α^{P2}) is the only local extrema (namely local maximum) and if the objective function at the border of (P2) does not exceed its value at (C_1^{P2}, α^{P2}), then this point is also a global maximum of (P2) when $\lambda > \lambda_1^{\Omega<0}$.

The first order conditions are

$$\left.\begin{array}{lll}
\frac{d\mathbb{E}(U)}{dC_1} = & (C_1 - \bar{C}_1)^{-\gamma} & -\delta p\left[(1+r_f)(Y_1 - C_1) + Y_2 - \bar{C}_2 + (r_g - r_f)\alpha\right]^{-\gamma}(1+r_f) \\
 & & -\lambda\delta(1-p)\left[\bar{C}_2 - Y_2 - (1+r_f)(Y_1 - C_1) + (r_f - r_b)\alpha\right]^{-\gamma}(1+r_f) = 0 \\
 & & \\
\frac{d\mathbb{E}(U)}{d\alpha} = & & \delta p\left[(1+r_f)(Y_1 - C_1) + Y_2 - \bar{C}_2 + (r_g - r_f)\alpha\right]^{-\gamma}(r_g - r_f) \\
 & & -\lambda\delta(1-p)\left[\bar{C}_2 - Y_2 - (1+r_f)(Y_1 - C_1) + (r_f - r_b)\alpha\right]^{-\gamma}(r_f - r_b) = 0
\end{array}\right\} \quad (43)$$

$\frac{d\mathbb{E}(U)}{d\alpha} = 0$ from (43) implies the following

$$\begin{aligned}
& p\left[\bar{C}_2 - Y_2 - (1+r_f)(Y_1 - C_1) + (r_f - r_b)\alpha\right]^{\gamma}(r_g - r_f) \\
& = \lambda(1-p)\left[(1+r_f)(Y_1 - C_1) + Y_2 - \bar{C}_2 + (r_g - r_f)\alpha\right]^{\gamma}(r_f - r_b)
\end{aligned}$$

which gives

$$\left(\frac{1}{K_0}\right)^{\frac{1}{\gamma}}\left[\bar{C}_2 - Y_2 - (1+r_f)(Y_1 - C_1) + (r_f - r_b)\alpha\right] = \lambda^{\frac{1}{\gamma}}\left[(1+r_f)(Y_1 - C_1) + Y_2 - \bar{C}_2 + (r_g - r_f)\alpha\right]$$

This implies that

$$\begin{aligned}\alpha &= \frac{\lambda^{\frac{1}{\gamma}} + \left(\frac{1}{K_0}\right)^{\frac{1}{\gamma}}}{\lambda^{\frac{1}{\gamma}}(r_g - r_f) - \left(\frac{1}{K_0}\right)^{\frac{1}{\gamma}}(r_f - r_b)}(\bar{C}_2 - Y_2 - (1+r_f)(Y_1 - C_1)) \\ &= \frac{\lambda^{\frac{1}{\gamma}} + \left(\frac{1}{K_0}\right)^{\frac{1}{\gamma}}}{\left(\lambda^{\frac{1}{\gamma}} - \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right)(r_g - r_f)}(\bar{C}_2 - Y_2 - (1+r_f)(Y_1 - C_1)) \qquad (44)\end{aligned}$$

If we plug the last expression for α into the C_1 part of the FOC in (43) we obtain

$$\begin{aligned}\frac{(C_1 - \bar{C}_1)^{-\gamma}}{\delta(1+r_f)} &= \left[\bar{C}_2 - Y_2 - (1+r_f)(Y_1 - C_1)\right]^{-\gamma} \\ &\times \left[p\left(\frac{\left(\frac{1}{K_0}\right)^{\frac{1}{\gamma}} + \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}}{\lambda^{\frac{1}{\gamma}} - \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}}\right)^{-\gamma} + \lambda(1-p)\left(1 + \frac{\lambda^{\frac{1}{\gamma}} + \left(\frac{1}{K_0}\right)^{\frac{1}{\gamma}}}{\lambda^{\frac{1}{\gamma}} - \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}}\frac{r_f - r_b}{r_g - r_f}\right)^{-\gamma}\right]\end{aligned}$$

After some simplifications we obtain

$$\begin{aligned}\bar{C}_2 - Y_2 - (1+r_f)(Y_1 - C_1) &= (C_1 - \bar{C}_1)\frac{r_g - r_f}{r_g - r_b}\left[\delta(1+r_f)(1-p)\frac{r_g - r_b}{r_g - r_f}\right]^{\frac{1}{\gamma}}\left[\lambda^{\frac{1}{\gamma}} - \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right] \\ &= (C_1 - \bar{C}_1)M_1(\lambda) \qquad (45)\end{aligned}$$

which gives (24). In addition, after plugging C_1^{P2} into (44) we obtain α^{P2} as given in (25). Note that

$$\bar{C}_2 - Y_2 - (1+r_f)(Y_1 - C_1^{P2}) = \frac{M_1(\lambda)(-\Omega)}{M_1(\lambda) - 1 - r_f}$$

and thus assumption $C_1 > Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$ is satisfied for $C_1 = C_1^{P2}$ only if $\Omega < 0$ which happens to be our assumption.

Note that (24), the assumptions $\Omega < 0$ and $\lambda > \lambda_1^{\Omega<0}$ (which gives $M_1(\lambda) > 1 + r_f$) imply that $C_1^{P2} > \bar{C}_1$.

What remains to be shown is when is the expected utility function strictly concave at (C_1^{P2}, α^{P2}). For this to hold it is sufficient to show that the following holds at (C_1^{P2}, α^{P2}):

$\frac{d^2\mathbb{E}(U)}{d\alpha^2} < 0$ and $D \equiv \nabla^2 \mathbb{E}(U(C_1, C_2)) = \frac{d^2\mathbb{E}(U)}{dC_1^2}\frac{d^2\mathbb{E}(U)}{d\alpha^2} - \left(\frac{d^2\mathbb{E}(U)}{dC_1 d\alpha}\right)^2 > 0$. Note that

$$\begin{aligned} C_{2g}^{P2} - \bar{C}_2 &= \frac{\left(\frac{1}{K_0}\right)^{\frac{1}{\gamma}}(r_g - r_b)}{(r_g - r_f)\left(\lambda^{\frac{1}{\gamma}} - \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right)} \frac{M_1(\lambda)}{M_1(\lambda) - 1 - r_f}\left(\bar{C}_2 - Y_2 - (1 + r_f)(Y_1 - \bar{C}_1)\right) \\ &= k\frac{-\Omega}{M_1(\lambda) - 1 - r_f}\frac{r_g - r_b}{r_g - r_f}\left(\frac{1}{K_0}\right)^{\frac{1}{\gamma}} \end{aligned} \tag{46}$$

$$\begin{aligned} \bar{C}_2 - C_{2b}^{P2} &= \frac{\lambda^{\frac{1}{\gamma}}(r_g - r_b)}{(r_g - r_f)\left(\lambda^{\frac{1}{\gamma}} - \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right)} \frac{M_1(\lambda)}{M_1(\lambda) - 1 - r_f}\left(\bar{C}_2 - Y_2 - (1 + r_f)(Y_1 - \bar{C}_1)\right) \\ &= k\frac{-\Omega}{M_1(\lambda) - 1 - r_f}\frac{r_g - r_b}{r_g - r_f}\lambda^{\frac{1}{\gamma}} \end{aligned} \tag{47}$$

and thus $\bar{C}_2 - C_{2b}^{P2} = (K_0\lambda)^{\frac{1}{\gamma}}(C_{2g}^{P2} - \bar{C}_2)$. Using (43), (46) and (47) we obtain the following

$$\begin{aligned} \frac{1}{\gamma}\frac{d^2\mathbb{E}(U)}{dC_1^2}|_{(C_1,\alpha)=(C_1^{P2},\alpha^{P2})} &= \left[\frac{-\Omega}{M_1(\lambda) - 1 - r_f}\right]^{-1-\gamma} \\ &\quad \times\left[-1 + \frac{1 + r_f}{k}\left(\frac{r_g - r_f}{r_g - r_b}\right)^2\left(\lambda^{-\frac{1}{\gamma}} - K_0^{\frac{1}{\gamma}}\frac{r_f - r_b}{r_g - r_f}\right)\right] \end{aligned} \tag{48}$$

