THE NUEL

by

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I. INTRODUCTION

Does the Fittest Necessarily Survive ?

The startling outcome of a three-person shooting match gave rise to this paper. In their hitherto unpublished manuscript "Competition, Welfare and the Theory of Games" Shapley and Shubik (1) showed that a contestant with a shooting accuracy superior to those of his two opponents may end up with the smallest probability to survive a shooting match ("truel") is carried out under the following rules :

Contestants A, B, and C fire at each other's balloons with pistols, from fixed positions. At the beginning, and after each shot, the players with unbroken balloons decide by lot who is to shoot next. The surviving balloon determines the winner.

The player who is to shoot first faces the problem at which of his opponents he should fire. He knows that as soon as one player has been eliminated the game becomes mechanical and a direct calculation can be derived.

Let us suppose C has been eliminated, A and B are the remaining players with shooting accuracies a and b respectively (a and b in the closed interval $[0, 1]$). The next shot will result in one of the following : an immediate win for A (probability a/2), an immediate win for B (b/2) or a repetition of the status quo. Writing $P_{A,B}$ for the probability that A will win against B alone, we have

$$P_{A,B} = \frac{a}{2} - (1 - \frac{a}{2} - \frac{b}{2}) \cdot P_{A,B}$$

[+] We wish to thank Gerhard Schwödiauer for valuable comments.

Solving the equation gives $P_{A,B} = \frac{a}{a+b}$. Similarily, $P_{A,C} = \frac{a}{a+c}$, and likewise for the other four possibilities.

Now suppose that the players are ranked in skill by $a > b > c$. Then it is clear that A will aim at B if he has the first shot. His chances of an immediate hit are the same but he would definitely prefer to shoot it out later with C, since $P_{A,C} > P_{A,B}$. Similarily, B and C both aim at A's balloon.

Thus, the strategic question is settled, a "rationality" rule was derived: Shoot at the stronger opponent first !

We can proceed now to calculate the surviving probabilities for the players.

A can only win if he shoots B and C himself.

B will either win i) by shooting A and C himself, or

ii) by having C shot A at first, and hitting C himself afterwards.

C will either win i) by shooting A and B himself, or

ii) by having B shot A at first, and hitting B himself afterwards, or

iii) by having A shot B at first, and hitting A himself afterwards.

Writing $P_A^{(3)}$ for the probability that A will survive this three-person game we obtain

$$P_A^{(3)} = (\tfrac{a}{3}) P_{A,C} + (1 - \tfrac{a}{3} - \tfrac{b}{3} - \tfrac{c}{3}) P_A$$

which can be simplified to

$$P_A^{(3)} = \frac{a^2}{(a+b+c)(a+c)} \qquad (1)$$

The corresponding expressions for the other contestants are

$$P_B^{(3)} = \frac{b}{a+b+c} \qquad (2)$$

$$P_C^{(3)} = \frac{c(2a+c)}{(a+b+c)(a+c)} \qquad (3)$$

If we assign shooting accuracies $a = .8$, $b = .6$, and $c = .4$ to players A, B, and C respectively a paradox "solution" arises :

$$P_A^{(3)} = .296, \quad P_B^{(3)} = .333, \quad P_C^{(3)} = .370. \tag{3a}$$

The order of strength has been reversed. The poorest shot has the best chance to survive.

A number of arguments can be found in critique of the solution. The "rational" strategy was evidently foolish for A and B. While they cut each other's throat, C has the laugh on his side. They can improve their surviving probabilities strikingly by "irrationally" shooting down C's balloon first. A is in a relatively unfavourable position. While he has to do all the work alone, both B and C have helpers that do part of the work out of their own impetus. A could, however, employ a strategy of deterrence by openly committing himself to shoot at C's balloon if C should shoot at him (and misses).

The paradoxy of the solution was one starting point for this study. Even if the striking outcome is due to the numerical values, i.e. the relative shooting accuracies in the example, the "trend" in favouring the weakest contestant cannot be overlooked.

An example may serve as illustration:

$a = .9$	$P_A^{(3)} = .506$	$a = .9$	$P_A^{(3)} = .375$
$b = .6$	$P_B^{(3)} = .375$	$b = .6$	$P_B^{(3)} = .333$
$c = .1$	$P_C^{(3)} = .119$	$c = .3$	$P_C^{(3)} = .292$
$a = .9$	$P_A^{(3)} = .328$	$a = .9$	$P_A^{(3)} = .289$
$b = .6$	$P_B^{(3)} = .316$	$b = .6$	$P_B^{(3)} = .300$
$c = .4$	$P_C^{(3)} = .356$	$c = .5$	$P_C^{(3)} = .411$

Given the "rational" strategy to shoot at the strongest opponent, it is clear that the weakest shot may have the best chance to survive no matter how many contestants there are in a game. In an attempt to generalize the shooting match by introducing n players we will develop formulae for the "nuel" to quantify the surviving probability of the player i ($i = 1, 2, \ldots n$).

Another object of this paper is the calculation of the winning probabilities if the formation of coalitions is permitted. The game will be treated in characteristic function form making use of the partion function form by Thrall and Lucas (Ref. 2) as generalization to the von Neumann - Morgenstern theory of games in characteristic function form (Ref. 3). In order to make a more complete comparison the notion of a transitory coalition will be introduced as opposed to the permanent alliance.

"VIELE HUNDE SIND DES HASEN TOD"
(Austrian Proverb)
Many dogs are the rabbit's death.

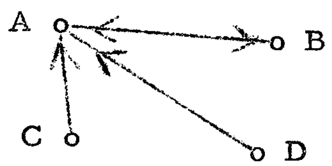
II. "VIELE HASEN SIND DES HUNDES TOD"

II. 1.

When we introduce a fourth player D it is no more possible that the contestants fire at each other from fixed positions. In the truel we can imagine the positions of the opponents at the corners of an equilateral triangle. As soon as $n > 3$ we abstract from the fact that the distances between the players are not equal and assume that distance has no effect on the shooting accuracies. Alternatively, the rules of the game could be reconstructed to ensure the equality of the distances.

If the players are ranked in skill, i.e. $a > b > c > d$, again the "rationality" rule holds. A, for instance, will prefer to shoot at B first. His chance of an immediate hit against B, C or D is always $\frac{a}{4}$, but the probability to win against C and D is superior to the winning probability against any of the other two combinations. Similarily, B will shoot at A, C first at A, then at B, while D will only shoot at C after A and B have been eliminated.

First shoot



Thus, A can only win if he shoots B, C, and D himself.

B will either win
i) by shooting A, C, and D himself, or
ii) by having C shot A at first, and eliminating C and D afterwards, or
iii) by having D shot A at first, and hitting C and D afterwards.

C will either win
i) by shooting A, B, and D himself, or
ii) by having B shot A, later on eliminating B and D himself, or
iii) by having D shot A, and eliminating B and D himself afterwards, or
iv) by having A shot B, shooting A and D himself.

Similarily,

D will win i) by eliminating A, B, and C himself, or

ii) if B shoots A, and he himself shoots B and C, or

iii) if C eliminates A, and B and C will be shot by him, or

iv) if A shoots B, and he himself wins against A and C.

