

**A LIKELIHOOD-RATIO TEST
FOR SEASONAL UNIT ROOTS**

Robert M. KUNST

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SUMMARY

A new test for the presence of seasonal unit roots in a quarterly time series, i.e. for seasonal integratedness, is constructed. A seasonally integrated process is characterized by a factor $1-L^4$ in its autoregressive representation. The test is based on the correlation between the series X_t and its seasonal differences $X_t - X_{t-4}$, adjusted for lagged differences. It is equivalent to the likelihood-ratio test against stationary alternatives.

If the series is taken from a seasonally integrated process indeed, the test statistic can be shown to converge towards a limit distribution. Fractiles of this distribution are given and finite-sample properties are studied via Monte Carlo. The use of correlations instead of second-order cross-moments around zero imposes a non-trivial bias whose influence is seen from simulations. If the series is stationary, a random walk, contains additional unit roots, or can be stationarized by seasonal moving averages, the test statistic can be shown to diverge.

1. Introduction

The last few years have seen a burgeoning literature on unit root tests and on co-integration mostly assuming the typical non-stationary model to be the integrated process which may be transformed into a stationary one by differencing once or several times. Some experience with quarterly national accounts or related series, however, reminds of the importance of seasonal cycles which quite frequently give rise to the presumption that they can be removed by applying fourth-order rather than first-order differences. Data with a different seasonal frequency - e.g. monthly data - entail similar properties. Here, we shall concentrate on the quarterly case.

Important studies in the area of unit roots and seasonal time series are the ones by Hasza & Fuller (1982) and Dickey, Hasza & Fuller (1984) who suggest modifications of the Dickey-Fuller test (Dickey & Fuller 1979, Said & Dickey 1984) in the presence of seasonal and/or trend unit roots. Alternatively, this paper is devoted to distributional results of a seasonal unit roots statistic which represents the analogon to Johansen's (1988) co-integration test. It can be shown to correspond to the likelihood-ratio test against stationary alternatives. Fractiles and means of the distribution will be listed based on Monte Carlo simulation. The statistic is easy to calculate and its asymptotic law is well approximated in medium samples.

Recently, Hylleberg, Granger, Engle, and Yoo (1988) presented a test for seasonal unit roots based on a similar design. Their test does not only take into account an eventual seasonal differencing factor $1-L^4$ in the autoregressive representation but also allows for the separate apparition of the component factors $1-L$, $1+L$, $1+L^2$. Except for the $1-L$ factor - and there are special tests available for this case - it is difficult to give meaning to these single factor hypotheses. The "seasonal moving average" $1+L+L^2+L^3=(1+L)(1+L^2)$ could be another

interesting feature which is touched upon in Section 7 of this paper.

2. A seasonal unit root test based on conditional correlation

The popular test by Dickey & Fuller (1979) for the presence of the factor $\Delta=1-L$ in the autoregressive (AR) representation of a time series relies on the fact that any p^{th} -order AR process

$$X_t = \sum_{i=1}^p \phi_i^* X_{t-i} + \epsilon_t \quad 1$$

can be written down in first differences keeping only one lagged level term ²

$$\Delta X_t = \sum_{i=1}^{p-1} \phi_i \Delta X_{t-i} + \phi_p X_{t-p} + \epsilon_t \quad (2.1)$$

If Δ is a factor of the AR polynomial

$$\Phi(L) = 1 - \phi_1^* L - \dots - \phi_p^* L^p$$

then ϕ_p must be zero. The likelihood-ratio (LR) test of the null hypothesis $\phi_p = 0$ against the general alternative $\phi_p \neq 0$ (under the environment assumptions of Gaussian innovations and no further roots of $\Phi(L) = 0$ being inside or on the unit circle) is equivalent to the t-value of ϕ_p in the (2.1) regression. Fractiles of the asymptotic and finite-sample distribution of this test statistic have been tabulated by Dickey & Fuller (1979). The asymptotic distribution is a non-standard law which can be represented by integrals over Brownian motion.

For the seasonal problem, a p^{th} -order AR process may be reparametrized in fourth differences ($\Delta_4 X_t = X_t - X_{t-4}$)

$$\Delta_4 X_t = \sum_{i=1}^{p-4} \phi_i \Delta_4 X_{t-i} + \sum_{i=1}^4 \phi_{p-4+i} X_{t-p+4-i} + \epsilon_t$$

1 Here and throughout the paper, ϵ_t denotes white noise.

2 Dickey and Fuller (1979) rather use a representation with a level term at lag $t-1$ instead of $t-p$. Both forms and the derived tests are equivalent.

i.e. by an AR process in differences of order $\hat{p}=p-4$ plus an influence of four level terms. Similar to the case of first differences, under the null hypothesis of Δ_4 being a factor of $\Phi(L)$ all Φ_{p-4+i} ($i=0, \dots, 3$) are zero and the corresponding LR test is equivalent to an F-test on these coefficients.

