IHS Economics Series
Working Paper 294
December 2012

Spatial System Estimators for Panel Models: A Sensitivity and Simulation Study

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Impressum

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Title:
Spatial System Estimators for Panel Models: A Sensitivity and Simulation Study

ISSN: Unspecified

2012 Institut für Höhere Studien - Institute for Advanced Studies (IHS)
Josefstädter Straße 39, A-1080 Wien
E-Mail: office@ihs.ac.at
Web: www.ihs.ac.at

294

Reihe Ökonomie

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Abstract

System of panel models are popular models in applied sciences and the question of spatial errors has created the recent demand for spatial system estimation of panel models. Therefore we propose new diagnostic methods to explore if the spatial component will change significantly the outcome of non-spatial estimates of seemingly unrelated regression (SUR) systems. We apply a local sensitivity approach to study the behavior of generalized least squares (GLS) estimators in two spatial autoregression SUR system models: a SAR model with SUR errors (SAR-SUR) and a SUR model with spatial errors (SUR-SEM). Using matrix derivative calculus we establish a sensitivity matrix for spatial panel models and we show how a first order Taylor approximation of the GLS estimators can be used to approximate the GLS estimators in spatial SUR models. In a simulation study we demonstrate the good quality of our approximation results.

Keywords

Seemingly unrelated regression models, panel systems with spatial errors, SAR and SEM models, generalized least-squares estimators, Taylor approximations

JEL Classification

G14, G15, C22

Contents

1 Introduction

Spatial models and applications in statistics and econometrics have been studied in the past decades; see e.g. Paelinck and Klaassen (1979), Anselin (1988, 2010), Florax and Van Der Vlist (2003), Haining (2003), LeSage and Polasek (2008), LeSage and Pace (2009), and Liu et al. (2012). On the other hand, panel models have become increasingly important and different estimators in such models with spatial components have also been studied; see e.g. Kapoor et al. (2007), Anselin et al. (2008), Baltagi (2008), Elhorst (2010), and Lee and Yu (2010). It is clearly useful to examine the sensitivity of these estimators in terms of a minor change in the spatial correlation parameter ρ.

The usual spatial auto-regression (SAR) model for the $n \times 1$ cross-sectional observations y is given by

$$y = \rho W_n y + X\beta + u, \quad u \sim N[0, \sigma^2 I_n]. \tag{1}$$

In a similar way we define the SEM model:

$$y = X\beta + e, \quad with \quad e = \theta W_n e + u, \tag{2}$$

where we assume a heteroskedastic error term for $u : \mathcal{E}(u) = \sigma^2 D_v, \quad t, s = 1, ..., T$.

Sensitivity analysis with respect to ρ means we are interested in the behavior of the estimators of β upon a small change of ρ. The numerical computation of the "spatial filter" estimator of spatial autoregression (SAR) models uses the spatial filter matrix $R = I - \rho W$, which acts as a filter in the reduced form of the SAR model. Spatial estimators are a function of the spatial neighborhood matrix W, which can become really large in large spatial panel systems, and the spatial correlation parameter ρ.

Our question is if 'good' approximations of simple spatial estimators exist to justify a reasonable sensitivity analysis (or making a spatial diagnostics without employing a time consuming spatial estimation procedure), and if so, what estimators and what GLS estimation approaches should be considered to use for diagnostics or approximations? A previous study of the sensitivity of spatial estimators, like the cross-sectional SAR and the SEM model, has been made in Liu et al. (2012). It was shown that good approximations exists for small values of the spatial correlation parameter.

The present paper extends a system of panel models to a seemingly unrelated regression (SUR) system with spatial errors in two ways: one is a SAR regression model with SUR errors (SAR-SUR) and the other is a SUR model with spatial errors model (SUR-SEM). First, we propose a system least squares (SUR based) estimator with spatially filtered variables, which is the SF-GLS estimator and we show that it can be expanded in a first order Taylor series around the non-spatial GLS estimator of a non-spatial regression model. The second type of system (or SUR based) estimator is the reduced form (RF) estimator, which is a GLS estimator that amounts to spatially transform all dependent and independent variables and is called RF-GLS estimator.

While in a cross-sectional SAR or SEM model we have to explore the sensitivity with respect to only one spatial parameter, we need in the system case a vector of spatial parameters, i.e. for each cross-sectional sample an own spatial correlation parameter. To get the sensitivity result using matrix derivatives for a vector of correlation parameters, we use a simple trick that is found in Magnus and Neudecker (1999). Because we are only interested in the derivative with respect to the diagonal matrix, we first derive the matrix-to-matrix derivative and then in a last step we employ the general result, that a diagonal derivative can be obtained by post-multiplying the matrix-to-matrix derivative with the selection matrix J, presented in Appendix.

Thus, we are also interested if simplifications of the system sensitivity results are possible if we make the simplifying assumptions, that there exists only one common spatial correlation parameter, briefly called the "cc-case". Luckily, in the SAR-SUR system case we gain no new insights by doing these simplifications, we find for the SAR-SEM model nice interpretations in the line of global sensitivity analysis as in Leamer (1978). The spatial correlation parameter traces out a hyper-curve between two simpler non-spatial estimators.

For the simulation study we develop a basic design involving the number of observations, the neighborhood matrix W and the SUR covariance matrix Σ to compute the average MSE or MPLS for the evaluation of the estimators. We also discuss how to measure the distance between the GLS estimator and its first order Taylor approximation in dependence of the spatial parameter ρ.

The structure of paper is as follows. In section 2, we consider the SAR-SUR model and their estimators. We derive the sensitivity results with respect to the two types of spatial correlation parameters (SAR or SEM). In section 3, we study the SUR-SEM model and the Taylor approximation sensitivity results. A simulation study for different sample sizes to check the quality of the approximate GLS estimates together with a comparison of the two estimators, the spatial filter estimator and the reduced form estimator are presented in section 4. Our concluding remarks are made in section 5. Finally, some basic definitions and their relevant mathematical properties are presented in the appendix, together with our detailed evaluation of the simulation study.

2 SAR-SUR models and their estimators

In this section, we consider the SAR model (1) and the extension of the model to a panel system. This leads to a system of regression equations and a seemingly unrelated regression (SUR) specification of the residual variance matrix, as in e.g. Anselin et al. (2008).

2.1 Simple GLS estimators of the SAR-SUR model

First we discuss the system or panel SAR model with a SUR error structure and the simple GLS estimators for this model. The following spatial autoregression (SAR) model is as in Anselin et al. (2008, pp 637-638) for time $t = 1, ..., T$

$$y_t = \rho_t W_N y_t + X_t\beta_t + e_t, \quad with \quad E(e_t e_s') = \sigma_{ts} I_N, \quad for \quad t \neq s, \tag{3}$$

where ρ_t is the spatial AR (SAR) correlation parameter for time t, y_t is the $N \times 1$ dependent variable, W_N is the $N \times N$ neighborhood matrix, X_t is the $N \times k$ regressor matrix, β_t is the $k \times 1$ regression coefficient, e_t is the $N \times 1$ error term, σ_{ts} is the temporal covariance parameter between time s and t (for convenience, the variance terms are expressed as σ_t^2), and I_N is the $N \times N$ identity matrix.

In compact form, the SAR-SUR model for T cross-sections is given by

$$y = (D_\rho \otimes W_N)y + X\beta + e, \quad with \quad e \sim N[0_{NT}, \Sigma_{sur}] \tag{4}$$

where $y = (y_1', ..., y_T')' = vecY$ is a $NT \times 1$ vector obtained from Y which is a $N \times T$ dependent panel matrix, vec is the vectorisation operator (see e.g. Neudecker et al. 1995a, 1995b), $D_\rho = diag(\rho_1, ..., \rho_T)$ is a $T \times T$ diagonal matrix, $X = diag(X_1, ..., X_T)$ is an $NT \times kT$ matrix with X_t of order $N \times k$, $\beta = (\beta_1', ..., \beta_T')'$ contains the $kT \times 1$ system regression coefficients, $e = (e_1', ..., e_T')'$ is the error vector and its covariance matrix is

$$\Sigma_{sur} = \mathcal{E}(ee') = \Sigma_T \otimes I_N \tag{5}$$

with a $T \times T$ positive definite covariance matrix Σ_T for the SUR system.

Assuming normal errors, the log-likelihood function (ignoring the constants) is given by

$$
\begin{aligned}
L &= ln\ |I_{NT} - D_\rho \otimes W_N| + \frac{1}{2} ln\ |\Sigma_{sur}^{-1}| - \frac{1}{2} e'(\Sigma_{sur}^{-1})e \\
&= \Sigma_{t=1}^{T} ln\ |I_N - \rho_t W_N| + \frac{N}{2} ln\ |\Sigma_T^{-1}| - \frac{1}{2} e'(\Sigma_T^{-1} \otimes I_N)e,
\end{aligned}
$$

where the error term is $e = (I_{NT} - D_\rho \otimes W_N)y - X\beta$. For further details see Anselin (1988, pp 145-146) and Anselin et al. (2008 p. 650).

Definition 1: The spatial filter SF-GLS estimator $\hat{\beta}_G$.
We consider the spatial filter (SF) form of the SAR-SUR panel model (4)

$$
R_{NT} y = X\beta + e \quad with \quad e \sim N[0, \Sigma_{sur}]. \tag{6}
$$

The generalized LS (GLS) estimator is then given by

$$
\hat{\beta}_G = (X'\Sigma_{sur}^{-1} X)^{-1} X'\Sigma_{sur}^{-1} R_{NT} y, \tag{7}
$$

where $\mathcal{E}(ee') = \Sigma_{sur} = \Sigma_T \otimes I_N$ is the SUR covariance matrix (5), and the system spread matrix is $R_{NT} = I_{NT} - D_\rho \otimes W_N$.

