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Abstract. The present paper models the process of issuing new equity as a three-stage game. First the issuing firm is allowed to negotiate with several investment-bankers on an underwriting contract. Then the shares are issued at a primary market to the public and, finally, investors can trade shares at a secondary market. Under symmetric information the costs of issuing shares in equilibrium consist of two parts: Underpricing at the primary market and the cost of an underwriting contract, where the incentive to conclude the latter stems from the danger of even more severe underpricing in a non-underwritten issue.

1. INTRODUCTION

A phenomenon which has attracted considerable attention in the literature on financial markets is the fact that new issues of shares are frequently underpriced, when measured against the price at the secondary market [**Asquith and Mullins**, 1986; **Kalay and Shimrat**, 1986; **Masulis and Korwar**, 1986; **Mikkelson and Partch**, 1986; **Ritter**, 1987; **Smith**, 1986; concerning initial public offerings see: **Ibbotson, Sindelair and Ritter**, 1988; **Tiniç**, 1988]. Recent explanations of this phenomenon draw on the literature on signalling games and present models, where asymmetric information is responsible for underpricing [**Rock**, 1986; **Beatty and Ritter**, 1986; **Gale and Stiglitz**, 1989; **Grinblatt and Hwang**, 1989; **Welch**, 1989], or where reputational effects are present [**Tiniç** , 1988]. A more fundamental reason for the phenomenon can be found in the fact that underpricing has to generate an incentive for investors to actually buy at the primary market, rather than wait for cheaper opportunities at the secondary market [**Parsons and Raviv**, 1985].

The model to be presented rules out any asymmetric information, but still generates the underpricing phenomenon, demonstrating it to be a much deeper rooted effect. Moreover, no assumption on price-parametric

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This is a first and preliminary version. Comments are welcome. The paper first took shape while K. Ritzberger was visiting fellow at the Australian National University and F. Milne was visiting fellow at the Australian Graduate School of Management.

behavior on the part of investors will be used. Rather a full game with rational players is specified which encompasses all of the stages of an equity issue. This full specification allows an identification of all costs of issuing shares, except out-of-the-pocket expenses (transactions costs) which would present frictions and are, therefore, ruled out.

Very informally the costs of issuing shares can be described as follows: Even if the issue is guaranteed to succeed ("underwritten"), multiplicity of equilibria at the secondary market may make it optimal for the issuer to underprice the issue at the primary market. The reason for this is that underpricing may be the only device to induce investors to order shares at the primary market, rather than wait for a (possibly) lower price at the secondary market. That it may indeed be preferable for the issuer to activate the primary market, even if strong underpricing is required to do so, can be due to discontinuities of the mapping associating outcomes of the primary market with an equilibrium at the secondary market: An inactive primary market may lead to a sharp drop of the price at the secondary market. If the issue is *not* underwritten, an extra source of underpricing emerges: Self-fulfilling expectations on the failure of the issue may only be avoidable by underpricing the issue at the primary market even more severely. The latter constitutes an incentive for the issuing firm to contract an underwriter and identifies the second major source of costs. Since underwriting-contracts are not traded at centralized markets, but are negotiated between the firm and investment-bankers, the latter can extract a reward for their service of guaranteeing the issue. This reward or rent may be non-vanishing, as long as there are only finitely many investment-bankers, despite the fact that potential underwriters engage in a kind of Bertrand-competition while bargaining with the issuing firm on the conditions of a contract. Thus the costs of issuing shares are due to the interdependence of three different types of markets: A bargaining market for underwriting-contracts, a primary market with a price-setting monopolistic supplier of new shares, and a secondary market which operates like stock-exchanges organized as so-called "specialist-markets".

This paper has no intention to reject explanations of underpricing based on asymmetric information. Quite on the contrary, these are valuable models of other sources of underpricing. The point here rather is that those costs of issuing shares identified in the sequel may emerge even under symmetric information (in fact even under complete information).

The plan of the paper is as follows: Section 2 introduces informally the three-stage game and Section 3 gives some heuristics on the results. Sections 4, 5, and 6 contain the rigorous definitions and the analysis of the subgames at the secondary market, the primary market, and the market for underwriting-contracts, respectively. Section 7 draws some conclusions and summarizes.

2. Description of the game

The game to be studied in the sequel is a three-stage game between $n + m + 1$ players, $n > m \geq 2$. The players are:

(i) A single firm, who wishes to finance an investment project by a new issue of seasoned shares (equity). The firm is risk neutral, not allowed to hold its own stock, and a profit maximizer, where profit is the profit from the operation of the business (held constant throughout) plus the revenue from the share-issue minus its cost. In order to successfully finance its project, the firm has to raise a given amount of money via the issue and, if it fails to do so, the project has to be cancelled (cancellation is costless).

(ii) There are $m \geq 2$ investment bankers, indexed for most of the time by $j \in \mathcal{M} = \{1, \ldots, m\}$, whose business is to potentially function as underwriters of the issue of new shares. They are also profit maximizers, but they have no interest in long-term holdings of stock. That is to say that investment bankers (frequently referred to as "underwriters") are true financial intermediaries, who command a very large budget (in any case larger than the cost of the firm's project), but who are *not* long-term shareholders. Formally the latter will be reflected by the fact that holdings of shares by investment bankers at terminal nodes of the game do not yield them any payoff.

(iii) Finally there are n, $n > m$, private investors, who will be the ultimate holders of shares. Investors are indexed by $i \in \mathcal{N} = \{1, \ldots, n\}$. Their preferences will be described below.

The economic environment is one with only two commodities, a consumption good and shares. The consumption good (which serves as the numeraire) can also be thought of as "money", when shares are thought of as "risky assets", or as "present consumption", when shares are thought of as (possibly risky claims on) "future consumption". The investors' preferences allow all these interpretations: Preferences for investor $i \in \mathcal{N}$ are represented by a utility function $u^i \colon \Re^2_+ \to \Re$ which is assumed smooth ($\mathcal{C}^\infty$), strictly monotone increasing in both its arguments and strictly concave,

$$u^i_h > 0, \quad u^i_{hh} < 0, \quad u^i_{hk} \geq 0, \; k \neq h,$$

for $h = 1, 2$, $k = 1, 2$, $i \in \mathcal{N}$, (where subscripts denote partial derivatives) and satisfies

$$u^i_1(x, .) \longrightarrow_{x \to 0} +\infty, \quad u^i_2(., x) \longrightarrow_{x \to 0} +\infty, \forall i \in \mathcal{N},$$

that is, indifference curves stay away from the boundary. Investors are maximizers of expected utility, and enter the game with endowments $(w_i, S_i) \in \Re_{++} \times \Re_+$, where $w_i > 0$ is the endowment with the first commodity (consumption good, money, or present consumption) and $S_i \geq 0$ denotes the endowment with seasoned shares. An investor, for whom $S_i > 0$ holds, is called a shareholder. Endowments may, of course, change in the course of the game. Denote $S = \sum_{i \in \mathcal{N}} S_i$. The firm has no endowment of either commodity.

In case the firm succeeds in financing its investment project, the return on (one unit of) a share will be (a function of) a random variable, $R(\tilde{\theta})$, with values in a compact interval in $\Re_+$, such that $R(\tilde{\theta}) \geq 0$, $\forall \tilde{\theta}$. The distribution of the state variable $\tilde{\theta} \in \Omega$ is given by $\mu\colon \Omega \to [0, 1]$, and satisfies

$$\int_\Omega R(\theta)\, d\mu(\theta) > 1$$

(such that at least risk neutral investors would wish to hold shares), and is continuous and common knowledge. This may sound slightly removed from the picture of an equity issue. But $R(\tilde{\theta})$ can simply be taken to be the firm's liquidation value in the future divided by the number of all previously issued shares and converted into real consumption. Thus there is no problem with evaluating shares by multiplication with $R(\tilde{\theta})$.

An investor $i \in \mathcal{N}$, who holds at a terminal node of the game an endowment x_i of the first commodity and ζ_i shares, derives a payoff of $\int_\Omega u^i(x_i, R(\theta)\zeta_i)\, d\mu(\theta)$.

The three stages of the game work as follows: The first stage is a T-period (T finite, but large) sequential non-cooperative bargaining game without discounting between the firm and the m investment bankers on an underwriting contract. The constituent bargaining steps are bilateral, but at each step both bargaining partners have an option to withdraw and contact an other bargaining partner. Since all of this happens under perfect information, potential underwriters effectively engage in a kind of Bertrand-competition. The details of the bargaining game are specified in Section 6. Suffice it here to say that a contract with a single underwriter is assumed sufficient to guarantee the issue. Thus the formation of syndicates of underwriters is not considered in the present paper, but rather it is assumed that each one of the investment bankers commands a sufficiently large budget to guarantee the whole issue. An underwriting contract is assumed to be a firm commitment underwriting contract (which is, however, not a restrictive assumption, as we argue in Section 5), such that the bargaining is on the price which the underwriter will have to pay for the shares. The number of new shares is exogeneously fixed at $s > 0$.

The second stage of the game is termed the primary market. If the bargaining at the first stage of the game ended with an agreement on a contract, one of the investment bankers (underwriters) will supply the shares to the primary market. If the bargaining failed, the firm will supply the shares to the primary market. In the latter case the firm must succeed in selling all new shares (or some prefixed proportion) at the primary market, because otherwise the issue will fail and the investment project will have to be cancelled. If the issue, on the other hand, is underwritten, then, in case not all new shares are sold at the primary market, the investment project is *not* cancelled (but financed from the underwriter's funds), and the underwriter will dump the remaining new shares onto the secondary market. That the underwriter does not simply hold the remaining shares is a consequence of the assumption that investment bankers are pure financial intermediaries and not institutional investors: They derive no profit from holding shares beyond the secondary market.

The issue at the primary market is a public offering (general cash-offer or a rights issue): The supplier of new shares sets a price, denoted p^o, and chooses an issuing method (e.g. general cash-offer or non- renounceable rights issue, among others) which are publicly announced and binding. Then investors can (simultaneously) take up the offer (if it is a rights issue, only as far as their rights allow them to) entirely, partially, or not at all. The price at which investors may take up the offer is the price announced by the supplier beforehand and *cannot* be changed in the light of conditions at the primary market. The extend to which investors take up the offer and the primary market price p^o constitutes the outcome of the primary market and determines the endowments with which investors move on to the secondary market.

The third stage of the game is the secondary market, where price-setting power is now reversed. If the supplier at the primary market did not sell the whole issue $s > 0$ (and the project did not have to be cancelled), the remaining new shares are dumped onto the secondary market by a market order, that is an offer to sell all remaining shares at the best price available. Investors, on the other hand, are allowed to play limit orders at the secondary market. A limit order is an offer to buy (sell) up to a specified maximum quantity at any price equal to or below (above) a specified bid- (ask-) price. Once investors have (simultaneously) submitted their limit orders, a market maker ("specialist") determines a market bid- and ask- price at which the (turnover maximizing and profit maximizing for the specialist) transactions are carried out. The spread is the specialist's profit. This game form indeed resembles quite closely what happens on real-world stock exchanges (namely so-called specialist-markets, like the New York Stock Exchange) which tend to be highly centralized markets with similar clearing devices.

Once transactions at the secondary market are carried out, the game terminates and players receive their payoffs. The firm's (net) payoff consists of what the underwriter paid for the new shares, if the bargaining at the first stage was successful, and of the proceeds from the primary and secondary market from selling the shares, if no underwriting contract was agreed upon and the issue did not fail. If the issue failed, the firm's payoff is zero, because cancellation of the project is costless. An underwriter, who did not succeed in concluding a contract, always receives zero payoff. A successful underwriter receives the revenue from selling the shares at the primary and secondary market minus what he had to pay to the firm. Investors derive their utility from their final holdings of the consumption good and shares.

The particular extensive forms for the three markets (stages) are specified in more detail in the corresponding sections. The solution concept for the game will be subgame perfect Nash equilibrium [**Selten**, 1965].

3. Some heuristics on the results

The present paper contains several messages, depending on how one reads it. The first and probably major one is on how the costs of issuing shares are generated. We identify and analyse two major sources of costs: The first is that new issues of shares may be underpriced at the primary market, when measured against the price at the secondary market. The second source is that, to guard against excessive underpricing, an underwriter may have to be employed. And the service of guaranteeing the issue is costly, even if potential underwriters engage in Bertrand-like competition, whenever this competition is via free bargaining. These identifications are nothing new as such. What may be new, is, how difficult it is to rigorously derive these conclusions in a completely specified model, where traders do not take prices as given, but are rational players in a game.

In fact this paper has grown unfortunately long and technical, because our original intuition was (mis-?) lead by considering worlds with price parametric behavior. We should have learned by now that many of these intuitions do not survive the tough test of non-cooperative game theory. Still it is interesting to see that some intuition from a world with price parametrically behaved individuals does indeed carry over to full games. Consider for the moment a world, where investors always take prices at the secondary market as given and do not - not even at the primary market - conceive any influence on the secondary market price. Then shares at the secondary and the primary market are perfect substitutes, such that investors would wish to buy shares exclusively at the primary (secondary) market, if $p^o < p^1$ ($p^o > p^1$), where p^1 denotes the price at the secondary market. For $p^o = p^1$ investors would be indifferent. Thus in order to sell anything at all at the primary market, shares have to be underpriced, $p^o \leq p^1$. The supplier of shares to the primary market will, consequently, sell all shares at the primary (secondary) market, if $p^o < p^1$ ($p^o > p^1$), and has thus, in (p^1, p^o)-space, Leontief-type preferences with the kink at the diagonal. Figure 1 illustrates.

(Insert Figure 1 about here)

In Figure 1 the bold folded curve is the graph of the correspondence mapping p^o into p^1 for a given allocation of new shares to investors which sums to s, i.e. such that all new shares are sold at the primary market. The price $\bar{p}$ denotes the price obtained at the secondary market, if nothing is sold at the primary market (it is located, where the folded curve intersects the diagonal at point B, because this will always be an equilibrium price at the secondary market, if nothing is sold at the primary market, though it may not be the only one). All indifference curves of the supplier of new shares will look like the Leontief-type indifference curve through point C at the diagonal. Point A, obviously, is the optimal (p^o, p^1)-combination for the supplier, and point A is an underpricing equilibrium, because $p^1 > p^o$ holds.

Have we explained underpricing by this simple example? The answer is, of course, negative, because point A is a critical equilibrium (where the partial derivative of excess demand at the secondary market with respect to p^1 vanishes). And, even if one is willing to concede that at large markets individual investors have so little influence that price parametric behavior is a good approximation, this certainly does *not* hold true for critical equilibria. But it was precisely the assumption of price parametric behavior which induced the Leontief-type perferences of the supplier and led to the selection of a critical equilibrium. (Of course, if the equilibrium correspondence does not have a fold, as in the example, then a regular equilibrium will be selected, but then $p^o = p^1$ in equilibrium.)

Thus it is essential to specify the game at the secondary market completely, in order to understand, whether an equilibrium like point A in Figure 1 may in fact obtain also, when investors do *not* take prices as given at the secondary market. On this issue we have interesting results to offer: Under the specification of the game adopted for the secondary market (which resembles closely the rules at real-world markets), it can be shown that all active equilibria are Walras-equilibria.

The price which has to be paid for this result is that, although investors at the secondary market now behave *as if* they would be price-takers, *at the primary market* they are now very much aware of their influence on the price at the secondary market. In particular, investors now do *not* conceive shares at the primary market and at the secondary market as perfect substitutes. None of the elegant arbitrage arguments does apply, once investors understand how their actions at the primary market will translate into the secondary market price. Still we can show that slightly weaker propositions apply, stating that, *under certain conditions*, *activity* of the primary market in equilibrium will require underpricing. Thus the first source of the costs of issuing shares may at least hold true also in a fully specified game.

Moreover, it turns out that underpricing can in principle be broken into two parts: There is an "unavoidable" degree of underpricing, stemming from the incentive to activate the primary market, which is present even if the issue is underwritten. But there may be an "extra" degree of underpricing, if the issue is not underwritten, stemming from the lack of public trust into the success of the issue. The second part can be avoided by contracting an underwriter which may explain the widespread use of this arrangement.

The second source of costs also survives the game-theoretic test. Since the situation here is reversed in that traditional wisdome would suggest that this cost should not exist, results here are even slightly stronger: The equilibrium, where it is costly to contract an underwriter, though not unique, has very desirable game-theoretic properties.

