UPDATING INPUT-OUTPUT MATRICES
AND STRUCTURAL CHANGE

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Forschungsbericht/
Research Memorandum No. 227

February 1986
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Abstract

The present paper develops an algorithm to update arbitrary Input-Output Tables from knowledge of row- and column-sums. The purpose is to generate unique Update-matrices, which are optimal in a specific sense. It turns out that the set of Lagrange multipliers used in this algorithm does possess economic significance.

Zusammenfassung

1. Introduction

Discussions on "structural change" frequently tend to focus on the quality of products and technology, on import and export values or even on income-distribution. Less attention seems to be paid to the "structure" of the economy, where the term "structure" is closer to what the layman would think it means. The latter notion of a "structure" of the economy relates to multisector-models introduced as an empirical device by Leontief (1951, 1966) and still widely applied to problems in studying indirect effects of, e.g., economic policy or growth (Carter, Brody, 1970, 1972; Ehret, 1970; Evers, 1974; Holub, Schnabl, 1982). These "Input-Output-models" or "linear-production models", however, are of definite interest to economics, since economic theory predicts - using the linear framework - that growth and the structure of an economy are fundamentally inter-linked (von Neumann, 1945/46). Investigations of linear models have even generated a long tradition of distribution-theory-discussion (Sraffa, 1974; Harcourt, 1969). Turning to empirical applications in Austria, Input-Output-Tables have proved useful in studying sectorwise multiplier effects and other "structural" issues (Richter, 1981; Richter, Teufelsbauer, 1970, 1974; Schmoranz, 1983) and will prove useful in the future since new and reliable data is available (USTZ, 1985).

In an earlier paper it has been shown that Input-Output Tables can be used to study deviations of the economy from an optimal path (Ritzberger, 1984) both with respect to intensities and relative prices, by using properties of eigensystems. These investigations, however, refer to a given period in time - specifically to the year to which the Input-Output Table refers - and cannot generate conclusions on the path of the economy. To obtain the latter, one would need time series of Input-Output Tables which, however, are beyond the capabilities of data collection. Consequently, already quite early in the tradition of Input-Output Analysis, there have been studies devoted to using other information, than new tables each year, to generate statistical updates of Input-Output Tables (Bacharach, 1970; Evers, 1974; Stäglin, 1972). This has led to algorithms

+) I am grateful to Bernhard Böhm and Peter Mitter for helpful comments and discussions. Remaining errors and shortcomings are, of course, mine.
known as RAS or modified forms of biproportional RAS-methods. All of these algorithms use data on the row- and column-sums of the matrices to be updated. Most of them share the disadvantage that the results of the update-algorithm, i.e. the limit of the iteration process, needs not be unique.

The present paper aims at developing a method of updating arbitrary matrices from knowledge of time-series data on row- and column-sums. This method is designed to result in unique updates under very weak assumptions which, moreover, are optimal in the sense that a weighted sum of the distances between the original matrix-entries and the updated ones is minimized. Throughout it is assumed that an initial matrix is given, e.g. an Input-Output Table from a certain year, and that data on the historical development of row- and column-sums is also given. There is, however, no assumption on the magnitude of the entries in the initial matrix, i.e. they may be either coefficients or absolute values, and no assumption on whether the initial matrix is rectangular or quadratic.

2. A weighted least-squares method

Let the given initial matrix be denoted \( A_0 \), which is an \((n \times m)\)matrix. Denote the given sequence of row-sums by \( u_t, \ t=1,2,\ldots \), the sequence of column-sums by \( v_t, \ t=1,2,\ldots \), \( u_t \in \mathbb{R}^n, v_t \in \mathbb{R}^m \), and denote the row- and column-sums of the initial matrix \( A_0 \) by \( u_0 \in \mathbb{R}^n, v_0 \in \mathbb{R}^m \). An \((n \times m)\)matrix \( G=[g_{ij}], \ g_{ij}>0 \), will denote the matrix of weights and the \((n \times m)\)matrix \( G^+=[g_{ij}^{-1}] \) will denote the matrix containing the inverses of the weights as its elements. Let \( e':=[1,1,\ldots,1] \), with the appropriate dimension, and write the elements of \( A_0 \) as \( a_{ij}(0), \ i=1,\ldots,n, \ j=1,\ldots,m \). The updating method aims at generating a matrix \( A_t=[a_{ij}(t)], \ t=1,2,\ldots \), which solves the following problem:

\[
\min_{A_t} \sum_{i=1}^{n} \sum_{j=1}^{m} g_{ij} (a_{ij}(t)-a_{ij}(0))^2
\]

s.t. \( A_te = u_t, \ e'A_t = v_t \).