With this background the derivations of the surviving probabilities for A, B, C, and D are very easy to obtain.

$$P_A^{(4)} = \frac{a}{4} P_{A,CD} + \left(1 - \frac{a}{4} - \frac{b}{4} - \frac{c}{4} - \frac{d}{4}\right) P_A^{(4)}$$

where $P_{A,CD} = P_A^{(3)} = \dfrac{a^2}{(a + c + d)(a + d)}$

Therefore,

$$P_A^{(4)} = \frac{a^3}{(a+b+c+d)(a+c+d)(a+d)}$$

$$P_B^{(4)} = \frac{b}{4} P_{B,CD} + \frac{c}{4} P_{B,CD} + \frac{d}{4} P_{B,CD} + \left(1 - \frac{a}{4} - \frac{b}{4} - \frac{c}{4} - \frac{d}{4}\right) P_B^{(4)}$$

As $P_{B,CD} = \dfrac{b^2}{(b+c+d)\,(b+d)}$, we get

$$P_B^{(4)} = \frac{b^2}{(a+b+c+d)(b+d)}$$

In a similar manner $P_C^{(4)}$ and $P_D^{(4)}$ are calculated.

$$P_C^{(4)} = \frac{c(2a+c+d)}{(a+b+c+d)\,(a+c+d)}$$

$$P_D^{(4)} = \frac{d(2b+d)(a+c+d)(a+d) + ad(2a+d)(b+d)}{(a+b+c+d)(a+c+d)(a+d)(b+d)}$$

These four probabilities add up to 1 as required.

II. 2

Proceeding from the findings of the four-person game, the existence of a "formation law" seemed to be indicated, we developed the formulae for the five-person game (by introducing player E with a shooting accuracy $e < d$) and the six-person game (player F with accuracy $f < e$).

Table 1 shows a synopsis of the surviving probabilities for the players A, B, C and D in the three- to six-person game.

Table 1: SURVIVING PROBABILITIES FOR PLAYERS A, B, C, D WHEN n = 3, 4, 5, 6

n = 3

$$P_A^{(3)} = \frac{a^2}{(a+b+c)\,(a+c)}$$

$$P_B^{(3)} = \frac{b}{(a+b+c)}$$

$$P_C^{(3)} = \frac{c(2a+c)}{(a+b+c)(a+c)}$$

n = 4

$$P_A^{(4)} = \frac{a^3}{(a+b+c+d)(a+c+d)(a+d)}$$

$$P_B^{(4)} = \frac{b^2}{(a+b+c+d)(b+d)}$$

$$P_C^{(4)} = \frac{c(2a+c+d)}{(a+b+c+d)(a+c+d)}$$

$$P_D^{(4)} = \frac{d(2b+d)(a+c+d)(a+d) + ad(2a+d)(b+d)}{(a+b+c+d)(b+d)(a+d)(a+c+d)}$$

n = 5

$$P_A^{(5)} = \frac{a^4}{(a+b+c+d+e)(a+b+d+e)(a+d+e)(a+e)}$$

$$P_B^{(5)} = \frac{b^3}{(a+b+c+d+e)(b+d+e)(b+e)}$$

$$P_C^{(5)} = \frac{c^2\,(2a+c+d+e)}{(a+b+c+d+e)(a+c+d+e)(c+e)}$$

$$P_D^{(5)} = \frac{d(2b+d+e)(a+c+d+e)(a+d+e) + ad(2a+d+e)(b+d+e)}{(a+b+c+d+e)(a+c+d+e)(b+d+e)(a+d+e)}$$

n = 6

$$P_A^{(6)} = \frac{a^5}{(a+\ldots+f)(a+c+d+e+f)(a+d+e+f)(a+e+f)(a+f)}$$

$$P_B^{(6)} = \frac{b^4}{(a+\ldots+f)(b+d+e+f)(b+e+f)(b+f)}$$

$$P_C^{(6)} = \frac{c^3\,(2a+c+d+e+f)}{(a+\ldots+f)(a+c+d+e+f)(c+e+f)(c+f)}$$

$$P_D^{(6)} = \frac{d^2\,(2b+d+e+f)(a+c+d+e+f)(a+d+e+f)}{(a+\ldots+f)(a+c+d+e+f)(a+d+e+f)(b+d+e+f)(d+f)} + \frac{ad^2\,(2a+d+e+f)(b+d+e+f)}{\ldots\ldots\ldots\ldots}$$

It can easily be checked that the developments of the formulae of the, say, 5-person game made use of the results of the preceding step (4-person game). This is the reason why the expressions get lengthier from step to step. In addition the combinatorics increases as the number of players does.

The existence of a "formation law" can indeed be derived from the obtained results when one additional player after the other is introduced. It will be verified later on, but let us first redefine the probabilities.

Let P_k (1, 2,... k, ... n) be the probability that player k survives the Nuel. Thus, P_3 (2, 3, ... n) is the probability that player 3 survives the shooting match consisting of the players 2, 3, 4, 5, ... n.

In the Nuel, players 1, 2, 3, ... n are competing against each other. They are ranked according to their shooting accuracies α_j, j = 1, 2, ... n where $\alpha_1 > \alpha_2 > \alpha_3 > \ldots\ldots > \alpha_n$.

It should be noted that a clear distinction between subjective and objective shooting accuracies must be made.

The objective shooting accuracies are statistical shooting results.

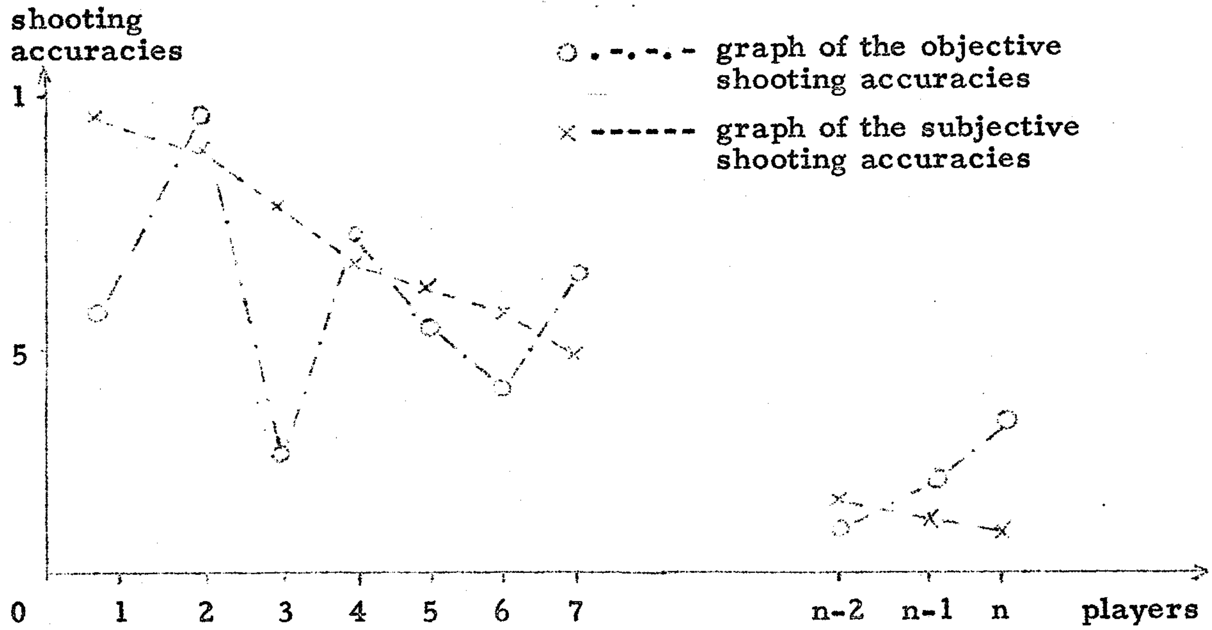
The subjective shooting accuracies are the result of the unanimous ordering of the supposed strength of each player. They can be represented by a strictly decreasing monotonous function, i.e. a mapping one-one, of any subset N of the set of positive integers into the closed interval $[0, 1]$.

For the objective shooting accuracies only a function needs to be assumed. The requirement of decreasing monotony can be dropped.

This means, for instance, that a contestant is aimed at the first shot even though his objective shooting accuracy is not necessarily the highest. He could, theoretically, be the lousiest shot (but the best bluff) in the group.