Rather than to calculate this F statistic directly, we shall exploit the property that the F-test can be interpreted as a description of the multiple correlation between $\Delta_4 X_t$ and X_{t-p+i} ($i=0, \dots, 3$), conditional on $p-1$ lagged seasonal differences. The calculations by Johansen (1988) can be used to establish this equivalence and allow for a potential extension of the test idea to the multivariate setting. This entails that the following primary regression equations can be used as an intermediate step:

$$\Delta_4 X_t = \sum_{j=1}^{p-4} a_j \Delta_4 X_{t-j} + r_{0t} \quad (2.2)$$

$$X_{t-p+i} = \sum_{j=1}^{p-4} b_{ij} \Delta_4 X_{t-j} + r_{it} \quad i=0, 1, 2, 3 \quad (2.3)$$

and the residual sums-of-squares and cross-sums matrices can be calculated: the scalar s_{00} , the 4-vector s_{0p} , the 4x4-matrix S_{pp} . The maximized log-likelihood under the general case is seen to be

$$-\frac{1}{2}T \log(2\pi) - \frac{1}{2}T \log\{(s_{00} - s_{0p} S_{pp}^{-1} s_{p0})/T\} - \frac{1}{2}T$$

Under the null hypothesis of $b_i = 0$, $i=0, \dots, 3$, the corresponding maximum is the outcome from a simple regression (2.2), i.e.

$$-\frac{1}{2}T \log(2\pi) - \frac{1}{2}T \log(s_{00}/T) - \frac{1}{2}T$$

Consequently, the log of the likelihood-ratio test statistic is

$$-\frac{1}{2}T \log(1-s_{00}^{-1}s_{0p}s_{pp}^{-1}s_{p0})$$

The test statistic of this paper will be defined as twice this expression, i.e

$$J_S = -T \log(1-s_{00}^{-1}s_{0p}s_{pp}^{-1}s_{p0}) \quad (2.4)$$

This J_S represents the equivalent of the J statistic by Johansen (1988) to the fourth-difference case in the case of scalar X_t .

The representation (2.4) facilitates the calculation of some theoretical results in the next sections but there are hardly any numerical finite-sample differences to a straightforward F-test which needs only two instead of five auxiliary regressions. Note, however, that these two regressions cannot use the same regressor matrix and the difference in computer time between the two versions is almost zero. The two versions can be viewed as the same test.

3. Distributional properties of the J_S statistic under the null

This section and the succeeding ones focus on asymptotic distributional properties if only one lagged difference is included into the primary regressions (2*2) and (2*3). In most cases, it will be shown that J_S will follow analogous patterns with higher orders. Let us repeat the primary regression equations in the first-order case to determine notational conventions:

$$\Delta_4 X_t = a \Delta_4 X_{t-1} + r_0 t \quad (3*1)$$

$$X_{t-i} = b_i \Delta_4 X_{t-1} + r_{it} \quad i=2,3,4,5 \quad (3*2)$$

The asymptotic properties of conventional unit root tests for the factor Δ rely on the convergence of random elements like

$$T^{-1/2} \sum_{t=1}^T \epsilon_t \quad (3*3)$$

to Wiener processes (to $B(s)$ if T is replaced by $\text{entier}(Ts)$) where ϵ_t denotes white noise. This generalizes to stationary ϵ_t under mild restrictions and e.g. implies that for a random walk X_t

$$T^{-3/2} \sum_{t=1}^T X_t \quad \text{converges to} \quad \int_0^1 B(t) dt$$

In the context of seasonal unit roots, the basic process is the seasonal random walk (SRW)

$$X_t = X_{t-4} + \epsilon_t \quad (\epsilon_t \text{ white noise})$$

rather than the random walk. In its strict definition, the t^{th} observation is independent of $X_{t-1}, X_{t-2}, X_{t-3}$. The SRW allows the representation

$$X_t = \sum \epsilon_{t-4j} + X^*$$

with X^* a starting value whose time index depends on the remainder from an integer division of t by 4. The sum only comprises about $\text{entier}(t/4)$ white noise increments. The SRW can be considered as consisting of four independent random walks. The elementary processes $(e_{4t-i}), i=0, \dots, 3$ are white noise just as (ϵ_t) .