$$\begin{aligned} \frac{1}{\gamma}\frac{d^2\mathbb{E}(U)}{d\alpha^2}|_{(C_1,\alpha)=(C_1^{P2},\alpha^{P2})} &= \frac{(r_f - r_b)^2}{k(1 + r_f)}\left(\frac{r_g - r_f}{r_g - r_b}\right)^2\left[\frac{-\Omega}{M_1(\lambda) - 1 - r_f}\right]^{-1-\gamma}\left(\lambda^{-\frac{1}{\gamma}} - K_\gamma^{\frac{1}{\gamma}}\right) \\ \frac{1}{\gamma}\frac{d^2\mathbb{E}(U)}{dC_1 d\alpha}|_{(C_1,\alpha)=(C_1^{P2},\alpha^{P2})} &= \frac{r_f - r_b}{k}\left(\frac{r_g - r_f}{r_g - r_b}\right)^2\left[\frac{-\Omega}{M_1(\lambda) - 1 - r_f}\right]^{-1-\gamma}\left(\lambda^{-\frac{1}{\gamma}} + K_0^{\frac{1}{\gamma}}\right) \end{aligned} \tag{49}$$

Note that (49) and $\lambda > \frac{1}{K_\gamma}$ imply that $\frac{d^2\mathbb{E}(U)}{d\alpha^2}|_{(C_1,\alpha)=(C_1^{P2},\alpha^{P2})} < 0$. In addition,

$$\begin{aligned} \frac{1}{\gamma^2}\left[\frac{-\Omega}{M_1(\lambda) - 1 - r_f}\right]^{2(1+\gamma)} D &= \left[-1 + \frac{1 + r_f}{k}\left(\frac{r_g - r_f}{r_g - r_b}\right)^2\left(\lambda^{-\frac{1}{\gamma}} - \frac{r_f - r_b}{r_g - r_f}K_0^{\frac{1}{\gamma}}\right)\right] \\ &\quad \times\frac{r_f - r_b}{k(1 + r_f)}\left(\frac{r_g - r_f}{r_g - r_b}\right)^2\left[(r_f - r_b)\lambda^{-\frac{1}{\gamma}} - (r_g - r_f)K_0^{\frac{1}{\gamma}}\right] \\ &\quad - \left(\frac{r_f - r_b}{k}\right)^2\left(\frac{r_g - r_f}{r_g - r_b}\right)^4\left(\lambda^{-\frac{1}{\gamma}} + K_0^{\frac{1}{\gamma}}\right)^2 \end{aligned}$$

and thus

$$\frac{1}{\gamma^2}\left[\frac{-\Omega}{M_1(\lambda) - 1 - r_f}\right]^{2(1+\gamma)}\left(\frac{r_g - r_b}{r_g - r_f}\right)^2\frac{k}{r_f - r_b}D = \frac{1}{1 + r_f}\left[(r_g - r_f)K_0^{\frac{1}{\gamma}} - (r_f - r_b)\lambda^{-\frac{1}{\gamma}}\right]$$

$$
\begin{aligned}
&+ \quad \frac{1}{k}\left(\frac{r_g-r_f}{r_g-r_b}\right)^2\left(\lambda^{-\frac{1}{\gamma}}-\frac{r_f-r_b}{r_g-r_f}K_0^{\frac{1}{\gamma}}\right)\left[(r_f-r_b)\lambda^{-\frac{1}{\gamma}}-(r_g-r_f)K_0^{\frac{1}{\gamma}}\right] \\
&- \quad \frac{r_f-r_b}{k}\left(\frac{r_g-r_f}{r_g-r_b}\right)^2\left(\lambda^{-\frac{1}{\gamma}}+K_0^{\frac{1}{\gamma}}\right)^2
\end{aligned}
$$

After some derivations we obtain

$$
\begin{aligned}
\frac{1}{\gamma^2}\left[\frac{-\Omega}{M_1(\lambda)-1-r_f}\right]^{2(1+\gamma)}\left(\frac{r_g-r_b}{r_g-r_f}\right)^2\frac{k}{r_f-r_b}D \quad &= \quad \frac{1}{1+r_f}\left[(r_g-r_f)K_0^{\frac{1}{\gamma}}-(r_f-r_b)\lambda^{-\frac{1}{\gamma}}\right] \\
&\quad -\frac{r_g-r_f}{k}\lambda^{-\frac{1}{\gamma}}K_0^{\frac{1}{\gamma}}
\end{aligned}
$$

Now it can be easily shown that if $\lambda > \left[\frac{1+r_f}{k}+\left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right]^\gamma$, which follows from assumption $\lambda > \lambda_1^{\Omega<0}$ and (21), then $D>0$.

Regarding the feasibility, note that (46) and (47) imply that $C_{2g}^{P2} > \bar{C}_2$ and $C_{2b}^{P2} < \bar{C}_2$. In addition, (47), $\bar{C}_2 \leq \bar{C}_2^{P2}$ and $\lambda > \lambda_1^{\Omega<0}$ imply that $C_{2b}^{P2} \geq 0$.[28]

Note that

$$
\begin{aligned}
(1-\gamma)\mathbb{E}(U(C_1^{P2},\alpha^{P2})) \quad &= \quad \left(\frac{-\Omega}{M_1(\lambda)-1-r_f}\right)^{1-\gamma}\left[1+\frac{k}{1+r_f}\left(\left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}-\lambda^{\frac{1}{\gamma}}\right)\right] \\
&= \quad -\frac{(-\Omega)^{1-\gamma}}{1+r_f}\left[k\left(\lambda^{\frac{1}{\gamma}}-\left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right)-1-r_f\right]^\gamma \\
&= \quad -\frac{(-\Omega)^{1-\gamma}}{1+r_f}k^\gamma\left[\lambda^{\frac{1}{\gamma}}-(1-\tilde{c}^{P2})\left(\lambda_1^{\Omega<0}\right)^{\frac{1}{\gamma}}\right]^\gamma \\
&= \quad -\frac{(-\Omega)^{1-\gamma}}{1+r_f}\left(M_1(\lambda)-1-r_f\right)^\gamma \qquad (50)
\end{aligned}
$$

What remains to show is that feasible solutions at the border do not exceed the expected utility at (C_1^{P2},α^{P2}), where (P2) obtains its local maximum. The feasible solution at the border that come into consideration are: (i) $C_{2g}=\bar{C}_2$, (ii) $C_{2b}=\bar{C}_2$, (iii) $C_{2b}=0$ and (iv) $C_1=\bar{C}_1$.

Case (i). $C_{2g}=\bar{C}_2$ when $C_1 = Y_1+\frac{Y_2-\bar{C}_2}{1+r_f}+\frac{r_g-r_f}{1+r_f}\alpha$ for $\frac{-\Omega}{r_g-r_f}\leq\alpha\leq\frac{\bar{C}_2}{r_g-r_b}$ and thus

$$
\begin{aligned}
(1-\gamma)\mathbb{E}\left(U\left(Y_1+\frac{Y_2-\bar{C}_2}{1+r_f}+\frac{r_g-r_f}{1+r_f}\alpha,\,\alpha\right)\right) \quad &= \quad \left(\frac{\Omega+(r_g-r_f)\alpha}{1+r_f}\right)^{1-\gamma} \\
&\quad -\lambda\delta(1-p)(r_g-r_b)^{1-\gamma}\alpha^{1-\gamma}
\end{aligned}
$$

[28] Note that if we would not want to guarantee $C_{2b}\geq 0$ then it would be sufficient to have $\lambda_1^{\Omega<0} = \left[\frac{1+r_f}{k}+\left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right]^\gamma$. The more complicated expression of $\lambda_1^{\Omega<0}$ defined by (21) follows from the constraint $C_{2b}\geq 0$. Also condition $\bar{C}_2 < \bar{C}_2^{P2}$ is implied by the constraint $C_{2b}\geq 0$.