Are we thus back to the old story? Not quite, because first of all we only study one particular (class of) equilibrium (equilibria) and it is possible that there are others which make issuing shares virtually costless. This should be a warning against to much hope put onto the search for empirical regularities. Second, the results are generated by the particular interplay of three different types of markets: A bargaining market with price-competition, but bilateral negotiations, a centralized market, where only one supplier has price-setting power, and a centralized market, where all investors can set limit orders. Changing one of these institutions may change the results. If, for example, primary market issues are sold by sealed bid auctions (as in France) or by tender (as in the UK or Austria), equilibria with overpricing may result. In fact, since the institutional arrangements at the primary market are crucial, this may be a potential field for designing optimal institutions via regulations on primary market issues. The present analysis can also be understood as a first step towards understanding the mechanisms involved.

The latter may even be read as another message of the present paper: The outcomes of markets depend on the institutional details of the market-organization and the interdependence among different types of markets [a point already made by **Shaked**, 1988]. Finally we should remark against a possible misinterpretation of the term "costs of issuing shares": There is nothing in the analysis to follow that allows welfare conclusions, i.e. costs of issuing shares are not necessarily welfare losses (or gains). The next section starts the analysis of the game by working backwards, i.e. starting with the secondary market.

4. The Secondary Market

The secondary market is the spot market for shares which opens after the primary market has closed and the allocation on the primary market has become public knowledge. Investors come to the secondary market endowed with their endowments of old shares plus their purchases of new shares at the primary market and with their original endowments of the first commodity (the consumption good) minus what they paid for new shares at the primary market. There are two crucial differences on the secondary market as compared to the primary market: Now any investor may also supply shares to the market and the supplier of new shares at the primary market (the underwriter) is now forced to supply any number of new shares which he did not yet sell at the primary market and he cannot set a price. But, on the other hand, on the secondary market any investor has an influence on the price by being allowed to submit a limit order.

The operation of the secondary market is given by the following game: First a market maker (a "specialist") announces an opening price. The opening price is, however, *not* binding, but is a pure coordination device which the investors may well ignore, if they wish so. Once the opening price has been announced, each investor $i \in \mathcal{N}$ submits, simultaneously with the others, a limit order which is a (maximum) quantity of shares offered or demanded at the secondary market plus a limit price. A limit price is a bid-price, if the quantity demanded from the market is positive, and is to be interpreted as the maximum price which the bidder is willing to pay for the quantity demanded. A limit price is an ask-price, if the quantity offered on the market is negative, and is to be interpreted as the minimum price which the supplier stays ready to accept for the quantity offered. A (pure) choice for an investor on the secondary market is thus a duple

$$(p_i,\, s_i^1) \in \Re_+ \times [-S_i - s_i^o,\, \infty),$$

consisting of a limit price $p_i \geq 0$ and a maximum quantity s_i^1 supplied to or demanded from the market, where s_i^o denotes the purchases of new shares from the primary market by investor i.

Once all limit orders by investors are submitted, the market maker (the "specialist") determines the trades which maximize turnover in terms of quantities of shares and announces a corresponding market bid-price and a market ask-price (which may not be equal), such as to maximize the spread. If the market bid-price strictly exceeds the market ask-price, the spread is the specialist's profit. Transactions are carried out at these market prices. Since the specialist's behavior is completely determined by the rules of the game, he will not be considered as an active player, but rather as a mechanism.

The underwriter has no move at the secondary market: He supplies the quantity $s - \sum_{i \in \mathcal{N}} s_i^o \geq 0$ to the market by a market order (again $\sum_{i \in \mathcal{N}} s_i^o$ denotes the total quantity of new shares sold at the primary market). A market order is an offer to sell (at most) the quantity $s - \sum_{i \in \mathcal{N}} s_i^o$ at the best possible price. That the underwriter is not allowed to set an ask-price is meant to portray that on the secondary market he is forced to sell anything he did not yet sell at the primary market.

Let $p = (p_i)_{i \in \mathcal{N}}$, $s_i^1 = \zeta_i^1 - S_i - s_i^o$, where ζ_i^1 denotes final holdings of shares by investor $i \in \mathcal{N}$, and let $s^1 = (s_i^1)_{i \in \mathcal{N}}$. Say that the secondary market *can be active*, if $\exists (i, j) \in \mathcal{N} \times \mathcal{N}$: $s_i^1 > 0$, $s_j^1 < 0$, and $p_i \geq p_j$, or if $s - \sum_{i \in \mathcal{N}} s_i^o > 0$ and $\exists i \in \mathcal{N}$: $s_i^1 > 0$. If the secondary market can be active, define functions

$$Z_d(p^1, (p, s^1)) = \sum_{j \in \{i \in \mathcal{N} \mid p_i \geq p^1\}} \max[0, s_j^1],$$
$$Z_s(p^1, (p, s^1)) = -\sum_{j \in \{i \in \mathcal{N} \mid p_i \leq p^1\}} \min[0, s_j^1] + s - \sum_{i \in \mathcal{N}} s_i^o.$$

Also define

$$\underline{b}(p, s^1) = \sup\{p^1 \in \Re_+ \mid Z_d(p^1, (p, s^1)) \geq Z_s(p^1, (p, s^1))\},$$
$$\bar{a}(p, s^1) = \inf\{p^1 \in \Re_+ \mid Z_d(p^1, (p, s^1)) \leq Z_s(p^1, (p, s^1))\},$$
$$\bar{b}(p, s^1) = \max\{p^1 \in \Re_+ \mid Z_d(p^1, (p, s^1)) = Z_d(\underline{b}(p, s^1), (p, s^1))\},$$
$$\underline{a}(p, s^1) = \min\{p^1 \in \Re_+ \mid Z_s(p^1, (p, s^1)) = Z_s(\bar{a}(p, s^1), (p, s^1))\},$$

and observe that by definition of $\underline{b}$ and $\bar{a}$ also

$$\underline{b}(p, s^1) = \inf\{p^1 \in \Re_+ \mid Z_d(p^1, (p, s^1)) < Z_s(p^1, (p, s^1))\},$$
$$\bar{a}(p, s^1) = \sup\{p^1 \in \Re_+ \mid Z_d(p^1, (p, s^1)) > Z_s(p^1, (p, s^1))\},$$

and $\bar{b} \geq \underline{b}$, $\bar{a} \geq \underline{a}$ holds. The bid-price $\underline{b}$ is the minimal bid among demanders, who will actually trade non-zero quantities at the turnover-maximizing trades. The ask-price $\bar{a}$ is the maximal ask among suppliers, who will actually trade non-zero-quantities at the turnover-maximizing trades. The bid- and ask-prices $\bar{b}$ and $\underline{a}$ are the market bid- and ask-prices, maximizing the spread. By definition Z_d is (a) monotone decreasing (step function) and Z_s is (a) monotone increasing (step function) in p^1.

The rules of the secondary market game are formalized as follows: If, given (p, s^1), the secondary market cannot be active, no transactions take place. If the market can be active, then:

(1) If $Z_d(\underline{b}(p, s^1), (p, s^1)) = Z_s(\bar{a}(p, s^1), (p, s^1))$ then each investor submitting some limit order (p_i, s_i^1) with $p_i \geq \underline{b}(p, s^1)$ and $s_i^1 > 0$, or $p_i \leq \bar{a}(p, s^1)$ and $s_i^1 < 0$, will get the quantity he ordered at a price $\bar{b}(p, s^1)$, if $s_i^1 > 0$, or sell his quantity at a price $\underline{a}(p, s^1)$, if $s_i^1 < 0$. The underwriter (or the firm) will get $\underline{a}(p, s^1)[s - \sum_{i \in \mathcal{N}} s_i^0]$ in return for supplying $s - \sum_{i \in \mathcal{N}} s_i^0$ to the market.

(2) If $\underline{b}(p, s^1) = \bar{a}(p, s^1)$, then either **(1)** is applicable, or one side of the market is longer than the other, but trade is possible. Suppose $Z_d(\underline{b}(p, s^1), (p, s^1)) > Z_s(\bar{a}(p, s^1), (p, s^1))$ (the rule, if the inequality is reversed, is exactly analogous). Then all investors in the set $\{i \in \mathcal{N} \mid p_i = \underline{b}(p, s^1),\, s_i^1 > 0\}$ will be rationed with a certain probability, i.e. a random mechanism, the distribution of which is supported on a set contained in

$$\prod_{j \in \{i \in \mathcal{N} \mid p_i = \bar{b}(p,\, s^1),\, s_i^1 > 0\}} [0, s_j^1],$$

decides on who is rationed to what extend. (Higher bidders are always served first.) The random rationing mechanism may depend on arbitrary public information in the game, but must be such that for each $j \in \{i \in \mathcal{N} \mid p_i = \underline{b}(p, s^1),\, s_i^1 > 0\}$ the probability of $[s_j^1 - \varepsilon, s_j^1]$ and $[0, \varepsilon]$ is strictly positive for any $\varepsilon > 0$ (i.e. everybody in this set has a chance to get nothing and a chance to get what he demanded). Moreover, the rationing mechanism must ensure that all of the supply is distributed. In particular this implies that, if there is only one $i \in \mathcal{N}$ bidding $\bar{b}$, then this investor will be rationed deterministically. Demanders ($s_i^1 > 0$) will again pay $\bar{b}(p, s^1)$ per unit of whatever they are allocated and suppliers ($s_i^1 < 0$) will obtain $\underline{a}(p, s^1)$ per unit supplied to the market.

Investors on the secondary market do not pay or obtain the price they specified in their limit orders, but demanders pay $\bar{b}(p, s^1)$ and suppliers obtain $\underline{a}(p, s^1)$ per unit traded. The reason for this is that the market is made by a specialist, who has to maximize turnover, but must treat all investors symmetrically. That is the specialist may, by setting market bid- and ask-prices, maximize his profit (viz. the difference between the turnover evaluated at the market bid-price and the turnover evaluated at the market ask-price), but is not allowed to discriminate against individual investors. Thus only marginal traders are charged their limit prices.

That the rules **(1)** and **(2)** completely specify the rules of the game is demonstrated in the first lemma. (Proofs of lemmas and theorems are gathered in the Appendix).

LEMMA 1. *(i)* $\bar{b}(p,\, s^1) \geq \underline{b}(p,\, s^1) \geq \bar{a}(p,\, s^1) \geq \underline{a}(p,\, s^1)$.
(ii) If the market can be active, then $\underline{b}(p,\, s^1) > \bar{a}(p,\, s^1)$ *implies*

$$\text{(ii.a)} \qquad Z_d(p^1,\, (p,\, s^1)) = Z_s(p^1,\, (p,\, s^1)),\ \forall p^1 \in (\bar{a},\, \underline{b}),$$

$$\text{(ii.b)} \qquad \begin{aligned} &Z_d(\bar{b}(p,\, s^1),\, (p,\, s^1)) = Z_d(\underline{b}(p,\, s^1),\, (p,\, s^1)) = \\ &= Z_s(\bar{a}(p,\, s^1),\, (p,\, s^1)) = Z_s(\underline{a}(p,\, s^1),\, (p,\, s^1)). \end{aligned}$$

When investors come to the secondary market, they come with endowment vectors $(w_i - p^o s_i^o,\, S_i + s_i^o) \in \Re_+^2$, where w_i is the original endowment of i with the first commodity (consumption good, money, or present consumption), $p^o s_i^o$ is the quantity of the first commodity which i had to give up in order to purchase s_i^o new shares on the primary market, and $S_i \geq 0$ is the investors original endowment with old shares. Let for the moment p^1 denote the price which investor $i \in \mathcal{N}$ pays for his (allocated) demand or obtains for his (allocated) supply at the secondary market. Let s_i^1 be the quantity which i specified in his limit order. Then the payoffs to investors $i \in \mathcal{N}$ from the secondary market are given by

$$\begin{aligned} &U_i((p,\, s^1),\, (w_i - p^o s_i^o,\, S_i + s_i^o)) = \\ &= \int_{[\min(0,\, s_i^1),\, \max(0,\, s_i^1)]} \int_\Omega u^i(w_i - p^o s_i^o - p^1 \zeta, \\ &\qquad R(\theta)(S_i + s_i^o + \zeta))\, d\mu(\theta)\, dF(\zeta | (p,\, s^1)), \end{aligned}$$

where F is the marginal distribution induced by the random rationing mechanism on $[\min(0,\, s_i^1),\, \max(0,\, s_i^1)]$ which degenerates to a unit mass at s_i^1, if the player is on the short side of the market (or supply equals demand) and otherwise is non-degenerately supported on $[0,\, s_i^o]$ or $[s_i^o,\, 0]$. The random mechanism is common knowledge as a function of $(p,\, s^1)$.

The *direct* payoff from the secondary market to the underwriter is exactly what he gets in return for his supply, i.e. it is $\underline{a}(p,\, s^1)[s -$

$\sum_{i\in\mathcal{N}} s_i^o]$, if the market is active, and nothing otherwise. Since the underwriter gets no utility from holding shares at the end of the game, his utility function has just one argument (final wealth in terms of the first commodity) and is strictly monotone increasing in this direct payoff. His utility function can thus, without loss of generality, taken to be linear.

LEMMA 2. *In any pure strategy equilibrium on the secondary market, where the market can be active,*
(i) $\bar{b}(p, s^1) = \underline{b}(p, s^1)$, *and* $\underline{a}(p, s^1) = \bar{a}(p, s^1)$,
(ii) $\underline{b}(p, s^1) = \bar{a}(p, s^1)$.

By Lemma 2 the price configuration at any equilibrium on the secondary market, where the market can be active, can be described by a single number

$$p^1 = \bar{b} = \underline{b} = \bar{a} = \underline{a}.$$

From Lemma 2 it also follows that the market maker's profit will in any equilibrium be zero. Hence the market operates efficiently in the sense that specialists cannot earn money by making the market.

A *Walras equilibrium* on the secondary market is an n-tuple of quantities $(\bar{s}_i^1)_{i\in\mathcal{N}} \in \prod_{i\in\mathcal{N}}[-S_i - s_i^o, \infty)$ plus a price $p^1 \geq 0$ such that

$$\begin{aligned} &(w_i - p^o s_i^o - p^1 \bar{s}_i^1,\, \bar{s}_i^1) \in \\ &\quad \in \operatorname{argmax}_{p^1 s + x \leq w_i - p^o s_i^o} \int_\Omega u^i(x,\, R(\theta)(S_i + s_i^o + s))\, d\mu(\theta), \\ &\text{and} \quad \sum_{i\in\mathcal{N}} \bar{s}_i^1 = s - \sum_{i\in\mathcal{N}} s_i^o. \end{aligned}$$

In this "Walras-economy" investors take p^1 as given and optimize under their budget constraints, and there is an exogeneous supply of $s - \sum_{i\in\mathcal{N}} s_i^o \geq 0$. [A very similar result to the one to follow, though in a different game, where the market maker discriminates against individual traders by charging limit prices, has been demonstrated by **Simon**, 1984.]

THEOREM 1. *Any pure strategy (Nash) equilibrium on the secondary market, where the market can be active, must be a Walras equilibrium, and any Walras equilibrium can be supported as a pure strategy (Nash) equilibrium.*

Theorem 1 ensures that the set of Nash-equilibria on the secondary market coincides with the set of Walras equilibria of the corresponding Walras-economy, whenever the market can be active. The latter qualification is needed, because whenever $s - \sum_{i \in \mathcal{N}} s_i^o = 0$, there are Nash-equilibria at which every investor specifies a zero quantity in his limit order which is as good as anything else, given that nobody else wishes to trade. For the rest of the paper we will rule out such inactive equilibria, even when the underwriter's market order is zero. Then Theorem 1 allows to identify the equilibria of the subgame on the secondary market with the set of Walras equilibria which can be characterized.

When characterizing the set of Walras equilibria an assumption on a reference economy will be used. A *reference economy* is a "Walrasian" economy, where endowments of investors are given by $(w_i, S_i) \in \Re_{++} \times \Re_+$, $\forall i \in \mathcal{N}$, and there is an exogeneous supply of new shares $s > 0$. The reference economy, therefore, is an economy with price-taking investors, where no primary market existed prior to the operation of this economy, but new shares are sold in a perfectly competitive way. The assumption on the reference economy is that it is *regular*, i.e. that for all Walras equilibria of the reference economy the partial derivative of aggregate excess demand with respect to the price of shares (at the equilibrium) is non-zero. That this assumption indeed holds for almost all economies in the space of economies (parametrized by endowments) is a well known result from General Equilibrium Theory [**Dierker**, 1982].

To study the set of Walras equilibria redefine variables by setting $\zeta_i^1 = S_i + s_i^o + s_i^1$, $\forall i \in \mathcal{N}$, and consider an investor's problem in the Walrasian economy:

$$
(1) \qquad \begin{aligned} &\max_{(\zeta_i^1,\, x_i) \in \Re_+^2} \int_\Omega u^i(x_i,\, R(\theta)\zeta_i^1)\, d\mu(\theta) \\ &\text{s.t.} \quad p^1\zeta_i^1 + x_i \leq w_i - p^o s_i^o + p^1(S_i + s_i^o), \end{aligned}
$$

and define $\zeta_i^1 = \zeta_i^1(p^1, (w_i - p^o s_i^o, S_i + s_i^o))$ as the demand function of investor $i \in \mathcal{N}$ for shares.