For reference note the following definitions:
\( A_0 := [a_{ij}(0)], \quad A_t := [a_{ij}(t)], \quad G := [g_{ij}], \quad g_{ij} > 0, \)

\( G^+ := [g_{ij}^{-1}], \quad e' := [1, 1, \ldots, 1]; \)

\( A_0 e =: u_0, \quad e' A_0 =: \nu_0, \quad A_t e =: u_t, \quad e' A_t =: \nu_t, \)

\( G^+ e =: u, \quad e' G^+ =: \nu, \)

\( \Lambda_t := \text{diag } \lambda_t, \quad \lambda_t \in \mathbb{R}^n, \quad U_t := \text{diag } \mu_t, \quad \mu_t \in \mathbb{R}^m \)

where \( \lambda_t \in \mathbb{R}^n \) and \( \mu_t \in \mathbb{R}^m \) will denote Lagrange multipliers.

Writing down the Lagrangian for problem (1)

\[
L = \sum \sum g_{ij} (a_{ij}(t) - a_{ij}(0))^2 + \sum \lambda_i(t) (u_i(t) - \sum a_{ij}(t)) + \\
+ \sum \mu_j(t) (v_j(t) - \sum a_{ij}(t))
\]

it is straightforward to obtain the necessary condition for an optimum

\( 2g_{ij}(a_{ij}(t) - a_{ij}(0)) - \lambda_i(t) - u_j(t) = 0, \)

which clearly also satisfies the second order condition. Suppressing the \( t \)-notation on the Lagrange multipliers one may rewrite (2) as

\( 2' \quad A_t = A_0 + 1/2 \Lambda G^+ + 1/2 G^+ U \)

which implies

\( 3 \quad u_t = u_0 + 1/2(\text{diag } u)\lambda + 1/2 G^+ \cdot, \)

\( 4 \quad v_t = v_0 + 1/2 \lambda' G^+ + 1/2 \mu (\text{diag } v), \)

where \( \lambda \in \mathbb{R}^n \) has been defined as a column vector and \( \mu \in \mathbb{R}^m \) as a row vector. Substitution now yields


(5) \( u = [2(v_t' - v_0') - \lambda' G^t] (\text{diag } v)^{-1} \)

and

\[
\begin{align*}
    u_t - u_0 &= \frac{1}{2}(\text{diag } u)\lambda + \frac{1}{2} G^t (\text{diag } v)^{-1} [2(v_t' - v_0') - G_t' \lambda] = \\
    &= \frac{1}{2}(\text{diag } u)\lambda - \frac{1}{2} G^t (\text{diag } v)^{-1} G_t' \lambda + G^t (\text{diag } v)^{-1} (v_t' - v_0').
\end{align*}
\]

Defining now

\[
B := [(\text{diag } u) - G^t (\text{diag } v)^{-1} G_t']
\]

\[
\varepsilon_t := 2(u_t - u_0) - 2G^t (\text{diag } v)^{-1} G_t' (v_t' - v_0')
\]

one obtains

(6) \([(\text{diag } u) - G^t (\text{diag } v)^{-1} G_t'] \lambda = 2(u_t - u_0) - 2G^t (\text{diag } v)^{-1} G_t' (v_t' - v_0')\)

from (5) which may now be rewritten, using the above definitions, as

(6') \( B \lambda = \varepsilon_t. \)

Since \( u \) can be calculated from knowledge of \( \lambda \) equation (6') is the fundamental system of equations to be solved in order to obtain the Lagrange multipliers needed to generate the updated matrix \( A_t. \)