The following can be easily shown

$$\lim_{\alpha\to^+\frac{-\Omega}{r_g-r_f}} \frac{d\mathbb{E}\left(U\left(Y_1+\frac{Y_2-\bar{C}_2}{1+r_f}+\frac{r_g-r_f}{1+r_f}\,\alpha,\,\alpha\right)\right)}{d\alpha}=+\infty$$

and

$$\left.\frac{d\mathbb{E}\left(U\left(Y_1+\frac{Y_2-\bar{C}_2}{1+r_f}+\frac{r_g-r_f}{1+r_f}\,\alpha,\,\alpha\right)\right)}{d\alpha}\right|_{\alpha=\alpha_1}=0$$

where

$$\alpha_1=\frac{k\lambda^{\frac{1}{\gamma}}(-\Omega)}{\left(k\lambda^{\frac{1}{\gamma}}-1-r_f\right)(r_g-r_f)}$$

Note in addition that $\alpha_1\leq\frac{\bar{C}_2}{r_g-r_b}$ for $\lambda>\left[\frac{1+r_f}{k(1-\tilde{c}^{P2})}\right]^{\gamma}$, where $\tilde{c}^{P2}=\frac{(r_g-r_b)(-\Omega)}{(r_g-r_f)\bar{C}_2}$ and that

$$\left.\frac{d^2\mathbb{E}\left(U\left(Y_1+\frac{Y_2-\bar{C}_2}{1+r_f}+\frac{r_g-r_f}{1+r_f}\,\alpha,\,\alpha\right)\right)}{d\alpha^2}\right|_{\alpha=\alpha_1}<0$$

Thus, for $\lambda>\left[\frac{1+r_f}{k(1-\tilde{c}^{P2})}\right]^{\gamma}$ is the maximum reached at α_1. As

$$(1-\gamma)\mathbb{E}\left(U\left(Y_1+\frac{Y_2-\bar{C}_2}{1+r_f}+\frac{r_g-r_f}{1+r_f}\,\alpha_1,\,\alpha_1\right)\right)=-\frac{(-\Omega)^{1-\gamma}}{1+r_f}\left(k\lambda^{\frac{1}{\gamma}}-1-r_f\right)^{\gamma} \tag{51}$$

then based on this it can be shown that for $\lambda>\lambda_1^{\Omega<0}$

$$(1-\gamma)\mathbb{E}\left(U\left(Y_1+\frac{Y_2-\bar{C}_2}{1+r_f}+\frac{r_g-r_f}{1+r_f}\,\alpha_1,\,\alpha_1\right)\right)<(1-\gamma)\mathbb{E}\left(U\left(C_1^{P2},\,\alpha^{P2}\right)\right)$$

where $(1-\gamma)\mathbb{E}\left(U\left(C_1^{P2},\,\alpha^{P2}\right)\right)=-\frac{(-\Omega)^{1-\gamma}}{1+r_f}\left[k\left(\lambda^{\frac{1}{\gamma}}-\left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right)-1-r_f\right]^{\gamma}$, see (50). In summary, there are only two possible candidates for the maximum: (1) $\alpha=\frac{\bar{C}_2}{r_g-r_b}$, which is tackled in case (iii) and (2) $\alpha=\alpha_1$ for $\lambda>\left[\frac{1+r_f}{k(1-\tilde{c}^{P2})}\right]^{\gamma}$ where we have shown that expected utility at $\left(C_1=Y_1+\frac{Y_2-\bar{C}_2}{1+r_f}+\frac{r_g-r_f}{1+r_f}\,\alpha_1,\,\alpha_1\right)$ is smaller than the expected utility at $\left(C_1^{P2},\,\alpha^{P2}\right)$.

Case (ii). $C_{2b}=\bar{C}_2$ when $C_1=Y_1+\frac{Y_2-\bar{C}_2}{1+r_f}-\frac{r_f-r_b}{1+r_f}\alpha$. This case has no feasible solution as C_1 can not exceed $Y_1+\frac{Y_2-\bar{C}_2}{1+r_f}$.

Case (iii). $C_{2b}=0$ when $C_1=Y_1+\frac{Y_2}{1+r_f}-\frac{r_f-r_b}{1+r_f}\alpha$ for $\frac{\bar{C}_2}{r_g-r_b}\leq\alpha\leq\frac{(1+r_f)(Y_1-\bar{C}_1)+Y_2}{r_f-r_b}$ and thus

$$\begin{aligned}(1-\gamma)\mathbb{E}\left(U\left(Y_1+\frac{Y_2}{1+r_f}-\frac{r_f-r_b}{1+r_f}\,\alpha,\,\alpha\right)\right) \quad &= \quad \left(\frac{(1+r_f)(Y_1-\bar{C}_1)+Y_2-(r_f-r_b)\alpha}{1+r_f}\right)^{1-\gamma}\\ &+\delta p\,\left((r_g-r_b)\alpha-\bar{C}_2\right)^{1-\gamma}-\lambda\delta(1-p)\bar{C}_2^{1-\gamma}\end{aligned}$$

It can be shown that this expected utility function is concave and thus its maximum is reached either at α^{P20} where

$$\left.\frac{d\mathbb{E}\left(U\left(Y_1+\frac{Y_2}{1+r_f}-\frac{r_f-r_b}{1+r_f}\alpha,\,\alpha\right)\right)}{d\alpha}\right|_{\alpha=\alpha^{P20}}=0$$

or at one of the end-points. After some derivations we obtain

$$\alpha^{P20}=\frac{1+r_f}{k_2+1+r_f}\frac{\bar{C}_2}{r_g-r_b}+\frac{k_2}{k_2+1+r_f}\frac{(1+r_f)(Y_1-\bar{C}_1)+Y_2}{r_f-r_b} \tag{52}$$

which is a convex combination of the end-points and thus the maximum is reached at $\alpha=\alpha^{P20}$ such that $\frac{\bar{C}_2}{r_g-r_b}<\alpha^{P20}<\frac{(1+r_f)(Y_1-\bar{C}_1)+Y_2}{r_f-r_b}$. The last inequalities imply that optimal C_1 is strictly above $\bar{C}_1$ and optimal C_{2g} is strictly above $\bar{C}_2$. Further derivations give

$$\begin{aligned}
&(1-\gamma)\mathbb{E}\left(U\left(C_1^{P20}=Y_1+\frac{Y_2}{1+r_f}-\frac{r_f-r_b}{1+r_f}\,\alpha^{P20},\,\alpha^{P20}\right)\right)=\\
&=\frac{k^\gamma}{1+r_f}\left(\frac{r_g-r_f}{r_g-r_b}\,\bar{C}_2\right)^{1-\gamma}\left[\left(\left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}+\frac{1+r_f}{k}\right)^\gamma(1-\tilde{c}^{P2})^{1-\gamma}-\lambda\right]\\
&=\delta(1-p)\,\bar{C}_2^{1-\gamma}\left[\lambda_1^{\Omega<0}(1-\tilde{c}^{P2})-\lambda\right]
\end{aligned} \tag{53}$$

Finally, based on this and (50) it can be shown that for $\lambda>\lambda_1^{\Omega<0}$

$$(1-\gamma)\mathbb{E}\left(U\left(C_1^{P20},\,\alpha^{P20}\right)\right)<(1-\gamma)\mathbb{E}\left(U\left(C_1^{P2},\,\alpha^{P2}\right)\right) \tag{54}$$

if

$$\left(\lambda^{\frac{1}{\gamma}}-\left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}-\frac{1+r_f}{k}\right)^\gamma\left(\tilde{c}^{P2}\right)^{1-\gamma}<\lambda-\left(\left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}+\frac{1+r_f}{k}\right)^\gamma\left(1-\left(\tilde{c}^{P2}\right)\right)^{1-\gamma}$$

or if

$$F(\lambda)\equiv\lambda-\left(\left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}+\frac{1+r_f}{k}\right)^\gamma\left(1-\left(\tilde{c}^{P2}\right)\right)^{1-\gamma}-\left(\lambda^{\frac{1}{\gamma}}-\left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}-\frac{1+r_f}{k}\right)^\gamma\left(\tilde{c}^{P2}\right)^{1-\gamma}>0$$

The last inequality holds as $F(\lambda)$ is a convex function with the minimum being reached at $\lambda=\lambda_1^{\Omega<0}$, see (21), where $F(\lambda_1^{\Omega<0})=0$.