LEMMA 3. *For all $i \in \mathcal{N}$ the demand function ζ_i^1 is a continuously differentiable function of all arguments, and*
(i) $\zeta_i^1 \geq S_i + s_i^o \implies \partial\zeta_i^1/\partial p^1 < 0,\ \forall i \in \mathcal{N}$;
(ii) $p^1 \to 0 \implies \zeta_i^1 \to +\infty$ *and* $\partial\zeta_i^1/\partial p^1 < 0,\ \forall i \in \mathcal{N}$;
(iii) $\mathit{sign}(\partial\zeta_i^1/\partial s_i^o) = \mathit{sign}(p^1 - p^o),\ \forall i \in \mathcal{N}$;
(iv) $S_i + s_i^o = 0 \implies \zeta_i^1 > 0$.

The market clearing condition at the Walrasian equilibrium can be rewritten as

$$\sum_{i \in \mathcal{N}} \zeta_i^1 = S + s.$$

Let $S_s^o = \{(s_i^o)_{i\in\mathcal{N}} \in \Re_+^n \,|\, 0 \le \sum_{i\in\mathcal{N}} s_i^o \le s\}$ and define the mapping $\beta\colon [0, \sum_{i\in\mathcal{N}} w_i/s] \to S_s^o$ by

$$\beta(p^o) = \{(s_i^o)_{i\in\mathcal{N}} \in S_s^o \,|\, p^o s_i^o \le w_i,\ \forall i \in \mathcal{N}\}.$$

The graph of this mapping

$$\mathcal{G}(\beta) = \{(p^o, (s_i^o)_{i\in\mathcal{N}}) \in [0, \sum_{i\in\mathcal{N}} w_i/s] \times S_s^o \,|\, (s_i^o)_{i\in\mathcal{N}} \in \beta(p^o)\}$$

is a connected and compact set of dimension $n+1$ and can serve as the parameter space for the correspondence assigning Walras equilibria. Letting G, $G \supset \mathcal{G}(\beta)$, be a neighbourhood of $\mathcal{G}(\beta)$ one may define the mapping $\zeta_1 : \Re_{++} \times G \to \Re$ by

$$\zeta_1(p^1, (p^o, (s_i^o)_{i\in\mathcal{N}})) = \sum_{i\in\mathcal{N}} \zeta_i^1(p^1, (w_i - p^o s_i^o, S_i + s_i^o))$$

and view the Walras equilibria as the preimage of $S + s > 0$, i.e. as $\zeta_1^{-1}(S+s)$. To make things sensible it will be assumed from now on that all investors together have sufficient budget to buy all shares in any of the Walras economies in $\mathcal{G}(\beta)$: $\sum_{i\in\mathcal{N}} w_i > p^* s$, where p^* denotes the largest equilibrium price at the secondary market over $\mathcal{G}(\beta)$.

LEMMA 4. *If the reference economy (with endowments $(w_i, S_i)_{i\in\mathcal{N}}$) is regular, then the preimage of $S+s>0$ under ζ_1, i.e. $\zeta_1^{-1}(S+s) \subset \Re_{++} \times G$, is a differentiable manifold of dimension $n+1$. Moreover, $\partial\zeta_1/\partial p^1 \neq 0$ for almost all $(p^o, (s_i^o)_{i\in\mathcal{N}}) \in G$.*

There is precious little extra information to be had on $\zeta_1^{-1}(S+s)$ which will, however not be formally proved here [**Dierker**, 1972]:
i] For almost all $(p^o, (s_i^o)_{i\in\mathcal{N}}) \in G$ the set of equilibrium prices p^1 is a finite set of odd cardinality.
ii] For almost all economies the sum of sign$(-\partial\zeta_1/\partial p^1)$ over all equilibrium prices p^1 must be $+1$. (This follows essentially from Lemma 4 and the boundary behavior of aggregate excess demand, $\zeta_1 \to_{p^1\to 0} +\infty$, and $p^1 \to +\infty \implies \zeta_1 < S + s$.)
iii] Both for the largest and the smallest equilibrium price p^1 one has generically $\partial\zeta_1/\partial p^1 < 0$. (This follows essentially from Lemma 3, (ii), together with the preceeding statement.)

The preimage $\zeta_1^{-1}(S+s) \cap \Re_{++} \times \mathcal{G}(\beta)$ gives all $(n+2)$-tuples of the form $(p^1, (p^o, (s_i^o)_{i\in\mathcal{N}}))$ of prices and allocations such that $\sum_{i\in\mathcal{N}} s_i^o \le s$, $p^o s_i^o \le w_i$, $\forall i \in \mathcal{N}$, and

$$\sum_{i\in\mathcal{N}} \zeta_i^1(p^1, (w_i - p^o s_i^o, S_i + s_i^o)) = S + s,$$

and, by Theorem 1, the set of all equilibria in the subgame at the secondary market, where the market can be active, for any feasible constellation on the primary market. An equilibrium of the overall game will specify a selection from $\zeta_1^{-1}(S+s)$. In other words: An equilibrium of the overall game will associate a *single* p^1 to any $(p^o, (s_i^o)_{i\in\mathcal{N}})$ on the primary market. For the moment we will, however, be especially interested in how the prices on the secondary market change, when parameters change.

LEMMA 5. *For all active equilibria at the secondary market*

$$\frac{\partial\zeta_1}{\partial p^1} < 0 \implies sign(\frac{dp^1}{ds_i^o}) = sign(p^1 - p^o),\ \forall i \in \mathcal{N},$$
$$\frac{\partial\zeta_1}{\partial p^1} > 0 \implies sign(\frac{dp^1}{ds_i^o}) = sign(p^o - p^1),\ \forall i \in \mathcal{N}.$$

Lemma 5 shows that in (s_i^o, p^1)-space the slope of p^1 with respect to s_i^o depends on the sign of $p^o - p^1$: If $p^o > p^1$, then "perverse" equilibria, with $\partial\zeta_1/\partial p^1 > 0$, will be upward sloping in s_i^o, and the "good" equilibria, with $\partial\zeta_1/\partial p^1 < 0$, will be downward sloping in s_i^o. If $p^o < p^1$, then "perverse" equilibria will be downward sloping and the "good" equilibria will be upward sloping in s_i^o. This completes the investigation of potential equilibria on the secondary market.

The market maker ("specialist") has not been considered as an active player here, because by Lemma 2 his profit is always zero in equilibrium. The market maker may, however, serve as an important coordination authority, when there are multiple equilibria, by communicating through his opening price to the investors which equilibrium will be played. If he does this in the interest of market participants, who play market orders ("best price orders"), this will select particular equilibria. In the present case only the underwriter places a supply-side market order. If the market maker indeed tries to obtain the best price for the underwriter, he will try to coordinate investors on the Walras equilibrium with the highest equilibrium price. This will yield a selection

$$p^1 = \bar{P}_1(p^o, (s_i^o)_{i\in\mathcal{N}}) = \max_{(p^1, (p^o, (s_i^o)_{i\in\mathcal{N}}))\in\zeta_1^{-1}(S+s)} p^1.$$

This particular selection will play a prominent role and is referred to as *efficiency with respect to market orders.*

5. THE PRIMARY MARKET

The primary market operates after the bargaining between the firm and the potential underwriters has ended and before the secondary market opens. On the primary market an underwriter or, if the bargaining ended without the conclusion of a contract, the firm attempts to sell the number $s > 0$ of new shares to investors via a public issue. The first case to be studied in the sequel is the case, where the bargaining was concluded successfully and the issue cannot fail anymore, because it is guaranteed by an underwriter with sufficient budget. The crucial effect of this is that trading opportunities at the secondary market are rationally expected by all investors in this case, for any constellation of trades at the primary market. The latter holds true, even if the primary market stays inactive, because all shares unsold at the primary market must then be dumped onto the secondary market by a market order. For the case of an underwritten issue only (subgame perfect) equilibria will be studied which assign an active equilibrium of the secondary market to outcomes of the primary market.

The alternative case of a non-underwritten issue, where the bargaining was unsuccessful and the firm itself supplies the new shares to the primary market, will be studied later in this section. This case still involves the risk of failure of the issue, leading to the possibility of a cancellation of the investment project of the firm.

Concerning information of investors, when they enter the primary market, they only know whether the issue is underwritten (the bargaining was successful) or not (the firm supplies the shares without a guarantee), but they are uninformed on any details of the preceeding bargaining process. In particular investors do *not* know at which price the underwriter and the firm have concluded a contract. This rules out that investors may condition their behavior at the primary market on details of the equilibrium path of the bargaining process and seems a rather realistic assumption.

The procedure at the primary market is as follows: First the underwriter publicly announces a price $p^o \in [0, \sum_{i\in\mathcal{N}} w_i/s]$ and an issuing method (from a restricted class of issuing methods). The constraint $p^o \leq (1/s)\sum_{i\in\mathcal{N}} w_i$ ensures that the underwriter does not ask a price at which it would be impossible to sell all new shares. The price p^o is understood as an offer to sell at the price p^o a maximum quantity of $s > 0$ shares. That the price p^o forms a binding commitment is part of the rules of the game at least at the US-market [**Ritter**, 1987, p.270; **Smith**, 1986, p.15]. An issuing method is a vector $r \in \Re^n_+$ which satisfies $\sum_{i\in\mathcal{N}} r_i \geq s$, $r_i \in [0, s]$, $\forall i \in \mathcal{N}$, and a rationing mechanism ϕ which is a probability measure on (a set contained in) $\Re^n_+$ conditional upon $(z_i^o)_{i\in\mathcal{N}} \in \Re^n_+$ such that

$$0 \leq \sum_{i\in\mathcal{N}} z_i^o \leq s \implies \phi(\{(z_i^o)_{i\in\mathcal{N}}\}) = 1,$$

and

$$\sum_{i\in\mathcal{N}} z_i^o > s \implies \phi(\{(s_i)_{i\in\mathcal{N}} \mid \sum_{i\in\mathcal{N}} s_i = s,\ s_i \in [0, z_i^o],\ \forall i \in \mathcal{N}\}) = 1,$$

which is continuous and (weakly) monotone increasing in all its (conditioning) arguments. The z_i^o, $i \in \mathcal{N}$, on which the rationing rule is conditional, are the investors' market orders of shares. The r_i, $i \in \mathcal{N}$ can be interpreted as non-renounceable rights (i.e. non-tradable rights), if $\sum_{i\in\mathcal{N}} r_i = s$ and $r_i = (s/S)S_i$, $\forall i \in \mathcal{N}$, or as the a-priori exclusion of particular investors (for whom $r_i = 0$) from participation in a general cash-offer, if $\sum_{i\in\mathcal{N}} r_i \geq s$ and $r_i = 0$ for some i.

This class of issuing methods is chosen to include both general cash-offers and non-renounceable rights issues, but it excludes renounceable (i.e. tradable) rights issues. The reason for this restriction is that public (underwritten) issues via general cash-offers of seasoned shares have dominated at least the (largest) US-market during the past decade (hence their inclusion), while rights issues are more common on smaller markets like the UK- or the Australian market (at the latter about one third of the rights issues were non- renounceable). And the underpricing phenomenon seems to be considerably more common at the US-market [**Asquith and Mullins**, 1986; **Kalay and Shimrat**, 1987; **Masulis and Korwar**, 1986; **Mikkelson and Partch**, 1986; **Smith**, 1986] than at smaller, rights-dominated markets; hence the exclusion of renounceable rights issues.

That renounceable rights issues are excluded from consideration simplyfies the analysis considerably, but is *not* a truely restrictive assumption. The reason, why this assumption only involves a minor loss of generality, works as follows: Suppose rights can be traded. We claim that there is always an equilibrium, where rights are not actually traded at the equilibrium path. Since transactions in rights are simply a reallocation of the investors' endowments which sums to zero (i.e. holding total endowments constant at $\sum_{i\in\mathcal{N}} w_i$ and $S + s$) the non- existence of an inactive equilibrium of the rights market would imply that there exists such a reallocation among investors which pareto-dominates the allocation without trades in rights. But this contradicts the first theorem of welfare economics, because the equilibria at the secondary market are Walras equilibria. Thus the exclusion of renounceable rights issues from the analysis can also be understood as the assumption that on the market for renounceable rights the inactive equilibrium will be played.

A general cash-offer will be taken to mean $r_i \geq w_i/p^o$, $\forall i \in \mathcal{N}$, and a rights issue can be taken to mean $r_i = (s/S)S_i$, $\forall i \in \mathcal{N}$. The class of allowable issuing methods is potentially much larger than just the two abovementioned methods for the mere sake of generality. The rationing mechanism for excess demand of new shares is chosen endogeneously to avoid problems of the sensitivity of the solutions with respect to rationing mechanisms [**cf. Kreps and Scheinkman**, 1983]. It will be shown, however, that the chosen issuing method has not a very large role to play. Letting effectively the underwriter supply the new shares even in a rights issue is without loss of generality, because the subscription of the issue by the underwriter is effectively a purchase of the new shares with the option to resell on the market; hence the assumption of a firm commitment underwriting contract. An alternative justification for assuming a firm commitment underwriting contract is that this is consistent with the present assumption of symmetric information: In the absence of asymmetric information the theory of contract choice predicts firm commitment contracts [**Mandelker and Raviv**, 1977; **Ritter**, 1987]. Summarizing, a choice (pure behavior strategy) for the underwriter at the primary market is a triple $(p^o, (r_i)_{i\in\mathcal{N}}, \phi)$ consisting of a price and an issuing method.

The underwriter's choice of $(p^o, (r_i)_{i\in\mathcal{N}}, \phi)$ defines a simultaneous move game among investors, in which the choices of investors are quantities of shares which they wish to buy at p^o (market orders). For technical reasons we have to assume that the possible quantity choicec of investors at the primary market form a finite set.[1] Although this assumption is made for technical reasons, it is by no means an unrealistic assumption: On most markets for primary issues shares are sold only in discrete quantities. Formally, let

$$Z_\varepsilon = \{0, \varepsilon, \dots, z\varepsilon\}, \; z\varepsilon = s > 0,$$

where $z = s/\varepsilon$ is assumed to be an integer number. Then the choices of investors at the primary market are

$$z_i^o \in [0, \min(r_i, w_i/p^o)] \cap Z_\varepsilon,$$

where the z_i^o are interpreted as market orders to buy, if $z_i^o > 0$.

[1] The reason is as follows: Since the function mapping outcomes of the primary market into equilibria of the secondary market may have discontinuities, one may encounter the phenomenon that an investor wishes to increase (decrease) his purchases at the primary market up to a point, where p^1 jumps and makes the investor strictly worse off. Depending on where the discontinuity point is mapped to, the investor may or may not have a (pure) best response.- Moreover, the properties of the Nash-equilibrium correspondence will have to be extensively used, and these are much better understood for finite games than for infinite action games.

Given an n-tuple $(z_i^o)_{i\in\mathcal{N}}$, either $0 \leq \sum_{i\in\mathcal{N}} z_i^o \leq s$ in which case transactions $(s_i^o)_{i\in\mathcal{N}} = (z_i^o)_{i\in\mathcal{N}}$ are carried out at the price p^o, or $\sum_{i\in\mathcal{N}} z_i^o > s$ in which case the rationing rule ϕ assigns $(s_i^o)_{i\in\mathcal{N}}$ such that $\sum_{i\in\mathcal{N}} s_i^o = s$, and $s_i^o \in [0,\, z_i^o]$, $\forall i \in \mathcal{N}$, and the quantities $(s_i^o)_{i\in\mathcal{N}}$ are traded at p^o by the investors.

An outcome of the primary market is an $(n+1)$-tuple $(p^o,\, (s_i^o)_{i\in\mathcal{N}})$ which is a point in $\mathcal{G}(\beta)$. Note that the current assumptions rule out short trading and credit facilities. The crucial difference of the primary market as compared to the secondary market is that on the former only the underwriter is allowed to play a publicly known limit order.