Table 2 provides an example how the functions for subjective and objective shooting accuracies could be different.

Table 2



The "formation laws" can now be formulated.

For k = 1, we have:

$$P_1(1,\ldots,n) = \frac{a_1}{n} P_1(1,3,4,\ldots,n) + \Big(1 - \sum_{j=1}^{n} \frac{a_j}{n}\Big) P_1(1,\ldots,n)$$

For k = 2:

$$P_2(1,\ldots,n) = \Big(\frac{1}{n} \sum_{j=2}^{n} a_j\Big) . P_2(2,3,\ldots n) + \Big(1 - \frac{1}{n} \sum_{j=1}^{n} a_j\Big) P_2(1,\ldots,n)$$

For k = 3:

$$P_k(1,\ldots,n) = \Big(\frac{1}{n} \sum_{j=2}^{n} a_j\Big) P_k(2,3,\ldots n) + \frac{a_1}{n} P_k(1,3,\ldots,n) +$$

$$+ \Big(1 - \frac{1}{n} \sum_{j=1}^{n} a_j\Big) P_k(1,\ldots,n)$$

From this follow the recursion formulae:

$$P_1(1,\ldots,n) = \frac{1}{\sum_{j=1}^{n} a_j} a_1 P_1(1,3,4,\ldots,n) \qquad (4)$$

$$P_2(1,\ldots n) = \frac{1}{\sum_{j=1}^{n} a_j} \sum_{j=2}^{n} a_j \, P_2(2,3,\ldots,n) \tag{5}$$

and for k - 3

$$P_k(1,\ldots,n) = \frac{1}{\sum_{j=1}^{n} a_j} \sum_{j=2}^{n} a_j \, P_k(2,3,\ldots,n) + a_1 P_k(1,3,4,\ldots,n) \tag{6}$$

These probabilities must add up to 1.

$$\sum_{j=1}^{n} P_j(1,\ldots,n) = \frac{1}{\sum_{j=1}^{n} a_j} \Big[\alpha_1 \overbrace{\sum_{\substack{j=1\\ j\neq 2}}^{n} P_j(1,3,4,\ldots,n)}^{=1} + \sum_{j=2}^{n} a_j \underbrace{\sum_{j=2}^{n} P_j(2,\ldots,n)}_{=1}$$

Since the expressions in the winged brackets are each equal to 1 we obtain

$$\sum_{j=1}^{n} P_j(1,\ldots,n) = \frac{1}{\sum_{j=1}^{n} a_j} (a_1 + \sum_{j=2}^{n} a_j) = 1 \qquad \text{q.e.d.}$$

By means of iterative inserting formulae (1), (2) and (3) can be restated in a form that discloses immediate quantitative values.

$$P_1(1,\ldots n) = \frac{a_1^{n-1}}{\prod_{l=2}^{n} (a_1 + \sum_{j=1}^{n} a_j)} \tag{7}$$

$$P_2(1,\ldots,n) = \frac{a_2^{n-2}}{\sum_{j=1}^{n} a_j \prod_{l=4}^{n} (a_2 + \sum_{j=2}^{n} a_j)} \tag{8}$$

For k= 3 the surviving probability is

$$P_k(1,\ldots,n) =$$

$$= \frac{1}{\sum_{j=1}^{n} a_j} \left[\frac{a_3^{n-3}}{\prod_{l=5}^{n} (a_3 + \sum_{j=1}^{n} a_j)} + \frac{a_1 a_3^{n-3}}{(a_1 + \sum_{j=3}^{n} a_j) \prod_{l=5}^{n} (a_3 + \sum_{j=2}^{n} a_j)} \right] \tag{9}$$

Please note that (6) is the corresponding formula for (3) if $k = 3$. While formula ("formation law") (3) is valid for $k \geqq 3$ in the Nuel, formulas of type (4) to (6) for $k > 3$ are lengthy expressions. As n gets larger they are increasingly cumbersome to obtain.

It should be observed that in (1) to (3) the surviving probabilities of the player's i ($i = 1, 2, \ldots, n$) are expressed in terms of surviving probabilities <u>and</u> shooting accuracies while in (4) to (6) the surviving probabilities for the contestants 1, 2 and 3 are expressions in terms of shooting accuracies only.

The validity of formula (7) will be proved by complete induction: Suppose (7) has been proved already. If we can also prove that the corresponding formula

$$P_1(1, 2, \ldots, n) = \frac{a_1^n}{\prod_{1=2}^{n+1} (a_1 + \sum_{1=j}^{n+1} a_j)}$$

holds for the shooting match with $n+1$ participants, the "formation law" that has led to (7) is valid and likewise (7).

$$P_1(1, 2, \ldots, n+1) = \frac{a_1}{n+1} P_1(1, 3, 4, \ldots, n+1) + (1 - \frac{a_1 + a_2 + \ldots + a_{n+1}}{n+1}) P_1(1, \ldots n+1)$$

But $$P_1(1, 3, \ldots, n+1) = \frac{a_1^{n-1}}{\prod_{1=3}^{n+1} (a_1 + \sum_{1=j}^{n+1} a_j)}$$

Therefore $$P_1(1, 2, \ldots n+1) = \frac{a_1^n}{\sum_{j=1}^{n+1} a_j \prod_{1=3}^{n+1} (a_1 + \sum_{1=j}^{n+1} a_j)}$$

Since we can write $$\sum_{j=1}^{n+1} a_j = a_1 + \sum_{j=2}^{n+1} a_j$$

we obtain $$P_1 (1, 2, \ldots n+1) = \frac{a_1^n}{\prod_{1=2}^{n+1} a_1 + \sum_{1=j}^{n+1} a_j)}$$ q. e. d.

"But many that are first
shall be last
and the last shall be first."
Matthew 19.30

II. 3. A special solution

If $a_k = p \;\; \forall k = 1, 2, \ldots n$ an interesting result arises.
It is of importance, of course, that the opponents maintain the ranking of the players according to their supposed skills (subjective shooting accuracies).

A little pencil work reveals (using (4), (5) and (6))

$$P_1(1, 2, \ldots, n) = \frac{1}{n!} \tag{10}$$

$$P_k(1, 2, \ldots, n) = \frac{n-k+1}{(n-k+2)!} \qquad \text{for } k \geqq 2 \tag{11}$$

Given the special circumstance the surviving probabilities of all shots depend only on the number of contestants in the shooting match irrespective of their own accuracies.

In order to varify (10) and (11) for $2 \leq k \leq n$ we must show that

$$\sum_{k=1}^{n} P_k(1, 2, \ldots, n) = 1$$

Thus, it is to be proved that

$$\frac{1}{n!} + \sum_{k=2}^{n} \frac{n-k+1}{(n-k+2)!} = 1 \tag{12}$$

The proof will be given by the device of complete induction. At first, however, (12) will be reformulated by multiplying both sides by n ! and setting $j = n-k+2$ to give

$$1 + \sum_{j=2}^{n} n! \frac{j-1}{j!} = n! \tag{13}$$

This expression holds for $n = 1$. We make the assumption (13) has been proved for n. If it can be shown that the expression holds also for n + 1, formula (13) holds for every n.
Therefore

$$1 + \sum_{j=2}^{n+1} (n+1)!\ \frac{j-1}{j!} \stackrel{?}{=} (n+1)!$$

$$1 + (n+1)(n!-1) + (n!+1)\frac{n}{(n+1)!} = (n+1)!$$

$$1 + (n+1)! - (n+1) + n = (n+1)!$$

$$(n+1)! = (n+1)! \qquad \text{q.e.d.}$$

A quantitative conception should be transmitted through the following example where $n = 5$ and $a_i = p$, $\forall\, i$, $i = 1, 2, \ldots, 5$.