<u>Case (iv)</u>. $C_1=\bar{C}_1$ for $\frac{-\Omega}{r_g-r_f}\leq\alpha\leq\frac{(1+r_f)(Y_1-\bar{C}_1)+Y_2}{r_f-r_b}$ and thus

$$(1-\gamma)\mathbb{E}\left(U\left(\bar{C}_1,\,\alpha\right)\right)\;=\;\delta p\left(\Omega+(r_g-r_f)\alpha\right)^{1-\gamma}-\lambda\delta(1-p)\left(-\Omega+(r_f-r_b)\alpha\right)^{1-\gamma} \tag{55}$$

The following can be easily shown

$$\lim_{\alpha \to^+ \frac{-\Omega}{r_g - r_f}} \frac{d\mathbb{E}\left(U\left(\bar{C}_1, \alpha\right)\right)}{d\alpha} = +\infty \tag{56}$$

and

$$\frac{d\mathbb{E}\left(U\left(\bar{C}_1, \alpha\right)\right)}{d\alpha}\Big|_{\alpha=\alpha_3} = 0$$

for

$$\alpha_3 = \frac{\lambda^{\frac{1}{\gamma}} + \left(\frac{1}{K_0}\right)^{\frac{1}{\gamma}}}{\lambda^{\frac{1}{\gamma}} - \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}} \frac{(-\Omega)}{r_g - r_f}$$

and $\lambda > \frac{1}{K_\gamma}$. As

$$(1-\gamma)\mathbb{E}\left(U\left(\bar{C}_1, \alpha_3\right)\right) = -\frac{k^\gamma(-\Omega)^{1-\gamma}}{1+r_f}\left(\lambda^{\frac{1}{\gamma}} - \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right)^\gamma \tag{57}$$

then based on this and (50) it can be shown that for $\lambda > \lambda_1^{\Omega<0}$

$$(1-\gamma)\mathbb{E}\left(U\left(\bar{C}_1, \alpha_3\right)\right) < (1-\gamma)\mathbb{E}\left(U\left(C_1^{P2}, \alpha^{P2}\right)\right)$$

Note that the maximum can not be reached at $\frac{-\Omega}{r_g - r_f}$, see (56), and another end-point, $\alpha = \frac{(1+r_f)(Y_1 - \bar{C}_1) + Y_2}{r_f - r_b}$, is tackled in case (iii).

There is no feasible solution for **problem (P3)** when $\Omega < 0$.

Problem (P4). There is no feasible solution for $C_1 \leq Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$. Let $C_1 \geq Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$. At first we show that no local extreme can be a local maximum. This implies that, as the function is continuous, a maximum will occur at the border of the set of feasible solutions.

In more detail

$$\begin{aligned}
\frac{d\mathbb{E}(U)}{d\alpha} &= \lambda\delta p\left[\bar{C}_2 - Y_2 - (1+r_f)(Y_1 - C_1) - (r_g - r_f)\alpha\right]^{-\gamma}(r_g - r_f) \\
&\quad -\lambda\delta(1-p)\left[\bar{C}_2 - Y_2 - (1+r_f)(Y_1 - C_1) + (r_f - r_b)\alpha\right]^{-\gamma}(r_f - r_b)
\end{aligned}$$

and thus

$$\begin{aligned}
\frac{d^2\mathbb{E}(U)}{d\alpha^2} &= \lambda\gamma\delta p\left[\bar{C}_2 - Y_2 - (1+r_f)(Y_1 - C_1) - (r_g - r_f)\alpha\right]^{-\gamma-1}(r_g - r_f)^2 \\
&\quad +\lambda\gamma\delta(1-p)\left[\bar{C}_2 - Y_2 - (1+r_f)(Y_1 - C_1) + (r_f - r_b)\alpha\right]^{-\gamma-1}(r_f - r_b)^2 > 0
\end{aligned}$$

This excludes the possibility of the objective function of (P4) to obtain its maximum in the interior and thus it would occur at the border of the feasible solutions of problem (P4).

Now we will consider feasible solutions at the border, namely: (i) $C_{2g} = \bar{C}_2$, (ii) $C_{2b} = \bar{C}_2$, (iii) $C_{2g} = 0$, (iv) $C_{2b} = 0$ and (v) $C_1 = \bar{C}_1$. Note that case (i) coincides with case (i) when proving (P2) and there is only one feasible solution in cases (ii) and (iii), namely $\left(C_1 = Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}, \alpha = 0\right)$, which is feasible also for (P2).

Case (iii). The only feasible solution for $C_{2g} = 0$ is $(C_1 = Y_1 + \frac{Y_2}{1+r_f}, \alpha = 0)$ with the utility function being

$$(1-\gamma)\mathbb{E}\left(U\left(Y_1 + \frac{Y_2}{1+r_f}, 0\right)\right) = \left(Y_1 - \bar{C}_1 + \frac{Y_2}{1+r_f}\right)^{1-\gamma} - \lambda\delta\bar{C}_2^{1-\gamma}$$

which is dealt in case (iv) below.

$$\begin{aligned}(1-\gamma)\mathbb{E}(U(C_1,\alpha)) \quad = \quad & \left(\frac{(1+r_f)(Y_1-\bar{C}_1) + Y_2 - (r_f - r_b)\alpha}{1+r_f}\right)^{1-\gamma} - \lambda\delta p(\bar{C}_2 - (r_g - r_b)\alpha)^{1-\gamma} \\ & -\lambda\delta(1-p)\bar{C}_2^{1-\gamma} \end{aligned} \tag{58}$$

Case (iv). $C_{2b} = 0$ when $C_1 = Y_1 + \frac{Y_2}{1+r_f} - \frac{r_f - r_b}{1+r_f}\alpha$ and $0 \leq \alpha \leq \frac{\bar{C}_2}{r_g - r_b}$ and the utility function is given by (41). The potential candidates for maximum are $\alpha = 0$, $\alpha = \frac{\bar{C}_2}{r_g - r_b}$ and $\alpha = \bar{\alpha}$ where $\bar{\alpha}$ is a unique stationary point such that $\left.\frac{d\mathbb{E}(U)}{d\alpha}\right|_{\alpha=\bar{\alpha}} = 0$ where

$$\bar{\alpha} = \frac{(\lambda\delta p(r_g - r_b))^{\frac{1}{\gamma}}\left(Y_1 - \bar{C}_1 + \frac{Y_2}{1+r_f}\right) - \left(\frac{r_f - r_b}{1+r_f}\right)^{\frac{1}{\gamma}}\bar{C}_2}{(\lambda\delta p(r_g - r_b))^{\frac{1}{\gamma}}\frac{r_f - r_b}{1+r_f} - \left(\frac{r_f - r_b}{1+r_f}\right)^{\frac{1}{\gamma}}(r_g - r_b)}$$

Note that for $\bar{C}_2 < \bar{C}_2^{P2}$ is $\bar{\alpha}$ infeasible and for $\bar{C}_2 = \bar{C}_2^{P2}$is $\bar{\alpha} = \frac{\bar{C}_2}{r_g - r_b}$. For $\alpha = \frac{\bar{C}_2}{r_g - r_b}$ is the point $\left(C_1 = Y_1 + \frac{Y_2}{1+r_f} - \frac{r_f - r_b}{1+r_f}\frac{\bar{C}_2}{r_g - r_b}, \alpha = \frac{\bar{C}_2}{r_g - r_b}\right)$ feasible for (P2). Finally, we show that the utility function at $\alpha = 0$ is below the utility function at $\left(C_1^{P2}, \alpha^{P2}\right)$; i.e., that

$$\mathbb{E}\left(U\left(Y_1 + \frac{Y_2}{1+r_f}, 0\right)\right) \leq \mathbb{E}\left(U\left(C_1^{P2}, \alpha^{P2}\right)\right) \tag{59}$$