Since an outcome of the game at the primary market is a point in $\mathcal{G}(\beta)$ and after each of these points (nodes of the extensive form) a subgame at the secondary market starts, it is straightforeward to define payoffs for players at the primary market. Subgame perfection allows to substitute equilibrium payoffs from subgames for the subgames ["Backward Induction", cf. **Kohlberg and Mertens**, 1986], such that payoffs to players at the primary market can be defined directly for any equilibrium price p^1 at the secondary market (where p^1 is, of course, a function mapping points in $\mathcal{G}(\beta)$ into equilibria of the secondary market). The payoff to an investors $i \in \mathcal{N}$ from purchasing at the primary market a quantity s_i^o at the price p^o can thus be written as the investor's indirect utility function

$$V_i(p^1,\, (w_i - p^o s_i^o, S_i + s_i^o)) =$$
$$= \max_{p^1\zeta + x \leq w_i - p^o s_i^o + p^1(S_i + s_i^o)} \int_\Omega u^i(x,\, R(\theta)\zeta)\, d\mu(\theta).$$

The underwriter's payoff from the operation of the primary market is given by

$$\pi(p^1,\, (p^o,\, (s_i^o)_{i\in\mathcal{N}})) = p^o \sum_{i\in\mathcal{N}} s_i^o + p^1(s - \sum_{i\in\mathcal{N}} s_i^o),$$

where for both types of payoffs $(p^1,\, (p^o,\, (s_i^o)_{i\in\mathcal{N}})) \in \zeta_1^{-1}(S+s)$ must hold. Again observe that the underwriter's payoff can be taken to be linear without loss of generality, because the underwriter derives no utility from holding shares beyond the secondary market.

The indirect utility functions of investors, V_i, $i \in \mathcal{N}$, can be used to define induced preferences of investors on the space of pairs $(z_i^o,\, p^1) \in [-S_i,\, w_i/p^o] \times \Re_{++}$. The properties of these preferences are summarized in the following lemma:

LEMMA 6. *(i) For each $i \in \mathcal{N}$ indifference curves on $[-S_i, w_i/p^o] \times \Re_{++}$ for given $p^o > 0$ are described by*

$$\left.\frac{dp^1}{ds_i^o}\right|_{V_i=\text{const.}} = \frac{p^o - p^1}{S_i + s_i^o - \zeta_i^1},$$

where $\zeta_i^1 = \zeta_i^1(p^1, (w_i - p^o s_i^o, S_i + s_i^o))$ is the demand function at the secondary market.

(ii) For each $i \in \mathcal{N}$ there exists a continuously differentiable function $p_i^1 : (s_i^o, p^o) \mapsto p_i^1$ which is strictly monotone decreasing in s_i^o and strictly monotone decreasing (increasing) in p^o for all $s_i^o > 0$ ($s_i^o < 0$), such that

$$\zeta_i^1(p^1, (w_i - p^o s_i^o, S_i + s_i^o)) \begin{cases} > S_i + s_i^o, & \text{if } p^1 < p_i^1(s_i^o, p^o), \\ = S_i + s_i^o, & \text{if } p^1 = p_i^1(s_i^o, p^o), \\ < S_i + s_i^o, & \text{if } p^1 > p_i^1(s_i^o, p^o). \end{cases}$$

Moreover, if the parameter $S_i \geq 0$ increases, the function $p_i^1(s_i^o, p^o)$ shifts downward in (s_i^o, p^o)-space and

$$p_i^1(s_i^o, p^o) \rightarrow_{s_i^o \rightarrow w_i/p^o} 0, \quad \forall p^o > 0,$$
$$p_i^1(s_i^o, p^o) \rightarrow_{s_i^o \rightarrow -S_i} +\infty, \quad \forall p^o > 0.$$

By Lemma 6 the space of pairs (s_i^o, p^1) is partitioned into four regions by the graph of the function $p_i^1(s_i^o, p^o)$, for fixed p^o, and the horizontal line $p^1 = p^o$: The first region is to the South-West, where $p^1 \leq \min(p^o, p_i^1(s_i^o, p^o))$ and, consequently, $\zeta_i^1 \geq S_i + s_i^o$. This first region extends to the right until $s_i^o = w_i/p^o$ holds. The second region is to the South-East, where $p_i^1(s_i^o, p^o) < p^1 < p^o$ and $\zeta_i^1 < S_i + s_i^o$. The third region is to the North-East, where $p^1 \geq \max(p^o, p_i^1(s_i^o, p^o))$ and $\zeta_i^1 \leq S_i + s_i^o$. The fourth region is to the North-West, where $p^o < p^1 < p_i^1(s_i^o, p^o)$ and $\zeta_i^1 > S_i + s_i^o$. In the first and third region indifference curves are downward sloping, while in the second and fourth region indifference curves are upward sloping. Along the graph of $p_i^1(s_i^o, p^o)$, for fixed p^o, indifference curves are vertical. For $p^1 > p^o$ the direction of increasing utility is to the right, for $p^1 < p^o$ it is to the left, and the horizontal line $p^1 = p^o$ is itself an indifference curve. Finally, any pair (s_i^o, p^1) in the second or fourth region is strictly worse than any pair on the horizontal line $p^1 = p^o$. Monotonicity and the boundary behavior of $p_i^1(s_i^o, p^o)$ imply that there exists a unique point in the interior, where $p_i^1(s_i^o, p^o) = p^o$. The value of s_i^o at this point can be determined from

$$\zeta_i^1(p^o, (w_i - p^o s_i^o, S_i + s_i^o)) = \zeta_i^1(p^o, (w_i, S_i)) = S_i + s_i^o$$
$$\Longrightarrow s_i^o = \zeta_i^1(p^o, (w_i, S_i)) - S_i.$$

Hence $(\zeta_i^1(p^o, (w_i, S_i)) - S_i, p^o)$ is a saddle point of the indirect utility function V_i.

Part of any subgame perfect equilibrium of the overall game is some function $P_1 : \mathcal{G}(\beta) \to \Re_{++}$ defined by the equilibrium condition at the secondary market,

$$(P_1(p^o, (s_i^o)_{i\in\mathcal{N}}), (p^o, (s_i^o)_{i\in\mathcal{N}})) \in \zeta_1^{-1}(S+s),$$

i.e. the graph of P_1 satisfies $\mathcal{G}(P_1) \subset \zeta_1^{-1}(S+s)$. Two qualifications are in order here: Strictly speaking, if $s = \sum_{i\in\mathcal{N}} s_i^o$, then P_1 may also map into inactive equilibria of the secondary market. Since this is not a very interesting case, it has been ruled out for all what follows. Second, P_1 may also depend on the chosen issuing method at the primary market, rather than only on the price p^o and the allocation $(s_i^o)_{i\in\mathcal{N}}$. But since the (Walrasian) equilibria of the secondary market only depend on the investors' endowments, there is no loss of generality for the results to be presented, when P_1 is assumed to depend exclusively on the outcome of the primary market (in fact any dependence on the issuing method, or other information in the game, can be packed into the way $\mathcal{G}(P_1)$ is selected from $\zeta_1^{-1}(S+s)$).

The function P_1 can potentially have very awkward properties. In principle, for all subsets of $\mathcal{G}(\beta)$ for which multiple (active) equilibria exist at the secondary market, P_1 can potentially be a nowhere differentiable function. This, however, would imply that, given some point in $\mathcal{G}(\beta)$ for which multiple equilibria of the secondary market do exist, for any close point in $\mathcal{G}(\beta)$ the equilibrium of the secondary market (selected by P_1) may be quite far away from the original one. Hence small "trembles" in the strategy choice of any player at the primary market could alter the equilibrium in the ensuing subgame dramatically. Although equilibria with such strongly discontinuous P_1- functions may lend some substance to the notion that financial markets may be unstable, it is hard to see, why such phenomena should be peculiar to financial markets. The game theoretic consequences of allowing excessively many discontinuities of P_1 (though some discontinuities may be unavoidable) are rather unattractive: Potentially a proper choice of discontinuities may support all sorts of fancy behavior as equilibrium behavior. On the other hand, P_1 also enters directly the underwriter's payoff function which may, therefore, not be continuous in his pure strategies. The latter may even prohibit the existence of equilibria in pure strategies. Thus the following theorem implicitely defines restrictions on P_1 and is, therefore, of some and not merely technical importance. Effectively the following theorem shows that an equilibrium of the primary market exists, if the secondary market is efficient with respect to market orders.

THEOREM 2. *There always exists a subgame perfect equilibrium for the game at the primary market.*

The proof of Theorem 2 indicates what a sufficient condition on P_1 to ensure existence is: The set of discontinuities of P_1 must be "small" in $\mathcal{G}(\beta)$, viz. P_1 should be "almost everywhere" continuous. The particular choice of P_1 in the proof of Theorem 2, $P_1 = \bar{P}_1$, ensures this stability property in a natural way. The selection $P_1 = \bar{P}_1$ is also desirable from a purely game theoretic point of view, because it ensures that the property, that small trembles do not change the equilibrium in the ensuing subgame too much, holds for most of $\mathcal{G}(\beta)$.

But there is also some economic significance of $P_1 = \bar{P}_1$. Recall that, in case $s > \sum_{i \in \mathcal{N}} s_i^o$, the underwriter will be forced to dump the remaining shares onto the secondary market by a market order. A market order is an order to sell the specified quantity at the *best* price available. If the underwriter, or the market maker at the secondary market, could somehow communicate to investors what the best price is, at which the market order can be carried out, this would always result in $P_1 = \bar{P}_1$. Of course, a communication mechanism by which the underwriter (or the market maker) could coordinate investors on a particular equilibrium was not specified in such a way that it would be binding for investors. The opening price of the market maker could serve as such a coordination device, but nothing prohibits the investors from ignoring the opening price. Still an *interpretation* of $P_1 = \bar{P}_1$ in this spirit is possible: As in Section 4, say that the secondary market is *efficient with respect to market orders*, if $P_1 = \bar{P}_1$. (This definition is not vacuous, because Theorem 2 demonstrates existence precisely for this selection.)

The next and core underpricing result of the analysis says two things: First it states that, if an equilibrium of the primary market is an inactive equilibrium with $s_i^o = 0$, $\forall i \in \mathcal{N}$, then the primary market can be made active by underpricing the issue. The second part states that, if the secondary market is efficient with respect to market orders, then any active equilibrium of the primary market must be an underpricing equilibrium.

PROPOSITION 1. *(i) If there exists an equilibrium at the primary market such that $s_i^o = 0$, $\forall i \in \mathcal{N}$, then there exists an equilibrium with $\sum_{i \in \mathcal{N}} s_i^o > 0$ and $p^1 \geq p^o$.*

(ii) If the secondary market is efficient with respect to market orders, $P_1 = \bar{P}_1$, then for any equilibrium at the primary market

$$\exists\, i \in \mathcal{N}\colon s_i^o > 0 \implies p^1 \geq p^o,$$

for all $\varepsilon > 0$ sufficiently small.

PROOF: (i) Assume there exists an equilibrium with $s_i^o = 0$, $\forall i \in \mathcal{N}$, as its outcome. Let $\bar{p}$ be the largest equilibrium price in the reference economy, i.e. the largest p which solves

$$\sum_{i \in \mathcal{N}} \zeta_i^1(p, (w_i, S_i)) = S + s. \tag{2}$$

By definition, $\bar{p} \geq P_1(p^o, (0)_{i\in\mathcal{N}})$, $\forall p^o \in (0, \sum_{i\in\mathcal{N}} w_i/s]$, because clearly $P_1(p^o, (0)_{i\in\mathcal{N}})$ also solves (2). Thus the underwriter's profit at the conjectured equilibrium is

$$\pi(P_1, (p^o, (0)_{i\in\mathcal{N}})) = P_1 s \leq \bar{p}s.$$

Now suppose the underwriter sets, instead of p^o, the price at the primary market equal to $\bar{p}$. Then alter P_1 at $\bar{p}$ to evaluate to

$$P_1(\bar{p}, (s_i^o)_{i\in\mathcal{N}}) = \bar{p}, \quad \forall (s_i^o)_{i\in\mathcal{N}} \in \prod_{i\in\mathcal{N}} [0, w_i/\bar{p}],$$

which is always possible, because

$$\zeta_i^1(\bar{p}, (w_i - \bar{p}s_i^o, S_i + s_i^o)) = \zeta_i^1(\bar{p}, (w_i, S_i)), \ \forall s_i^o,$$

leaving the rest of P_1 intact. By Lemma 6 investors are now indifferent among all allocations of $(s_i^o)_{i\in\mathcal{N}}$ and thus there exists an allocation $(s_i^o)_{i\in\mathcal{N}}$ such that $\sum_{i\in\mathcal{N}} s_i^o > 0$, $s_i^o \in [0, w_i/p^o]$, $\forall i \in \mathcal{N}$, which is an equilibrium of the subgame starting after the underwriter has announced $\bar{p}$ and an issuing method which makes the new allocation feasible. But now the underwriter's profit is $\bar{p}s$, such that, if the original configuration was an equilibrium, then the new one must be an equilibrium as well.

(ii) To demonstrate the second part, set $P_1 = \bar{P}_1$ and suppose that $p^1 = \bar{P}_1(p^o, (s_i^o)_{i\in\mathcal{N}}) < p^o$ and $\exists i \in \mathcal{N}$: $s_i^o > 0$. Lemma 5 shows that, wherever $\bar{P}_1$ is differentable,

$$\text{sign}(\frac{dp^1}{ds_i^o}) = \text{sign}(p^1 - p^o) = -1, \ \forall i \in \mathcal{N}.$$

We first claim that the conjectured equilibrium pair (s_i^o, p^1), $s_i^o > 0$, cannot be located in the second region of (s_i^o, p^1)-space as specified by Lemma 6, where $p^o > p^1 > p_i^1(s_i^o, p^o)$. To see this, first observe that, if $\bar{P}_1$ is differentiable on the whole interval $[0, w_i/p^o]$, then it must be decreasing in $s_i^o \in [0, w_i/p^o]$. But $\bar{P}_1$ may have discontinuities on $[0, w_i/p^o]$. Such a discontinuity can be an upward or a downward jump (from the left). If a discontinuity is a downward jump from the left, then $\bar{P}_1$ remains strictly monotone decreasing. If the discontinuity is an upward jump from the left, the value of $p^1 = \bar{P}_1(\,.\,)$ to which $\bar{P}_1$ jumps upward must be located (weakly) above p^o: If the value of p^1 to which $\bar{P}_1$ jumps upward (from the left) would be located below p^o, then from this value p^1 two branches (extending to the right) of

$$\zeta_1^{-1}(S+s) \cap \Re_{++} \times (\{p^o\} \times \{(s_j^o)_{j\in\mathcal{N}\setminus\{i\}}\} \times [0,\, w_i/p^o])$$

emerge, because possible discontinuities only occur at "critical" Walras economies, where the projection of $\zeta_1^{-1}(S+s)$ into $\mathcal{G}(\beta)$ is not surjective. By Lemma 5 the higher one of these two branches must have $\partial\zeta_1/\partial p^1 > 0$ and can, therefore, not be the location of $\bar{P}_1$ (see the third remark following Lemma 4) which contradicts the assumption of an upward jump leading to a value $p^1 < p^o$. Thus any upward jump from the left must lead to a value of p^1 (weakly) above p^o. If this is the case and the value of $z_i^o \in [0,\, w_i/p^o]$, at which this upward jump of $\bar{P}_1$ occurs, satisfies $z_i^o \leq r_i$, then this z_i^o is certainly preferable to $s_i^0 < z_i^o$ by Lemma 6, such that (from the equilibrium assumption) it cannot be feasible under $((r_i)_{i\in\mathcal{N}},\, \phi)$. Consequently, for all feasible values of z_i^o the function $\bar{P}_1$ must be strictly decreasing with respect to z_i^o. But by Lemma 6 indifference curves in the second region are upward sloping and the direction of increased utility is to the North- West, such that, if $\bar{P}_1$ is monotone decreasing in the feasible region, the optimal value is $z_i^o \leq \zeta_i^1(p^o,\, (w_i,\, S_i)) - S_i$, contradicting the assumption that $(s_i^o,\, p^1)$, $s_i^o > 0$, is located in the second region. Hence $(s_i^o,\, p^1)$ must be located in the first region.