For comparison another five - person game is treated, this time with shooting accuracies a = .6, b = .5, c = .4, d = .3 and e = .2 for the opponents A, B, C, D and E respectively (see table 3).

Table 3

Player	a_i	$P^{(5)}_{a_i}$	a_i	$P^{(5)}_{a_i}$
A	p	$\frac{1}{120}$ = .008	.6	.049
B	p	$\frac{1}{30}$ = .033	.5	.089
C	p	$\frac{1}{8}$ = .125	.4	.187
D	p	$\frac{1}{3}$ = .333	.3	.287
E	p	$\frac{1}{2}$ = .500	.2	.388

It immediately follows that the smaller the difference of the objective shooting accuracies between the individually best and weakest shot the better (worse) is the winning probability for player n (for player A).

III. CHARACTERISTIC FUNCTION VS. PARTITION FUNCTION

The solution of the competitive Nuel (when everybody is playing against everybody) for n ≧ 3 implies a frustrating outcome for the contestants with the subjective highest shooting accuracies. They will question the rationality of the "rationality" rule and will deliberate whether the formation of a coalition might not more advantageous than a cutthroat competition even though they do not have the satisfaction to eliminate excluding themselves, at least temporarily.

This requires that we allow the players to know the payoffs of the game, i.e. we treat a game with complete information (Ref. 4). It is further implied that the players are allowed to form coalitions. The rules of the game will be modified accordingly. While the members of the coalition ar prohibited to shoot at each other, they will still find it an optimal strategy to aim at the strongest opponent outside the coalition. Likewise, the contestants outside the coalition will consider it advantageous to shoot at the strongest opponent whether he is in a coalition with another player or not.

By this last sentence we have touched on the generalization of the von Neumann-Morgenstern theory of games in characteristic function form by Thrall and Lucas (Ref. 2). The von Neumann-Morgenstern real valued characteristic function of n-person games (Ref. 3) is defined on the set of all subsets of the set $N = \{1, 2, \ldots, n\}$ of players 1, 2, ..., n. The partition function (Ref. 2) is defined on the set of all partitions of the set of players. This formulation assigns a real numbered outcome to each coalition (coset) in each partition of the set of players.

In addition, the theory of games in partition function form also avoids some of the restrictive assumptions made in the von Neumann-Morgenstern theory. The characteristic function in the classical theory is defined only in terms of coalitions and their complements, whereas the partition function theory allowes the complement to split into coalitions in an arbitrary manner. Another advantage which, however, is not relevant to our game of survival is given inasmuch as the requirement of superadditivity as in the classical case can be dropped. The partition theory also covers the case where certain partitions of N are not allowed to form.

The full scope of the theory of n-person cooperative games in terms of a partition function cannot be exploited in the game of survival, as (Ref. 2) was constructed for the side-payment case. In our game, however, the notion of side payment is not applicable as probabilities are our payoffs. And probabilities are non-transferable utilities.[1)]

The solution concept, therefore, implies the search for the set of imputations with the desired properties of external and internal stability. We will return to this problem lateron after we have introduced the payoffs for the partitions of the three- and four-person game, and all coalitions in the von Neumann-Morgenstern n-person game.

It is evident that games without transferable utilities can only properly be represented by a generalized characteristic function, i.e. by means of payoff vectors. Table 4, however, indicates how the truel can be formalized in characteristic function form, where v denotes the characteristic function (Ref. 1). It is assumed that the players A, B, C have accuracies of a = .8, b = .6 and c = .4 respectively and are competing for a prize of $ 27,--. S denotes the subsets of the set of players.

The essentiality of the game, the property that it is constant-sum is pointed out by this example as well as the superadditivity of the function v.

Table 4

CHARACTERISTIC FUNCTION OF THE TRUEL

S	v (S)
A	$ 8.00
B	5.40
C	2.40
A, B	24.60
A, C	21.60
B, C	19.00
A, B, C	27.00

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1) In another paper we will attempt to analyze the economic implications of the nuel where $n \geq 3$ and side-payments are allowed. A player, then, may survive a game, even if he is "out".

Table 5

THE TRUEL IN 0, 1 NORMALIZATION

S	v (S)
A	$ 0.00
B	0.00
C	0.00
A, B	1.00
A, C	1.00
B, C	1.00
A, B, C	1.00

Another feature of the example is its hidden symmetry. This can be shown by redefining the levels for the single coalitions and the full coalition in the normalized characteristic function (Table 5).
"We see that the worth of any coalition now depends only on its size. In other words, the three players are on equal footing in negotiating for partners." (Ref. 1)

III. 1. The payoffs in the three-person game

In the truel the von Neumann-Morgenstern characteristic function coincides with the partition of Thrall and Lucas with the exception of the case where three single coalitions play against each other. The characteristic function is not defined for this event whereas it is covered by the partition function.

We have the following partitions:

$$p^1 = \{\{A\}, \{B\}, \{C\}\}$$
$$p^2 = \{\{AB\}, \{C\}\}$$
$$p^3 = \{\{AC\}, \{B\}\}$$
$$p^4 = \{\{BC\}, \{A\}\}$$
$$p^5 = \{\{ABC\}\}$$

The payoffs (surviving probabilities) for p^1 were given above ((1), (2), (3)). For p^5 the payoff is, of course, 1 for each of the players.

The payoffs for p^2, p^3 and p^4 are not difficult to calculate:

p^2 :

$$P_{AB}^{(3)} = \frac{a+b}{a+b+c}$$

$$P_{B}^{(3)} = \frac{bc}{(b+c)(a+b+c)}$$

$$P_{C}^{(3)} = \frac{c^2}{(b+c)(a+b+c)}$$

where $P_{AB}^{(3)}$ is the probability that A and B survive the game. $P_{B}^{(3)}$ is the probability that only B survives. (A cannot win alone according to the "rationality" rule.) Thus, A's winning probability is $P_{AB}^{(3)}$. B's surviving chance is $P_{AB}^{(3)} + P_{B}^{(3)}$. $P_{C}^{(3)}$ is C's probability to win.

The corresponding formulae for p^3 and p^4 are:

p^3:

$$P_{AC}^{(3)} = \frac{a+c}{a+b+c}$$

$$P_{C}^{(3)} = \frac{bc}{(b+c)(a+b+c)}$$

$$P_{B}^{(3)} = \frac{b^2}{(b+c)(a+b+c)}$$

p^4:

$$P_{BC}^{(3)} = \frac{b+c}{a+b+c}$$

$$P_{C}^{(3)} = \frac{ac}{(a+c)(a+b+c)}$$

$$P_{A}^{(3)} = \frac{a^2}{(a+c)(a+b+c)}$$

The sum of the probabilities in p^2 to p^4 is 1 as required.
They have served, of course, as basis for the calculation of the characteristic function in table 4.