We proceed in two steps. If $\tilde{\lambda}^{\Omega<0} \geq \lambda_1^{\Omega<0}$ then we show that

$$\mathbb{E}\left(U\left(Y_1 + \frac{Y_2}{1+r_f}, 0\right)\right) \leq \mathbb{E}\left(U\left(C_1^{P20}, \alpha^{P20}\right)\right) \tag{60}$$

where $\left(C_1^{P20}, \alpha^{P20}\right)$ and $\mathbb{E}\left(U\left(C_1^{P20}, \alpha^{P20}\right)\right)$ are given by (52) and (53). Inequality (60) holds for $\lambda \geq \tilde{\lambda}^{\Omega<0}$ which implies that also (59) holds as for $\lambda > \lambda_1^{\Omega<0}$ is $\mathbb{E}\left(U\left(C_1^{P20}, \alpha^{P20}\right)\right) < \mathbb{E}\left(U\left(C_1^{P2}, \alpha^{P2}\right)\right)$. On the other hand, if $\tilde{\lambda}^{\Omega<0} < \lambda_1^{\Omega<0}$ then (59) can be shown directly.

Let $\tilde{\lambda}^{\Omega<0} \geq \lambda_1^{\Omega<0}$. Then (60) holds if

$$\left(Y_1 - \bar{C}_1 + \frac{Y_2}{1+r_f}\right)^{1-\gamma} - \lambda\delta\,\bar{C}_2^{1-\gamma} \leq \delta(1-p)\bar{C}_2^{1-\gamma}\left[\left(1-\tilde{c}^{P2}\right)\lambda_1^{\Omega<0} - \lambda\right]$$

which holds if

$$\lambda \geq \tilde{\lambda}^{\Omega<0} \equiv \frac{1}{p}\left[\frac{1}{\delta}\left(\frac{Y_1 - \bar{C}_1 + \frac{Y_2}{1+r_f}}{\bar{C}_2}\right)^{1-\gamma} - (1-p)\left(1-\tilde{c}^{P2}\right)\lambda_1^{\Omega<0}\right]$$

Let $\tilde{\lambda}^{\Omega<0} > \lambda_1^{\Omega<0}$. Then for $\lambda > \lambda_1^{\Omega<0}$ (59) holds if

$$\frac{(-\Omega)^{1-\gamma}}{1+r_f}k^{\gamma}\left(\lambda^{\frac{1}{\gamma}} - \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}} - \frac{1+r_f}{k}\right)^{\gamma} \leq \lambda\delta\,\bar{C}_2^{1-\gamma} - \left(Y_1 - \bar{C}_1 + \frac{Y_2}{1+r_f}\right)^{1-\gamma}$$

Let

$$G(\lambda) \equiv \frac{(-\Omega)^{1-\gamma}}{1+r_f}k^{\gamma}\left(\lambda^{\frac{1}{\gamma}} - \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}} - \frac{1+r_f}{k}\right)^{\gamma} - \lambda\delta\,\bar{C}_2^{1-\gamma} + \left(Y_1 - \bar{C}_1 + \frac{Y_2}{1+r_f}\right)^{1-\gamma} \tag{61}$$

As $G(\lambda)$ is a continuous decreasing function in λ[29] and as $G(\tilde{\lambda}^{\Omega<0}) \leq 0 \leq G(\lambda_1^{\Omega<0})$ then there exists $\lambda_0^{P2} \in \left[\lambda_1^{\Omega<0}, \tilde{\lambda}^{\Omega<0}\right]$ such that $G(\lambda_0^{P2}) = 0$. Thus, is obtained at $\left(C_1 = C_1^{P2},\ \alpha = \alpha^{P2}\right)$ if $\lambda > \lambda_0^{P2}$.

Thus, (59) holds for $\lambda > \max\left\{\tilde{\lambda}^{\Omega<0}, \lambda_1^{\Omega<0}\right\}$.

Case (v). The utility function of (P4) with $C_1 = \bar{C}_1$ is

$$\begin{aligned}\mathbb{E}(U(C_1,\alpha)) \quad &= \quad -\lambda\delta p\frac{(-\Omega-(r_g-r_f)\alpha)^{1-\gamma}}{1-\gamma} - \lambda\delta(1-p)\frac{(-\Omega+(r_f-r_b)\alpha)^{1-\gamma}}{1-\gamma} \\ &\qquad \text{for} \quad 0 \leq \alpha \leq \frac{-\Omega}{r_g - r_f}\end{aligned} \tag{62}$$

when $\bar{C}_2 \leq \bar{C}_2^{P2}$. It can be easily shown that utility function (62) is convex and thus its maximum is reached at either $\alpha = 0$, for which is $C_1 = \bar{C}_1, \alpha = 0$ infeasible, or $\alpha = \frac{-\Omega}{r_g - r_f}$, for which is $\left(\bar{C}_1, \frac{-\Omega}{r_g-r_f}\right)$ feasible for (P2).

Problem (P5). Note that for $C_1 \geq Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$ there is only one feasible solution, namely $(C_1 = Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}, \alpha = 0)$, which is thus feasible for case when $C_1 \leq Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}$ (see below).

When $C_1 \leq Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}$ it is easy to show that for any fixed $\tilde{C}_1$ such that $0 \leq \tilde{C}_1 \leq$

[29] This follows from $\lambda \geq \lambda_1^{\Omega<0} \geq \left(\frac{\left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}} + \frac{1+r_f}{k}}{1-(1-p)^{\frac{1}{1-p}}\tilde{c}^{P2}}\right)^{\gamma}$ where the latter inequality follows from $\lambda_1^{\Omega<0} \leq \tilde{\lambda}^{\Omega<0}$.

$Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$ is the expected utility of (P5) concave and thus its maximum is achieved at

$$\tilde{\alpha} = \frac{1 - K_0^{\frac{1}{\gamma}}}{r_f - r_b + K_0^{\frac{1}{\gamma}}(r_g - r_f)}((1+r_f)(Y_1 - \tilde{C}_1) + Y_2 - \bar{C}_2) \geq 0 \tag{63}$$

Note also that $(\tilde{C}_1, \tilde{\alpha})$ is feasible for (P5). Thus, the candidates for the maximum for (P5) are $(\tilde{C}_1, \tilde{\alpha})$ with $0 \leq \tilde{C}_1 \leq Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$ and $\tilde{\alpha}$ given by (63). By plugging this point into the expected utility of (P5) we obtain (after some derivations)

$$(1-\gamma)\mathbb{E}(U) = -\lambda(\bar{C}_1 - \tilde{C}_1)^{1-\gamma} + \hat{\lambda}\left(Y_1 - \tilde{C}_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}\right)^{1-\gamma} \tag{64}$$

where

$$\hat{\lambda} = \left[\frac{k_2\left(1 + K_\gamma^{\frac{1}{\gamma}}\right)}{1+r_f}\right]^\gamma = \left[\frac{k\left(1 + \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right)}{1+r_f}\right]^\gamma$$

As this expected utility is not monotone or concave – in $\tilde{C}_1$ such that $0 \leq \tilde{C}_1 \leq Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$ – the maximum of (64) can be reached at either end points (see cases (i) and (ii) below) or at the point (see case (iii) below) where $\left.\frac{d\mathbb{E}(U)}{d\tilde{C}_1}\right|_{\tilde{C}_1 = C_1^{P5}} = 0$ with $\mathbb{E}(U)$ being given by (64). Thus, the cases under consideration are