We claim that $(s_i^o,\, p^1)$, $s_i^o > 0$, $p^1 = \bar{P}_1(\,.\,) < p^o$, cannot be located in the first region either. To see this, observe that by Lemma 5 for $P_1 = \bar{P}_1$ any differentiable branch of $\bar{P}_1$ in the first region must be downward sloping in s_i^o. By Lemma 4 $\bar{P}_1$ can only have finitely many discontinuities, when s_i^o varies, but it may have some: Suppose it has an upward jump (from the left) at $s_i^o > 0$. Then again this jump must lead to a value of p^1 satisfying $p^1 \geq p^o$. Since by definition of $\bar{P}_1$ this $p^1 \geq p^o$ is the value of $\bar{P}_1$, $s_i^o > 0$ cannot be located exactly at the point, where $\bar{P}_1$ jumps upward by the hypothesis, but must be located either to the left of the discontinuity, in a differentiable region, or at a downward jump (from the left) of $\bar{P}_1$. Hence, if s_i^o sits below a discontinuity of $\bar{P}_1$, then the discontinuity must be a downward jump (from the left). Again, because discontinuities occur at "critical economies", this implies that there is some $\hat{s}_i^o < s_i^o$ arbitrary close to s_i^o for $\varepsilon > 0$ sufficiently small, such that at $\hat{s}_i^o$, $\hat{p}^1 = \bar{P}_1(p^o, (\sum_{j \in \mathcal{N} \setminus \{i\}} s_j^o, \hat{s}_i^o)) < p^o$, $\sum_{j \in \mathcal{N}} \partial\zeta_j^1/\partial p^1 \big|_{p^1=\hat{p}^1} < 0$,

$$(3) \qquad 0 > \frac{p^o - \hat{p}^1}{S_i + \hat{s}_i^o - \hat{\zeta}_i^1} \geq \frac{(p^o - \hat{p}^1)\partial\hat{\zeta}_i^1/\partial\hat{y}_i}{\sum_{j \in \mathcal{N}} \partial\hat{\zeta}_j^1/\partial\hat{p}^1},$$

(with the obvious notation and $\hat{y}_i = w_i - p^o\hat{s}_i^o + \hat{p}^1(S_i + \hat{s}_i^o)$) which is equivalent to

$$(4) \qquad \begin{aligned} &\sum_{j \in \mathcal{N}} \frac{\partial\hat{\zeta}_j^1}{\partial\hat{p}^1} \geq (S_i + \hat{s}_i^o - \hat{\zeta}_i^1)\frac{\partial\hat{\zeta}_i^1}{\partial\hat{y}_i} \iff \\ \iff &\sum_{j \in \mathcal{N} \setminus \{i\}} \frac{\partial\hat{\zeta}_j^1}{\partial\hat{p}^1} \geq \frac{1}{\Delta_i}\int u_1^i \, d\mu, \end{aligned}$$

where $\Delta_i > 0$ is the determinant defined in the proof of Lemma 3. But this implies that there is some $j \in \mathcal{N} \setminus \{i\}$, such that $\partial\hat{\zeta}_j^1/\partial\hat{p}^1 > 0$ and by continuity $\partial\zeta_j^1/\partial p^1 \geq 0$, for $\varepsilon > 0$ sufficiently small. The latter implies that for this $j \in \mathcal{N} \setminus \{i\}$ it must be true by Lemma 3, (i), that $\zeta_j^1 < S_j + s_j^o$. But, combined with $p^1 < p^o$, this is the statement that the pair (s_j^o, p^1) for this investor j is located in the second region of Lemma 6, contradicting the finding above.

Thus $s_i^o > 0$ must be somewhere in the first region, where $\bar{P}_1$ is differentiable. But this implies from $dV_i/ds_i^o \geq 0$ and the hypothesis $p^1 = \bar{P}_1(.) < p^o$ again the inequalities (3) which yield, via the equivalent inequalities (4), again the conclusion that for some investor $j \in \mathcal{N} \setminus \{i\}$ the pair (s_j^o, p^1) must be in the second region - a contradiction. The conclusion is that for all equilibria $\exists i \in \mathcal{N}$: $s_i^o > 0$ implies $p^1 = \bar{P}_1(p^o, (s_i^o)_{i \in \mathcal{N}}) \geq p^o$. ∎

Proposition 1 shows that there is a close relationship between underpricing of new issues and activity of the primary market. Still this relationship is much less rigid as it would be, if investors would take prices (at the secondary market) as given: In case investors conceive no influence on the price at the secondary market, shares at the primary and the secondary market would be perfect substitutes. Then $p^1 < p^o$ (overpricing) would imply that all investors wish to buy exclusively at the secondary market, while only $p^1 \geq p^o$ (underpricing) can ensure that the primary market is active.

Proposition 1, (i), also reveals an interesting aspect of the underpricing phenomenon. If, in fact, strict underpricing, $p^1 > p^o$, occurs at an active primary market, then it is underpricing with respect to the equilibrium price at the secondary market *after* the (non- zero) trades at the primary market. It is *not* underpricing with respect to the (largest) equilibrium price of shares in the reference economy, in the sense that no equilibrium would exist which would yield the underwriter the (largest) equilibrium price in the reference economy. On the contrary, there always exists an equilibrium in which the underwriter gets at least what he would earn as a passive supplier to the reference economy.

The present theory also yields a straightforward explanation of why *initial* public offerings (of unseasoned shares) are quite systematically underpriced [**Ibbotson, Sindelair and Ritter**, 1988; for alternative explanations under asymmetric information, see: **Gale and Stiglitz**, 1989; **Grinblatt and Hwang**, 1989; **Welch**, 1989.]

COROLLARY 1. *In any equilibrium of the primary market for an initial public offering,* $S_i = 0$, $\forall i \in \mathcal{N}$,

$$\sum_{i \in \mathcal{N}} s_i^o = s \implies p^1 \geq p^o.$$

PROOF: For an initial public offering $S_i = 0$, $\forall i \in \mathcal{N}$, implies that $\zeta_i^1(p^o, (w_i, S_i)) = \zeta_i^1(p^o, (w_i, 0)) > S_i = 0$, such that the saddle point of V_i in (s_i^o, p^1)-space from Lemma 6 occurs for some strictly positive value of s_i^o. This implies that the pair (s_i^o, p^1) can for no investor be located in the closure of the second region of Lemma 6, if $p^1 < p^o$, because setting $z_i^o = \zeta_i^1(p^o, (w_i, 0)) < s_i^o$ would always dominate s_i^o and is always feasible. Thus for all investors the pairs (s_i^o, p^1) must be located in the interior of the first region of Lemma 6, if $p^1 < p^o$. But this implies $\zeta_i^1 > S_i + s_i^o = s_i^o$, $\forall i \in \mathcal{N}$. The latter, however, yields the contradiction $S + s = \sum_{i \in \mathcal{N}} \zeta_i^1 > S + \sum_{i \in \mathcal{N}} s_i^o$ to the hypothesis $s = \sum_{i \in \mathcal{N}} s_i^o$. ∎

The Corollary is especially important, if the firm rather than an underwriter supplies the unseasoned shares to the primary market. In this case it may be forced to sell all shares at the primary market to avoid a failure of the issue. Then it can never avoid underpricing. Thus an underwritten initial public offering *may* be underpriced, a non-underwritten issue of unseasoned shares *will* be underpriced (if the investment project is cancelled, in case not all new shares are sold at the primary market).

The reader may wonder, when strict underpricing, $p^1 > p^o$, can be an equilibrium. We have no general result to offer on this problem. The only thing we can say is that examples suggest that multiplicity of equilibria in the neighbourhood of the largest equilibrium price of the reference economy (yielding discontinuities of P_1) seems to do the job. The issue remains, however, an important field for future research.

Turning to the case, where the issue is not underwritten, i.e. the bargaining failed, it will from now on be assumed that there is a unique equilibrium price $\bar{p}^o$ in the subgame, where the issue is offered underwritten at the primary market. This may seem like a restrictive assumption at first sight, but in fact there is no loss of generality for what we will have to say: Any equilibrium of the subgame, where the issue is offered underwritten at the primary market, is part of some solution of the overall game. Since the characterization of the solution to the subgame, where the issue is offered non-underwritten, will depend exclusively on the price which can be obtained in the other subgame (where the issue is underwritten), the equilibria can be parametrized by the price obtained in the subgame, where the issue is underwritten. (Recall that all the information which investors have, when they enter the primary market, is whether the issue is underwritten or not, and nothing more.) Thus any statement which refers to "the" price $\bar{p}^o$ in the subgame, where the issue is underwritten, can also be read as a statement conditional upon the value of the equilibrium price in the subgame, where the issue is offered underwritten at the primary market.

In the subgame, where the issue is non-underwritten, the procedure is basically the same as in the case, where it is underwritten, except that now, if not all new shares are sold at the primary market, the project is cancelled and nothing is traded at the primary market. Alternatively one could fix a quantity $\bar{s} \in (0,\, s)$, such that, if not at least $\bar{s}$ new shares are sold at the primary market, the issue fails and no transactions take place (or: investors, who bought new shares, get fully reimbursed). Since it is quite obvious, how the proposition to follow has to be adapted to the alternative case, there is no true loss of generality in assuming $\bar{s} = s$.

Thus, if the issue is non-underwritten, the firm chooses a price and an issuing method, $(p^o,\, (r_i)_{i\in\mathcal{N}},\, \phi)$, which is publicly announced and binding, and then investors choose market orders $z_i^o \in [0,\, \min(r_i,\, w_i/p^o)] \cap Z_\epsilon$, as before. The investors' payoff functions also remain unchanged. The firm's payoff is $p^o s$, if $\sum_{i\in\mathcal{N}} s_i^o = s$ ($\sum_{i\in\mathcal{N}} z_i^o \geq s$), and zero otherwise, because, if the issue fails, the firm cannot dump the remaining shares onto the secondary market. (If $\bar{s} < s$, then it would be $p^o \sum_{i\in\mathcal{N}} s_i^o + p^1(s - \sum_{i\in\mathcal{N}} s_i^o)$, if $\sum_{i\in\mathcal{N}} s_i^o \geq \bar{s}$, and zero otherwise.)

PROPOSITION 2. *If the firm offers the new shares non- underwritten at the primary market, then any price p^o which satisfies*

$$\frac{1}{s}\, max_{i\in\mathcal{N}}\, w_i < p^o \leq \bar{p}^o$$

can be supported as a subgame perfect equilibrium, where $\bar{p}^o$ denotes the *equilibrium price in the subgame, where the issue is offered underwritten at the primary market.*

PROOF: The proof is by construction. Clearly $\bar{p}^o$ can be supported by imitating the behavior in the subgame, where the issue is underwritten. Let $p_o \in (\max_{i\in\mathcal{N}} w_i/s,\ \bar{p}^o]$ and assume that investors play the following strategies: If $p^o > p_o$, then $z_i^o = 0$, and, if $p^o \leq p_o$, then

$$z_i^o \in \ \mathrm{argmax}_{z\in[0,\min(r_i,\,w_i/p^o)]\cap Z_i} \int V_i(P_1(p^o,\,(s_j^o)_{j\in\mathcal{N}}),$$
$$(w_i - p^o s_i^o,\, S_i + s_i^o))\, d\phi((s_j^o)_{j\in\mathcal{N}} | (z_j^o)_{j\in\mathcal{N}\setminus\{i\}},\, z),$$

for some fixed function $P_1 : \mathcal{G}(\beta) \to \Re_{++}$. We claim that this is subgame perfect behavior, because given some $p^o > p_o$ and $z_j^o = 0$, $\forall j \in \mathcal{N} \setminus \{i\}$, the maximum that investor i can invest in new shares is $\min(r_i,\, w_i/p^o) \leq w_i/p^o < s$, by $p^o > p_o > \max_{j\in\mathcal{N}} w_j/s \geq w_i/s$. Hence investor i cannot guarantee the issue, such that it will fail under $p^o > p_o$. Given that the issue fails and no transactions will be carried out at the primary market, any z_i^o is a best response for investor i. Thus the investor's behavior is subgame perfect behavior. Given the investors' behavior, the firm's best choice is to set $p^o = p_o$ which is then the equilibrium price at the primary market. ∎

The method of the proof of Proposition 2 is familiar from the literature on bank-runs [**Diamond and Dybvig**, 1983; **Eichberger and Milne**, 1990] in that it uses the dependence of the value of an asset on the public's expectation on its value. Although the proof of the proposition follows directly from the fact that no individual investor can guarantee the issue, it is - at least in our opinion - a method of proof indeed peculiar to financial markets (as opposed to markets for real consumption): A financial asset has only value, if sufficiently many players believe it to have value. Hence the valuation of a financial security is not underpinned by the utility which holders derive from its consumption, but rather rests on equilibrium expectations on how it will convert into real commodities at a later stage, viz. on a Nash-equilibrium rather than on technological fundamentals. If the project, presumably financed by the issue of new shares, would be perfectly divisible (with constant returns to scale) and each investor could somehow persuade the firm to erect the piece of the project which the investor can finance, the proof of Proposition 2 would not go through anymore. The reason for this is that in the latter case each share translates automatically (without being dependend on the other investors' decisions) into real consumption. The point of Proposition 2 is that precisely, because the latter case is *not* typical for equity issues, there may be extra costs (in the form of even more severe underpricing) associated with an issue, if it is non-underwritten. And this is a possible explanation, why firms do indeed, in the vast majority of cases, use underwriters.

This does *not* imply that a non-underwritten issue will always be more severely underpriced than an underwritten issue. Proposition 2 only states that this can happen. In fact, an outside observer of games with the above structure is likely to observe less underpricing of non-underwritten issues than of underwritten issues. The reason is as follows: If the firm expects strong underpricing in the subgame, where the issue is non-underwritten, it has a strong incentive to conclude a contract with an underwriter. If, on the other hand, the firm expects no underpricing in the subgame, where the issue is offered without the aid of an underwriter, it may in fact decline the underwriter's service. But in equilibrium this must be "rational expectations", i.e. if the firm does not conclude a contract with an underwriter, then in equilibrium the issue will not fail and will not be "too much" underpriced.

The above discussion already makes it clear on what the firm and the potential underwriters will have to bargain at the first stage of the overall game: Let $\bar{\pi}$ denote the expected equilibrium profit of the underwriter in the subgame, where the issue is offered underwritten. Again this is mere terminology, because the arguments to follow hold for all equilibrium payoffs to underwriters from the corresponding subgames. Since investors at the primary market only know, whether the issue is underwritten or not, $\bar{\pi}$ is constant across potential underwriters. The net return for an underwriter from an underwritten issue is $\bar{\pi}$ minus what the underwriter has to pay to the firm for the new shares which will be denoted by qs. The firm's revenue from an underwritten issue is qs. If the issue is non-underwritten, any potential underwriter obtains zero and the firm obtains $p_o s$ which is understood as the equilibrium revenue from the primary market, if the issue does not fail, and $p_o = 0$, if the issue will fail in a non- underwritten issue. (If $\bar{s} < s$, then $p_o = 0$, if the issue fails in equilibrium, and $p_o s = p^o \sum_{i \in \mathcal{N}} s_i^o + p^1(s - \sum_{i \in \mathcal{N}} s_i^o)$, otherwise.) Since the assumptions on the investors' information structure imply that only the fact, whether the issue is underwritten or not, matters, $\bar{\pi}$ and $p_o s$ can be held constant for all what follows. If $p_o s > \bar{\pi}$, then, since the firm will never accept a contract with $qs < p_o s$, $\bar{\pi} - qs < 0$ such that no underwriter is willing to conclude a contract. Hence for $p_o s > \bar{\pi}$ there is nothing to bargain on. If $p_o s = \bar{\pi}$, then $qs \geq p_o s = \bar{\pi} \geq qs$ implies that, if the bargaining ends with a contract, the $q = p_o$. Thus the case to be studied in the next section is the case, where $p_o s < \bar{\pi}$, i.e. where the underwriter can indeed earn a larger return on the market than the firm can do on its own.

6. Bargaining on an underwriting contract

The first of the three stages of the game is a finite bargaining process between the firm and $m \geq 2$ potential underwriters. The underwriters are all identical, that is they all derive utility only from the first commodity (consumption good, or money), with constant marginal utility, and will have no utility from holding shares beyond the final stage of the game. Moreover, each underwriter has sufficient budget to potentially finance the firm's project without the aid of investors, that is, each underwriter can unilaterally guarantee the issue. Underwriters, however, compete with each other for the contract with the firm, if the contract is profitable. The profitability of an underwriting contract is measured against alternative uses of the underwriter's funds in the economy. These alternative uses of funds are either contracts with other firms in the background, or investments of the funds into other securities. The payoff to an underwriter from investing his budget into these alternative opportunities is normalized to zero.

The firm wishes to finance a given indivisible project by the issue of new shares and the cost of this project is fixed exogeneously. But the firm also derives utility from earnings in excess of the cost of the investment project. Denoting by $c > 0$ the cost of the investment project, the firm's payoff from the issue is whatever it earns in return for the new shares minus c. Let $\underline{\pi} = \max(c, p_o s)$ denote the firm's "reservation-revenue", where $p_o s$ is the firm's revenue from a non-underwritten issue, and denote by $\bar{\pi}$ an underwriter's revenue from an underwritten issue. Then there is room for a mutually profitable contract between the firm and an underwriter, if and only if $\bar{\pi} > \underline{\pi}$, because, if $\bar{\pi} \leq \underline{\pi}$, then either $\bar{\pi} \leq c$, in which case the underwriter cannot even raise the cost of the project, or $\bar{\pi} \leq p_o s$, in which case the firm has no incentive to conclude a contract with an underwriter (or both cases hold), and $\bar{\pi} > \underline{\pi}$ implies both $\bar{\pi} - c > 0$ and $\bar{\pi} - p_o s > 0$. For $\bar{\pi} > \underline{\pi}$ the surplus to be shared between the firm and the underwriter is given by $\bar{\pi} - c > 0$. For convenience this surplus will be normalized to 1 (i.e. the equilibrium division of the surplus has to be multiplied by $\bar{\pi} - c$ to obtain the true values).