III. 2. The partitions of the four-person game

By adding a single player the partition function explodes to cover 15 partitions. Partition 2, for instance, is already a case not included in the von Neumann-Morgenstern development of the characteristic function. We write in an abbreviated notation:

p^1 = A, B, C, D	p^8 = AB, CD
p^2 = A, B, CD	p^9 = AC, BD
p^3 = A, C, BD	p^{10} = AD, BC
p^4 = A, D, BC	p^{11} = ABC, D
p^5 = B, C, AD	p^{12} = ABD, C
p^6 = B, D, AC	p^{13} = ACD, B
p^7 = C, D, AB	p^{14} = BCD, A

p^{15} = ABCD

We will give the payoffs for the partitions p^8 and p^7 by means of recursive formulae where, for instance, P_A (AB, D) is A's probability to survive when his coalition with B is opposed by D.

p^8:

$$P_A^{(4)} = \frac{(a+b)\, P_A(AB,\ D)}{a+b+c+d}$$

$$P_B^{(4)} = \frac{(a+b)\, P_B(AB,\ D) + (c+d)\, P_B(B,\ CD)}{a+b+c+d}$$

$$P_C^{(4)} = \frac{(c+d)\, P_C(CD,\ D)}{a+b+c+d}$$

$$P_D^{(4)} = \frac{(a+b)\, P_D(AB,\ D) + (c+d)\, P_D(CD,\ B)}{a+b+c+d}$$

p^7:

$$P_A^{(4)} = \frac{(a+b)\, P_A(AB,\ D)}{a+b+c+d}$$

$$P_B^{(4)} = \frac{(a+b)\, P_B(AB,\ D) + (c+d)\, P_B(B,\ C,\ D)}{a+b+c+d}$$

$$P_C^{(4)} = \frac{(c+d)\ P_C(B,\ C,\ D)}{a+b+c+d}$$

$$P_D^{(4)} = \frac{(a+b)\ P_D(AB,\ D) + (c+d)\ P_D(B,\ C,\ D)}{a+b+c+d}$$

It can easily be shown that $P_A^{(4)} + P_B^{(4)} + P_C^{(4)} + P_D^{(4)} = 1$ in both cases.

III. 3. The partition of the n-person game

It is one of the hitherto unsolved problems in combinatorics to enumerate the number of all possible partitions of the n-person game in a closed form. A solution by means of the characteristic function is given in Ref. 5

We have developed, however, a recursive formula for every possible partition in the n-person game for n = 2.

Let p again be a partition of the set N of the players $\{1, 2, 3, \ldots, n\}$ N is a totally ordered set. Let C_i be the set of the i-th coalition. There are l coalitions in N, $l \leqq n$. Thus $N = \cup C_i$, $C_i \cap C_j = \emptyset$, $i, j = 1, 2, 3, \ldots, l$.

The partition p is an ordered set of coalition-sets C_i. It is ordered by the strongest member of each coalition, e.g. C_1 is the coalition which includes (1, 1), the strongest member of C_1 and the strongest contestant over all, C_2 is the coalition which includes the strongest contestant of the set $\overline{C}_1 = N - C_1$ etc.

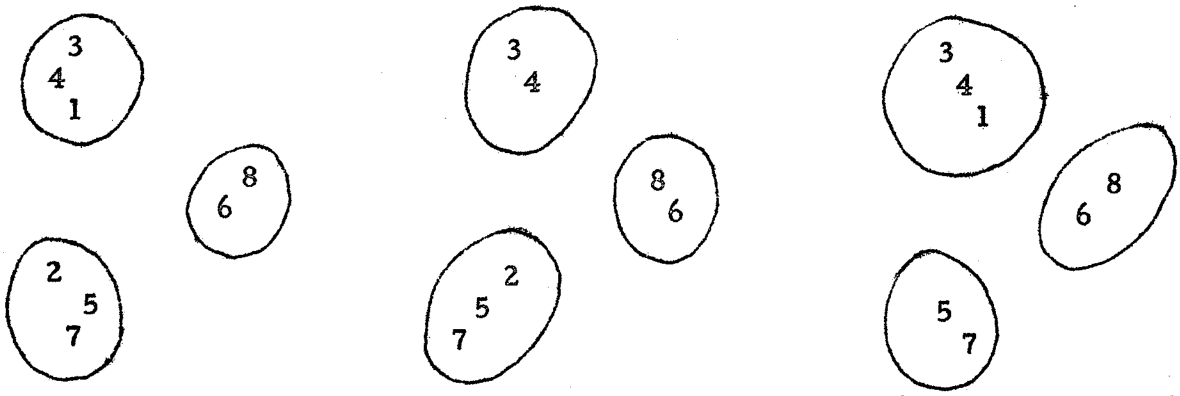
In the following two partitions p' and p'' will be defined when the set of n players is reduced to a set of n-1 players.

a) p': If we drop (1, 1), the strongest member of C_1, we obtain a new set C_i' which is defined by $C_1' = C_1 \quad (1, 1)$. We obtain p' if we replace the set C_1 in p by C_1'.

b) p'': If (2, 1), the strongest member of C_2, is dropped, similarily the outcome is p''.

An example will cast light on the notation.

p : p' : p'' :



A shooting accuracy α_{ij} corresponds to each player i, j of coalition C_i. Each coalition C_i corresponds to a set of shooting accuracies A_i.

$C_1, C_2, C_3, \ldots \quad \ldots C_l$	Coalitions (partition p)
$A_1, A_2, A_3, \ldots \quad \ldots A_l$	Sets of shooting accuracies

$P_{ij}(p)$ shall be the winning probability of player j in coalition C_i. This definition implies that all players in C_i with shooting accuracies inferior to α_{ij} survive too , while all other players have been eliminated.

Since we have given so few definitions in this section, we will add two more.

$$P_k(p) = 0 \text{ if } k \in N, \quad k \in C_i \notin p \tag{14}$$

where p is a partition of a set with less than n members

$$P_{ij}(C_i) = \begin{cases} 1 \text{ if } j = 1 \\ 0 \text{ if } j \geq 2 \end{cases} \tag{15}$$

Now we are in a position to present the formula

$$P_{ij}(p) = \frac{P_{ij}(p')(\sum_{\alpha \in \bar{A}_1} \alpha) + P_{ij}(p'')(\sum_{\alpha \in A_1} \alpha)}{\sum_{\alpha \in \cup A_i} \alpha}$$

where $\bar{A}_1 = \bigcup_{j=1}^{l} A_j - A_1$.

Formula (16) is applicable for both the cooperative and noncooperative games. It, thus, comprises all cases treated above. It can easily be verified that it covers all partitions of an n-person game. The partition consisting of only two coalitions (von Neumann-Morgenstern characteristic function form) are only special cases among all possible partitions.

As an example we shall obtain equivalent results of p^7 in III. 2. by means of formula (16).

Partition p^7 consists of following coalitions :

$$C_1 = \{1, 2\}$$
$$C_2 = \{3\}$$
$$C_3 = \{4\}$$

The following matrix sybolizes which players in which coalitions survive where a "1" indicates survival and a "0" indicates death.
Therefore in P_1 ($\{1, 2\}$, $\{3\}$, $\{4\}$) players 1 and 2 survive,
P_2 ($\{1\!\!\!/, 2\}$, $\{3\}$, $\{4\}$) is the case when 1 is out and only 2 survives.
The total probability of survival is obtained as the sum of the probabilities indicated by "1"s in the column of player i. In our case the sum $P_1 + P_2$ is the total probability of survival of player 2.

Table 6

Probability	Player			
	1	2	3	4
$P_1(p)$	1	1	0	0
$P_2(p)$	0	1	0	0
$P_3(p)$	0	0	1	0
$P_4(p)$	0	0	0	1

Player 1 :

Step 1 : $$P_{11}(12,3,4) = \frac{(a_3+a_4)\ P_{11}(2,3,4) + a_1+a_2)\ P_{11}(12,4)}{a_1+a_2+a_3+a_4}$$

$$P_{11}(12,3,4) = \frac{(a_1+a_2)\ P_{11}(12,4)}{a_1+a_2+a_3+a_4}$$

Step 2 : $P_{11}(2,3,4) = 0$ because of formula (14)

$$P_{11}(12,4) = \frac{a_4 . P_{11}(2,4) + (a_1+a_2)\ P_{11}(1,2)}{a_1+a_2+a_4}$$

Step 3 : $P_{11}(2,4) = 0$ because of formula (14)

$P_{11}(1,2) = 1$ because of formula (15)

Step 4 : $$P_{11}(12,3,4) = \frac{(a_1+a_2)^2}{(a_1+a_2+a_3+a_4)(a_1+a_2+a_4)}$$

Player 2 :