Denote the $T \times 1$ vector of spatial correlations $p = (\rho_1, ..., \rho_T)'$ of the SAR-SUR system in (4). When $p = 0$ we get $D_\rho = 0$ and the simplification of the filter matrix $R_{NT} = I_{NT}$ to the homoskedastic case, such that the spatial GLS estimator reduces to the non-spatial (panel) GLS estimator, which is the SUR estimator for equation systems

$$
\hat{\beta}_{gls} = (X'(\Sigma_T^{-1} \otimes I_N)X)^{-1} X'(\Sigma_T^{-1} \otimes I_N)y. \tag{8}
$$

Next, we derive the sensitivity of the GLS estimator and the Taylor approximation of $\hat{\beta}_G$ with respect to p, using Magnus and Neudecker's (1999) matrix differential calculus; for their differential idea, see part A1 in Appendix 6.

Theorem 1 *The $kT \times T$ sensitivity matrix of the GLS estimator (7) of the spatial SAR-SUR panel model (4) with respect to $p = (\rho_1, ...\rho_T)'$ is*

$$
\begin{aligned}
S_G &= \partial \hat{\beta}_G / \partial p' \\
&= -(X'\Sigma_{sur}^{-1} X)^{-1} X'\Sigma_{sur}^{-1} (I_T \otimes W_N Y)\ J
\end{aligned} \tag{9}
$$

where J is the $T^2 \times T$ selection matrix given in (10) and $Y = (y_1, ..., y_T)$ is the $N \times T$ de-vectorized (or stacked) panel matrix such that vec $Y = y$.

Proof: In the SAR-SUR system the spread matrix is $R_{NT} = I_{NT} - D_\rho \otimes W_N$ and we find the derivative with respective to p

$$\begin{aligned} d\ R_{NT}y &= -d(D_\rho \otimes W_N) vec\ Y \\ &= -vec\ (W_N Y\ dD_\rho) \\ &= -(I_T \otimes W_N Y) vec\ (dD_\rho) \\ &= -(I_T \otimes W_N Y) J\ dp, \end{aligned}$$

where vec is the vectorisation operator, J is the $T^2 \times T$ selection matrix for diagonal derivatives, defined as

$$J = (i_1 \otimes i_1, ..., i_T \otimes i_T) \tag{10}$$

with $I_T = (i_1, ..., i_T)$ being a $T \times T$ identity matrix, and $\otimes$ denotes the matrix Kronecker product; for these definitions and the relevant properties, see part A1 of the Appendix 6.

Therefore, for the differential of the SAR-SUR GLS estimator (7) of the spatial filter form we find

$$\begin{aligned} d\hat{\beta}_G &= d\ [(X'\Sigma_{sur}^{-1}X)^{-1}X'\Sigma_{sur}^{-1}R_{NT}y] \\ &= -(X'\Sigma_{sur}^{-1}X)^{-1}X'\Sigma_{sur}^{-1}(I_T \otimes W_N Y) J\ dp. \end{aligned} \tag{11}$$

This establishes the theorem.

Note that S_G is free of the correlations in p and hence we obtain a simple expression of the derivative matrix $S_{G0} = S_G|_{p=0} = S_G$.

Theorem 2 *The first order Taylor approximation of the SF-GLS estimator* $\hat{\beta}_G$ *in (7) of the SAR-SUR panel model is*

$$\hat{\beta}_G \approx \hat{\beta}_{gls} + S_{G0}p, \tag{12}$$

where $\hat{\beta}_G$ *is the* $kT \times 1$ *vector and* S_{G0} *is the* $kT \times T$ *sensitivity matrix in (9).*

Proof: Use the Taylor series expansion.

2.2 The reduced form of the SAR-SUR model

From the SF form of the SAR-SUR panel model

$$R_{NT}y = X\beta + e \quad with \quad e \sim N[0, \Sigma_{sur}] \tag{13}$$

with $\mathcal{E}(ee') = \Sigma_{sur} = \Sigma_T \otimes I_N$, we get the reduced form of the SAR-SUR model with $R_{NT} = I_{NT} - D_\rho \otimes W_N$ to be

$$y = R_{NT}^{-1}X\beta + R_{NT}^{-1}e \quad with \quad R_{NT}^{-1}e \sim N[0, \widetilde{\Sigma}_{NT}] \tag{14}$$

and

$$\widetilde{\Sigma}_{NT} = var(R_{NT}^{-1}e) = R_{NT}^{-1}\Sigma_{sur}R_{NT}'^{-1} = (R_{NT}'\Sigma_{sur}^{-1}R_{NT})^{-1}. \tag{15}$$

Note that the GLS estimator in the reduced form of the SAR-SUR model is the same as $\hat{\beta}_G$ in (7). Next we consider a GLS estimator.

Definition 2: The GLS estimator in the reduced form SAR-SUR model $\hat{\beta}_z$. The reduced form GLS estimator of the SAR-SUR model (14) is just the OLS estimator in terms of the transformed regressor

$$\begin{aligned} \hat{\beta}_z &= (Z'Z)^{-1}Z'y \quad with\ Z = R_{NT}^{-1}X \quad or \\ \hat{\beta}_z &= Z^+y \quad with \quad Z^+ = (X'R_{NT}^{-1}{}'R_{NT}^{-1}X)^{-1}X'R_{NT}^{-1}{}'. \end{aligned} \tag{16}$$

For the special case with $p = 0$ we get $R_{NT} = I_{NT}$ and then the GLS estimator simply becomes the OLS estimator of the untransformed panel system

$$\hat{\beta}_{ols} = (X'X)^{-1}X'y.$$

Theorem 3 *In the SAR-SUR model in (14), the sensitivity of the reduced form GLS estimator (16) is the $kT \times T$ matrix*

$$\begin{aligned} S_z &= \partial\hat{\beta}_z/\partial p' \\ &= Z^+(I_T \otimes W_N'E_1)J - Z^+R_{NT}^{-1}(I_T \otimes W_N\hat{Y}_{gls})J \\ &= Z^+Q\ J \end{aligned} \tag{17}$$

with

$$Q = (I_T \otimes W_N'E_1) - R_{NT}^{-1}(I_T \otimes W_N\hat{Y}_{gls})$$

where $Z = R_{NT}^{-1}X$ is the transformed regressor, E_1 is de-vectorized residual matrix obtained from $vecE_1 = R_{NT}'^{-1}(I - ZZ^+)y = R_{NT}'^{-1}(y - \hat{y}_{gls})$, and the GLS fit $\hat{Y}_{gls}$ is computed from $vec\hat{Y}_{gls} = \hat{y}_{gls} = ZZ^+y$.

Proof: Because the differential of the inverse matrix of $R_{NT} = I_{NT} - D_\rho \otimes W_N$ with respect to p is $d\ R_{NT}^{-1} = -R_{NT}^{-1}(d\ R_{NT})R_{NT}^{-1}$, $dR_{NT} = -(dD_\rho) \otimes W_N$ and vec $dD_\rho = Jdp$, we get for (16)

$$
\begin{aligned}
d\hat{\beta}_z &= d((Z'Z)^{-1}Z'y) \\
&= -(Z'Z)^{-1}(d(Z'Z))(Z'Z)^{-1}Z'y + (Z'Z)^{-1}(dZ')y \\
&= Z^{+}(dD_\rho \otimes W_N')R_{NT}'^{-1}(I - ZZ^{+})y - Z^{+}R_{NT}^{-1}(dD_\rho \otimes W_N)ZZ^{+}y \\
&= Z^{+}(dD_\rho \otimes W_N')vecE_1 - Z^{+}R_{NT}^{-1}(dD_\rho \otimes W_N)vec\hat{Y}_{gls} \\
&= Z^{+}vec(W_N'E_1dD_\rho) - Z^{+}R_{NT}^{-1}vec(W_N\hat{Y}_{gls}dD_\rho) \\
&= Z^{+}(I_T \otimes W_N'E_1)vec(dD_\rho) - Z^{+}R_{NT}^{-1}(I_T \otimes W_N\hat{Y}_{gls})vec(dD_\rho) \\
&= Z^{+}(I_T \otimes W_N'E_1)J\ dp - Z^{+}R_{NT}^{-1}(I_T \otimes W_N\hat{Y}_{gls})J\ dp.
\end{aligned}
$$

We establish the theorem by rearranging terms.

Theorem 4 *The first order Taylor approximation of the RF-GLS estimator of the SAR-SUR model in (16) is*

$$\hat{\beta}_z \approx \hat{\beta}_{ols} + S_{z0}p, \tag{18}$$

where

$$S_{z0} = (X'X)^{-1}X'[I_T \otimes \tilde{E}_0]J \tag{19}$$

is the first order sensitivity matrix S_z *evaluated at the origin* $p = (\rho_1, ..., \rho_T)' = 0$, $\tilde{E}_0 = W_N'Y - (W_N' + W_N)\hat{Y}_0$, $\hat{Y}_0$ *is de-vectorized from the OLS fit* $vec\hat{Y}_0 = X(X'X)^{-1}X'y$, *and* $vec\ Y = y$ *is the vectorized panel matrix.*

Proof: From Theorem 3 we get

$$
\begin{aligned}
S_{z0} &= \partial\hat{\beta}_z/\partial p'|_{p=0} \\
&= (X'X)^{-1}X'[I_T \otimes W_N'Y]J - (X'X)^{-1}X'[I_T \otimes W_N'\hat{Y}_0]J \\
&\quad -(X'X)^{-1}X'[I_T \otimes W_N\hat{Y}_0]J
\end{aligned} \tag{20}
$$

and then the Taylor approximation follows.

2.3 Special case: common spatial SAR correlation coefficient

We get a special case of the SAR-SUR model (4) when all the spatial correlations across the system are equal $\rho_1 = ... = \rho_T = \rho$. In this case we have $D_\rho = \rho I_T$ with ρ being the common correlation coefficient and the model reduces to the simple SAR regression model as in Anselin et al. (2008)

$$y = \rho(I_T \otimes W_N)y + X\beta + u \tag{21}$$

where the error term $u = (u_1', ..., u_T')'$ is a $N \times 1$ error vector and follows a normal distribution with a $NT \times 1$ mean vector centered at 0 and a $NT \times NT$

variance matrix $\sigma^2 I_{NT}$. Also $y = vecY$ is an $NT \times 1$ vectorized panel vector, ρ is the common spatial autocorrelation parameter (a scalar), $W_{NT} = I_T \otimes W_N$ is a system common neighborhood matrix W_N is the $N \times N$ spatial weight matrix normalized with row sums 1, X is an $NT \times kT$ regressor matrix, and β is a $kT \times 1$ regression coefficient vector.