**Lemma 1:**

(i) \(|B| = 0.\)

(ii) \( \text{rk } (B) = n-1. \)

(iii) \( \inf_i u_i \leq \xi (G^t (\text{diag } v)^{-1} G_t') \leq \sup_i u_i, \) where \( \xi(.). \) denotes the dominant eigenvalue of the matrix,

(iv) \( \sum_i \varepsilon_i = 0. \)

**Proof:**

(i) \( Be = B' e = [(\text{diag } u) - G^t (\text{diag } v)^{-1} G_t'] e = u - G^t e = 0, \)

such that the homogeneous system \( Bx = 0, x \in \mathbb{R}^n, \) has a solution, and therefore \(|B| = 0. \)
(ii) From the definition of $B$ it follows that all its off-diagonal elements $b_{ij}$, $i \neq j$, are negative, and $Be = 0$ implies

$$b_{jj} = \sum_{i \neq j} |b_{ij}|.$$  

By taking out the $k$-th column from $B$ one obtains

$$b_{jj} > \sum_{i \neq j} |b_{ij}|$$  

such that the matrix $B$ reduced by one dimension has a dominant diagonal and is consequently regular. Thus it follows that $rk(B) = n - 1$.

(iii) $G^+(\text{diag } v)^{-1}G^+$ is a (symmetric) positive square matrix which is consequently irreducible, such that (Gantmacher, II, pp.55)

$$\inf_{i} u_i \leq \epsilon(x(G^+(\text{diag } v)^{-1}G^+)) \leq \sup_{i} u_i$$

as required.

(iv) $e'[2(u_t - u_0) - 2 G^+(\text{diag } v)^{-1}(v_t - v_0')] = e' \epsilon_t =

= 2 e'A_t e - 2e'A_0 e - 2e'A_0 e + 2e'A_0 e = 0.$

**Lemma 2:** (i) $B\lambda = \epsilon_t \iff B(\lambda + he) = \epsilon_t$, $h \in \mathbb{R}$

(ii) $\exists \lambda \in \mathbb{R}^n : B\lambda = \epsilon_t$.

**Proof:** (i) $B\lambda = \epsilon_t \iff B(\lambda + he) = B\lambda + he = B\lambda = \epsilon_t$.

(ii) $Be = 0 \iff b_{j}^{(i)} + B_{(ij)}e = 0$,

where $b_{j}^{(i)}$ denotes the $j$-th column of $B$ with the $i$-th element deleted and $B_{(ij)}$ denotes $B$ with the $i$-th row and the $j$-th column deleted. By Lemma 1, (ii), $B_{(ij)}$ is nonsingular and possesses therefore a unique inverse; thus

$$B_{(ij)}^{-1} b_{j}^{(i)} = -e.$$
and analogously \( b_{i}^{(j)} b_{(ij)}^{-1} = -e' \), where \( b_{i}^{(j)} \) denotes the \( i \)-th row of \( B \) with the \( j \)-th element deleted. Now reduce (6') by the \( i \)-th equation, such that

\[
(7) \quad \lambda_{j} b_{j}^{(i)} + B_{(ij)} \lambda^{(j)} = e^{(i)}_{\mathcal{E}},
\]

where \( \lambda_{j} \in \mathbb{R} \), \( \lambda^{(j)} \) denotes \( \lambda \in \mathbb{R}^{n} \) with the \( j \)-th component deleted, \( \lambda^{(j)} \in \mathbb{R}^{n-1} \), and \( e^{(i)}_{\mathcal{E}} \in \mathbb{R}^{n-1} \) denotes \( e_{\mathcal{E}} \) with the \( i \)-th component deleted. Now (7) will be satisfied by \( \lambda^{(j)} \) if and only if

\[
(8) \quad \lambda^{(j)} = B_{(ij)}^{-1} e^{(i)}_{\mathcal{E}} + \lambda_{j} e
\]

using \( B_{(ij)}^{-1} b_{j}^{(i)} = -e \). Substituting into the \( i \)-th equation of (6') yields

\[
\lambda_{j} b_{i} + b_{j}^{(j)} B_{(ij)}^{-1} e^{(i)}_{\mathcal{E}} + \lambda_{j} b_{i}^{(j)} e = -e' e^{(i)}_{\mathcal{E}} = e_{\mathcal{E} i}
\]

by Lemma 1, (iv).