Then the bargaining problem reduces to the problem of splitting a unit surplus between the firm and any of the m potential underwriters. This problem will be treated in the fashion of the literature on non-cooperative bargaining [**Rubinstein**, 1982; **Shaked and Sutton**, 1984; **Wolinsky**, 1985; for an overview see: **Osborne and Rubinstein**, 1990].

The precise extensive form of the bargaining stage works as follows: It is a T-period bargaining game without discounting, where $m < T < \infty$ is a finite number (taken to be very large). The set of potential underwriters, $\mathcal{M} = \{1, \ldots, m\}$, is indexed in such a way as to reflect the order in which the firm contacts the various underwriters (the indexing of underwriters together with the set $\mathcal{M}$ will sometimes be referred to as a "queue"). In the first of the T periods, the firm is matched with underwriter $j = 1$ and a chance move decides whether the firm (with probability $\alpha \in (0, 1)$) or underwriter 1 (with probability $(1 - \alpha) \in (0, 1)$) will make an offer. Then the chosen proposer makes his offer, which is a number $\omega \in [0, 1]$, if the firm proposes, and $\pi \in [0, 1]$, if underwriter 1 proposes. The responder can then choose to accept the offer or to reject it. If the offer is accepted, the first stage of the game terminates immediately and the underwriter offers the new shares at the primary market. If the offer is rejected, then the proposer can choose whether to continue bargaining or to split and then the responder can decide whether to continue or quit. If both partners decide to continue bargaining, then in the next period the firm and underwriter 1 repeat the same game (but now with a time horizon of $T - 1$). If any one of the two partners decides to split, then underwriter 1 chooses whether to invest his funds somewhere else (yielding him a payoff of zero) or to wait for a chance to renegotiate with the firm after it has declined contracts with all other underwriters $j \in \mathcal{M} \setminus \{1\}$, and the firm moves on to bargain with underwriter $j = 2$ in the next period. In the latter case, the firm repeats the same structure of bargaining in period $t = 2$ with underwriter $j = 2$. If underwriter 1 has decided not to withdraw, but to wait for renegotiation with the firm, he has to queue in behind the other underwriters, who still wait for a chance to bargain with the firm, at the *end* of the queue. This procedure is repeated until either an agreement between the firm and some underwriter has been reached, or period T is reached, or all underwriters have withdrawn by investing their funds somewhere else. In the first case, if an agreement has been reached, the firm obtains ω, if it proposed, or $1 - \pi$, if the underwriter proposed, and the successful underwriter obtains $1 - \omega$, if the firm proposed, of π, if he himself proposed, i.e. the subgame, where the underwriter supplies to the primary market, is reached. If period T is reached without an agreement, then the firm tries to issue the shares non-underwritten and obtains its payoff from the equilibrium in this subgame, and all underwriters obtain zero. The latter also holds true, in case all underwriters withdraw from potential bargains with the firm.

THEOREM 3. *There exists a subgame perfect equilibrium of the bar-*

gaining stage, such that the firm's share of the surplus is given by

$$\bar{\omega} = 1 - (1 - \alpha)^m \in (0, 1).$$

The equilibrium constructed in the proof of Theorem 3 shows that the firm will conclude a contract with underwriter $j = 1$ in the first period. Hence the reader may wonder, why the other underwriters $j \in \mathcal{M} \setminus \{1\}$ do not withdraw immediately. The answer is that they may well do so, once the contract is concluded, without altering anything in Theorem 3. In fact, the set $\mathcal{M}$ of potential underwriters is meant as a pool of, say, investment bankers having frequent contacts with the firm and thus staying potentially ready to subscribe to the firm's issue of new shares. Once the firm has chosen one of them, all the others are free to invest their funds in alternative uses. An investment banker, who is not willing to contact the firm from the very outset, is simply eliminated from the set $\mathcal{M}$.

The point which we wish to make with Theorem 3 is that the use of an underwriter constitutes a potential extra cost of issuing shares, though this cost is unavoidable, if $\bar{\pi} > \underline{\pi}$. Since m is a finite integer number (there cannot be more potential underwriters than investors), $1 - (1 - \alpha)^m < 1$, i.e. despite the fact that potential underwriters engage in a Bertrand-type competition, a successful underwriter obtains a non-vanishing share of the surplus in the equilibrium constructed in Theorem 3. And the equilibrium constructed in Theorem 3, although not unique in the class of subgame perfect equilibria, has very desirable game theoretic properties. Without explicitely giving a formal proof, we claim that in particular the equilibrium constructed in the proof of Theorem 3 satisfies ***Strategic Equilibrium*** [**Leininger**, 1988] and a proper version of ***Forward Induction*** [**van Damme**, 1989].

7. Conclusions

The present paper has studied a three stage game which resembles the sequencing of events, when a firm attempts to raise funds for an investment project by issuing new shares. At the first stage the firm bargains with potential underwriters on a subscription of the issue. Then a primary market opens, where private investors can take up some or all of the new stock. Finally trades at a secondary market are allowed. It turns out that the cost of issuing shares split into two principle parts: Underpricing of the issue at the primary market (when measured against the price at the secondary market) and the costs for the services of an underwriter. Both types of costs may be unavoidable (i.e. equilibrium costs), because by declining the services of an underwriter, the firm risks even more severe underpricing at the primary market as it would with the support of an underwriter. But the notion of underpricing as a (part of the) cost of issuing shares also carries a slight ambiguity. Though an issue may be underpriced, when measured against the price at the secondary market *after* the (non-zero) transactions at the primary market, this does *not* imply underpricing as measured against the price which would obtain, if no primary market existed, but shares are sold in a perfectly competitive world (the "reference economy").

The main point of the paper is to carry through these arguments in a rigorous game-theoretic framework: Underpricing is a device to generate an incentive for investors to take up the issue at the primary market, rather than to wait for the secondary market, despite the fact that investors at the primary market are aware of their influence on the secondary market price. And the services of an underwriter are costly, because, if the underwriter would not get more than with an alternative use of his funds, he will reject the firm's approach; this threat (being subgame perfect) forces the firm to concede a non-vanishing part of the surplus to the underwriter.

Results generated in the course of the analysis are: The secondary market, modelled to resemble closely the rules of real-world stock exchanges, has all active equilibria Walrasian, despite the fact that all investors have price-setting power. Activity of the primary market will require underpricing, if the secondary market is efficient with respect to market orders. And to sell the whole issue of an initial public offering at the primary market will require underpricing in any equilibrium. The bargaining on an underwriting contract will result in an underwritten issue, if and only if a non-underwritten issue would require even more severe underpricing than an underwritten issue. And the latter always happens in at least some of the equilibria, precisely because a non-underwritten issue still carries the risk of failure. Thus the costs of issuing shares are generated by the very nature of shares: Since shares are financial claims, their valuation depends on an equilibrium of expectations on how the financial asset will translate into real consumption in the future. But the way the financial asset converts into real commodities does itself depend on expectations. Equity issues are just one example for this - distinguishing - feature of financial instruments.

APPENDIX

PROOF OF LEMMA 1: (i) Since $Z_d(q, .) < Z_s(q, .)$, $\forall q > \underline{b}$, and $Z_d(q, .) > Z_s(q, .)$, $\forall q < \bar{a}$, an assumption $\underline{b} < \bar{a}$ would imply that $\exists \hat{q} \in (\underline{b}, \bar{a})$ such that both $Z_d(\hat{q}, .) < Z_s(\hat{q}, .)$ and $Z_d(\hat{q}, .) > Z_s(\hat{q}, .)$, which is clearly impossible.

(ii.a) First suppose that there exists no $\hat{q} \in (\bar{a}, \underline{b})$: $Z_d(\hat{q}, .) = Z_s(\hat{q}, .)$. Then from

$$Z_d(q, .) \leq Z_s(q, .), \quad \forall q > \bar{a}, \text{ and}$$
$$Z_d(q, .) \geq Z_s(q, .), \quad \forall q < \underline{b},$$

the hypothesis implies

$$Z_d(q, .) < Z_s(q, .), \quad \forall q \in (\bar{a}, \underline{b}), \text{ and}$$
$$Z_d(q, .) > Z_s(q, .), \quad \forall q \in (\bar{a}, \underline{b}),$$

which is clearly not possible. Consequently $\exists\, \hat{q} \in (\bar{a}, \underline{b})$, such that $Z_d(\hat{q}, .) = Z_s(\hat{q}, .)$. But then by the monotonicity properties of Z_d and Z_s

$$Z_d(q, .) \leq Z_s(q, .), \quad \forall q \in [\hat{q}, \underline{b}], \text{ and}$$
$$Z_d(q, .) \geq Z_s(q, .), \quad \forall q \in [\bar{a}, \hat{q}],$$

which from the definition of $\underline{b}$ and $\bar{a}$ implies

$$Z_d(q, .) = Z_s(q, .), \quad \forall q \in (\bar{a}, \underline{b}).$$

(ii.b) Consider a sequence $q^k \to \bar{a}$, $q^k \in (\bar{a}, \underline{b}), \forall k$. Since Z_s is continuous from the right, i.e.

$$q^k \searrow q^o \text{ and } Z_s(q^k, .) \searrow z_s^o \implies z_s^o = Z_s(q^o, .),$$

it must be true that

$$\lim_{k\to\infty} Z_s(q^k, .) = Z_s(\bar{a}, .) = Z_d(q, .),$$

for all $q \in (\bar{a}, \underline{b})$. For a sequence $q^k \to \underline{b}$, $q^k \in (\bar{a}, \underline{b}), \forall k$, left hand continuity of Z_d, i.e.

$$q^k \nearrow q^o \text{ and } Z_d(q^k, .) \searrow z_d^o \implies z_d^o = Z_d(q^o, .),$$

implies that

$$\lim_{k\to\infty} Z_d(q^k, .) = Z_d(\underline{b}, .) = Z_s(q, .),$$

for all $q \in (\bar{a}, \underline{b})$. Using (i) this yields the desired result. ∎

Proof of Lemma 2: (i) Suppose first that $\bar{b} > \underline{b}$. Then there must be some bidder $i \in \mathcal{N}$ on the demand side, who bids $p_i = \bar{b}$, and who can reduce his bid without risking any change in the allocation of shares. Since this contradicts optimality, this cannot be an equilibrium.

Now suppose that $\underline{a} < \bar{a}$. This case is somewhat more involved, because on the supply side the underwriter has to play a market order and he may be the only supplier (otherwise an analogous argument as above holds): From the definition of $\bar{a}$ it follows that $Z_d(p^1, .) > Z_s(p^1, .)$, $\forall p^1 \in [\underline{a}, \bar{a})$. But $Z_s(p^1, .)$ is constant on $[\underline{a}, \bar{a}]$ by the definition of $\underline{a}$, and Z_d is continuous from the left, such that $\underline{a} < \bar{a}$ implies $Z_d(\bar{a}, .) > Z_s(\bar{a}, .)$. From the right-hand continuity of Z_s it follows that

$$\exists \hat{p}^1 > \bar{a}: \; Z_s(p^1, .) = Z_s(\bar{a}, .), \; \forall p^1 \in [\bar{a}, \hat{p}^1].$$

But from the definition of $\bar{a}$ also

$$\bar{a} = \sup\{p^1 \in \Re_+ \mid Z_d(p^1, .) > Z_s(p^1, .)\}$$

which implies that $Z_d(p^1, .) \leq Z_s(p^1, .)$, $\forall p^1 \in (\bar{a}, \hat{p}^1]$, such that $Z_d(p^1, .) \leq Z_s(\bar{a}, .)$, $\forall p^1 \in (\bar{a}, \hat{p}^1]$. Consequently there are only two possible cases:

(a) $Z_d(p^1, .) = Z_s(\bar{a}, .)$ for some $p^1 \in (\bar{a}, \hat{p}^1]$, or

(b) $Z_d(p^1, .) < Z_s(\bar{a}, .)$ for all $p^1 \in (\bar{a}, \hat{p}^1]$.

If case (a) holds, the definition of $\underline{b}$ implies $\underline{b} > \bar{a}$, and, therefore, $\bar{b} \geq \underline{b} > \bar{a}$. But then again there is some investor $i \in \mathcal{N}$ on the demand side, who bids $p_i = \bar{b} > \bar{a}$, and who can lower his bid, thereby reducing the price which he will have to pay, without risking a change in the allocation (by Lemma 1). Since this investor would be strictly better off with a lower bid, case (a) cannot hold in equilibrium.

If case (b) holds, the definition of $\underline{b}$ implies $\underline{b} = \bar{a}$ and one has

$$Z_d(\underline{b}, .) = Z_d(\bar{a}, .) > Z_s(\underline{b}, .) = Z_s(\bar{a}, .),$$

from the argument above. Since there is excess demand at $\underline{b} = \bar{a}$, there must be some bidder $i \in \mathcal{N}$ on the demand side, who will be rationed. If there is more than one bidder, who will be rationed, then risk- aversion (strict concavity of u^i, $\forall i \in \mathcal{N}$) implies that each of these bidders is better off with reducing his demanded quantity to the expected value $\int_0^{s_i^1} \zeta \, dF(\zeta|(p, s^1))$, holding limit prices constant. But this again contradicts the equilibrium assumption. If there is only one bidder, who will be rationed (with certainty), then this bidder can strictly improve his payoff by lowering his limit price from $p_i = \bar{b} \geq \underline{b}$ to $p_i' = \underline{a} < \underline{b} = \bar{a}$. Since Z_s is constant on $[\underline{a}, \bar{a}]$ this will not change his allocated quantity, but the price which he will have to pay is lowered.

Since both the implications (a) and (b) contradict the equilibrium assumption, this completes the demonstration of (i).

(ii) Suppose $\underline{b} > \bar{a}$ holds. Then from Lemma 1 above it follows that $Z_d(\underline{b}, .) = Z_s(\bar{a}, .)$, and $Z_d(p^1, .) = Z_s(\bar{a}, .)$, for all $p^1 \in (\bar{a}, \underline{b})$. Consequently there must be some bidder $i \in \mathcal{N}$ on the demand side, who bids $p_i = \underline{b} = \bar{b}$ (by (i)), and who will not be rationed. This bidder can lowerd his limit price without risking a change in the allocation of shares, contradicting optimality. ∎

PROOF OF THEOREM 1: The first part, $p^1 = \underline{b} = \bar{a}$, follows from Lemma 2. Now consider the indifference curves of an investor $i \in \mathcal{N}$ in $(p^1, s_i^1) \in \Re_+ \times \Re$ space, given by

$$-s_i^1 \int_\Omega u_1^i(\,.\,)\,d\mu\,dp^1 + [\int_\Omega u_2^i(\,.\,)R\,d\mu - p^1 \int_\Omega u_1^i(\,.\,)\,d\mu]\,ds_i^1 = 0$$
$$\Longrightarrow \frac{dp^1}{ds_i^1} = \frac{\int u_2^i R\,d\mu - p^1 \int u_1^i\,d\mu}{s_i^1 \int u_1^i\,d\mu}.$$

Step 1: Each indifference curve has a unique maximum in $\Re_+ \times \Re_{++}$, i.e. when $s_i^1 > 0$, and a unique minimum in $\Re_+ \times \Re_{--}$, i.e. when $s_i^1 < 0$: The second derivative of indifference curves is given by

$$\frac{d^2p^1}{d(s_i^1)^2} = \frac{(\int u_2^i R\,d\mu)^2}{s_i^1(\int u_1^i\,d\mu)^3} \int u_{11}^i\,d\mu - \frac{\int u_2^i R\,d\mu}{s_i^1(\int u_1^i\,d\mu)^2} \times$$
$$\times [\int u_{12}^i R\,d\mu + \int u_{21}^i R\,d\mu] - \frac{2}{s_i^1}\frac{dp^1}{ds_i^1} + \frac{\int u_{22}^i R^2\,d\mu}{s_i^1 \int u_1^i\,d\mu},$$

such that

$$\frac{dp^1}{ds_i^1} = 0 \text{ and } s_i^1 > 0 \Longrightarrow \frac{d^2p^1}{d(s_i^1)^2} < 0,$$
$$\frac{dp^1}{ds_i^1} = 0 \text{ and } s_i^1 < 0 \Longrightarrow \frac{d^2p^1}{d(s_i^1)^2} > 0.$$

First consider the case $s_i^1 > 0$ and a point $\bar{s}_i^1 > 0$, where $dp^1/ds_i^1 = 0$. To the left of $\bar{s}_i^1$ for some $s_i^1 \in (0, \bar{s}_i^1)$ sufficiently close to $\bar{s}_i^1$ it will be true that $dp^1/ds_i^1 > 0$. But then at this point $d^2p^1/d(s_i^1)^2 < 0$. Iterating this argument shows that for any $s_i^1 \in (0, \bar{s}_i^1)$ one has $dp^1/ds_i^1 > 0$. As a consequence $\bar{s}_i^1$ must be unique.