The panel spatial filter (SF-GLS) estimator $b_\rho : kT \times 1$ of the SAR-SUR model (4) can be written in analogy to the non-system case as a linear combination of 2 simpler GLS estimators

$$\begin{aligned} b_\rho &= (X'\Sigma_{sur}^{-1}X)^{-1}X'\Sigma_{sur}^{-1}R_{NT}y \\ &= (X'\Sigma_{sur}^{-1}X)^{-1}X'\Sigma_{sur}^{-1}(I_{NT} - \rho W_{NT})y \\ &= (X'\Sigma_{sur}^{-1}X)^{-1}X'\Sigma_{sur}^{-1}y - \rho(X'\Sigma_{sur}^{-1}X)^{-1}X'\Sigma_{sur}^{-1}W_{NT}y \\ &= b_0 - \rho(X'\Sigma_{sur}^{-1}X)^{-1}X'\Sigma_{sur}^{-1}W_{NT}y \\ &= b_0 - \rho b_1. \end{aligned} \tag{22}$$

We see that the difference between the SUR-GLS estimator

$$\hat{\beta}_{sur} = (X'\Sigma_{sur}^{-1}X)^{-1}X'\Sigma_{sur}^{-1}y$$

and the spatial lag estimator $\hat{\beta}_{sur} - b_\rho = \rho\hat{\beta}_{lag}$ is proportional in size to the spatial parameter ρ and the first order spatial lag estimator

$$\hat{\beta}_{lag} = (X'\Sigma_{sur}^{-1}X)^{-1}X'\Sigma_{sur}^{-1}W_{NT}y. \tag{23}$$

Next we compute the derivative of b_ρ with respect to the common ρ, which measures the sensitivity of b_ρ with respect to a small change in ρ. For analytical and mathematical convenience, we use the differential notation from where the derivative can be obtained equivalently and more easily; see Magnus and Neudecker (1999) and Liu and Neudecker (2009).

Theorem 5 *The sensitivity or first derivative of the spatial filter b_ρ estimator in the common correlation model with respect to the ρ parameter is the negative spatial lag estimator in the linear model for explaining the first order spatial lag*

$$\partial b_\rho / \partial \rho = -(X'\Sigma_{sur}^{-1}X)^{-1}X'\Sigma_{sur}^{-1}W_{NT}y = -\hat{\beta}_{lag}, \tag{24}$$

where the spatial lag estimator $\hat{\beta}_{lag}$ is given in (23).

Proof The matrix differential of the b_ρ estimator with respect to ρ in (22) is

$$db_\rho = -\hat{\beta}_{lag} d\rho$$

and by rearranging terms, we establish the derivative.

2.4 The common correlation (cc) SAR model

In some cases it might be interesting to look at SAR models that have a common correlation and common coefficient. For practical applications like model choice this can be a good starting point.

The stacked panel SUR-SAR model has the same structure as the model (4) before

$$y = (D_\rho \otimes W_N)y + X\beta + e, \quad with \quad e \sim N[0_{NT}, \Sigma_{sur}] \tag{25}$$

where $y = (y_1', ..., y_T')' = vecY$ is a $NT \times 1$ vector, Y is an $N \times T$ dependent panel matrix, the diagonal matrix $D_\rho : T \times T$ reduces in the cc case to ρI_T, $X = (X_1', ..., X_T')'$ is an $NT \times k$ stacked regressor matrix with X_t of order $N \times k$, $\beta = (\beta_1, ..., \beta_k)'$ is a $k \times 1$ vector of stacked regression coefficients.

Assuming a common correlation and common coefficient (cc&cc), we simplify the model structure of the GLS estimator, because $R_{NT} = I_T \otimes R_N$ with $R_N = I_N - \rho W_N$, $\Sigma_{sur} = \Sigma_T \otimes I_N$, and then the covariance matrix (15) reduces to

$$\Sigma_c = (\Sigma_T^{-1} \otimes R_N' R_N)^{-1} \tag{26}$$

Therefore the GLS estimator in the cc case takes the form

$$\hat{\beta}_c = (X' \Sigma_c^{-1} X)^{-1} X' \Sigma_c^{-1} y, \tag{27}$$

which is the SUR estimator with common spatial heteroskedasdicity of the form $R_N' R_N$, because Σ_{sur}^{-1} is the covariance/correlation matrix across the equations of the panel system. This estimator is a SUR-GLS estimator for the spatial transformed (SEM filtered) variables $X_* = (I_T \otimes R_N)X$ and $y_* = (I_T \otimes R_N)y$: $\hat{\beta}_c = (X_*' \Sigma_{sur}^{-1} X_*)^{-1} X_*' \Sigma_{sur}^{-1} y_*$.

3 Approximating the GLS estimator in the SUR-SEM model

This section computes the sensitivity of the GLS estimator in the SUR-SEM model and describes the first order Taylor approximation using the sensitivity results.

3.1 SUR models with SEM errors

In this section we look at a system generalization of the spatial error model (SEM), which can be found as alternative to the SAR-SUR model in e.g.

LeSage and Pace (2009).

Consider the following panel SUR-SEM model

$$y_t = X_t\beta + e_t, \quad with \quad e_t = \theta_t W_N e_t + u_t, \tag{28}$$

where we assume a centered homoskedastic error term for u : $\mathcal{E}(u_t, u_s') = \sigma_{ts} I_N \quad (t, s = 1, ..., T)$. Now the error term in the SUR-SEM model can be written as

$$e_t = (I_N - \theta_t W_N)^{-1} u_t = B_{N,t}^{-1} u_t, \tag{29}$$

where $B_{N,t} = I_N - \theta_t W_N$ is the t-th component of the SEM filter matrix B_{NT}.

In the cc case the cross-equation covariance matrix between the error vectors e_t and e_s then becomes

$$\mathcal{E}[e_t \ e_s'] = B_N^{-1} \mathcal{E}[u_t \ u_s'] B_N^{-1'} = \sigma_{ts} B_N^{-1} B_N^{-1'} = \sigma_{ts}(B_N' B_N)^{-1}. \tag{30}$$

Definition 3: The spatial SUR-SEM model.
In matrix form, the SUR-SEM model with the $NT \times 1$ error vector e can be written as

$$\begin{aligned} &y = X\beta + e, \quad with \quad e = B_{NT}^{-1} u, \quad or \\ &y \sim N[X\beta, \Sigma_{sem}] \quad with \quad \Sigma_{sem} = B_{NT}^{-1}(\Sigma_T \otimes I_N) B_{NT}'^{-1} \end{aligned} \tag{31}$$

where the error term u contains the SUR correlation matrix $\mathcal{E}(uu') = \Sigma_T \otimes I_N$, and the SUR-SEM system filter matrix

$$B_{NT} = I_{NT} - D_\theta \otimes W_N \quad with \quad D_\theta = diag(\theta_1, ..., \theta_T) \tag{32}$$

is a $T \times T$ diagonal SUR-SEM correlation parameter matrix. Furthermore, $y = (y_1', ..., y_T')' = vecY : NT \times 1$ is the vectorized panel matrix, $X = diag(X_1, ..., X_T)$ is a $NT \times KT$ block-diagonal regressor matrix, and $e = (e_1', ..., e_T')'$ is the error vector.

Note that the log-likelihood function of the SUR-SEM model (31) is

$$\begin{aligned} L &= \frac{1}{2} ln \ |B_{NT}'(\Sigma_T^{-1} \otimes I_N) B_{NT}| - \frac{1}{2} e' B_{NT}'(\Sigma_T^{-1} \otimes I_N) B_{NT} e \\ &= \frac{1}{2} ln \ |\Sigma_{sem}^{-1}| - \frac{1}{2} e' \Sigma_{sem}^{-1} e \end{aligned} \tag{33}$$

Definition 4: The spatial SEM-GLS estimator $\hat{\beta}_{sem}$.
The SEM-GLS estimator in the SUR-SEM model (31) is the $kT \times 1$ vector

$$\hat{\beta}_{sem} = [X'\Sigma_{sem}^{-1}X]^{-1}X'\Sigma_{sem}^{-1}y. \tag{34}$$

Define the SEM correlation parameter vector as $q = (\theta_1, ..., \theta_T)'$. For zero correlation $q = 0$, the SEM filter vanishes to $B_{NT} = I_{NT}$, and the GLS estimator in the SUR-SEM model reduces to the GLS type SUR estimator (8).

Theorem 6 (The sensitivity of $\hat{\beta}_{sem}$) *Let $q = (\theta_1, ..., \theta_T)'$ be the correlation vector, then will use the classical sensitivity results. The $kT \times T$ sensitivity matrix of the reduced form RF-GLS estimator in the SUR-SEM model is*

$$\begin{aligned}
S_{sem} &= \partial\hat{\beta}_{sem}/\partial q' \\
&= -[X'\Sigma_{sem}^{-1}X]^{-1}X'(I_T \otimes W_N' M \Sigma_T^{-1})\ J \\
&\quad -[X'\Sigma_{sem}^{-1}X]^{-1}X'B_{NT}'(\Sigma_T^{-1} \otimes W_N(Y - \hat{Y}_{sem}))\ J \\
&= -[X'\Sigma_{sem}^{-1}X]^{-1}\hat{Q}_{sem}\ J \\
&\quad \textit{with} \\
\hat{Q}_{sem} &= (I_T \otimes W_N'[\hat{E} - W_N\hat{E}D_\theta]\Sigma_T^{-1}) + (\Sigma_T^{-1} \otimes W_N\hat{E}),
\end{aligned} \tag{35}$$

where $\hat{E} = Y - \hat{Y}_{sem}$ is the SEM panel residual matrix and $\hat{Y}_{sem}$ is the fit of $vec\hat{Y}_{sem} = \hat{y}_{sem} = X\hat{\beta}_{sem}$.