(i) $\tilde{C}_1 = 0$ where

$$\mathcal{U}^{P5i} \equiv (1-\gamma)\mathbb{E}(U) = -\lambda\bar{C}_1^{1-\gamma} + \hat{\lambda}\left(Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}\right)^{1-\gamma}$$

(ii) $\tilde{C}_1 = Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$ where

$$\mathcal{U}^{P5ii} \equiv (1-\gamma)\mathbb{E}(U) = -\lambda\left(\frac{-\Omega}{1+r_f}\right)^{1-\gamma}$$

(iii) $\tilde{C}_1 = C_1^{P5} \equiv \frac{\left(Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}\right)\lambda^{\frac{1}{\gamma}} - \bar{C}_1\hat{\lambda}^{\frac{1}{\gamma}}}{\lambda^{\frac{1}{\gamma}} - \hat{\lambda}^{\frac{1}{\gamma}}}$ for $\lambda \geq \lambda_2^{\Omega<0}$ where

$$\lambda_2^{\Omega<0} = \hat{\lambda}\left(\frac{\bar{C}_1}{Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}}\right)^\gamma$$

to guarantee that $0 \leq \tilde{C}_1 \leq Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$. Note in addition that C_1^{P5} is the only stationary point of $\mathbb{E}(U)$ given by (64). Thus,

$$\mathcal{U}^{P5} \equiv (1-\gamma)\mathbb{E}(U) = -\left(\frac{-\Omega}{1+r_f}\right)^{1-\gamma}\left(\lambda^{\frac{1}{\gamma}} - \frac{k_2\left(1+K_\gamma^{\frac{1}{\gamma}}\right)}{1+r_f}\right)^{\gamma} \tag{65}$$

$$= -\left(\frac{-\Omega}{1+r_f}\right)^{1-\gamma}\left(\lambda^{\frac{1}{\gamma}} - \hat{\lambda}^{\frac{1}{\gamma}}\right)^{\gamma}$$

It can be shown that $\mathbb{E}(U)$ given by (64) is concave at $C_1 = C_1^{P5}$ given by case (iii) as C_1^{P5} is the only stationary point there. Thus, the maximum for (P5) with $\lambda \geq \lambda_2^{\Omega<0}$ is reached in case (iii), i.e., at point

$$\left(0 < C_1^{P5} = \frac{\left(Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}\right)\lambda^{\frac{1}{\gamma}} - \bar{C}_1\hat{\lambda}^{\frac{1}{\gamma}}}{\lambda^{\frac{1}{\gamma}} - \hat{\lambda}^{\frac{1}{\gamma}}} < \bar{C}_1,\ \alpha^{P5} = \frac{1-K_0^{\frac{1}{\gamma}}}{r_f - r_b + K_0^{\frac{1}{\gamma}}(r_g - r_f)}\frac{\hat{\lambda}^{\frac{1}{\gamma}}}{\lambda^{\frac{1}{\gamma}} - \hat{\lambda}^{\frac{1}{\gamma}}}(-\Omega) > 0\right)$$

Let

$$\mathcal{U}^{P2} \equiv (1-\gamma)\mathbb{E}(U(C_1^{P2}, \alpha^{P2})) = -\frac{(-\Omega)^{1-\gamma}}{1+r_f}\left[k\left(\lambda^{\frac{1}{\gamma}} - \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right) - 1 - r_f\right]^{\gamma}$$

Note that (65) can be written also as

$$\mathcal{U}^{P5} = (1-\gamma)\mathbb{E}\left(U\left(C_1^{P5}, \alpha^{P5}\right)\right) = -\left(\frac{-\Omega}{1+r_f}\right)^{1-\gamma}\left(\lambda^{\frac{1}{\gamma}} - \frac{k\left(1+\left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right)}{1+r_f}\right)^{\gamma}$$

Then for $\lambda \geq \lambda_2^{\Omega<0}$ the utility function of problem (P2) at its maxima is related to the utility function of (P5) at its maximum as follows: $\mathcal{U}^{P2} > \mathcal{U}^{P5}$ for $k < 1 + r_f$, $\mathcal{U}^{P2} = \mathcal{U}^{P5}$ for $k = 1 + r_f$ and $\mathcal{U}^{P2} < \mathcal{U}^{P5}$ for $k > 1 + r_f$. Note in addition that condition $k \leq 1 + r_f$ is equivalent to $\delta \leq \delta^+$ and condition $k_2 \leq 1 + r_f$ is equivalent to $\delta \leq \delta^-$.

Problem (P6). We proceed in two steps: for case $C_1 \geq Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}$ we show that there is no interior local maximum or minimum for (P6) which implies that the maximum will occur at the border of the set of feasible solutions for (P6). Then we check all potential feasible solutions at the border. For case $C_1 \leq Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}$ we show that all possible candidates for maximum are also feasible solutions of (P5) which is the case we have already dealt with.

Let $C_1 \geq Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$. The first order conditions are

$$\left.\begin{aligned}
\frac{d\mathbb{E}(U)}{dC_1} = \; & \lambda(\bar{C}_1 - C_1)^{-\gamma} \; -\delta p\left[(1+r_f)(Y_1 - C_1) + Y_2 - \bar{C}_2 + (r_g - r_f)\alpha\right]^{-\gamma}(1+r_f) \\
& -\lambda\delta(1-p)\left[\bar{C}_2 - Y_2 - (1+r_f)(Y_1 - C_1) + (r_f - r_b)\alpha\right]^{-\gamma}(1+r_f) = 0 \\
\frac{d\mathbb{E}(U)}{d\alpha} = \; & \delta p\left[(1+r_f)(Y_1 - C_1) + Y_2 - \bar{C}_2 + (r_g - r_f)\alpha\right]^{-\gamma}(r_g - r_f) \\
& -\lambda\delta(1-p)\left[\bar{C}_2 - Y_2 - (1+r_f)(Y_1 - C_1) + (r_f - r_b)\alpha\right]^{-\gamma}(r_f - r_b) = 0
\end{aligned}\right\} \tag{66}$$

$\frac{d\mathbb{E}(U)}{d\alpha} = 0$ from (66) implies the expression for α given by (44) and if we plug it into the C_1 part of the FOC in (66) we obtain after some simplifications

$$\begin{aligned}
\lambda^{\frac{1}{\gamma}}(\bar{C}_2 - Y_2 - (1+r_f)(Y_1 - C_1)) &= (\bar{C}_1 - C_1)\frac{r_g - r_f}{r_g - r_b}\left[\delta(1+r_f)(1-p)\frac{r_g - r_b}{r_g - r_f}\right]^{\frac{1}{\gamma}}\left[\lambda^{\frac{1}{\gamma}} - \left(\frac{1}{K_\gamma}\right)^{\frac{1}{\gamma}}\right] \\
&= (\bar{C}_1 - C_1)M_1(\lambda)
\end{aligned}$$

which gives

$$C_1^+ = \frac{\bar{C}_1 M_1(\lambda) + \lambda^{\frac{1}{\gamma}}[(1+r_f)Y_1 + Y_2 - \bar{C}_2]}{M_1(\lambda) + \lambda^{\frac{1}{\gamma}}(1+r_f)} = \bar{C}_1 + \frac{\Omega}{\frac{M_1(\lambda)}{\lambda^{\frac{1}{\gamma}}} + 1 + r_f} \tag{67}$$

In addition, after plugging C_1^+ from (67) into (44) we obtain

$$\alpha^+ = \frac{k}{r_g - r_f}\left[\left(\frac{1}{K_0}\right)^{\frac{1}{\gamma}} + \lambda^{\frac{1}{\gamma}}\right]\frac{-\Omega}{M_1(\lambda) + \lambda^{\frac{1}{\gamma}}(1+r_f)}$$