Step 1 : $$P_{12}(12,3,4) = \frac{(a_3+a_4)\ P_{12}(2,3,4) + (a_1+a_2)\ P_{12}(12,4)}{a_1+a_2+a_3+a_4}$$

Step 2 : $$P_{12}(2,3,4) = \frac{(a_3+a_4)\ P_{12}(3,4) + a_2 . P_{12}(2,4)}{a_2+a_3+a_4}$$

$$P_{12}(12,4) = \frac{a_4 . P_{12}(2,4) + (a_1+a_2)\ P_{12}(12)}{a_1+a_2+a_4}$$

Step 3 : $P_{12}(3,4) = 0$ because of formula (14)

$$P_{12}(2,4) = \frac{a_2}{a_2+a_4}$$

$P_{12}(12) = 0$ because of formula (15)

Step 4 : $$P_{12}(12,3,4) = \frac{(a_3+a_4)\ a_2^2}{(a_2+a_3+a_4)(a_2+a_4)(a_1+a_2+a_3+a_4)} + \frac{(a_1+a_2) \cdot \dfrac{a_2a_4}{(a_2+a_4)}}{(a_1+a_2+a_4)(a_1+a_2+a_3+a_4)}$$

Player 3 :

Step 1 : $$P_{21}(12,3,4) = \frac{(a_3+a_4)\, P_{21}(2,3,4) + (a_1+a_2)\, P_{21}(12,4)}{a_1+a_2+a_3+a_4}$$

Step 2 : $P_{21}(12,4) = 0$ by means of formula (14)

$$P_{21}(2,3,4) = \frac{(a_3+a_4)\, P_{21}(3,4) + a_2 P_{21}(2,4)}{a_2+a_3+a_4}$$

Step 3 : $P_{21}(2,4) = 0$

Step 4 : $$P_{21}(12,3,4) = \frac{(a_3+a_4)\, a_3}{(a_1+a_2+a_3+a_4)(a_2+a_3+a_4)}$$

Player 4 :

Step 1 : $$P_{31}(12,3,4) = \frac{(a_3+a_4)\, P_{31}(2,3,4) + (a_1+a_2)\, P_{31}(12,4)}{a_1+a_2+a_3+a_4}$$

Step 2 : $$P_{31}(12,3,4) = \frac{(a_3+a_4)\, P_{31}(3,4) + a_2 P_{31}(2,4)}{a_2+a_3+a_4}$$

$$P_{31}(12,4) = \frac{a_4 P_{31}(3,4) + (a_1+a_2)\, P_{31}(12)}{a_1+a_2+a_4}$$

Step 3 : $P_{31}(12) = 0$ by means of formula (14)

$$P_{31}(2,4) = \frac{a_4}{a_2+a_4}$$

$$P_{31}(3,4) = \frac{a_4}{a_3+a_4}$$

Step 4 : $$P_{31}(12,3,4) = \frac{(a_3+a_4)^2 \dfrac{a_4}{a_3+a_4} + (a_3+a_4)\dfrac{a_2}{a_2+a_4}}{(a_2+a_3+a_4)(a_1+a_2+a_3+a_4)} + \frac{(a_1+a_2)\dfrac{a_4^2}{a_2+a_4}}{(a_1+a_2+a_3+a_4)(a_1+a_2+a_4)}$$

It can easily be verified that the sum of the probabilities $P_{ij}(12,3,4)$ <u>is</u> 1 .

The total probabilities for survival are :

$$TP_{11}(12,3,4) = P_{11}(12,3,4)$$

$$TP_{12}(12,3,4) = P_{11}(12,3,4) + P_{12}(12,3,4)$$

$$TP_{21}(12,3,4) = P_{21}(12,3,4)$$

$$TP_{31}(12,3,4) = P_{31}(12,3,4)$$

Don't fight alone against two opponents.
Align at first with one of them to eliminate the other.
Then proceed to eliminate your coalition partner.

(Mao Tse-tung)

IV. THE SOLUTION CONCEPT

In consistency with preceding paragraphs the definitions will be given in terms of games in partition function form (Ref. 2).

Let $$N = \{1, \ldots, n\}$$

be a set of players who are represented by 1, ..., n.

Let $$P = \{P_1, \ldots, P_h\}$$

be an arbitrary partition of N into coalitions $P_1, \ldots, P_h$.
The set of all partitions of N is denoted by

$$\pi = \{P\}.$$

Denote the real numbers by R^1. Then for each partition assume there is an outcome function,

$$F_P : P_{ij} \rightarrow R^1,$$

which assigns the real numbered outcome $F_P(P_{ij})$ to player j of the coalition P_i when the partition P forms. The function

$$F : \pi \rightarrow \{F_P\} \in R^n$$

which assigns to each partition its outcome function, is called the pay-off function or partition function of the game. The definition of the outcome function causes the resultant of the pay-off function to appear as a pay-off vector. (It will be remembered that games without transferable utilities can only be treated adequately, by a generalized function, by means of a pay-off vector (Ref. 6)). Finally, the ordered pair,

$$\Gamma = (N, F),$$

is the mathematical representation of an n-person game in partition function form.

In our game we call a vector $\underline{x} = (x_1, \ldots, x_n)$, $\underline{x} \in R^n$, a pay-off vector, if following conditions are fulfilled :

a) $0 \leq x_i \leq 1 \,, \forall i \in N$

b) $\sum_{i \in N} x_i \geq 1$

where x_i menas the total probability of surviving of player i.

We define individual rationality by

$$x_i \geq v_i \qquad \underline{v} = (v_1, \ldots, v_n),$$

$$\forall i \in N \qquad v_i = \min_{(P,\, i \in P)} F_p(\{i\})$$

This condition implies that player i will only be induced to cooperate if the resulting pay-off assures him an amount at least what he can obtain by forming a single coalition.

We introduce an n-person characteristic function as a pair (N, v), where v is a function that carries each $S \subset N$ into a vector $v(S) \in R^n$ so that, if $\underline{x} \in v(S)$ and $x_i^S \geq y_i^S \, \forall i \in S$ then $\underline{y} \in v(S)$.

If $\underline{x}$ and $\underline{y}$ are pay-off vectors and S is a non-empty subset of N, then $\underline{x}$ dominates $\underline{y}$ via S, denoted $\underline{x} \text{ dom}_S \, \underline{y}$, means that

$$x_i > y_i \qquad \forall i \in S \subset N$$

$$\underline{x} \in v(S) = (v_1^S, \ldots, v_n^S),$$

$$v_i^S = \begin{cases} +\infty & \forall i \notin S \\ \min\limits_{(P,\, S \in P)} F_P(S) & \forall i \in S \end{cases}$$

v_i^S are extended real numbers, i.e. a real number or $+\infty$ or $-\infty$.

The domination $\underline{x}$ dominates $\underline{y}$, denoted $\underline{x}$ dom $\underline{y}$, therefore is given if there exists an S such that $\underline{x} \text{ dom}_S \, \underline{y}$. Let R be the set of all pay-off vectors.

If $X \subset R$, let $\qquad \text{dom}_S X = \{\underline{y} \in R \mid \underline{x} \text{ dom}_S \, \underline{y} \text{ for some } \underline{x} \in X\}$

and $\qquad \text{dom } X = \{\underline{y} \in R \mid \underline{x} \text{ dom } \underline{y} \text{ for some } x \in X\}$.

A set of pay-offs K is a solution if

$$K \cap \text{dom } K = \emptyset$$

and $$K \cup \text{dom } K = R.$$

These two conditions are equivalent to the one condition

$$R - \text{dom } K = K.$$

In words, these two equations say that:

if x̲ and y̲ are in K, then neither dominates the other (internal stability),

and if z̲ is not in K, then there exists an x̲ in K which dominates z̲ (external stability).