For SRW, the least-squares estimate of a in (3*1) is root-consistent for the true value 0, and s_{00} necessarily converges to the innovations variance if downweighted by T . The b_i estimates in (3*2) do not converge but their distribution approaches a well-defined limit law. This implies that S_{pp} is asymptotically equivalent to the cross-products matrix of four consecutive observations from a seasonally integrated process. All matrix elements have to be weighted by T^2 to warrant convergence to random integrals. Let $B_i, i=1, \dots, 4$ denote the Wiener processes corresponding to the mutually independent random walks constituting the SRW. We have

$$S_{pp}/T^2 \rightarrow H = (h_{ij})$$

$$h_{ii} = \frac{1}{16} \sum_{i=1}^4 \left(\int_0^1 B_i^2(t) dt \right) = A$$

$$h_{i,i+1} = \frac{1}{16} \int_0^1 (B_1 B_2(t) + B_2 B_3(t) + B_3 B_4(t) + B_4 B_1(t)) dt = B$$

$$h_{i,i+2} = \frac{1}{8} \int_0^1 (B_2 B_4(t) + B_1 B_3(t)) dt = C$$

$$h_{14} = h_{i,i+1} \quad \text{and generally} \quad h_{ij} = h_{ji}$$

giving a Toeplitz matrix

$$H = \begin{bmatrix} A & B & C & B \\ B & A & B & C \\ C & B & A & B \\ B & C & B & A \end{bmatrix}$$

The inverse of H needed for calculating J_S is the product of its inverted determinant and its adjoint:

$$\det(H) = (A-C)^2((A+C)^2-4B^2)$$

$$H^{-1} = \frac{1}{(A-C)((A+C)^2-4B^2)} \begin{bmatrix} D & E & F & E \\ E & D & E & F \\ F & E & D & E \\ E & F & E & D \end{bmatrix}$$

where

$$D = A^2+AC-2B^2 \quad E = BC-AB \quad F = 2B^2-AC-C^2$$

The eigenvalues of H are $C-A$ (twice) and $A+C \pm 2B$ but one $A-C$ factor cancels with the adjoint. As the coefficient estimates in the $(3 \cdot 2)$ regressions approach a well-defined limit law, the elements of s_{0p} will behave asymptotically like the sums

$$\sum X_{t-i} \epsilon_t \quad i=2, \dots, 5$$

Taking the nature of the SRW into account, $T^{-1}S_{0p}$ converges toward a limit vector $h=(h_1, \dots, h_4)'$

$$h_1 = \frac{1}{4} \left(\int B_1 dB_3 + \int B_2 dB_4 + \int B_3 dB_1 + \int B_4 dB_2 \right)$$

$$h_2 = \frac{1}{4} \left(\int B_1 dB_4 + \int B_2 dB_1 + \int B_3 dB_2 + \int B_4 dB_3 \right)$$

$$h_3 = \frac{1}{4} \left(\sum_{i=1}^4 B_i dB_i \right)$$

$$h_4 = \frac{1}{4} \left(B_1 dB_2 + \int B_2 dB_3 + \int B_3 dB_4 + \int B_4 dB_1 \right)$$

Ignoring the variances of e_t and B_i which cancel, the limit of $Ts_{00}^{-1}s_{0p}s_{pp}^{-1}s_{p0}$ is given by a quadratic form

$$\frac{[D(\Sigma h_i^2) + 2E(h_1 + h_3)(h_2 + h_4) + 2F(h_1 h_3 + h_2 h_4)]}{[(A-C)((A+C)^2 - 4B^2)]}$$

If Y_t is a general seasonally integrated process of higher order, properties of martingale differences may be used to straightforwardly establish asymptotic equivalence to the SRW case. See, for example, Phillips (1987). To ensure convergence of s_{00}/T to the residual variance, it is necessary to insert at least the correct number of lags into the first preliminary equation. As this is not known in practice, an identification stage is required before estimating (2*1) and (2*2). An automatic approach for this aim can incorporate information criteria, preferring slightly overparametrizing criteria like AIC to more restrictive ones like BIC. An alternative used in some simulations of this paper is to base the decision on the Ljung-Box Q statistic. As uncorrelated residuals are more important for the procedure than minimized residual variances, this Q approach seems somehow more natural.

The empirical fractiles and means of the distribution of J_S given in Table 1 for selected seasonally integrated processes and alternatives using samples of size $T=100$ are based on a variant which uses regressions with constant terms in both (3*1) and (3*2). In the case of pure SRW, larger samples of $T=1000$ and $T=10000$ were also investigated. It is seen that fractiles hardly change at all if T is increased. Table 2 compares empirical and theoretical fractiles for the homogeneous regression variant tackled in the text. The marked differences between Tables 1 and 2 are due to the influence of the constant estimate in (2*3) or (3*2) which does not go to zero asymptotically. On the other hand, insertion of a constant

term in (3'1) does not affect the limit distribution. If one is not sure whether or not the data process contains a non-zero "drift" it is recommended to use inhomogeneous regression in (3'1) in order to retain asymptotically correct residuals r_{0t} and to avoid misspecification of lag orders.