Next we show that the expected utility function is indifferent at (C_1^+, α^+), namely, we show that at (C_1^+, α^+) are $\frac{d^2\mathbb{E}(U)}{d\alpha^2} < 0$, and $D_3 \equiv \nabla^2\mathbb{E}(U(C_1, C_2)) = \frac{d^2\mathbb{E}(U)}{dC_1^2}\frac{d^2\mathbb{E}(U)}{d\alpha^2} - \left(\frac{d^2\mathbb{E}(U)}{dC_1 d\alpha}\right)^2 < 0$. Note that

$$C_{2g}^+ - \bar{C}_2 = k\frac{-\Omega}{M_1(\lambda) + \lambda^{\frac{1}{\gamma}}(1+r_f)}\frac{r_g - r_b}{r_g - r_f}\left(\frac{1}{K_0}\right)^{\frac{1}{\gamma}} \tag{68}$$

$$\bar{C}_2 - C_{2b}^+ = k\frac{-\Omega}{M_1(\lambda) + \lambda^{\frac{1}{\gamma}}(1+r_f)}\frac{r_g - r_b}{r_g - r_f}\lambda^{\frac{1}{\gamma}} \tag{69}$$

and thus $\bar{C}_2 - (C_{2b}^+) = (K_0\lambda)^{\frac{1}{\gamma}}((C_{2g}^+) - \bar{C}_2)$. Using (66), (68) and (69) we obtain the following

$$\frac{1}{\gamma}\frac{d^2\mathbb{E}(U)}{dC_1^2}|_{(C_1^+,\alpha^+)} = \left[\frac{-\Omega}{M_1(\lambda)+\lambda^{\frac{1}{\gamma}}(1+r_f)}\right]^{-1-\gamma} \times \left[\frac{1}{\lambda^{\frac{1}{\gamma}}} + \frac{1+r_f}{k}\left(\frac{r_g-r_f}{r_g-r_b}\right)^2\left(\lambda^{-\frac{1}{\gamma}} - K_0^{\frac{1}{\gamma}}\frac{r_f-r_b}{r_g-r_f}\right)\right] \tag{70}$$

$$\frac{1}{\gamma}\frac{d^2\mathbb{E}(U)}{d\alpha^2}|_{(C_1^+,\alpha^+)} = \frac{(r_f-r_b)^2}{k(1+r_f)}\left(\frac{r_g-r_f}{r_g-r_b}\right)^2\left[\frac{-\Omega}{M_1(\lambda)+\lambda^{\frac{1}{\gamma}}(1+r_f)}\right]^{-1-\gamma}\left(\lambda^{-\frac{1}{\gamma}} - K_\gamma^{\frac{1}{\gamma}}\right) \tag{71}$$

$$\frac{1}{\gamma}\frac{d^2\mathbb{E}(U)}{dC_1 d\alpha}|_{(C_1^+,\alpha^+)} = \frac{r_f-r_b}{k}\left(\frac{r_g-r_f}{r_g-r_b}\right)^2\left[\frac{-\Omega}{M_1(\lambda)+\lambda^{\frac{1}{\gamma}}(1+r_f)}\right]^{-1-\gamma}\left(\lambda^{-\frac{1}{\gamma}} + K_0^{\frac{1}{\gamma}}\right) \tag{72}$$

Note that (71) and $\lambda > \frac{1}{K_\gamma}$ implies that $\frac{d^2\mathbb{E}(U)}{d\alpha^2}|_{(C_1^+,\alpha^+)} < 0$. In addition,

$$\begin{aligned}\frac{1}{\gamma^2}\left[\frac{-\Omega}{M_1(\lambda)+\lambda^{\frac{1}{\gamma}}(1+r_f)}\right]^{2(1+\gamma)} D &= \left[\frac{1}{\lambda^{\frac{1}{\gamma}}} + \frac{1+r_f}{k}\left(\frac{r_g-r_f}{r_g-r_b}\right)^2\left(\lambda^{-\frac{1}{\gamma}} - \frac{r_f-r_b}{r_g-r_f}K_0^{\frac{1}{\gamma}}\right)\right] \\ &\quad \times \frac{(r_f-r_b)^2}{k(1+r_f)}\left(\frac{r_g-r_f}{r_g-r_b}\right)^2\left[\lambda^{-\frac{1}{\gamma}} - K_\gamma^{\frac{1}{\gamma}}\right] \\ &\quad - \left(\frac{r_f-r_b}{k}\right)^2\left(\frac{r_g-r_f}{r_g-r_b}\right)^4\left(\lambda^{-\frac{1}{\gamma}} + K_0^{\frac{1}{\gamma}}\right)^2\end{aligned}$$

where $D = \nabla^2\mathbb{E}(U(C_1,C_2))_{(C_1^+,\alpha^+)} = \frac{d^2\mathbb{E}(U)}{dC_1^2}\frac{d^2\mathbb{E}(U)}{d\alpha^2} - \left(\frac{d^2\mathbb{E}(U)}{dC_1 d\alpha}\right)^2|_{(C_1^+,\alpha^+)}$. Thus,

$$\begin{aligned}\frac{1}{\gamma^2}\left[\frac{-\Omega}{M_1(\lambda)+\lambda^{\frac{1}{\gamma}}(1+r_f)}\right]^{2(1+\gamma)}\left(\frac{r_g-r_b}{r_g-r_f}\right)^2\frac{k}{r_f-r_b}D &= \frac{r_f-r_b}{1+r_f}\left[\lambda^{-\frac{1}{\gamma}} - K_\gamma^{\frac{1}{\gamma}}\right] \\ &\quad - \frac{r_g-r_f}{k}\lambda^{-\frac{1}{\gamma}}K_0^{\frac{1}{\gamma}} < 0\end{aligned}$$

for $\lambda > \frac{1}{K_\gamma}$ which gives that $D = \nabla^2\mathbb{E}(U(C_1,C_2)) = \frac{d^2\mathbb{E}(U)}{dC_1^2}\frac{d^2\mathbb{E}(U)}{d\alpha^2} - \left(\frac{d^2\mathbb{E}(U)}{dC_1 d\alpha}\right)^2 < 0$. Thus, the expected utility is indifferent at (C_1^+,α_1), also for $\lambda \leq \frac{1}{K_\gamma}$, which is the only point satisfying the FOC and thus the maximum will occur at the border.

The feasible solutions at the border for (P6) that come into consideration are given by: (i) $C_{2g} = \bar{C}_2$, (ii) $C_{2b} = \bar{C}_2$, (iii) $C_{2b} = 0$, (iv) $C_1 = \bar{C}_1$ and (v) $C_1 = 0$.

Case (i): $C_{2g} = \bar{C}_2$ when $C_1 = Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f} + \frac{r_g - r_f}{1+r_f}\alpha$ and $0 \leq \alpha \leq \frac{-\Omega}{r_g - r_f}$. It can be seen that

$$(1-\gamma)\mathbb{E}\left(U\left(Y_1+\frac{Y_2-\bar{C}_2}{1+r_f}+\frac{r_g-r_f}{1+r_f}\alpha,\alpha\right)\right) \;=\; -\lambda\left(\frac{-\Omega-(r_g-r_f)\alpha}{1+r_f}\right)^{1-\gamma} -\lambda\delta(1-p)(r_g-r_b)^{1-\gamma}\alpha^{1-\gamma}$$

is a convex function in α and thus its maximum is reached either for $\alpha=0$ or $\alpha=\frac{-\Omega}{r_g-r_f}$. Thus, the potential candidates for maximum in this case are $\left(C_1=Y_1+\frac{Y_2-\bar{C}_2}{1+r_f},\alpha=0\right)$ or $\left(C_1=\bar{C}_1,\alpha=\frac{-\Omega}{r_g-r_f}\right)$. Note that point $\left(C_1=Y_1+\frac{Y_2-\bar{C}_2}{1+r_f},\alpha=0\right)$ is feasible also for (P5) and is also the only feasible solution for case (ii). On the other hand, point $\left(C_1=\bar{C}_1,\alpha=\frac{-\Omega}{r_g-r_f}\right)$ is feasible solution for (P2).