We see that the derivative consists of the scaling matrix times the residual quantity matrix Q_{sem}, which is a complicated mixture of two components, because the covariance matrix Σ_T^{-1} appears at both sides of the Kronecker product.

Proof: First we get the derivative of the covariance matrix using (41), $d\Sigma_{sem}^{-1}$ with respect to the vector q

$$\begin{aligned}
d\Sigma_{sem}^{-1} &= d(B_{NT}'(\Sigma_T \otimes I_N)B_{NT}) \\
&= -(dD_\theta \otimes W_N')(\Sigma_T^{-1} \otimes I_N)(I_{NT} - D_\theta \otimes W_N) - B_{NT}'(\Sigma_T^{-1} \otimes I_N)(dD_\theta \otimes W_N) \\
&= -(dD_\theta)\Sigma_T^{-1} \otimes W_N' + dD_\theta\Sigma_T^{-1}D_\theta \otimes W_N'W_N - B_{NT}'(\Sigma_T^{-1}dD_\theta \otimes W_N)
\end{aligned}$$

Inserting $d\Sigma_{sem}^{-1}$ into the derivative of $\hat{\beta}_{sem}$ in (34) we get

$$
\begin{aligned}
d\hat{\beta}_{sem} =& -(X'\Sigma_{sem}^{-1}X)^{-1}X'd\Sigma_{sem}^{-1}X(X'\Sigma_{sem}^{-1}X)^{-1}X'\Sigma_{sem}^{-1}y \\
& +(X'\Sigma_{sem}^{-1}X)^{-1}X'd\Sigma_{sem}^{-1}y \\
=& \,(X'\Sigma_{sem}^{-1}X)^{-1}X'(d\Sigma_{sem}^{-1})(y-\hat{y}_{sem}) \\
=& -(X'\Sigma_{sem}^{-1}X)^{-1}X'(dD_\theta)\Sigma_T^{-1}\otimes W_N')vec\hat{E} \\
& +(X'\Sigma_{sem}^{-1}X)^{-1}X'(dD_\theta\Sigma_T^{-1}D_\theta\otimes W_N'W_N)vec\hat{E} \\
& -(X'\Sigma_{sem}^{-1}X)^{-1}X'B_{NT}'(\Sigma_T^{-1}dD_\theta\otimes W_N)vec\hat{E} \\
=& -(X'\Sigma_{sem}^{-1}X)^{-1}X'(dD_\theta)\Sigma_T^{-1}\otimes W_N')vec\hat{E} \\
& +(X'\Sigma_{sem}^{-1}X)^{-1}X'(dD_\theta\Sigma_T^{-1}D_\theta\otimes W_N'W_N)vec\hat{E} \\
& -(X'\Sigma_{sem}^{-1}X)^{-1}X'B_{NT}'(\Sigma_T^{-1}dD_\theta\otimes W_N)vec\hat{E} \\
=& -(X'\Sigma_{sem}^{-1}X)^{-1}X'(I_T\otimes W_N'[\hat{E}-W_N\hat{E}D_\theta]\Sigma_T^{-1})\ Jdp \\
& -(X'\Sigma_{sem}^{-1}X)^{-1}X'[\Sigma_T^{-1}\otimes W_N\hat{E}]\ Jdp \\
=& -(X'\Sigma_{sem}^{-1}X)^{-1}X'[(I_T\otimes W_N'[\hat{E}-W_N\hat{E}D_\theta]\Sigma_T^{-1})+(\Sigma_T^{-1}\otimes W_N\hat{E})]\ Jdp
\end{aligned}
$$

This proves the theorem.

Theorem 7 *The Taylor approximation of the reduced form GLS estimator (34) is*

$$\hat{\beta}_{sem} \approx \hat{\beta}_{sur} + S_{sur}q, \tag{36}$$

where we briefly write $V = (X'\Sigma_{sur}^{-1}X)^{-1}$ *and* $\Sigma_{sur} = \Sigma_T \otimes I_N$ *to get the the sensitivity matrix* S_{sur} *for the SEM-SUR model*

$$
\begin{aligned}
S_{sur} =& \,\partial\hat{\beta}_{sem}/\partial q'|_{q=0} \\
=& \,V\ X'[(\Sigma_T^{-1}\otimes W_N(Y-\hat{Y})) - (I_T\otimes W_N'(Y-\hat{Y})\Sigma_T^{-1})]J,
\end{aligned}
$$

where $\hat{Y}$ *is de-vectorized from the regression fit* $vec\hat{Y} = X\hat{\beta}_{sur}$ *with* $\hat{\beta}_{sur}$ *being the SUR system estimator.*

We see that the central quantity for this evaluation at point zero is the panel residual $Y - \hat{Y}$, which is scaled in 2 different ways in the Kronecker product.

Proof: If $q = 0$ then $B_{NT} = I_{NT}$ with residual $\hat{E} = Y - \hat{Y}$, and S_{sur} is obtained from S_{sem} evaluated at $q = 0$.

3.2 Special case: a common θ correlation coefficient in the SUR-SEM model

A special case of the SUR-SEM model (31) is obtained for $q_1 = ... = q_T = \theta$, which leads to the homogeneity model

$$
\begin{aligned}
&y = X\beta + e, \quad with \quad e = B_{NT}^{-1}u, \quad or \\
&y \sim N[X\beta, \Sigma_s] \quad with \quad \Sigma_s = (\Sigma_T^{-1} \otimes B_N' B_N)^{-1}
\end{aligned} \tag{37}
$$

where y is a $NT \times 1$ observation vector, X is a $NT \times kT$ regressor matrix, β is a $kT \times 1$ coefficient vector, and

$$
B_{NT} = I_T \otimes (I_N - \theta W_N) = I_T \otimes B_N \quad with \quad B_N = I_N - \theta W_N \tag{38}
$$

is the system spatial filter of the SUR-SEM model.

For the non-SUR case $\Sigma_T = \sigma_u^2 I_T$ we get a simple common correlation (cc) GLS estimator, which is exactly the 'associated' GLS estimator in the formula (19.37) for the model (19.9) given in Anselin et al. (2008):

$$
\hat{\beta}_{cc} = [X'(I_T \otimes B_N' B_N)X]^{-1} X'(I_T \otimes B_N' B_N)y. \tag{39}
$$

The non-SUR GLS estimator is a simple OLS estimator for the spatial transformed (SEM filtered) variables $X_* = (I_T \otimes B_N)X$ and $y_* = (I_T \otimes B_N)y$:

$$
\hat{\beta}_{cc} = (X_*' X_*)^{-1} X_*' y_*. \tag{40}
$$

If $\theta = 0$ we have $B_N = I_N$ and we get a special case the OLS estimator $\hat{\beta}_{ols} = (X'X)^{-1}X'y$.

Theorem 8 *The sensitivity of the $kT \times 1$ non-SUR GLS estimator $\hat{\beta}_{cc}$ in (39) of the SEM panel system model with respect to θ is*

$$
\begin{aligned}
S_{cc} &= \partial \hat{\beta}_{cc} / \partial \theta \\
&= V \ X'(I_T \otimes W_B)X \ V \ X'(I_T \otimes B_N' B_N)y - V \ X'(I_T \otimes W_B)y, \\
&= -V \ X'(I_T \otimes W_B)(y - \hat{y}_b),
\end{aligned}
$$

*where we use $V = [X'(I_T \otimes B_N' B_N)X]^{-1}$, $W_B = W_N' B_N + B_N' W_N$ and $\hat{y}_b = X \ V \ X'(I_T \otimes B_N' B_N)y$ is the Anselin estimator in (**??**).*

Again the central quantity of this sensitivity result is the residual of the Anselin estimator $(y - \hat{y}_b)$ of the SUR-SEM model.

Proof: Using the SEM filter matrix $B_N = I_N - \theta W_N$, we find for the matrix differential of the $\hat{\beta}_{cc}$ estimator with respect to θ

$$
\begin{aligned}
d\hat{\beta}_{cc} = &V \ X'(I_T \otimes W_N' B_N)XV \ X'(I_T \otimes B_N' B_N)y \ d\theta \\
&+V \ X'(I_T \otimes B_N' W_N)XV \ X'(I_T \otimes B_N' B_N)y \ d\theta \\
&-V \ X'(I_T \otimes W_N' B_N)y \ d\theta - V \ X'(I_T \otimes B_N' W_N)y \ d\theta
\end{aligned} \tag{41}
$$

By rearranging the differential, we get the result.

Theorem 9 looks at the Taylor approximation of the non-SUR GLS estimator $\hat{\beta}_{cc}$ for the SEM panel system.

Theorem 9 (The first order Taylor approximation of the cc case) *The first order Taylor approximation of the cc GLS estimator $\hat{\beta}_{cc}$ of the panel SEM model is*

$$\hat{\beta}_{cc} \approx \hat{\beta}_{ols} + S_{cc0}\theta \tag{42}$$

where

$$S_{cc0} = -(X'X)^{-1}X' vec((W_N' + W_N)\hat{E}_0)$$
$$\hat{E}_0 = Y - \hat{Y}$$

where S_{cc0} is the SEM sensitivity matrix S_{cc} evaluated around zero and $\hat{y} = vec\hat{Y} = X(X'X)^{-1}X'y$ is the OLS fit. The central quantity of this result is the OLS residual, which gets scaled twice, the matrix $(X'X)^{-1}X'$ and the symmetric weighting matrix $W_N' + W_N$.