The "cyclically integrated chi-square with four degrees of freedom" and its finite-sample "cyclically integrated F" are bell-shaped with modi around 4 (around 6 for the inhomogeneous variant) and left and right tails. Neither the size of the first-order autoregression coefficient in a seasonal AR model (0 represents the SRW case) nor the sample size change the fractiles too much. Note that even an additional unit factor seems not to matter either, but compare Table 3 and Section 6. For Tables 1 and 2, the lag orders p were fixed at the true value of one. Some control simulations for estimated p produced very similar results.

An Example: European temperature readings

Quarterly temperature averages provide a good example for severely seasonally infested series. Here, time series over two decades from 1960 to 1980 were investigated. We considered meteorological stations in Austria, France, Germany (Federal Republic), Italy, Sweden, Switzerland, and the United Kingdom. For none of the series, the hypothesis of a seasonal unit factor could be rejected, J_S values ranging from 1.25 (Italy) to 7.67 (Sweden). The models suggested by Q varied considerably between near-SRW (United Kingdom) and a fourth-order process for seasonal differences (Italy and Switzerland). Generally, explaining power of the models remained low, the best model being that for Switzerland with a corrected R^2 of 0.297.

4. Distributional properties under stationary alternatives

The most common alternative to the seasonal unit root process is a (covariance-) stationary one without unit roots, in particular with complex roots in its lag polynomial which are known to generate cyclical behavior. In this case, $\Delta_4 X_t$ will also be stationary, the first-stage regressions produce parameter estimates which are convergent of order \sqrt{T} , and all s_{ij} elements will converge if weighted by T . It follows that, unless any element converges towards zero, $s_{00}^{-1} s_{0p} s_{pp}^{-1} s_{p0}$ converges and J_S necessarily diverges of order T towards infinity.

The asymptotic forms may be calculated analytically. Here, this will be done for the case of first-order regressions (3.1) and (3.2) but the analysis may be extended to higher-order cases with slightly more intricate results. For the following, assume that X_t has an infinite MA representation

$$X_t = \sum \theta_i \epsilon_{t-i}$$

with white noise innovation (ϵ_t). Note the notation

$$R_i = \sum \theta_j \theta_{j-i} \quad R_{-i} = R_i$$

with j running over all possible coefficients. First, we see that the regression coefficients a , b_2 , b_3 , b_4 , b_5 converge to the following limits

$$a \rightarrow \alpha = (2R_1 - R_3 - R_5) / (2R_0 - 2R_4)$$

$$b_2 \rightarrow \beta_2 = (R_1 - R_3) / (2R_0 - 2R_4)$$

$$b_3 \rightarrow 0$$

$$b_4 \rightarrow -\beta_2$$

$$b_5 \rightarrow -1/2$$

Convergence is of order \sqrt{T} , so the asymptotic covariance matrix of the residuals is equivalent to the errors covariance matrix, particularly:

$$s_{00} \rightarrow 2(R_0 - R_4) - (2R_1 - R_3 - R_5)^2 / (2R_0 - 2R_4)$$

$$S_{p0} \rightarrow (\alpha(R_3 - R_1), R_3 - R_1, -\alpha(R_3 - R_1) + R_4 - R_0, (R_5 - R_3)/2)$$

$$R^* = (R_1 - R_3)^2 / (2R_0 - 2R_4)$$

$$S_{pp} \rightarrow \begin{bmatrix} R_0 - R^* & R_1 & R_2 + R^* & (R_1 + R_3)/2 \\ R_1 & R_0 & R_1 & R_2 \\ R_2 + R^* & R_1 & R_0 - R^* & (R_1 + R_3)/2 \\ (R_1 + R_3)/2 & R_2 & (R_1 + R_3)/2 & (R_0 + R_4)/2 \end{bmatrix}$$

These properties may be derived by straightforward calculation. Since S_{pp} has to be inverted to calculate J_S but has no obvious simple structure, it makes no sense to calculate a general closed form for the limit of J_S . However, this is fairly easy for any given MA process. Note that S_{pp} is slightly different from a Toeplitz autocovariance matrix and that the statistic approaches the convergence boundary for the fourth-order autocorrelation approaching one, i.e. the seasonal unit root case. For the special case of a white noise process, J_S/T converges to $\log(\frac{1}{2})$. More generally, for any fourth-order AR process of the form

$$X_t = \phi X_{t-4} + \epsilon_t \quad |\phi| < 1$$

it is seen that J_S behaves asymptotically like $T \log(\frac{1}{2} + \frac{1}{2}\phi)$. However, from Tables 1 and 2 we see that this approximation - for any fixed T - becomes poor if ϕ approaches 1. Moreover, it seems that J_S stays "above" the null distribution for all stationary cases. Note from Table 2 that 100 observations do not allow discrimination between the null hypothesis and the "near-seasonal unit root" model $X_t = 0.9X_{t-4} + \epsilon_t$ whereas 1000 observations enable safe decisions. Thus, the power of J_S is comparable to the unit root tests of Dickey & Fuller (1979).