The subset of undominated pay-offs in R is called the core and is defined by

$$C = R - \text{dom } R.$$

Clearly, the set C is more stabile than a solution set. Also, $C \subset K$.

IV.2. Permanent and transitory coalition

Before the profitability of the formation of a coalition for an individual player or sets of players shall be investigated, a clear distinction of two notions of the term "coalition" is appropriate.

A permanent coalition is not expected to fall asunder after all of the opponents have been eliminated.

A transitory coalition is expected to break up into new coalitions after the opponents have been eliminated in the "first round". The game will be continued until one player only survives.

No difficulty arises when we apply our solution concept to the case of the transitory coalition. For the case of the permanent coalition an additional feature would have to be added in order to derive a solution. It can cleary be expected that the transition of a coalition with n players to a coalition with n+1 players will result in a rise of the pay-offs (surviving probabilities) of all n+1 players provided no other coalition is enlarged.

The increment in surviving probability is, however, conditioned by the renunciation of the satisfaction to eliminate the partners in the coalition. The additional feature must be, therefore a system of indifference curves, i.e. functions where the increment in surviving probability (ΔP) is dependent on the number of players (r) a contestant must forego to eliminate. Such an indifference curve for a player i could perhaps be represented by the graph of parabolas as in tables 7 and 8.

Table 7

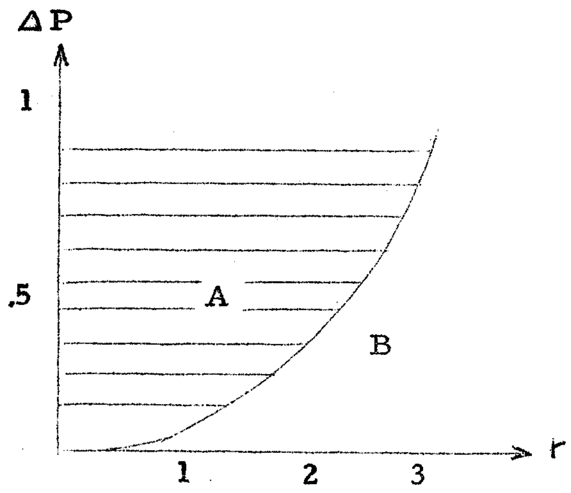
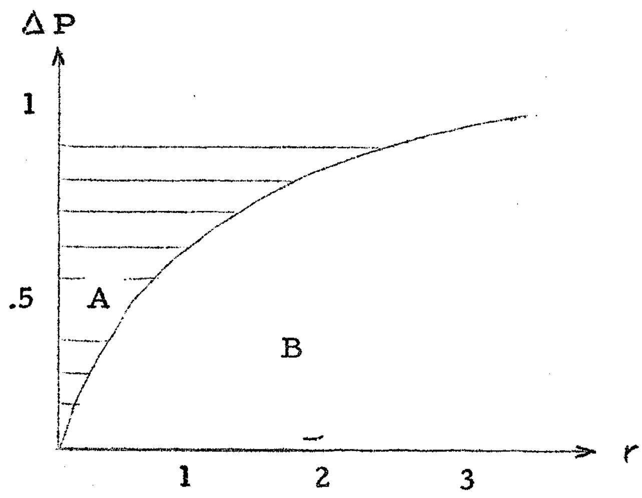


Table 8



In table 7 (8) the last (first) player our contestant i must forego to eliminate is the most important to him as measured in the required relief ΔP. The indifference curve of table 8 can perhaps be ascribed to a reluctantly cooperative outsider. In table 7 player i's willingness to cooperate diminishes as he is supposed to survive with an increasing number of players.

Clearly, if a point (r, ΔP) lies in the horizontally hatched region (A), player i will prefer a higher surviving probability to the urge to eliminate, and vice versa if a point lies in B.

Not only will the shape of the indifference curve vary from player to player, it also will differ according to the number of players in the coalition.

As, therefore, assumptions on the set of the set of indifference curves would be inevitable in order to arrive at a solution of the game, we skip these difficulties by treating the transitory coalition case only.

IV. 3. The solution for the three-person game

Referring to p^2 to p^4 of section III. 1., we have as the enforcable (minimum) pay-off vector for players A, B, and C respectively

$$\underline{x}_{MIN} = \left(\frac{a^2}{(a+c)(a+b+c)}, \frac{b^2}{(b+c)(a+b+c)}, \frac{c^2}{(b+c)(a+b+c)}\right)$$

Treating briefly the permanent coalition case and writing $x_{AB,C}$ for the pay-off when AB play against C, (likewise for the other cases) we have

$$\underline{x}_{AB,C} = \left(\frac{a+b}{a+b+c}, \frac{ab+ac+b^2+2bc}{(b+c)(a+b+c)}, \frac{c^2}{(b+c)(a+b+c)}\right)$$

$$\underline{x}_{AC,B} = \left(\frac{a+c}{a+b+c}, \frac{b^2}{(b+c)(a+b+c)}, \frac{ab+ac+c^2+2bc}{(b+c)(a+b+c)}\right)$$

$$\underline{x}_{BC,A} = \left(\frac{a^2}{(a+c)(a+b+c)}, \frac{b+c}{(a+b+c)}, \frac{ab+2ac+bc+c^2}{(a+c)(a+b+c)}\right)$$

In order to obtain a solution for this case, certain assumptions about the forms of the indifference must be made, for instance that for all three functions $\Delta P = 0$. Clearly, the imputation $\underline{x}_{ABC}$ would dominate all others, but it can be precluded for there would not be much point in discussing games of survival at all.

It is immediately evident that all vectors $\underline{x}_{AB,C}$, $\underline{x}_{AC,B}$ and $\underline{x}_{BC,A}$ are pay-offs as defined above.
Furtherm., $\underline{x}_{AB,C}$ dominates the other two because $\underline{x}_{AB,C} \underset{AB}{\text{dom}}$ $\underline{x}_{AC,B}$ and $\underline{x}_{BC,A}$. It is a core solution since p^5 was precluded.

A lot more confidence, however, can be derived from the application of the solution concept to the transitory coalition case. The transitory coalition solution concept will, of course be also appropriate in those cases where lack of confidence in the durability of the permanent coalition prevails.

In the three-person game the solution is not difficult to calculate. The match is carried out in one or two rounds: in the first round two players shoot against one. In case the single player wins, the game is over.

If not, the final winner is ascertained from the players in the two-person coalition. The imputations for the three possibilities are:

$$\underline{x}_{AB,C} = \left(\frac{a}{a+b+c}, \frac{b(b+2c)}{(b+c)(a+b+c)}, \frac{c^2}{(b+c)(a+b+c)}\right)$$

$$\underline{x}_{AC,B} = \left(\frac{a}{a+b+c}, \frac{b^2}{(b+c)(a+b+c)}, \frac{c(2b+c)}{(b+c)(a+b+c)}\right)$$

$$\underline{x}_{BC,A} = \left(\frac{a^2}{(a+c)(a+b+c)}, \frac{b}{a+b+c}, \frac{c(2a+c)}{(a+c)(a+b+c)}\right)$$

The results are not without interest:

1) A is indifferent between a temporary coalition with B or C.
2) B prefers $\underline{x}_{AB,C}$ over the other omputations for

$$\frac{(b+2c)}{b+c} > 1 > \frac{b}{b+c}$$

3) C prefers $\underline{x}_{BC,A}$ over the other two. He is only indifferent to $\underline{x}_{AC,B}$ if $a = b$. Furtheron, he is, however, indifferent between $\underline{x}_{BC,A}$ and the competitive case (when the formation of coalitions is prohibited) as indicated by a comparison with the results of formulae (1), (2) and (3).

What is the rationale of this outcome ? Which coalition-structure is most likely to occur ?