If $s_i^1 < 0$ again consider some $\bar{s}_i^1 < 0$ such that $dp^1/ds_i^1 = 0$ at $\bar{s}_i^1$. By $d^2p^1/d(s_i^1)^2 > 0$ at $\bar{s}_i^1$, for some $s_i^1 \in (\bar{s}_i^1, 0)$ one must have $dp^1/ds_i^1 > 0$ which implies that $d^2p^1/d(s_i^1)^2 > 0$ at $s_i^1 \in (\bar{s}_i^1, 0)$ and consequently for all s_i^1 to the right of $\bar{s}_i^1$. This again implies that $\bar{s}_i^1$ is unique.

Step 2: Define for a given investor $i \in \mathcal{N}$ the sets

$$D_i(p_i, (p, s^1)) = \{j \in \{0\} \cup \mathcal{N} \setminus \{i\} \mid p_j > p_i\},$$
$$T_i(p_i, (p, s^1)) = \{j \in \{0\} \cup \mathcal{N} \setminus \{i\} \mid p_j = p_i\},$$
$$S_i(p_i, (p, s^1)) = \{j \in \{0\} \cup \mathcal{N} \setminus \{i\} \mid p_j < p_i\},$$

and set $s_0^1 = \sum_{i \in \mathcal{N}} s_i^o - s \leq 0$ and the implicit ask-price of the underwriter $p_0 = 0$. Let

$$\underline{b}_i(p, s^1) = \sup\{p_i \in \Re_+ \mid \sum_{j \in D_i(p_i,(p,s^1)) \cup T_i(p_i,(p,s^1))} \max[0, s_j^1] \geq$$
$$\geq - \sum_{j \in S_i(p_i,(p,s^1)) \cup T_i(p_i,(p,s^1))} \min[0, s_j^1]\},$$
$$\bar{a}_i(p, s^1) = \inf\{p_i \in \Re_+ \mid \sum_{j \in D_i(p_i,(p,s^1)) \cup T_i(p_i,(p,s^1))} \max[0, s_j^1] \leq$$
$$\leq - \sum_{j \in S_i(p_i,(p,s^1)) \cup T_i(p_i,(p,s^1))} \min[0, s_j^1]\}.$$

Define the correspondences $\Upsilon_i \colon \Re_+ \to \Re_+$ and $\Psi_i \colon \Re_+ \to \Re_-$ by

$$\Upsilon_i(p_i) = \begin{cases} \{0\}, \quad \text{if} \quad p_i < \underline{b}_i(p, s^1), \\ [0, -\sum_{j \in S_i(p_i,(p,s^1)) \cup T_i(p_i,(p,s^1))} \min(0, s_j^1) - \\ \quad - \sum_{j \in D_i(p_i,(p,s^1))} \max(0, s_j^1)], \\ \quad \text{if } p_i \geq \underline{b}_i(p, s^1), \ \nexists j \in T_i(p_i, (p, s^1)) \colon s_j^1 > 0, \\ [0, -\sum_{j \in S_i(p_i,(p,s^1)) \cup T_i(p_i,(p,s^1))} \min[0, s_j^1] - \\ \quad - \sum_{j \in D_i(p_i,(p,s^1))} \max[0, s_j^1]), \\ \quad \text{if } p_i \geq \underline{b}_i(p, s^1), \ \exists j \in T_i(p_i, (p, s^1)) \colon s_j^1 > 0, \end{cases}$$

and analogously

$$\Psi_i(p_i) = \begin{cases} \{0\}, \quad \text{if} \quad p_i > \bar{a}(p, s^1), \\ [-\sum_{j \in D_i(p_i,(p,s^1)) \cup T_i(p_i,(p,s^1))} \max(0, s_j^1) - \\ \quad - \sum_{j \in S_i(p_i,(p,s^1))} \min(0, s_j^1), \ 0], \\ \quad \text{if } p_i \leq \bar{a}(p, s^1), \ \nexists j \in T_i(p_i, (p, s^1)) \colon s_j^1 < 0, \\ (-\sum_{j \in D_i(p_i,(p,s^1)) \cup T_i(p_i,(p,s^1))} \max(0, s_j^1) - \\ \quad - \sum_{j \in S_i(p_i,(p,s^1))} \min(0, s_j^1), \ 0], \\ \quad \text{if } p_i \leq \bar{a}(p, s^1), \ \exists j \in T_i(p_i, (p, s^1)) \colon s_j^1 < 0. \end{cases}$$

In Step 2 we claim that the union of the graphs of Υ_i and Ψ_i, $\mathcal{G}(\Upsilon_i) \cup \mathcal{G}(\Psi_i)$, is exactly the set of feasible trades for investor $i \in \mathcal{N}$. In other words: We claim that, by choosing an appropriate strategy, an investor $i \in \mathcal{N}$ can secure for himself any price- quantity pair contained in $\mathcal{G}(\Upsilon_i) \cup \mathcal{G}(\Psi_i)$, but nothing outside. Consider the demand side first: If investor $i \in \mathcal{N}$ bids $p_i < \underline{b}_i$, then he will clearly not get any shares. If he bids $p_i = \underline{b}_i$, then $T_i(p_i, .) \neq \emptyset$ by definition. From the left hana continuity of demand it follows that there exists some $p_i' > p_i = \underline{b}_i$, but smaller than the lowest bid of any bidder j other than i, whose limit price satisfies $p_j > \underline{b}_i$. Since at $p_i' > \underline{b}_i$ supply must exceed demand, investor i will, by bidding p_i', be able to get any quantity in $\Upsilon(p_i')$. But as long as he demands some s_i^1 strictly smaller than the smallest upper bound of $\Upsilon(p_i')$, he will only have to pay $\underline{b}_i$, because by definition demand will fall short of supply at p_i' for any such s_i^1. However, investor i will *not* be able to get the *upper bound* of $\Upsilon_i(p_i')$ at the price $\underline{b}_i$: If he only bids $p_i = \underline{b}_i$ and demands a quantity equal to the upper bound of $\Upsilon_i(p_i')$ (which equals the smallest upper bound of $\Upsilon_i(\underline{b}_i)$), then $T_i(\underline{b}_i, .) \neq \emptyset$ implies that at $p_i = \underline{b}_i$ there are rival demands and all investors in $\{i\} \cup T_i(\underline{b}_i, .)$ will be rationed. A quantity equal to the upper bound of $\Upsilon_i(p_i')$ can only be had by raising the effective market price to $p_i' > \underline{b}_i$. By the rationing mechanism no quantity above the upper bound of $\Upsilon_i(p_i')$ can be had.

A quantity above the upper bound of $\Upsilon_i(p_i')$ can only be achieved by bidding higher than the next bidder $j \in \mathcal{N}\backslash\{i\}$, who bids the lowest limit price $p_j > \underline{b}_i$. To this step the same arguments apply: By overbidding, i can get access to the half open interval without raising the effective market price; to get access to the (upper or right) boundary of the interval, i has to raise the effective market price. The arguments on the supply side are completely analogous, except that "overbidding" has to be substituted by "undercutting" and Υ_i has to be substituted by Ψ_i. We have shown that the feasibility set for any investor on the demand (supply) side is the area above (below) an increasing (decreasing) step-function, where the "corners" to the South-East (North-West) are left out.

Step 3: Now substitute the allocation (inclusive of the price) resulting from an equilibrium strategy combination for the strategy combination itself. We claim that at any equilibrium the allocation must be such that the slope of the indifference curve through the allocation for an individual investor is zero.

Step 1 has shown that for any investor on the demand (supply) side the indifference curve through the equilibrium allocation has a unique maximum (minimum) and is strictly concave (convex) to the left (right) of the maximum (minimum). Consequently the equilibrium allocation can never be on a vertical piece ("vertical" means a fixed quantity, but several prices) of the feasibility set, because in this case the intersection of the upper contour set with the interior of the feasibility set is non- empty, and the investor can improve upon the allocation. The equilibrium allocation can also not be on a corner of the feasibility set to the South-East (North-West), because these points do not belong to the feasibility set. Consequently, the equilibrium allocation must be on a horizontal piece ("horizontal" means fixed price, but variable quantity) of the step-function, because it can also not lie in the interior of the feasibility set. But this implies that for any investor, who trades in equilibrium,

$$\int_\Omega u_2^i(w_i - p^o s_i^o - p^1 s_i^1,\ R(\theta)(S_i + s_i^o + s_i^1))R(\theta)\, d\mu(\theta) =$$
$$= p^1 \int_\Omega u_1^i(w_i - p^o s_i^o - p^1 s_i^1,\ R(\theta)(S_i + s_i^o + s_i^1))\, d\mu(\theta),$$

where $p^1 = \underline{b}(p,\, s^1) = \bar{a}(p,\, s^1)$, from Lemma 2. That none of the investors, who trade in equilibrium, will be rationed follows from the definition of the horizontal pieces on the boundary of the feasibility set. Hence any investor, who trades in equilibrium, is on his Walrasian demand (supply) function. Since no one is rationed, supply equals demand and all investors are on their excess demand curves, the equilibrium allocation is a Walrasian equilibrium.

Step 4: To show the converse, let a Walras-equilibrium $(p^1,\, (\bar{s}_i^1)_{i\in\mathcal{N}})$ be given. Consider the following strategy combination:

$$(p_i,\, s_i^1) = (p^1,\, \bar{s}_i^1), \quad \forall i \in \mathcal{N}.$$

The feasibility set for an investor $i \in \mathcal{N}$ on the demand side is now

$$\{[0,\, - \sum_{j\in\mathcal{N}\setminus\{i\}} \min(0,\, \bar{s}_i^1) - s + \sum_{j\in\mathcal{N}} s_j^o] \times$$
$$\times [p^1, \infty)\} \setminus \{(- \sum_{j\in\mathcal{N}\setminus\{i\}} \min(0,\, \bar{s}_j^1) - s + \sum_{j\in\mathcal{N}} s_j^o,\, p^1)\}$$

and his optimal choice from this set is exactly his Walrasian demand. For the supply side, again, an analogous argument holds. Consequently, no one has an incentive to deviate, verifying that a Nash-equilibrium has been constructed. ∎

PROOF OF LEMMA 3: Since the budget set is compact and convex und u^i is continuous, the maximum-theorem yields that ζ_i^1 is an UHC correspondence. Strict concavity implies that ζ_i^1 is a function (single valued), such that it is a continuous function. The Kuhn-Tucker conditions for problem (1) are

$$\int_\Omega u_2^i(x_i, R(\theta)\zeta_i^1)R(\theta)\,d\mu(\theta) - \lambda p^1 \leq 0, \text{ compl. } \zeta_i^1 \geq 0,$$

$$\int_\Omega u_1^i(x_i, R(\theta)\zeta_i^1)\,d\mu(\theta) - \lambda \leq 0, \text{ compl. } x_i \geq 0,$$

$$w_i - p^o s_i^o + p^1(S_i + s_i^o) - x_i - p^1\zeta_i^1 \geq 0, \text{ compl. } \lambda \geq 0.$$

By the assumption that $u_2^i(\,.\,, x) \rightarrow_{x\rightarrow 0} +\infty$ and $u_1^i(x, \,.\,) \rightarrow_{x\rightarrow 0} +\infty$, the optimal choice can never be on the boundary and from $u_1^i > 0$ one concludes $\lambda > 0$, such that at the optimum the system of Kuhn-Tucker conditions has to hold with equalities, implicitely defining ζ_i^1, x_i, and λ.

The determinant of the Jacobian matrix of this system, viz. the bordered Hessian, is given by

$$\Delta_i = \begin{vmatrix} \int u_{22}^i R^2 d\mu & \int u_{21}^i R d\mu & -p^1 \\ \int u_{12}^i R d\mu & \int u_{11}^i d\mu & -1 \\ -p^1 & -1 & 0 \end{vmatrix} =$$
$$= p^1 \int u_{21}^i R d\mu + p^1 \int u_{12}^i R d\mu - (p^1)^2 \int u_{11}^i d\mu - \int u_{22}^i R^2 d\mu > 0$$

which verifies the s.o.c. Moreover, this implies that ζ_i^1 and x_i are continuously differentiable functions of p^1 and all other parameters, by the implicit function theorem.

Implicitely differentiating the f.o.c.'s yields

$$\frac{d\zeta_i^1}{dp^1} = \frac{1}{\Delta_i}[(p^1 \int u_{11}^i d\mu - \int u_{21}^i R d\mu)(\zeta_i^1 - S_i - s_i^o) - \lambda],$$
$$\frac{dx_i}{dp^1} = \frac{1}{\Delta_i}[\lambda p^1 + (\int u_{22}^i R^2 d\mu - p^1 \int u_{12}^i R d\mu)(\zeta_i^1 - S_i - s_i^o)],$$
$$\frac{d\zeta_i^1}{dp^o} = \frac{s_i^o}{\Delta_i}[p^1 \int u_{11}^i d\mu - \int u_{21}^i R d\mu],$$
$$\frac{d\zeta_i^1}{ds_i^o} = \frac{p^o - p^1}{\Delta_i}[p^1 \int u_{11}^i d\mu - \int u_{21}^i R d\mu],$$

from which (i) and the first part of (ii) follows directly. The second part of (ii) follows from strict monotonicity of u^i and (iii), (iv) follow from the above differentials and $u_2^i(\,.\,, x) \rightarrow_{x\rightarrow 0} +\infty$ directly. ∎

PROOF OF LEMMA 4: The gradient of ζ_1 is given by

$$\text{grad}(\zeta_1) = (\sum_{i\in\mathcal{N}} \frac{\partial \zeta_i^1}{\partial p^1}, \sum_{i\in\mathcal{N}} \frac{\partial \zeta_i^1}{\partial p^o}, (\frac{\partial \zeta_i^1}{\partial s_i^o})_{i\in\mathcal{N}}).$$

We wish to show that grad$(\zeta_1) \neq 0$. Suppose grad$(\zeta_1) = 0$. Then $(\partial\zeta_i^1/\partial s_i^o)_{i\in\mathcal{N}} = 0$ and $\sum_{i\in\mathcal{N}} \partial\zeta_i^1/\partial p^o = 0$ imply from Lemma 3, (iii), that $p^1 = p^o$. But then $\zeta_i^1(p^1, (w_i - p^o s_i^o, S_i + s_i^o)) = \zeta_i^1(p^o, (w_i, S_i))$ implies from

$$\sum_{i\in\mathcal{N}} \frac{\partial\zeta_i^1}{\partial p^1}\bigg|_{p^1=p^o} = \sum_{i\in\mathcal{N}} \frac{\partial\zeta_i^1}{\partial p^1}\bigg|_{s_i^o=0,\, \forall i\in\mathcal{N},\, p^1=p^o} - \sum_{i\in\mathcal{N}} \frac{\partial\zeta_i^1}{\partial p^o}$$

that $\sum_{i\in\mathcal{N}} \partial\zeta_i^1/\partial p^1 \neq 0$, from the assumption that the reference economy is regular.

Hence grad$(\zeta_1) \neq 0$, $\forall(p^1, (p^o, (s_i^o)_{i\in\mathcal{N}})) \in \Re_{++} \times G$, ensures that $S + s > 0$ is a regular value of ζ_1. The preimage theorem [**Guillemin and Pollack**, 1974, p.21] implies that $\zeta_1^{-1}(S+s)$ is a smooth manifold of dimension $n+1$. The second part of the Lemma follows from the parametric transversality theorem [**Hirsch**, 1976, p.79] which states that the set of parameter values $(p^o, (s_i^o)_{i\in\mathcal{N}}) \in G$ for which $\partial\zeta_1/\partial p^1 \neq 0$ at the equilibrium price is open and dense in G. ▮

PROOF OF LEMMA 5: Implicitely differentiating the market clearing condition yields

$$\frac{dp^1}{ds_i^o} = -\frac{\partial\zeta_i^1/\partial s_i^o}{\sum_{j\in\mathcal{N}} \partial\zeta_j^1/\partial p^1} = \frac{(p^o - p^1)\partial\zeta_i^1/\partial y_i}{\sum_{j\in\mathcal{N}} \partial\zeta_j^1/\partial p^1},\ \forall i \in \mathcal{N},$$

where $y_i = w_i - p^o s_i^o + p^1(S - i + s_i^o)$ and $\partial\zeta_i^1/\partial y_i > 0$. This completes the proof. ▮

PROOF OF LEMMA 6: (i) The first part follows directly from implicitely differentiating V_i and applying the envelope- theorem.