5. Distributional properties in the case of the random walk

This and the following sections analyze alternatives for which the test is not the LR test. Nevertheless, we shall see that J_S remains useful even in these cases.

An important alternative, especially if economic data are used, to the seasonal unit roots process is the integrated process with only one unit root at 1, whose seasonality is too "weak" to imply the factor $1-L^4$. We shall investigate into the properties of J_S assuming that X_t is the random walk (RW) but the results generalize to stationary increments.

First note that, for RW, $\Delta_4 X_t$ is stationary or, more precisely, is an MA process

$$\Delta_4 X_t = \epsilon_t + \epsilon_{t-1} + \epsilon_{t-2} + \epsilon_{t-3}$$

and that the coefficient of the first-stage regression of $\Delta_4 X_t$ on $\Delta_4 X_{t-1}$ converges to $3/4$. It follows that s_{00} , downweighted by T , converges to $7/4$ times the innovations variance.

In the other preliminary regressions, RW lags are regressed on MA processes. RW may be decomposed into a part which is uncorrelated with the MA process and another only consisting of $\epsilon_{t-1}, \dots, \epsilon_{t-4}$. This motivates that the regression coefficients converge to sums of Brownian integrals and scalars

$$(b_2, \dots, b_5) \rightarrow (3/4, 1/2, 1/4, 0) + \int_0^1 B(t) dB(u) (1, 1, 1, 1)$$

If the cross-products from the r_{it} ($i=2, \dots, 5$) residuals are downweighted by T^2 , only the cross-products between the RW lags do not disappear asymptotically. This implies that all S_{pp} elements converge to the identical distribution

$$\int_0^1 B^2(t) dt$$

and that S_{pp} converges to a singular matrix of rank one and that its inverse diverges to infinity. On the other hand, S_{0p} converges to the random vector

$$7/4 \int_0^1 B(t) dB(u) (1,1,1,1) - (1/4, 1/2, 3/4, 0)$$

which is not zero. Consequently, for RW and other integrated processes, J_S diverges towards infinity. (Also compare Table 1)

6. Seasonal double convolutes

Another model that sometimes represents an alternative to the seasonally integrated process is the seasonal double convolute. Letting Δ denote the unit factor $1-L$, such a process is defined by a factor of $\Delta\Delta_4$ in its AR representation. This factor can be separated into

$$\Delta\Delta_4 = (1-L)(1-L^4) = (1-L)^2(1+L+L^2+L^3)$$

This means that seasonal adjustment by forming a four-quarter moving average leaves a process with two unit roots at one, i.e. an integrated process of order two. Such processes are less frequent than seasonally integrated ones and some occasions where they were hypothesized presumably involved overdifferencing of the data.

6.1. An Example: The Series G of Box & Jenkins

When using the J_S test (with lag orders identified via the Ljung-Box Q) on quarterly aggregates of the famous Series G from Box & Jenkins (1976) representing the numbers of airline passengers from 1949-1960, some interesting results obtain. Box & Jenkins used a model of the form $\Delta\Delta_4 X_t = (1+aL)(1+bL^4)\epsilon_t$ on the logged data (originally seasonal factors of order 12 for monthly observations). Later critics objected that Box & Jenkins might have overdifferenced the data. This is corroborated by our analysis. The Q stage of the test stops at $\hat{p}=1$, reporting a coefficient around .68 for original as well as for logged data. The seasonal factor Δ_4 is rejected ($J_S=15.74$) for the original data model which is probably misspecified, a fact also pointed out by Box & Jenkins, but the test is unable to reject for the logged model ($J_S=4.66$). The test indicates that annual growth rates of the airline data are stationary and may be modeled according to an ARMA scheme, preferably a first-order AR process although some higher-order dependencies might have been masked by the summary Q statistic and an MA model might be more parsimonious.