Lemma 2, (i), shows that all solutions to (6') lie on a manifold in \( n \)-dimensional space, which is defined by adding a scalar multiple of the one-vector, \( e \), to any solution of (6').

Lemma 2, (ii), demonstrates the existence of such a solution.

Now consider the problem

\[
\min \left[ (\lambda - he)'(\lambda - he) + (u + he)'(u + he) \right], \quad h \in \mathbb{R}
\]

with the first order conditions

\[
\sum_{i=1}^{n} 2(\lambda_{i} - h) + \sum_{j=1}^{m} 2(\mu_{j} + h) = 0
\]

and the second order condition \( 2(m-n) > 0 \iff m > n \). (For the case \( m \leq n \) one defines \( \hat{h} = -h \) and minimizes with respect to \( \hat{h} \). This particular choice of a vector on the solution manifold of (6') is motivated by
\[ [2(v_t - v_o) - (\lambda' - he) G^+] (\text{diag } v)^{-1} = \mu + he \quad \text{from (5)} \]

and results in

\[ h = \frac{1}{n-m} \left[ e' \lambda + \mu e \right]. \quad (9) \]

This choice of $he \in \mathbb{R}$, therefore, leaves $(\lambda_i + \mu_j)$ unaffected but minimizes the Euclidian distance of Lagrange multipliers to the origin.

**Proposition 1:** There exists a unique solution to the problem

\[
\min (\lambda' \lambda + \mu \mu'), \quad \text{s.t. } B\lambda = \varepsilon_t.
\]

**Proof:** Existence of $\lambda \in \mathbb{R}^n$ is ensured by Lemma 2, (ii). One now has

\[
\lambda' \lambda + \mu \mu' = \lambda'u' + 4 v_t (\text{diag } v)^{-2} v_t' + 4 v_o (\text{diag } v)^{-2} v_o' -
\]

\[
- 8 v_t (\text{diag } v)^{-2} v_t' - 4 v_t'(\text{diag } v)^{-2} G^{++} \lambda +
\]

\[
+ 4 v_o (\text{diag } v)^{-2} G^{++} \lambda + \lambda' G^+ (\text{diag } v)^{-2} G^{++} \lambda
\]

using (5). Letting $\bar{e}_i$ denote a vector containing only zeros except for the $i$-th element, where it contains 1, the first order condition for the appropriate Lagrangian (with Lagrange multiplier vector $z$) reads

\[
2 \lambda_j - 4 (v_t - v_o)(\text{diag } v)^{-2} G^+ \bar{e}_j + 2 \lambda' G^+(\text{diag } v)^{-2} G^{++} \bar{e}_j - z B \bar{e}_j = 0
\]

with the second order condition

\[
2 \bar{e}_j G^+(\text{diag } v)^{-2} G^{++} \bar{e}_j + 2 = 2 + \sum \frac{g_{ik}^2}{v_k^2} > 0.
\]

Since the objective function in the problem is quadratic and the second derivative, consequently, a constant function with respect to $\lambda_j$ and positive, the convexity of the objective function and the linearity of the constraints imply a unique global minimum for the problem.
Within the set of vectors solving (6'), there is therefore a unique element which has minimal Euclidian distance to the origin. The proof of this conjecture has not directly been constructive, but it is easy to see that (9) gives the rule to find these "minimal" Lagrange multipliers. The proof of existence of \( \lambda \) satisfying (6') has, however, been constructive. Hence it suffices to proceed as in Lemma 2, (iii), then calculate \( h \) as given in (9) and shift the solution to (6') by this \( h \). This will determine the Lagrange multipliers \((\lambda^*, \mu^*)\) which are closest to the origin, and will thereby generate a unique update-matrix \( A_t \) from (2').