Proof The evaluation of the matrix differential S_{cc} around $\theta = 0$ is

$$\begin{aligned} S_{cc0} &= \partial\hat{\beta}_{cc}/\partial\theta|_{\theta=0} \\ &= [X'X]^{-1}X'(I_T \otimes W_N')X(X'X)^{-1}X'y \\ &\quad +(X'X)^{-1}X'(I_T \otimes W_N)X(X'X)^{-1}X'y \\ &\quad -(X'X)^{-1}X'(I_T \otimes (W_N' + W_N))\hat{y} \\ &= -(X'X)^{-1}X'(I_T \otimes (W_N' + W_N))(y - \hat{y}). \end{aligned} \tag{43}$$

4 Simulation study: how good are Taylor approximations of GLS estimates?

In this section, we use simulated data to compute the (generalized) least squares GLS estimates and their corresponding first order approximations, that were given in Theorems 2, 4, 7 and 9. We set the dimension of the panel system to $T = 2$, the number of observations to $N = 50, 100, 400$, and then I_{NT} as an $NT \times NT$ identity matrix, $D_\rho = diag(\rho_1, \rho_2)$, and $D_\theta = diag(\theta_1, \theta_2)$. We set $p = (\rho_1, \rho_2)' = (0.2, 0.15)'$ for Theorems 2 and 4, $q = (\theta_1, \theta_2)' = (0.1, 0.3)'$ for Theorem 7, and $\theta = 0.3$ for Theorem 9.

Two regressors X are drawn randomly from two $T \times 2$ matrices of uniformly distributed random numbers from a uniform distribution $U[0,1]$. We fix the regression coefficient values as $\beta_1 = (1,2)$ and $\beta_2 = (0.5, 1.5)$.

We choose two neighborhood matrices W_N as follows:

$$W_1 = \begin{pmatrix} 0 & 0.5 & 0.5 & 0 & \cdots & 0 \\ 0.5 & 0 & 0.5 & 0 & \cdots & 0 \\ 0 & 0.5 & 0 & 0.5 & \cdots & 0 \\ \vdots & \cdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0.5 & 0 & 0.5 \\ 0 & \cdots & 0 & 0.5 & 0.5 & 0 \end{pmatrix}, \quad W_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0.5 & 0 & 0.5 & 0 & \cdots & 0 \\ 0 & 0.5 & 0 & 0.5 & \cdots & 0 \\ \vdots & \cdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0.5 & 0 & 0.5 \\ 0 & \cdots & 0 & 0 & 1 & 0 \end{pmatrix} \tag{44}$$

We generate the error terms in e from a bivariate normal distribution $N[0,\Sigma]$ with mean 0 and covariance matrix Σ. The following two choices of Σ have been used, the first being the uncorrelated case and the second being the correlated:

$$\Sigma_1 = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.4 \end{pmatrix}, \quad \Sigma_2 = \begin{pmatrix} 0.5 & 0.3 \\ 0.3 & 0.4 \end{pmatrix} \tag{45}$$

Furthermore, we use the four combinations of the two choices of W_N and Σ matrices to generate the data for the response values y in the two models. For Theorems 2 and 4, we have y calculated using SUR model (3), i.e.

$$y = (I_{NT} - D_\rho \otimes W_N)^{-1}(X\beta + e). \tag{46}$$

For Theorems 7 and 9, we can simulate y using the reduced form model, i.e.

$$y = X\beta + (I_{NT} - D_\theta \otimes W_N)^{-1}e. \tag{47}$$

After generating the data, we calculate the corresponding GLS estimates $\hat{\beta}$ and their approximations $\tilde{\beta}$ as given in Theorems 2 and 4 for the SUR-SAR model, and Theorems 7 and 9 for the SEM model, respectively.

To get an overview of the approximation property of our sensitivity approach we calculate the mean squared error (MSE) between the estimators $\hat{\beta}$ and the true values β of the regression coefficients:

$$MSE(\hat{\beta}) = (\hat{\beta} - \beta)'(\hat{\beta} - \beta).$$

Since we are simulating a 2-dimensional system, we are estimating two spatial correlation coefficients, for each equation one parameter, over a 2-dimensional grid. We choose a grid between -0.8 and 0.8 with steps of 0.2, so we get a MSE for 81 points for each estimator. These MSE calculations are presented in part A2 of Appendix 6. To get a rough idea regarding what estimator is better in terms of MSE, we suggest computing the following average MSE over the grid:

$$AMSE_{\rho}(\hat{\beta}) = \sum_{i=1}^{N}\sum_{j=1}^{N} MSE_{ij}(\hat{\beta}; \rho_i, \rho_j)/NM$$

where $MSE_{ij}(\hat{\beta}; \rho_i, \rho_j)$ stands for the MSE of the estimator $\hat{\beta}$ evaluated at the grid point ρ_i, ρ_j. Similarly, we can evaluate an average MSE for the 4 possible design points of weights and covariance matrices:

$$AMSE_{\Sigma_i, W_j}(\hat{\beta}) = \sum_{i=1}^{2}\sum_{j=1}^{2} MSE_{ij}(\hat{\beta}; \Sigma_i, W_j)/4$$

To get an overview of the simulation results we use the AMSE as a rough guideline for a summary if the matrix combinations Σ and W_N matter or the values of the spatial autocorrelation.

Table 1 and 2 show the AMSE for Theorem 2 using our summary programs of the simulation. As we see, there are no differences between estimates and approximations, a result that we have also shown theoretically.

Table 1. The $AMSE_{\rho}$ of estimates and approximations for Theorem 2 (N=10)

$\hat{\beta}_{SF}$		$\hat{\beta}_{SF0} + S_{SF0}p$	
Σ_1, W_1	Σ_1, W_2	Σ_1, W_1	Σ_1, W_2
3.9325	3.1327	3.9325	3.1327
Σ_2, W_1	Σ_2, W_2	Σ_2, W_1	Σ_2, W_2
4.1473	4.9678	4.1473	4.9678

Table 2. The $AMSE_{\Sigma,W}$ of estimates and approximations for Theorem 2(N=10)

ρ_2 \ ρ_1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8
					$\hat{\beta}_{SF}$				
-0.8	9.8785	7.4475	5.6425	4.3905	3.7173	3.6207	4.0318	5.0363	6.6324
-0.6	9.0683	6.6232	4.8398	3.5732	2.8352	2.7177	3.1116	4.0871	5.6204
-0.4	8.5354	6.0524	4.2165	2.9113	2.1887	1.9952	2.3674	3.2997	4.8248
-0.2	8.1684	5.712	3.7833	2.4719	1.7157	1.5133	1.8674	2.7809	4.2703
0	8.0487	5.5195	3.6247	2.2658	1.4442	1.2327	1.5629	2.4372	3.8979
0.2	8.089	5.5759	3.6423	2.2296	1.4312	1.1657	1.4849	2.3423	3.7847
0.4	8.3733	5.8542	3.8685	2.4634	1.5972	1.328	1.5964	2.415	3.8201
0.6	8.8887	6.2621	4.3248	2.8963	1.9996	1.6764	1.9319	2.7318	4.1391
0.8	9.6054	6.9745	4.9366	3.5011	2.5884	2.2341	2.4856	3.2388	4.5929
					$\hat{\beta}_{SF0} + S_{SF0}p$				
-0.8	9.8785	7.4475	5.6425	4.3905	3.7173	3.6207	4.0318	5.0363	6.6324
-0.6	9.0683	6.6232	4.8398	3.5732	2.8352	2.7177	3.1116	4.0871	5.6204
-0.4	8.5354	6.0524	4.2165	2.9113	2.1887	1.9952	2.3674	3.2997	4.8248
-0.2	8.1684	5.712	3.7833	2.4719	1.7157	1.5133	1.8674	2.7809	4.2703
0	8.0487	5.5195	3.6247	2.2658	1.4442	1.2327	1.5629	2.4372	3.8979
0.2	8.089	5.5759	3.6423	2.2296	1.4312	1.1657	1.4849	2.3423	3.7847
0.4	8.3733	5.8542	3.8685	2.4634	1.5972	1.328	1.5964	2.415	3.8201
0.6	8.8887	6.2621	4.3248	2.8963	1.9996	1.6764	1.9319	2.7318	4.1391
0.8	9.6054	6.9745	4.9366	3.5011	2.5884	2.2341	2.4856	3.2388	4.5929

Table 3 shows the estimates and approximations for Theorem 4. Here we see that W_1 produces results to the approximations not as good as W_2 does. This shows that even slight deviations in the neighborhood matrices can have a large effect on the quality of the approximations.

Table 3. The $AMSE_\rho$ of estimates and approximations for Theorem 4 (N=10)

$\hat{\beta}_z$		$\hat{\beta}_{ols} + S_{z0}p$	
Σ_1, W_1	Σ_1, W_2	Σ_1, W_1	Σ_1, W_2
2.2003	2.7545	3.8691	3.8524
Σ_2, W_1	Σ_2, W_2	Σ_2, W_1	Σ_2, W_2
2.6051	2.3779	3.964	3.3368

Table 4 shows that the approximations for Theorem 4 do not perform well if it comes to the extremes of the correlation space: The largest deviations can be seen for the spatial correlation ± 0.8. These results are in line with univariate results for ρ, where we have found that the approximations will give good results in terms of MSE, only in the interval ± 0.3.

Table 4. The $AMSE_{\Sigma,W}$ of estimates and approximations for Theorem 4 (N=10)

ρ_2 \ ρ_1	−0.8	−0.6	−0.4	−0.2	0	0.2	0.4	0.6	0.8
					$\hat{\beta}_z$				
−0.8	2.6509	2.584	2.8292	2.521	2.0233	1.7562	1.9179	2.5616	3.7564
−0.6	2.9014	2.8403	3.0877	2.7941	2.2898	2.0273	2.1756	2.8317	4.0263
−0.4	2.801	2.7403	2.9724	2.678	2.1696	1.9183	2.0656	2.7104	3.8949
−0.2	2.5299	2.4818	2.7423	2.4282	1.9354	1.6424	1.8248	2.4606	3.6578
0	2.3078	2.2607	2.4902	2.2044	1.7122	1.4295	1.5803	2.2395	3.4319
0.2	2.2214	2.1574	2.3926	2.1104	1.5922	1.3393	1.4898	2.1286	3.3271
0.4	2.2966	2.2362	2.5021	2.1936	1.7008	1.4182	1.5638	2.2385	3.4284
0.6	2.6067	2.5543	2.8154	2.5205	2.0144	1.7382	1.8951	2.5518	3.7422
0.8	3.192	3.1484	3.3613	3.0733	2.5852	2.3201	2.4646	3.1293	4.3072
					$\hat{\beta}_{ols} + S_{z0}p$				
−0.8	4.1846	3.164	2.5525	2.3278	2.548	3.2104	4.2653	5.7536	7.6613
−0.6	3.6843	2.6376	2.0294	1.8367	2.045	2.7077	3.7463	5.2249	7.1559
−0.4	3.3968	2.3392	1.7267	1.5174	1.7278	2.3895	3.4686	4.9634	6.8299
−0.2	3.2666	2.194	1.5985	1.3909	1.6318	2.2505	3.3363	4.8279	6.7409
0	3.3198	2.3026	1.6658	1.4702	1.7122	2.3534	3.3954	4.8896	6.768
0.2	3.5818	2.5962	1.9576	1.7701	1.9563	2.6151	3.6924	5.1753	7.092
0.4	4.0684	3.0651	2.4356	2.2245	2.4475	3.0784	4.1366	5.6087	7.5192
0.6	4.74	3.6903	3.074	2.8873	3.117	3.7653	4.8177	6.2966	8.1996
0.8	5.5896	4.5524	3.9354	3.755	3.9505	4.6121	5.6943	7.2045	9.0915

Tables 5 and 6 show the quality of the approximations for Theorem 7. In terms of MSE, the difference between the estimates and approximations are the smallest if we compare them with the previous simulation results. This shows that the approximations work better for the SUR-SEM model.