6.2. J_S and seasonal double convolutes

Of course, some cases remain where the seasonal double is a good model. A typical example might be nominal wage series. Such series can be characterized by integrated behavior of their first differences and strong seasonal patterns. Table 1 shows that, at least for smaller samples, the properties of J_S under the null hypothesis remain valid. Table 3 reports further Monte Carlo on the basis of seasonal doubles of the form

$$(1-L)(1-L^4)X_t = (1+\theta L)(1+\theta L^4) \quad (6.1)$$

It is seen that the empirical distribution quickly moves away from the "cyclically integrated F" for $\theta < 0$ and reaches obviously divergent behavior for $\theta = -0.5$. Such models approach stationary cases which are explicitly reached at $\theta = -1$ when the unit roots cancel. On the other hand, $\theta > 0$ generates less pronounced differences to the null distribution.

The finite-sample Monte Carlo results do not coincide with asymptotical properties. It can be shown that $T^{-4}S_{pp}$ converges to a singular limit; $T^{-3/2}S_{op}$ approaches a well-defined limit law; and $T^{-1}S_{00}$ still converges to the innovations variance. Consequently, J_S diverges whatever the values of θ in (6.1) or whatever the stable ARMA part in a more complicated seasonal double. According to the simulations, this divergence is slow and J_S could still detect seasonal factors in smaller samples of double convolutes with low dependence in the stationarized series. For larger samples, it will become easier to discriminate a double convolute by the integrated behavior of its fourth differences.

Table 4 gives the empirical frequencies of the identification of lag orders in the models used in Table 3. With the exception of the basic model with $\theta = 0$, a true lag order does not exist. Note that 10 % of the replications in

the $\theta=0.5$ design had to be discarded because the variables reached the size limits of the mainframe. Such processes are less likely to show up in empirical data analysis.

7. J_S and seasonal series with constant long-run mean

Among the many models for seasonality whose treatment is outside the scope of this paper (see e.g. Hylleberg, 1986) there is a popular one expressing the seasonal process as a sum of a stationary component and a deterministic series with periodicity four

$$X_t = \theta(L)\epsilon_t + d_t \quad (7.1)$$

The d_t can be estimated by regressing on four dummy series s_i that are 1 in the i -th quarter and 0 otherwise. Such models are related in the same way to seasonally integrated processes as linear time trends are related to integrated processes and have comparable disadvantages. They seem to impose a seasonal pattern which is too constant and stable for most data series, even though they can be interpreted as solutions to difference equations with canceling factors.³

If the seasonal moving average (SMA) factor $1+L+L^2+L^3$ is applied to the process of (7.1), its seasonality disappears but a complicated dependence structure is imposed on the "adjusted" series. As the SMA factor is contained in the Δ_4 operator, it makes sense to focus attention on a similar class of processes which are stationarized by applying SMA

$$(1+L+L^2+L^3)\phi(L)X_t = \theta(L)\epsilon_t \quad (7.2)$$

where all $\phi(\cdot)$ roots are outside and $\theta(\cdot)$ roots are outside or on the unit circle.

Assuming $\phi(\cdot) = \theta(\cdot) \equiv 1$ and the first-order regressions (3.1) and (3.2), calculation of the asymptotic properties of J_S is easy. The a estimates converge to $-\frac{1}{2}$ of order \sqrt{T} ; r_{0t} behave asymptotically like $\epsilon_t - \frac{1}{2}(\epsilon_{t-1} + \epsilon_{t-2})$ and s_{00}/T converges to $3/2$ times the innovations variance. The b_i

³ For example, the solutions of $\Delta^2 X_t = \Delta^2 \epsilon_t$ differ from those of $X_t = \epsilon_t$ by allowing for a linear time trend $X_t = a + bt + \epsilon_t$.

estimates approach well-defined random distributions and s_{0p} and S_{pp} behave asymptotically like sums of squares and cross-sums of X_t and r_{0t} . In particular, S_{pp}/T^2 converges towards

$$\begin{bmatrix} 2(A-B) & 2B-A-C & 2(C-B) & 2B-A-C \\ 2B-A-C & 2(A-B) & 2B-A-C & 2(C-B) \\ 2(C-B) & 2B-A-C & 2(A-B) & 2B-A-C \\ 2B-A-C & 2(C-B) & 2B-A-C & 2(A-B) \end{bmatrix}$$

The elements A to D were defined in Section 3. The matrix is singular and has rank three. On the other hand, s_{0p}/T converges to a well-defined limit distribution vector whose elements are linear combinations of the h_i integrals also defined in Section 3. Consequently, J_S diverges for all (7*2) models. Particularly in larger samples, J_S can be used to discriminate seasonally integrated processes from series with constant seasonal means.

8. Outlook

The present test can be performed on a univariate time series. If the null hypothesis is accepted, the usual proceeding will be to apply seasonal differencing and to fit an ARMA structure to the differenced data.