By definition the vectors \((\lambda^*_t, \mu^*_t)\) give the partial derivatives with respect to the restrictions in problem (1). Thus they can be interpreted as measures of sensitivity with respect to the restriction of problem (1). Since they are chosen to stay in a neighborhood of the origin, \((\lambda^*_t, \mu^*_t)\) may, loosely speaking, be regarded as measures of the minimal sensitivity of the matrix elements with respect to the restrictions imposed by given row- and column-sums. Hence the values of these vectors may give an indication on the possible location of "structural change". The relative change of their minimal distances to the origin

\[
\Delta S^*_t := \frac{\left(\lambda^*_t, \lambda^*_t\right)^{1/2} + \left(\mu^*_t, \mu^*_t\right)^{1/2} - \left(\lambda^*_t-1, \lambda^*_t-1\right)^{1/2} - \left(\mu^*_t-1, \mu^*_t-1\right)^{1/2}}{\left(\lambda^*_t-1, \lambda^*_t-1\right)^{1/2} + \left(\mu^*_t-1, \mu^*_t-1\right)^{1/2}}
\]

may be used as a measure of the speed of "structural change".

The latter interpretation, however, depends crucially on how one defines "structural change" and indirectly, therefore, on the choice of the weights \( g_{ij}, i = 1, \ldots, n, j = 1, \ldots, m \). The question of how to choose the weights remains to be investigated.

**Proposition 2**: If \( G^+ = A_0 \geq 0 \), then

\[
u_t = (1+g)^t u_0 \quad \text{and} \quad v_t = (1+g)^t v_0 \quad \text{implies}
\]

\[
\Rightarrow A_t = (1+g)^t A_0.
\]
Proof: \( G^+ = A_0 > 0 \Rightarrow v_0 = v, u_0 = u \). By hypothesis \( u_t = (1+g)^tu_o \),
\( v_t = (1+g)^tv_o \), \( g \in \mathbb{R} \), and consequently
\[
\epsilon_t = [2(1+g)^tu-u] - 2g^t(\text{diag } v)^{-1}[(1+g)^tv'-v'] = \\
= 2[(1+g)^t-1] (u-u) = 0.
\]

Therefore (6') has the solution \( \lambda = re, r \in \mathbb{R} \), since
\[
Be = 0 \Rightarrow rB\dot{e} = 0 = \epsilon_t;
\]

moreover from (5) one obtains
\[
u = 2 [(1+g)^t-1] v(\text{diag } v)^{-1} - r e'G^t(\text{diag } v)^{-1} = \\
= 2 [(1+g)^t-1] e' - re' \\
\Leftrightarrow \mu_j = 2([(1+g)^t-1] - r, j=1,\ldots,m \\
\Rightarrow \frac{1}{2}\lambda_i + \mu_j = (1+g)^t - 1.
\]

From \( g_{ij} = 1/a_{ij}(0) \) one obtains
\[
a_{ij}(t) = a_{ij}(0) + \frac{1}{2g_{ij}} (\lambda_i + \mu_j) = (1+g)^t a_{ij}(0).
\]

The conclusion of Proposition 2 turns out also to hold, if \( A_0 \) is a non-negative matrix except for the \( j \)-th column, which contains only non-positive entries. In this case \( g_{ij} = 1/|a_{ij}(0)| \) yields the same result as in Proposition 2, except that for the negative \( a_{ij}(0) \)'s the last equation in the proofs reads
\[
a_{ij}(t) = - (1+g)^t|a_{ij}(0)|, \text{ as required.}
\]
Using $g_{ij} = 1/a_{ij}(0)$ one may consider (9) for this case:

$$h = \frac{1}{(n-m)} \left[ nr + 2m(1+g)^t - 1 \right] - mr =$$

$$= r + \frac{2m}{n-m} \left( 1+g \right)^t - \frac{2m}{n-m}$$

and consequently

$$\lambda^*_{it} = \text{re-}re- \frac{2m}{n-m} \left( 1+g \right)^t \epsilon + \frac{2m}{n-m} \epsilon = \frac{2m}{n-m} \left[ (1+g)^t - 1 \right] \epsilon$$