(ii) The existence of the function p_i^1 follows from Lemma 3, (i) ("once the price has risen sufficiently high for an investor to become a supplier, the investor will never again become a demander for any higher price"). Implicitely differentiating the equation

$$\zeta_i^1(p_i^1, (w_i - p^o s_i^o, S_i + s_i^o)) - S_i - s_i^o = 0$$

yields

$$\frac{dp_i^1}{ds_i^o}\bigg|_{\zeta_i^1=S_i+s_i^o} = \frac{1 + (p^o - p^1)\partial\zeta_i^1/\partial y_i}{\partial\zeta_i^1/\partial p_i^1},$$

where $y_i = w_i - p^o s_i^o + p^1(S_i + s_i^o)$. Using the calculations in the proof of Lemma 3 yields

$$\left.\frac{dp_i^1}{ds_i^o}\right|_{\zeta_i^1=S_i+s_i^o} = \frac{1}{\int u_1^i d\mu}[p^o p^1 \int u_{11}^i d\mu + \int u_{22}^i R^2 d\mu - p^1 \int u_{12}^i R d\mu -$$
$$-p^o \int u_{21}^i R d\mu] < 0,$$
$$\left.\frac{dp_i^1}{dp^o}\right|_{\zeta_i^1=S_i+s_i^o} = \frac{s_i^o}{\int u_1^i d\mu}[p^1 \int u_{11}^i d\mu - \int u_{21}^i R d\mu].$$

By the implicit function theorem, therefore, $p_i^1(s_i^o, p^o)$ is continuously differentiable with the required monotonicity properties. Next,

$$\left.\frac{dp_i^1}{dS_i}\right|_{\zeta_i^1=S_i+s_i^o} = \frac{1}{\int u_1^i d\mu}[\int u_{22}^i R^2 d\mu - p^1 \int u_{12}^i R d\mu] < 0$$

shows monotonicity in S_i. Finally, if $s_i^o \to w_i/p^o$, the assumption $u_1^i(x, .) \to_{x\to 0} +\infty$ implies that $S_i + s_i^o > \zeta_i^1$, $\forall p^1 > 0$, such that by definition of p_i^1, $p_i^1(s_i^o, p^o) < p^1$, $\forall p^1 > 0$, as $s_i^o \to w_i/p^o$. On the other hand, whenever $s_i^o \to -S_i$, the assumption $u_2^i(., x) \to_{x\to 0} +\infty$ implies that $\zeta_i^1 > S_i + s_i^o \to 0$, $\forall p^1 > 0$. This completes the proof. ∎

PROOF OF THEOREM 2: Let $P_1 = \bar{P}_1$, where $\bar{P}_1$ is defined by

$$\bar{P}_1(p^o, (s_i^o)_{i\in\mathcal{N}}) = \max_{(p^1, (p^o, (s_i^o)_{i\in\mathcal{N}}))\in\zeta_1^{-1}(S+s)} p^1$$

and fix some rationing mechanism ϕ. The latter is always possible, because for any profile of market orders $(z_i^o)_{i\in\mathcal{N}}$ which satifies $\sum_{i\in\mathcal{N}} z_i^o > s$ the underwriter is indifferent among rationing mechanisms, and for any $(z_i^o)_{i\in\mathcal{N}}$ satisfying $\sum_{i\in\mathcal{N}} z_i^o \le s$, $\phi(\{(z_i^o)_{i\in\mathcal{N}}\}) = 1$ for all rationing rules. Then, given $(p^o, (r_i)_{i\in\mathcal{N}})$, the investors $i \in \mathcal{N}$ play a finite simultaneous move game, because Z_ϵ is a finite grid. Now extend the pure strategy spaces of investors in these games to pure strategy spaces Z_ϵ for all $i \in \mathcal{N}$ by assigning strictly dominated payoffs to any $z_i^o \notin [0, \min(r_i, w_i/p^o)] \cap Z_\epsilon$ for all $i \in \mathcal{N}$. This operation does not eliminate any equilibrium of the original games and does not generate any new equilibria in which some investor uses any of the artificial new strategies. Then, given $(p^o, (r_i)_{i\in\mathcal{N}})$, investors play finite games with fixed pure strategy spaces Z_ϵ, all of which have equilibria (in possibly mixed strategies) by the Nash- theorem.

We do not show in greater detail here, but basically assert that for finite games the following theorem holds: Let G be a subspace of the space of normal form games with fixed pure strategy spaces for all players; then G can be partitioned into a closed set $G_0 \subset G$ with lower dimension than G and finitely many connected (relatively open) components G_t, $G_t \subset G \setminus G_0$, $t = 1, \ldots, T$, such that on each G_t, $t = 1, \ldots, T$, there exists a continuous function f_t (mapping G_t into the product of the simplices of probability distributions on pure strategy sets) assigning an equilibrium to each game in G_t. A formal proof of this theorem can be deduced from Theorem 4 and its Corollary 9 in **Ritzberger and Vogelsberger** [1990]. Alternatively it follows from the fact that the graph of the Nash- equilibrium correspondence is a semi-algebraic set [**Blume and Zame**, 1989, p.10]: Consider the projection of the graph of the Nash-equilibrium correspondence into G. Generic triviality [**Blume and Zame**, 1989, p.3] yields a subset G_0 and connected components G_t, $t = 1, \ldots, T$, of $G \setminus G_0$, semi-algebraic sets (fibres) F_t and homeomorphisms h_t, mapping $G_t \times F_t$ into the inverse image of the projection of G_t, such that by fixing a point in F_t for each $t = 1, \ldots, T$ a continuous function (the composition of the projection of the graph into mixed strategies with the homeomorphism h_t) can be defined on $\bigcup_{t=1}^{T} G_t$ which maps continuously into equilibria. This procedure can be repeated on G_0 and the critical set of G_0. Since the dimension of critical sets is strictly decreasing, this is a finite process. (This theorem has been proved by **Schanuel, Simon and Zame** [1990, p.13-14] for the whole space of games. That it also applies to a subspace follows from considering the restriction of the continuous selection of equilibria to the decomposed subspace, where the decomposition is obtained by intersecting with G_0 and G_t, $t = 1, \ldots, T$.)

In the present case the space of games under consideration, G, is generated by varying $(p^o, (r_i)_{i\in\mathcal{N}})$ and determining the corresponding payoffs V_i for any pure strategy combination $(z_i^o)_{i\in\mathcal{N}} \in Z_\varepsilon^n$ given ϕ (with the convention that pure strategies which are not strategies of the original games get assigned strictly dominated payoffs). Applying the above theorem then ensures that on finitely many subsets, the union of which is dense in G, continuous selections of equilibrium distributions on the $(s_i^o)_{i\in\mathcal{N}}$'s can be constructed. Let $\{G_t\}_{t=1}^{T}$ be the collection of these subsets and $\{f_t\}_{t=1}^{T}$ the corresponding collection of continuous equilibrium selections. Now extend each f_t continuously to the boundary of G_t. Since any finite game has at least one hyperstable component of equilibria [**Kohlberg and Mertens**, 1986] the continuous extension will select an equilibrium even for games at the boundary of G_t (in G_0) and, therefore, on the whole closure of G_t, $t = 1, \ldots, T$.

The choice of $P_1 = \bar{P}_1$ by Lemma 4 partitions $\mathcal{G}(\beta)$ into finitely many subsets of "regular economies" (for which all Walras equilibria are regular) on which $\bar{P}_1$ is continuous. The boundaries of these subsets are the sets of "critical Walras economies", i.e. the subsets of $\mathcal{G}(\beta)$ on which the projection of $\zeta_1^{-1}(S+s)$ into $\mathcal{G}(\beta)$ is not surjective. Now, on each of these sets of "regular economies" extend $\bar{P}_1$ continuously to the boundary.

With these constructions at hand the underwriter's pure strategy space can be partitioned into finitely many (relatively open) subsets, the union of which is dense. On the closure of each of these subsets both the Nash-equilibrium distributions of the $(s_i^o)_{i\in\mathcal{N}}$'s and the assignment of p^1 (the equilibrium at the secondary market) are continuous by construction. Hence a maximum of $E\,\pi(p^1, (p^o, (s_i^o)_{i\in\mathcal{N}}))$, where the expectation is taken with respect to the equilibrium distributions of the $(s_i^o)_{i\in\mathcal{N}}$'s, exists on the closure of each of these subsets, by the Weierstrass-theorem. Since there are only finitely many such subsets, a global maximum exists.

Since the general existence of hyperstable components ensures that at the global maximum an equilibrium distribution of the $(s_i^o)_{i\in\mathcal{N}}$'s has been used to evaluate the underwriter's expected payoff, it remains to show that the solution sits not at a "critical Walras economy", where P_1 was defined artificially by continuous extension of $\bar{P}_1$ (and does not coincide with $\bar{P}_1$). But the latter is impossible, because by definition of $\bar{P}_1$ the true value of $\bar{P}_1$ will always be at least as large as the value of the continuous extension. ▮

PROOF OF THEOREM 3: The proof is by construction.

Step 1: First consider the case, where $t = T$ (the current period is the final one) and there is at least one underwriter (the current bargaining partner) left to bargain with. Then by subgame perfection, if the firm makes the offer, the underwriter cannot reject any $\omega < 1$ (because his reservation payoff is zero and $1-\omega > 0$), and if the underwriter proposes, the firm cannot reject any offer $\pi < 1$. Consequently, by a standard argument (rejection at the boundary offer would force the proposer to optimize on an open set which is impossible), $\omega = 1$ is accepted, if the firm proposes, and $\pi = 1$ is accepted by the firm, if the underwriter proposes. Expected equilibrium payoffs are thus α for the firm and $(1-\alpha)$ for the underwriter.

Next consider the case, where all except one underwriter already have withdrawn from the game (invested somewhere else) and only the current bargaining partner is left. If $t = T$ the above applies and expected payoffs are α resp. $(1-\alpha)$ for the firm resp. the remaining underwriter. Now suppose that expected payoffs are α resp. $(1-\alpha)$ for the firm resp. the underwriter, if the remaining time horizon is $T - t \geq 1$. We claim that offers $\omega = \alpha$ and $\pi = 1-\alpha$, acceptance of these offers (rejection of all strictly higher offers), continuation of bargaining (if an offer should be rejected), and no withdrawal by the underwriter (if one partner should decide to split), form subgame perfect equilibrium behavior: Starting with the decision of the underwriter to withdraw or not, it is optimal to stay, because the firm will have to come back to the only underwriter (who has not yet withdrawn) which yields him $1 - \alpha > 0$. Given this behavior both partners are indifferent between splitting and continuation of bargaining after a rejection of an offer, verifying that continuation in response to a rejection is equilibrium behavior. Since the responder cannot reject any offer $\omega < \alpha$ ($\pi < 1 - \alpha$) acceptance of the offers $\omega = \alpha$ ($\pi = 1 - \alpha$) is the only equilibrium behavior and the optimal offers are $\omega = \alpha$ and $\pi = 1 - \alpha$. This yields expected equilibrium payoffs of

$$\alpha^2 + \alpha(1-\alpha) = \alpha, \quad \text{resp.}$$
$$\alpha(1-\alpha) + (1-\alpha)^2 = 1 - \alpha,$$

for the firm resp. the underwriter. Backward induction then yields that for all subgames, where either only one underwriter or only one period remains, expected equilibrium payoffs are α for the firm and $(1 - \alpha)$ for the underwriter.

Step 2: Let for some stage of the bargaining process k denote the number of remaining potential underwriters (inclusive of the firm's current bargaining partner) and denote by $T + 1 - t$ the length of the remaining time horizon in period t. Step 1 has shown that, whenever $k = 1$ or $t = T$, then the expected equilibrium payoffs to the firm resp. the (currently bargaining) underwriter are $\alpha = 1 - (1 - \alpha)$ resp. $(1 - \alpha)$.

As an induction hypothesis now assume that for $k = h - 1 \geq 1$ the expected equilibrium payoffs to the firm resp. the (currently bargaining) underwriter in period t are given by

$$1 - (1-\alpha)^{\min[T+1-t,\, h-1]} \quad \text{resp.} \quad (1-\alpha)^{\min[T+1-t,\, h-1]}.$$

Now consider a situation, where $k = h > 1$. For all periods $t \geq T + 1 - h$ we claim that the following constitutes subgame perfect equilibrium behavior: If the firm is chosen the proposer, it offers $\omega = 1$, and, if an underwriter is the proposer, he offers $\pi_t = (1 - \alpha)^{T-t}$; these offers are accepted and all strictly higher offers are rejected by the responder; if an offer is rejected, the firm quits the bargaining and the underwriter withdraws. Given that the firm quits and $k = h \geq T + 1 - t$, the underwriter is indifferent between withdrawing or not, because he will not have a chance to renegotiate with the firm, such that withdrawing is optimal. Since by $k = h \geq T+1-t$ the firm is never short of underwriters quitting in response to a rejection is also optimal. Consequently, the acceptability constraints for the offers of the firm resp. the underwriter are given by

$$1 - \omega \geq 0, \quad \text{resp.}$$
$$1 - \pi_t \geq 1 - (1-\alpha)^{T-t}.$$

The latter inequality follows by induction from $t = T \implies \pi_T = 1$ (Step 1) and $\pi_{\tau+1} = (1-\alpha)^{T-(\tau+1)} \implies \alpha + (1-\alpha)[1-(1-\alpha)^{T-\tau-1}] = 1-(1-\alpha)^{T-\tau}$. By the standard argument the acceptability constraints are equalities in equilibrium, such that expected payoffs are $\alpha + (1-\alpha)[1-(1-\alpha)^{T-t}] = 1-(1-\alpha)^{T+1-t}$ for the firm and $(1-\alpha)^{T+1-t}$ for the underwriter. Hence for $t \geq T+1-h$ the formula in the induction hypothesis follows also for $k = h$.

Next for all $t < T+1-h$ we claim that the following is subgame perfect equilibrium behavior: If the firm proposes, the offer is $\omega = 1-(1-\alpha)^h$ and, if an underwriter proposes, the offer is $\pi = (1-\alpha)^h$; these offers are accepted and all strictly larger ones are rejected; in response to a rejection both partners decide to continue bargaining; if indeed one partner should quit the bargaining, the underwriter withdraws. From $k = h > 1$ it follows that, after a quitting of the bargaining process, the underwriter will not have a chance to renegotiate, because the firm will conclude a contract with another underwriter next period. Hence the underwriter's withdrawal is optimal. Given that the underwriter withdraws after a quitting, the firm is faced with the choice between a situation, where (after quitting) the number of remaining underwriters is $k-1 = h-1$ and the remaining time horizon is $T-t$, and a situation, where (after continuation) the number of remaining underwriters is $k = h$ and the remaining time horizon is $T-t$. In the first situation (after quitting) the firm will obtain $1-(1-\alpha)^{h-1}$, from the induction hypothesis, and $h-1 < T-t$. To evaluate the second situation (after the decision to continue bargaining with the present partner) let first $t+1 = T+1-h$, such that $T-t = h$; then the firm will by continuing to bargain obtain $1-(1-\alpha)^h$ which is strictly larger that $1-(1-\alpha)^{h-1}$. Hence at least in $t = T-h$ it is optimal for the firm to continue bargaining. For the underwriter it is also optimal to continue, because in $t = T-h$ his expected payoff is $(1-\alpha)^h > 0$, if bargaining continues. Hence the expected equilibrium payoffs to the firm resp. the underwriter in $t = T-h$, when $k = h$, are from the acceptability constraints

$$\alpha[1-(1-\alpha)^h] + (1-\alpha)[1-(1-\alpha)^h] = 1-(1-\alpha)^h,$$
$$\text{resp.}\quad \alpha(1-\alpha)^h + (1-\alpha)(1-\alpha)^h = (1-\alpha)^h.$$

Suppose these are also the equilibrium payoffs for $\tau < T-h$, $\tau > t$. Then, in case the firm has in period t to decide on quitting or continuation of the bargaining (in response to a rejection), it is faced with choosing between $[1-(1-\alpha)^{h-1}]$ after quitting, because the underwriter withdraws, and $[1-(1-\alpha)^h]$ after continuation, such that continuation is strictly preferable. Obviously an analogous argument holds for the underwriter, such that the acceptability constraints

$$1-\omega \geq (1-\alpha)^h,$$
$$1-\pi \geq 1-(1-\alpha)^h,$$

yield the desired equilibrium payoffs by induction. Comparing these results with the induction hypothesis verifies that the formula in the induction hypothesis also holds for $k=h$, if it holds for $k=h-1$. In period $t=1$, then $T>m$ yield the statement of the theorem. ∎

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Figure 1