Perhaps even more interesting is the situation if the series at hand is embedded into a multivariate system of quarterly data. In most cases, J_S will reject for some series and accept for others. For those series containing $(1-L)(1+L+L^2+L^3)$ application of the seasonal moving average $1+L+L^2+L^3$ renders integrated processes. Again in most cases - not necessarily always - the remainder will contain integrated and stationary series and tests like the one by Dickey & Fuller (1979) can be used to decide upon this question. The generated and the original integrated series constitute an integrated system and tests for cointegration (see Engle & Granger, 1987, Johansen, 1988) should be carried out in order to avoid any loss of long-run information. The final form of the equations should then contain the differenced series, some error correction terms, and the original stationary cases. This proceeding has theoretical drawbacks as linear combinations of individual series may also reduce the amount of unit factors of the type $1+L$ or $1+L^2$. This feature, "seasonal cointegration", is the subject of recent work by Hylleberg et al. (1988).

The asymptotic distribution of J_S is similar but not identical to the chi-square distribution with four degrees of freedom. The variant based on inhomogeneous regression shows closer similarities with a chi-square of five to six degrees of freedom. Both laws are members of larger classes of probability laws generated by cyclically integrated processes. A further important case could be a law generated by 12 independent Brownian motion elements which is implied by testing for the factor $1-L^{12}$ in monthly data. For all of these cases, analogues to our LR-based J_S should

be preferable to the higher-order AR suggestion of Dickey, Hasza & Fuller (1979) but this presumption has to be subjected to further studies. Finally, factorizations like $1-L^4=(1-L)(1+L)(1+L^2)$ suggest testing for individual unit factors by tests similar to J_S even if J_S rejects. In general, behavior of such statistics is not robust to the existence of additional unit roots (compare Section 6) and knowledge of corresponding limit fractiles could empossible a descriptive "unit roots map" of tests on a variety of products of individual unit factors where the optimal description of the data is concluded from the least significant value. The work by Hylleberg et al. (1988) seems to point in a similar direction.

Most reports on unit root tests published in recent years include versions of the test which allow for certain deterministic components, like linear or even quadratic time trends. Unless insertion of such deterministic parts is suggested by the general solution of basic difference equations, like e.g. seasonal constants by $\Delta_4 X_t = \epsilon_t$, there is some arbitrariness around these models. Neither does it seem to be promising to perform tests of "deterministic" structures against "stochastic" ones (as linear trends against integrated processes) on the basis of rather small samples. Accordingly, this paper did not focus on this kind of problems.

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TABLE 1: Monte Carlo fractiles of J_S with \hat{p} set at 1 (100 observations) ¹

model	fractiles					mean		
	.1	.25	.5	.75	.9	.95	.975	.99
null hypothesis valid:								
$\Delta_4 X_t = \epsilon_t$	2.19	3.56	5.74	8.40	11.6	13.5	14.9	16.4 6.36
$\Delta_4 X_t = .1 \Delta_4 X_{t-1} + \epsilon_t$	2.17	3.56	5.65	8.44	11.6	13.7	15.0	16.9 6.38
$\Delta_4 X_t = .5 \Delta_4 X_{t-1} + \epsilon_t$	2.14	3.59	5.77	8.49	11.9	13.6	15.1	16.8 6.41
$\Delta_4 X_t = \epsilon_t$	2.09	3.56	5.75	8.24	11.3	13.7	15.0	18.0 6.36 2
$\Delta_4 X_t = \epsilon_t$	2.32	3.74	5.58	8.26	11.3	13.1	15.0	18.1 6.34 3
null hypothesis invalid:								
$X_t = .9X_{t-4} + \epsilon_t$	5.51	7.49	9.97	13.0	16.6	19.9	21.6	24.2 10.7
white noise	59.7	65.5	72.6	80.0	86.4	90.8	95.3	98.7 73.0
random walk	50.2	54.8	62.1	68.6	75.0	79.3	82.4	85.9 62.2
$\Delta \Delta X_t = \epsilon_t$	117	128	140	153	166	173	179	185 141
$\Delta \Delta_4 X_t = \epsilon_t$	1.88	3.37	5.42	8.09	11.2	13.1	15.1	18.1 6.11

¹ Fractiles and means based on a Monte Carlo experiment using 1000 replications via the NAGLIB random numbers generator. Innovations are specified as Gaussian. Δ denotes 1-L and Δ_4 denotes 1-L⁴ as in the rest of the paper.