Table 5. The $AMSE_\theta$ of estimates and approximations for Theorem 7 (N=10)

$\hat{\beta}_{sem}$		$\hat{\beta}_{sur} + S_{sur}q$	
Σ_1, W_1	Σ_1, W_2	Σ_1, W_1	Σ_1, W_2
2.3737	2.058	1.7048	1.5203
Σ_2, W_1	Σ_2, W_2	Σ_2, W_1	Σ_2, W_2
3.2821	3.7147	1.7609	2.9549

Table 6. The $AMSE_{\Sigma,W}$ of estimates and approximates for Theorem 7 (N=10)

$\theta_2 \backslash \theta_1$	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8
					$\hat{\beta}_{sem}$				
-0.8	5.7151	4.8568	3.9427	3.3249	3.1249	3.0734	3.1172	3.3758	3.8706
-0.6	5.3165	4.4214	3.5598	3.0534	2.7224	2.6841	2.7681	2.9402	3.4631
-0.4	4.9452	4.0754	3.204	2.6976	2.4362	2.328	2.3701	2.5887	3.0425
-0.2	4.5421	3.6855	2.9465	2.363	2.0918	2.0057	2.0133	2.2216	2.7185
0	4.3426	3.4556	2.7082	2.1504	1.8392	1.7481	1.7426	1.9023	2.4046
0.2	4.1975	3.423	2.6121	2.076	1.7572	1.621	1.6116	1.7746	2.2572
0.4	4.3257	3.448	2.6807	2.0833	1.7811	1.5967	1.6059	1.7407	2.2287
0.6	4.4737	3.7342	2.8552	2.2581	1.9114	1.7632	1.7325	1.8464	2.3199
0.8	4.8	4.098	3.226	2.6016	2.2669	2.0653	2.0049	2.1525	2.5969
					$\hat{\beta}_{sur} + S_{sur}q$				
-0.8	2.0308	2.0337	2.0408	2.0602	2.1427	2.2378	2.308	2.4432	2.5487
-0.6	1.9652	1.9699	1.942	2.0037	2.0359	2.0936	2.1728	2.285	2.3874
-0.4	1.9785	1.9354	1.8897	1.9479	1.9795	1.9939	2.0842	2.1769	2.2535
-0.2	1.9281	1.8844	1.8919	1.8672	1.9	1.9487	1.9951	2.0873	2.1724
0	1.9509	1.877	1.8491	1.8537	1.8392	1.8873	1.9178	1.9604	2.0619
0.2	1.9696	1.9338	1.8669	1.8505	1.8397	1.8476	1.8784	1.9272	2.0074
0.4	2.0416	1.9339	1.8968	1.8499	1.8454	1.7886	1.8365	1.8772	1.9519
0.6	2.0872	2.0198	1.9315	1.8742	1.8622	1.8415	1.8503	1.8585	1.8917
0.8	2.1982	2.0955	1.992	1.9328	1.9165	1.8894	1.8513	1.8799	1.8771

Table 7 and 8 show very good agreements between estimates and approximations. The common correlation (cc) case reduces the amount of spatial non-linearities and therefore linear approximations work quite well.

Table 7. The $AMSE_\theta$ of estimates and approximations for Theorem 9 (N=10)

$\hat{\beta}_{cc}$		$\hat{\beta}_{ols} + S_{cc0}\theta$	
Σ_1, W_1	Σ_1, W_2	Σ_1, W_1	Σ_1, W_2
2.7472	2.5135	2.8321	2.3878
Σ_2, W_1	Σ_2, W_2	Σ_2, W_1	Σ_2, W_2
2.5899	2.0581	2.4943	1.9629

Table 8. The $AMSE_{\Sigma,W}$ of estimates and approximations for Theorem 9 (N=10)

θ	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8
					$\hat{\beta}_{cc}$				
	4.5566	3.7408	3.1324	2.3765	1.4236	1.388	1.4277	1.7238	2.5251
					$\hat{\beta}_{ols} + S_{cc0}\theta$				
	4.5921	3.7067	3.0661	2.3631	1.4236	1.3909	1.4092	1.7919	2.0301

An alternative evaluation of the simulation study by distances can be found in part A2 of Appendix 6.

5 Conclusions

In this paper, we have considered two system panel spatial models and we have conducted a sensitivity analysis to study the approximation quality of the newly derived diagnostics based on MSE, Absolute and Eelative Distance measures. We have proposed (generalized) least squares estimators and established their sensitivity results with respect to the spatial correlation parameter in a SAR or SEM panel system. Based on the sensitivity matrices of the GLS estimators we have computed a first order Taylor approximation for two types of simple SUR based GLS estimators. By simulation comparisons we see that these sensitivity and approximation results perform well, at least for small vales of spatial correlations. We have found that the approximations work better for the SUR-SEM model than for the SUR-SAR model in terms of MSE. Also, for certain SUR-SAR models the neighborhood matrix seems to have more influence on the approximation than the SUR covariance matrix. Due to the multiplicity of potential influence factors, it is difficult to come up with an overall judgement of the approximations across all spatial system models.

Furthermore, the new approach might be useful for a Bayesian analysis using MCMC because it can be highly non-linear for spatial models. Generally, good proposal distributions are needed in a Metropolis step for the spatial correlation coefficients, and might avoid unnecessary long estimation time, because the simulation chain is better mixing if the proposal distribution generates less autocorrelations. In further research studies this new approach can be used to develop sensitivity results for space-time panel systems, which easily gets into high dimensions.

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6 Appendix

6.1 A1: Some mathematical definitions and properties

The following can be found in e.g. Neudecker et al. (1995a, 1995b), Magnus and Neudecker (1999) and Liu and Neudecker (2009).

1. The Kronecker product: $A \otimes B = (a_{jt}B)$, and the Hadamard product: $A \odot C = (a_{jt}c_{jt})$ where $A = (a_{jt})$ and $C = (c_{jt})$
2. The vectorisation operator vec: $\text{vec}(A) = (a_1', \ldots, a_T')'$, where $A = (a_1, \ldots, a_T)$ is an $m \times T$ matrix with a_t as the t^{th} $m \times 1$ column, $t = 1, ..., T$
3. The $T^2 \times T$ selection matrix: $J = (i_1 \otimes i_1, ..., i_T \otimes i_T) = (vecE_{11}, \ldots, vecE_{TT})$, where $E_{tt} = i_t i_t'$, $i_t = (0, \ldots, 1, \ldots, 0)'$ is the t^{th} $T \times 1$ unit vector i.e. t^{th} column of the $T \times T$ identity matrix I_T, $t = 1, \ldots, T$
Note that $(I_N \otimes A)J = (I_N \otimes A)(i_1 \otimes i_1, ..., i_T \otimes i_T) = (i_1 \otimes a_1, ..., i_T \otimes a_T)$. Similarly, for reversed matrices: $(A \otimes I_T)J = (A \otimes I_T)(i_1 \otimes i_1, ..., i_T \otimes i_T) = (a_1 \otimes i_1, ..., a_T \otimes i_T)$. 4. $\text{vec}D_\rho = Jp$, where $D_\rho = diag(p_1, \ldots, p_T)$ is an $T \times T$ diagonal matrix, and $p = (p_1, \ldots, p_T)'$ is an $T \times 1$ vector
5. $J'J = I_T$
6. $vec(ABC) = (C' \otimes A)vecB$, where C' is the transpose of C
7. $(A \odot B) = J'(A \otimes B)J$, where $\odot$ and $\otimes$ indicate the Hadamard and Kronecker products respectively
8. $df(x) = f'(x)dx$, where d indicates the differential operator and $f'(x)$ is the derivative of $f(x)$ with respect to x
9. $f(x) \approx f(0) + f'(x)x$, where the first order of Taylor approximation of an $k \times 1$ vector $f(x)$ is given, x is an $T \times 1$ vector, $f(0)$ is an $k \times 1$ vector and is $f(x)$ evaluated at $x = 0$, and $f'(x)$ is the $k \times T$ matrix of derivatives

6.2 A2: Alternative evaluation of the simulation study by distances

We define the absolute Euclidean distance between the estimate $\hat{\beta}$ and its approximate $\tilde{\beta}$ to be

$$abs.distance = |\hat{\beta} - \tilde{\beta}|^2$$

and the relative distance to be

$$rel.distance = |\hat{\beta} - \tilde{\beta}|^2 / |\hat{\beta}|^2.$$

The distance for the two SAR correlation models with $p = (\rho_1, \rho_2)'$ is plotted in a 2×2 panel of 3D plots for Theorems 2 and 4. The distance for the SEM correlation model with $q = (\theta_1, \theta_2)'$ is plotted in 3D for Theorem 7. The distance for the common correlation $q = \theta$ is plotted in 2D for Theorem 9. The 4 distance plots in a panel correspond to the following design cases: $(\Sigma_1, W_1), (\Sigma_2, W_1), (\Sigma_2, W_2)$ and (Σ_2, W_2), left to right, horizontally.