Case (iii). $C_{2b}=0$ when $C_1=Y_1+\frac{Y_2}{1+r_f}-\frac{r_f-r_b}{1+r_f}\alpha$ which is feasible for $\frac{(1+r_f)(Y_1-\bar{C}_1)+Y_2}{r_f-r_b}\le\alpha\le\frac{\bar{C}_2}{r_f-r_b}$. It can be seen that

$$(1-\gamma)\mathbb{E}\left(U\left(Y_1+\frac{Y_2}{1+r_f}-\frac{r_f-r_b}{1+r_f}\alpha,\alpha\right)\right) \;=\; -\lambda\left(\bar{C}_1-Y_1-\frac{Y_2}{1+r_f}+\frac{(r_f-r_b)}{1+r_f}\alpha\right)^{1-\gamma} +\delta p((r_g-r_b)\alpha-\bar{C}_2)^{1-\gamma}-\lambda\delta(1-p)\bar{C}_2^{1-\gamma} \tag{73}$$

The potential maximum of (73) thus can be reached either at the endpoints $\alpha=\frac{(1+r_f)(Y_1-\bar{C}_1)+Y_2}{r_f-r_b}$ and $\alpha=\frac{\bar{C}_2}{r_f-r_b}$ or at the point α^{P6} such that $\left.\frac{d\mathbb{E}\left(U\left(Y_1+\frac{Y_2}{1+r_f}-\frac{r_f-r_b}{1+r_f}\alpha,\alpha\right)\right)}{d\alpha}\right|_{\alpha^{P6}}=0$. Simple derivation gives

$$\alpha^{P6}\equiv\frac{\lambda^{1/\gamma}-\frac{k_2}{1+r_f}\frac{\bar{C}_2^{P2}}{\bar{C}_2}}{\lambda^{1/\gamma}-\frac{k_2}{1+r_f}}\frac{\bar{C}_2}{r_g-r_b} \tag{74}$$

which for $\bar{C}_2<\bar{C}_2^{P2}$ is below $\frac{(1+r_f)(Y_1-\bar{C}_1)+Y_2}{r_f-r_b}$ and thus infeasible. This implies then that for $\bar{C}_2<\bar{C}_2^{P2}$ the maximum of (73) can be reached only at the end points, namely $\left(C_1=\bar{C}_1,\alpha=\frac{(1+r_f)(Y_1-\bar{C}_1)+Y_2}{r_f-r_b}\right)$ or $\left(C_1=Y_1+\frac{Y_2-\bar{C}_2}{1+r_f},\alpha=\frac{\bar{C}_2}{r_f-r_b}\right)$ where the former is feasible also for (P2) and the latter for (P6) which will be dealt with later (in case when $C_1\le Y_1+\frac{Y_2-\bar{C}_2}{1+r_f}$).

In case (iv) any feasible solution is also feasible for (P2). There is no feasible solution in case (v).

Let $C_1\le Y_1+\frac{Y_2-\bar{C}_2}{1+r_f}$. Note that the utility of (P6) is a decreasing function in α for any fixed

C_1

$$\begin{aligned}\frac{d\mathbb{E}(U)}{d\alpha} &= \delta p\left[(1+r_f)(Y_1-C_1)+Y_2-\bar{C}_2+(r_g-r_f)\alpha\right]^{-\gamma}(r_g-r_f)\\ &\quad -\lambda\delta(1-p)\left[(1+r_f)(C_1-Y_1)+\bar{C}_2-Y_2+(r_f-r_b)\alpha\right]^{-\gamma}(r_f-r_b)<0\end{aligned}$$

if

$$\lambda > \frac{p(r_g-r_f)}{(1-p)(r_f-r_b)}\left[\frac{(1+r_f)(C_1-Y_1)+\bar{C}_2-Y_2+(r_f-r_b)\alpha}{(1+r_f)(Y_1-C_1)+Y_2-\bar{C}_2+(r_g-r_f)\alpha}\right]^{\gamma}$$

The latter is achieved if

$$\frac{1}{K_\gamma} \geq \frac{p(r_g-r_f)}{(1-p)(r_f-r_b)}\left[\frac{(1+r_f)(C_1-Y_1)+\bar{C}_2-Y_2+(r_f-r_b)\alpha}{(1+r_f)(Y_1-C_1)+Y_2-\bar{C}_2+(r_g-r_f)\alpha}\right]^{\gamma}$$

as it is assumed that $\lambda > \frac{1}{K_\gamma}$ where K_γ is given by (7). It can be shown that the above inequality holds if $C_1 \leq Y_1 + \frac{Y_2-\bar{C}_2}{1+r}$ which is our assumption. In more detail, the set of feasible solutions for (P6) can be written as

$$\begin{array}{rcl} 0 & \leq C_1 & \leq Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f} \\ Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f} - \frac{r_f-r_b}{1+r_f}\,\alpha & \leq C_1 & \leq Y_1 + \frac{Y_2}{1+r_f} - \frac{r_f-r_b}{1+r_f}\,\alpha \\ 0 & \leq \alpha & \leq \frac{(1+r_f)Y_1+Y_2}{r_f-r_b} \end{array}$$

Let $\tilde{C}_1$ be fixed and such that $0 \leq \tilde{C}_1 \leq Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}$. Based on the first inequality in the second row of the inequalities above and the fact that the utility of (P6) is decreasing in α it follows that the smallest possible $\tilde{\alpha}$ such that the feasible set is satisfied for $C_1 = \tilde{C}_1$ is given by

$$Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f} - \frac{r_f-r_b}{1+r_f}\,\tilde{\alpha} = \tilde{C}_1$$

and thus

$$\tilde{\alpha} = \frac{(1+r_f)(Y_1-\tilde{C}_1)+Y_2-\bar{C}_2}{r_f-r_b} \in \left[0, \frac{(1+r_f)Y_1+Y_2}{r_f-r_b}\right]$$

Note that $(\tilde{C}_1, \tilde{\alpha})$ completes $\tilde{C}_1 \leq Y_1 + \frac{Y_2}{1+r_f} - \frac{r_f-r_b}{1+r_f}\tilde{\alpha} = \tilde{C}_1 + \frac{\bar{C}_2}{1+r_f}$ as $\bar{C}_2 \geq 0$. Thus, for any given $\tilde{C}_1$ that satisfies $0 \leq \tilde{C}_1 \leq Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}$ is the point $\left(\tilde{C}_1, \frac{(1+r_f)(Y_1-\tilde{C}_1)+Y_2-\bar{C}_2}{r_f-r_b}\right)$ where the utility of (P6) achieves its maxima. As point $(\tilde{C}_1, \tilde{\alpha})$ is feasible also for (P5) and as utilities of (P5) and (P6) coincide at this point then the utility function of (P5) at its maximum is bigger or equal to the utility function of (P6) at any point $(\tilde{C}_1, \tilde{\alpha})$.

Problem (P7). The only feasible solution is $\left(Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}, 0\right)$ which is feasible also for (P5).

Problem (P8). Note that the only feasible solution for case when $C_1 \leq Y_1 + \frac{Y_2-\bar{C}_2}{1+r_f}$ is

$\left(C_1 = Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}, \alpha = 0\right)$. Let $C_1 \geq Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}$. As $\frac{d^2\mathbb{E}(U(C_1,\alpha))}{d\alpha^2} > 0$ then no local extreme can be a local maximum. Thus, an maximum will occur at the border of the set of feasible solutions. The feasible solutions at the border that come into considerations are: (i) $C_{2g} = \bar{C}_2$, (ii) $C_{2b} = \bar{C}_2$, (iii) $C_{2g} = 0$, (iv) $C_{2b} = 0$, (v) $C_1 = \bar{C}_1$ and (vi) $C_1 = 0$. Note that case (i) was already dealt with in case (i) of problem (P6) and the only feasible solution in case (ii) is $\left(Y_1 + \frac{Y_2 - \bar{C}_2}{1+r_f}, 0\right)$ which is feasible also for (P5). In addition, there are no feasible solutions in cases (iii) and (vi) and neither in case (iv) for $\bar{C}_2 < \bar{C}_2^{P2}$. Finally, if $C_1 = \bar{C}$, case (v), then any feasible solution will be feasible also for (P4). ■