First of all, there would be no difficulty if the $>$ sign in the dominance definition were replaced by the $\geqq$ sign. Clearly, $\underline{x}_{AB,C}$ would be the core solution, as p^5 is again precluded.

This is not the case, however, and A's indifference between a coalition with B or C seems to be the only difficulty. The difficulty is only apparent, however, as B prefers a coalition with A over one with C, and C will either prefer a coalition with B or the competitive case, but not a coalition with A. Since B is not willing to cooperate with C, all C can do is to prevent A and B from forming a coalition with each other. However, the partition p^2 is still most likely to result as p^1 causes frustrating results for A and B.

When two are quarreling the third has the laugh on his side.

(Austrian proverb)

III. 4.

Collecting results we have:

A) Assumption $a > b > c$

1) The vector $\underline{x}_{BC,A}$ is not in the solution space since $\underline{x}_{AB,C} \ \mathrm{dom}_{AB} \ \underline{x}_{BC,A}$.

2) Partition p^2 where A and B play against C is more likely to result than the other partition p^3 leading to a vector in the solution.

3) The transition from the permanent to the transitory coalitions case changed the game from a non-constant sum to a constant sum game. Therefore the core is empty.

4) Mao is wrong. In any event, his statement is not correct for all participating players. The weakest player in a three-person game will prefer the game to be played competitively, i.e. he must try to prevent his opponents from forming a coalition.

5) A's probability to win the game is irrespective of the choice of his partner in the transitory coalition.

6) B has the maximum maximorum of all surviving probabilities. It occurs when he forms a coalition with A.

B) Assumption $\alpha_i = \alpha \ \forall i \in N$

7) The special case where $\alpha_i = \alpha \ \forall i$, i.e. when the shooting order is based on subjective unanimous feelings about the skills of the players while their objective shooting accuracies are all equal, depicts the character of the solution concept. The payoff of p^4 is dominated by $\underline{x}_{AB,C}$ with respect to the coalition AB. $\underline{x}_{AB,C}$ and $\underline{x}_{AC,B}$ are the vectors in the solution. Again, $\underline{x}_{AB,C}$ is most likely to occur. The payoffs are irrespective of the level of α.

C) Assumption $a < b < c$

8) Taking up the distinction between subjective and objective shooting accuracies of section II.2., we have constructed a hypothetical example where $a = .4$, $b = .6$, $c = .8$. As a result we obtain that the pay-off of partition p^4 is dominated by both p^3, and p^2.

$$\underline{x}_{AB,C} \ \underset{AB}{\mathrm{dom}} \ x_{BC,A}$$

and

$$\underline{x}_{AC,B} \ \underset{AC}{\mathrm{dom}} \ \underline{x}_{BC,A}$$

Since the pay-offs of p^2 and p^3 don't dominate each other, they are both vectors in the solution.

Table 9 presents a synopsis of the numerical results of the cases A), B), and C).

Table 9

	p^1	p^2	p^3	p^4
A	a = .8, b = .6, c = .4			
P_A	.296	.445	.445	.296
P_B	.333	.466	.200	.333
P_C	.370	.089	.355	.370
B	a = b = c = d			
P_A	.167	.333	.333	.167
P_B	.333	.500	.167	.333
P_C	.500	.167	.500	.500
C	a = .4, b = .6, c = .8			
P_A	.074	.222	.222	.074
P_B	.333	.524	.143	.333
P_C	.593	.254	.635	.593

III. 5. The solution for the four-person game

When we apply formulae (17) to the four-person case, we obtain the total surviving probabilities for each of the players 1, 2, 3 and 4.
There are 14 possible partitions of the set of players, if we exclude the trivial coalition 1234. Table 10 shows the resulting pay-off vectors for each partition. In table 11 we collected the results of n-person characteristic functions v(S). This example was computed under the assumption that all shooting accuracies are equal, but there is a subjective order of the players. Player 1 is the strongest one, player 2 the second strongest, and so on.

We find immediately that p^{11}, p^{12} and p^{13} are the only partitions which have pay-off vectors in the solution. This three vectors are in the core, too. Therefore the union of $\underline{x}^{11}$, $\underline{x}^{12}$ and $\underline{x}^{13}$ is the solution and the core.

If the set of shooting accuracies A is an ordered set with the relation " "less than", in correspondence to the ordered set N of the players, $\underline{x}^{11}$ will dominate all other pay-off vectors, and is not dominated.

We suppose that this result is true for every game with n players, i.e. the coalition 123 .... (n-1) is most likely to occur and the pay-off vector $\underline{x}^{123 \dots (n-1), n}$ is the solution and the core. It will be tried to give the proof of this statement in another paper.

III. 6. Further problems

In extension to the concept of transitory coalitions it would be interesting to develop the solutions of games in the case that all or some coalitions after the first shot break to pieces of subcoalitions of one or more players or that new configurations after the first or k^{th} shot are formed.

Table 10 Pay-off Vectors $\underline{x}^{P}$

j	p^j	Player 1	Player 2	Player 3	Player 4
1	1, 2, 3, 4	1/24	1/8	1/3	1/2
2	12, 3, 4	1/3	1/6 + 1/3	1/6	1/3
3	1, 23, 4	1/24	1/2	5/24 + 1/2	1/4
4	1, 2, 34	1/24	1/8	2/3	1/6 + 2/3
5	13, 2, 4	1/3	1/6	1/6 + 1/3	1/3
6	14, 2, 3	1/3	1/6	1/3	1/6 + 1/3
7	1, 24, 3	1/24	1/2	1/4	5/24 + 1/2
8	12, 34	1/6	1/6 + 1/6	1/3	1/6 +1/3
9	13, 24	1/6	1/3	1/6 + 1/6	1/6 + 1/3
10	14, 23	1/6	1/3	1/6 + 1/3	1/6 + 1/6
11	123, 4	3/4	1/6 + 3/4	1/24 + 1/6 + 3/4	1/24
12	124, 3	3/4	1/6 + 3/4	1/24	1/24 + 1/6 + 3/4
13	134, 2	3/4	1/24	1/6 + 3/4	1/24 + 1/6 +3/4
14	1, 234	1/24	3/4	1/6 + 3/4	1/24 + 1/6 + 3/4

Table 11 n-person characteristic functions v(S)

S		Player 1	2	3	4
1		1/24	∞	∞	∞
2		∞	1/24	∞	∞
3		∞	∞	1/24	∞
4		∞	∞	∞	1/24
12		1/6	1/3	∞	∞
13		1/6	∞	1/3	∞
14		1/6	∞	∞	1/3
23		∞	1/3	1/2	∞
24		∞	1/3	∞	1/2
34		∞	∞	1/3	1/2
123		3/4	11/12	23/24	∞
124		3/4	11/12	∞	23/24
134		3/4	∞	11/12	23/24
234		∞	3/4	11/12	23/24

V. References

Ref. 1 : L. SHAPLEY and M. SHUBIK :
"Competition, Welfare and the Theory of Games", Chapter 1, (unpublished manuscript)

Ref. 2 : R. THRALL and W. LUCAS :
"n-Person Games in Partition Function Form", Naval Research Logistics Quarterly, 10, No. 4, Dec. 1963, Pp. 281-298.

Ref. 3 : J. VON NEUMANN and O. MORGENSTERN :
"Theory of Games and Economic Behavior ". Princeton University Press 1953.

Ref. 4 : M. SHUBIK :
"Strategy and Market Structure", New York, London : John Wiley & Sons, 1960.

Ref. 5 : J. RIORDAN :
"An Introduction to Combinatorial Analysis", John Wiley & Sons, Canada 1964.

Ref. 6 : LUCE and RAIFFA :
"Games and Decisions", John Wiley & Sons, USA, 1964.