These plots indicate the first-order approximates perform quite well, as they are close to their corresponding (generalized) least squares estimates. For a small sample size $N = 50$, we can see volatile variations reflected in the absolute and relative distances, but for a larger observation number $N = 400$ such variations are smoothed out and the curves become stable.

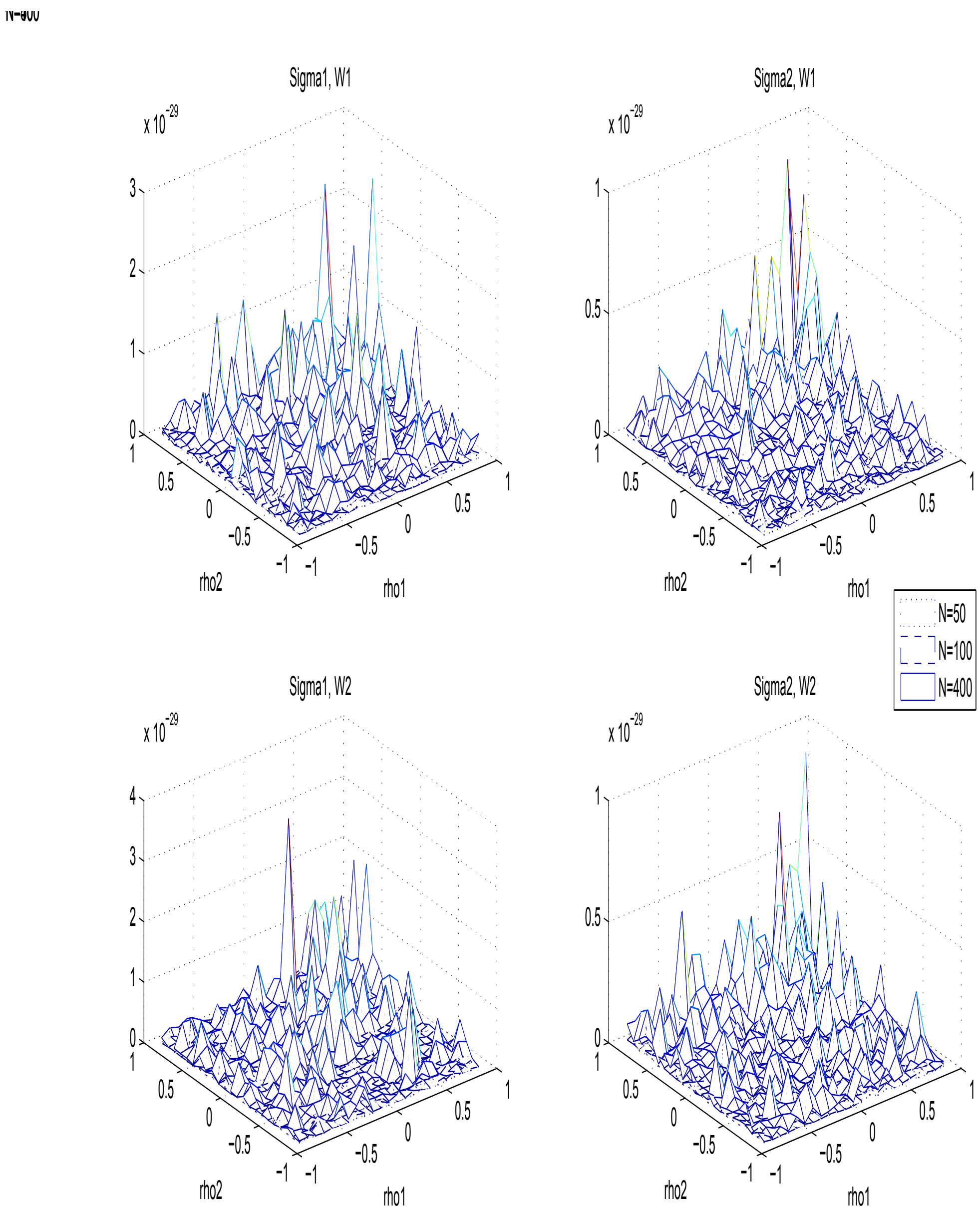

Fig. 1. Relative distances SF-GLS in the SAR-SUR panel model (Theorem 2)

The LS estimates and their approximates are presented in Tables 1 to 4 for a sample size $N = 100$. We can see from these tables that both the (generalized) least-squares estimates and their Taylor approximates are reasonably close to the "true" values of the parameters β. This indicates that both the (generalized) least-squares estimates and the Taylor approximates can be used, and especially the latter when the original least-squares estimates are available and the spatial correlation values are in a small range around the origin of $\rho = 0$ or $\theta = 0$. The distance values are small enough to be acceptable for the spatial parameter values between -0.3 and 0.3.

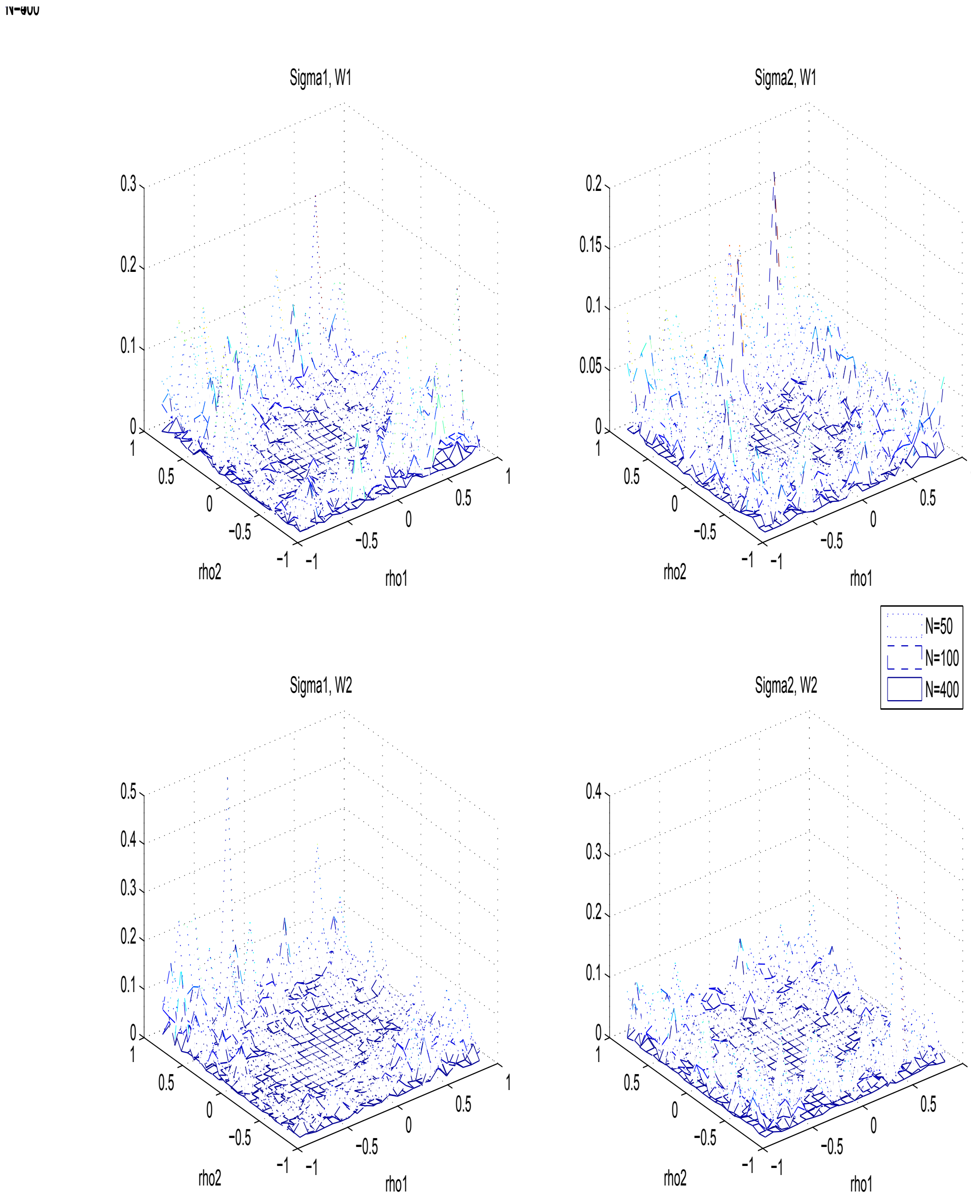

Fig. 2. Relative distances for GLS in the SAR-SEM panel model (Theorem 7)

Table 9. Estimates and approximates for Theorem 2 (N=100)

ρ_i	-0.8		-0.6		-0.4		-0.2		0		0.2		0.4		0.6		0.8	
	$\hat{\beta}_{SF}$	$\hat{\beta}_{G0}+S_{SF0}p$	$\hat{\beta}_{SF}$	$\hat{\beta}_{G0}+S_{SF0}p$	$\hat{\beta}_{SF}$	$\hat{\beta}_{G0}+S_{SF0}p$	$\hat{\beta}_{SF}$	$\hat{\beta}_{G0}+S_{SF0}p$	$\hat{\beta}_{SF}$	$\hat{\beta}_{G0}+S_{SF0}p$	$\hat{\beta}_{SF}$	$\hat{\beta}_{G0}+S_{SF0}p$	$\hat{\beta}_{SF}$	$\hat{\beta}_{G0}+S_{SF0}p$	$\hat{\beta}_{SF}$	$\hat{\beta}_{G0}+S_{SF0}p$	$\hat{\beta}_{SF}$	$\hat{\beta}_{G0}+S_{SF0}p$
	2.9226	2.9226	3.0778	3.0778	3.0956	3.0956	2.3480	2.3480	2.5806	2.5806	2.0320	2.0320	1.6669	1.6669	1.4072	1.4072	1.3011	1.3011
	2.6014	2.6014	1.9952	1.9952	1.8429	1.8429	1.7612	1.7612	0.9522	0.9522	1.1486	1.1486	0.6012	0.6012	0.4194	0.4194	0.0801	0.0801
	1.2836	1.2836	1.1332	1.1332	0.8932	0.8932	0.8609	0.8609	0.6748	0.6748	0.3458	0.3458	0.0545	0.0545	0.0663	0.0663	-0.0805	-0.0805
	1.8830	1.8830	1.7338	1.7338	1.6534	1.6534	1.8693	1.8693	1.4415	1.4415	1.5775	1.5775	1.5591	1.5591	1.2278	1.2278	1.0862	1.0862