² 1000 observations used in this experiment

³ 10000 observations used in this experiment

TABLE 2: Monte Carlo and theoretical J_S fractiles using $\hat{p}=1$ and homogeneous preliminary regressions

Theoretical fractiles ⁴	.1	.25	.5	.75	.9	.95	.975	.99	mean
	1.19	2.13	3.69	5.85	8.49	10.5	12.3	14.1	4.39

Monte Carlo fractiles ⁵

a) null hypothesis valid

model	T	.1	.25	.5	.75	.9	.95	.975	.99	mean
$\Delta_4 X_t = \epsilon_t$ (SRW)	100	1.32	2.22	3.86	5.86	8.50	10.0	11.6	14.0	4.42
	1000	1.34	2.32	3.82	5.92	8.30	10.4	12.7	14.4	4.48
$\Delta_4 X_t = 0.5 \Delta_4 X_{t-1} + \epsilon_t$	100	1.30	2.35	3.75	5.66	8.19	10.2	11.8	13.5	4.40
	1000	1.31	2.30	3.89	5.90	8.35	10.5	12.7	14.4	4.47

b) null hypothesis invalid:

model	T	.1	.25	.5	.75	.9	.95	.975	.99	mean
white noise	100	58.8	64.1	71.8	78.7	85.5	88.9	93.2	98.5	71.9
$X_t = 0.9X_{t-4} + \epsilon_t$	100	4.35	6.12	8.27	11.3	14.4	16.1	18.7	21.2	8.99
	1000	45.1	49.4	54.5	59.9	65.4	68.7	71.9	75.6	55.0
$(1+L+L^2+L^3)X_t = \epsilon_t$	100	34.4	38.5	43.8	49.5	55.0	58.5	60.4	64.2	44.3
	1000	380	394	409	426	442	452	462	467	410
$\Delta_4 X_t = \epsilon_t$	100	1.41	2.58	4.34	6.62	9.44	11.4	13.2	15.1	4.99

⁴ approximated by numerical integration

⁵ Fractiles and means based on a Monte Carlo experiment using 1000 replications via the NAGLIB random numbers generator. Innovations are specified as Gaussian.

TABLE 3: Monte Carlo fractiles of J_S with \hat{p} identified by Q (100 observations) ⁶

model	fractiles					mean			
	.1	.25	.5	.75	.9				
$\Delta\Delta X_t=(1-.5L)(1-.5L^4)\epsilon_t$	5.22	9.25	19.0	52.3	357	404	445	474	90.7
$\Delta\Delta X_t=(1-.4L)(1-.4L^4)\epsilon_t$	3.27	5.35	9.15	14.1	22.5	27.2	32.1	41.0	11.1
$\Delta\Delta X_t=(1-.3L)(1-.3L^4)\epsilon_t$	2.34	3.91	6.57	10.3	14.9	17.5	20.2	25.5	7.83
$\Delta\Delta X_t=(1-.2L)(1-.2L^4)\epsilon_t$	2.07	3.60	5.88	8.89	12.4	14.5	15.9	19.9	6.68
$\Delta\Delta X_t=\epsilon_t$	2.00	3.37	5.78	8.35	11.6	13.9	15.6	18.5	6.39
$\Delta\Delta X_t=(1+.2L)(1+.2L^4)\epsilon_t$	2.16	3.88	6.50	9.73	13.8	15.6	18.4	21.7	7.31
$\Delta\Delta X_t=(1+.5L)(1+.5L^4)\epsilon_t$	2.52	4.50	8.02	12.7	17.5	22.4	24.4	30.0	9.37

⁶ Fractiles and means based on a Monte Carlo experiment using 500 replications via the NAGLIB random numbers generator. Innovations are specified as Gaussian.

⁷ Only 450 replications could be used in this experiment.

TABLE 4: Frequency of \hat{p} identified in the experiments of Table 3

model	$\hat{p} = 1$	2	3	4	5	6	7	8	9	≥ 10
$\Delta\Delta X_t = (1-.5L)(1-.5L^4)\epsilon_t$.22	.01	.52	.16	.01	.02	.01	.00	.01	.03
$\Delta\Delta X_t = (1-.4L)(1-.4L^4)\epsilon_t$.36	.54	.02	.02	.01	.01	.00	.01	.01	.02
$\Delta\Delta X_t = (1-.3L)(1-.3L^4)\epsilon_t$.75	.16	.02	.01	.02	.01	.00	.01	.01	.02
$\Delta\Delta X_t = (1-.2L)(1-.2L^4)\epsilon_t$.85	.07	.01	.02	.01	.01	.00	.00	.01	.02
$\Delta\Delta X_t = \epsilon_t$.82	.04	.03	.02	.01	.01	.01	.01	.01	.03
$\Delta\Delta X_t = (1+.2L)(1+.2L^4)\epsilon_t$.60	.11	.04	.03	.11	.05	.02	.01	.01	.02
$\Delta\Delta X_t = (1+.5L)(1+.5L^4)\epsilon_t$.11	.15	.03	.16	.42	.06	.02	.01	.01	.03