$$\mu^*_{jt} = \frac{2n}{m-n} \left[ 1-(1+g)^t \right] \epsilon'.$$

Thus

$$\frac{\lambda^*_{it} + \mu^*_{jt}}{2} \left| a_{ij}(0) \right| = \begin{cases} 
\left[ (1+g)^t - 1 \right] \left| a_{ij}(0) \right| & \text{if } a_{ij}(0) > 0 \\
- \left[ (1+g)^t - 1 \right] \left| a_{ij}(0) \right| & \text{if } a_{ij}(0) < 0
\end{cases}$$

i.e. the sum of Lagrange multipliers in the balanced growth case - which could sensibly be regarded as a case, where "structural change" is absent - is

$$\frac{\lambda^*_{it} + \mu^*_{jt}}{2} = (1+g)^t - 1.$$

In empirical application one may therefore take

$$(1+g) := \frac{1}{T} \sum_{t=1}^{T} \frac{e'u_t}{e'u_{t-1}} \xrightarrow{T \to \infty} E(1+g)$$

as an estimate for the balanced growth rate. Using this estimate it is possible to analyse the deviations from the balanced growth path

$$\lambda^*_{it} = \frac{2m}{(m-n)} \left[ (1+g)^t - 1 \right] \text{ and } \mu^*_{jt} = \frac{2n}{(m-n)} \left[ (1+g)^t - 1 \right]$$
Of course, one could also use the inverse of the dominant eigenvalue of $A_0$ as the growth-factor $(1+g)$ instead of the proposed estimate.

Note that an Input-Output Table can be written in the form

$$ A_t = (\text{diag } p_t) C_t (\text{diag } x_t), $$

where $p_t$ denotes (relative) prices and $x_t$ denotes intensities, while $C_t$ denotes the matrix of technical coefficients. Working empirically "in real terms" amounts to holding prices constant, $p_t = p_0$, such that "structural change" can be split into two parts:

"structural change"

/  \ "sectoral shifts"  "technical changes"

The first part, "sectoral shifts", may be identified with variations in $x_t$, i.e. with shifts in the relative weights of the columns of $A_t$, while, with constant prices, the "technical change"-part should be reflected in shifts of the relative weights of the rows of $A_t$. Splitting "structural change" in this way, when working with constant prices, allows the following interpretation of Lagrange multipliers:

(i) The deviation $(\lambda_{it}^* - \frac{2m}{(m-n)} [(1+g)^t-1])$ indicates "technical changes", i.e. changes in the coefficients of the matrix $C_t$ — although in a rather vague way by summarizing coefficient-changes along rows.

(ii) The deviation $(\nu_{jt}^* + \frac{2n}{(m-n)} [(1+g)^t-1])$ indicates "structural shifts", i.e. changes in activity levels of the sectors.

If both deviations do not significantly differ from zero, then this is a clearcut notion of the absence of "structural change".

These last remarks demonstrate that the updating-method developed above cannot only be used to generate unique update-matrices, but does also allow for an analysis which may answer questions, more directly relevant to economic policy.
3. Summary

The present paper develops a method to generate update-matrices from a
given initial matrix using time-series data on row- and column-sums.
The method has two advantages in comparison with the well known RAS-
algorithm: It generates unique solutions for the update-matrices and,
second, these solutions are optimal in the sense that a weighted sum of
squared deviations from the initial matrix-entries is minimized. The
idea behind the approach is to calculate appropriate Lagrange-multiplier-

vectors. These multipliers, moreover, do possess an interpretation as
measures of "structural change" suitably defined. Furthermore, time-
series of the optimal Lagrange-multiplier-vectors could be analysed in
various ways to gain insight into the connection between the develop-
ment of row- and column-sums and the evolution of a conservative estimate
of the corresponding Input-Output-Tables. Forecasts of the Lagrange-

multiplier-vectors could also be used to generate projections of Input-
Output-Tables as an extra source of information besides data on the
forecasts of economic aggregates.
References


Evers, I., Input-Output Projektionen, Meisenheim 1974.