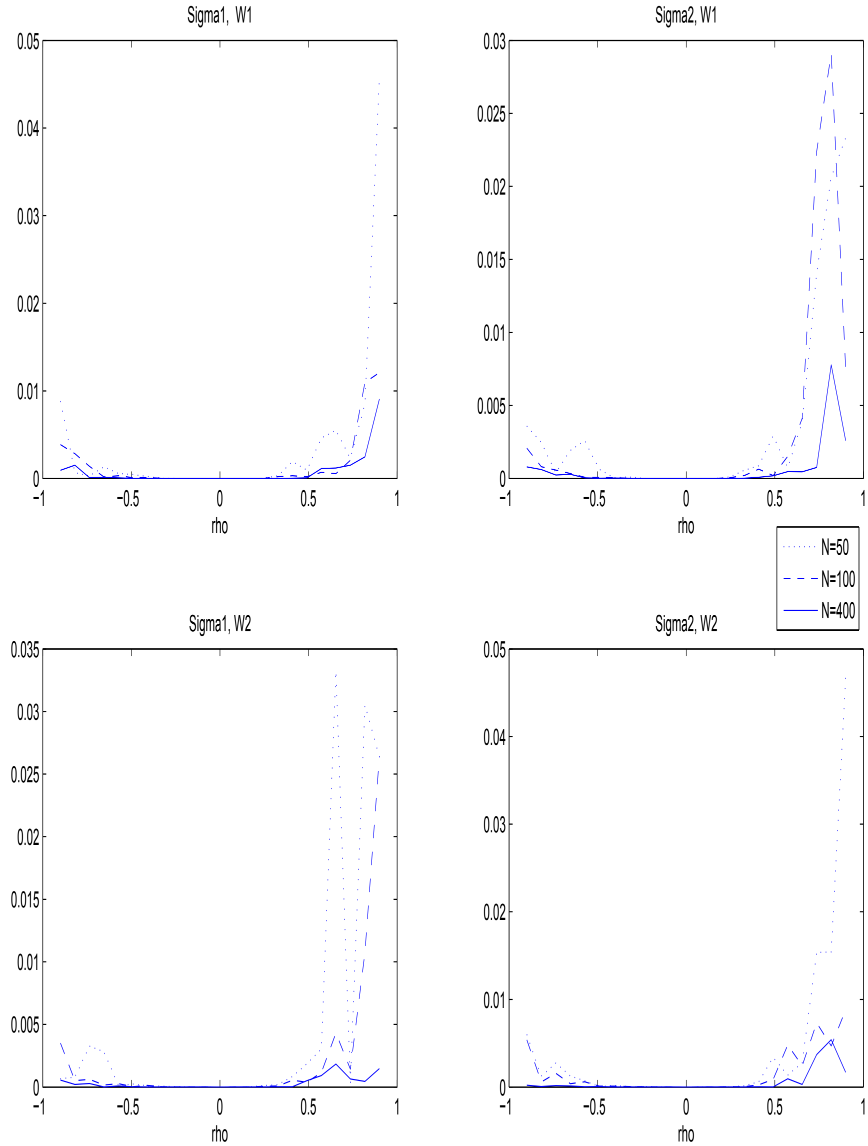

Fig. 3. Relative distances for GLS in the SAR-SEM equal correlation model (Theorem 9)

Table 10. Estimates and approximations for Theorem 4 (N=100)

ρ_i	-0.8		-0.6		-0.4		-0.2		0		0.2		0.4		0.6		0.8	
	$\hat{\beta}_z$	$\hat{\beta}_{ols}+S_{z0}p$	$\hat{\beta}_z$	$\hat{\beta}_{ols}+S_{z0}p$	$\hat{\beta}_z$	$\hat{\beta}_{ols}+S_{z0}p$	$\hat{\beta}_z$	$\hat{\beta}_{ols}+S_{z0}p$	$\hat{\beta}_z$	$\hat{\beta}_{ols}+S_{z0}p$	$\hat{\beta}_z$	$\hat{\beta}_{ols}+S_{z0}p$	$\hat{\beta}_z$	$\hat{\beta}_{ols}+S_{z0}p$	$\hat{\beta}_z$	$\hat{\beta}_{ols}+S_{z0}p$	$\hat{\beta}_z$	$\hat{\beta}_{ols}+S_{z0}p$
	1.0791	0.6836	2.0177	1.0900	2.9418	2.0665	3.0496	2.4414	2.2318	2.2318	1.5372	2.2347	1.5042	2.9686	0.6942	2.9045	0.5382	4.3435
	1.6998	0.5223	2.1247	0.7379	1.7789	0.4982	1.4161	0.8132	1.4387	1.4387	1.3895	1.8886	0.8036	1.9451	0.8374	2.4279	0.1919	1.7153
	0.7561	-0.3220	1.2228	0.0905	1.1046	0.2631	0.8111	0.2849	0.8179	0.8179	0.2491	0.7050	0.3848	1.2725	0.2033	1.4182	-0.0339	1.5561
	0.7632	1.0759	1.2362	1.0739	1.6842	1.2988	1.7903	1.6425	1.3581	1.3581	1.6347	1.9095	1.0475	1.7750	0.7281	1.8283	0.5132	2.3066

Table 11. Estimates and approximations for Theorem 7 (N=100)

ρ_i	-0.8		-0.6		-0.4		-0.2		0		0.2		0.4		0.6		0.8	
	$\hat{\beta}_{sem}$	$\hat{\beta}_{sur}+S_{sur}p$	$\hat{\beta}_{sem}$	$\hat{\beta}_{sur}+S_{sur}p$	$\hat{\beta}_{sem}$	$\hat{\beta}_{sur}+S_{sur}p$	$\hat{\beta}_{sem}$	$\hat{\beta}_{sur}+S_{sur}p$	$\hat{\beta}_{sem}$	$\hat{\beta}_{sur}+S_{sur}p$	$\hat{\beta}_{sem}$	$\hat{\beta}_{sur}+S_{sur}p$	$\hat{\beta}_{sem}$	$\hat{\beta}_{sur}+S_{sur}p$	$\hat{\beta}_{sem}$	$\hat{\beta}_{sur}+S_{sur}p$	$\hat{\beta}_{sem}$	$\hat{\beta}_{sur}+S_{sur}p$
	2.1592	1.9665	1.7140	1.8389	2.4112	2.2285	1.8119	1.8154	2.2570	2.2570	1.9244	1.9672	1.9170	1.9347	2.5284	2.2073	2.2882	2.3696
	0.8987	0.9893	1.5178	1.3074	0.7511	0.9605	0.8183	0.8044	0.8377	0.8377	0.9099	0.8612	1.2189	1.1460	0.8539	0.7583	0.9321	0.8881
	0.3265	0.3365	0.3605	0.3835	0.3603	0.4045	0.4371	0.4402	0.5678	0.5678	0.3398	0.3374	0.3616	0.3785	0.3247	0.1330	0.6591	0.6194
	1.6187	1.5356	1.6450	1.5804	1.6635	1.6332	1.5585	1.5525	1.4551	1.4551	1.6104	1.6194	1.6700	1.5667	1.5791	1.7023	1.5845	1.5544

Table 12. Estimates and approximations for Theorem 9 (N=100)

ρ_i	-0.8		-0.6		-0.4		-0.2		0		0.2		0.4		0.6		0.8	
	$\hat{\beta}_{cc}$	$\hat{\beta}_{ols}+ S_{cc0}p$	$\hat{\beta}_{cc}$	$\hat{\beta}_{ols}+ S_{cc0}p$	$\hat{\beta}_{cc}$	$\hat{\beta}_{ols}+ S_{cc0}p$	$\hat{\beta}_{cc}$	$\hat{\beta}_{ols}+ S_{cc0}p$	$\hat{\beta}_{cc}$	$\hat{\beta}_{ols}+ S_{cc0}p$	$\hat{\beta}_{cc}$	$\hat{\beta}_{ols}+ S_{cc0}p$	$\hat{\beta}_{cc}$	$\hat{\beta}_{ols}+ S_{cc0}p$	$\hat{\beta}_{cc}$	$\hat{\beta}_{ols}+ S_{cc0}p$	$\hat{\beta}_{cc}$	$\hat{\beta}_{ols}+ S_{cc0}p$
	1.7349	1.5443	2.5923	2.6481	2.1597	2.1579	2.0284	2.0324	1.6317	1.6317	2.2142	2.2026	2.1898	2.1253	2.0181	2.0557	1.0887	1.3350
	1.4214	1.5334	0.4975	0.4825	0.9315	0.9302	0.9361	0.9381	1.0821	1.0821	0.6500	0.6414	0.9524	0.9427	0.9963	1.0724	0.9039	1.0205
	0.6554	0.6558	0.6474	0.6803	0.5845	0.5721	0.3258	0.3309	0.1492	0.1492	0.5511	0.5517	0.4800	0.4714	0.5510	0.4901	0.3665	0.2698
	1.3259	1.2907	1.4927	1.5028	1.3914	1.3903	1.6419	1.6410	1.6726	1.6726	1.4227	1.4193	1.4959	1.4850	1.4699	1.4524	1.7487	1.7415

Table 13: The MSE of estimates and 1st order approximation for Theorem 2 (N=10)

ρ_2 \ ρ_1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8
	Σ_1, W_1									Σ_1, W_2								
-0.8	10.0743	7.6397	5.8352	4.5859	3.9367	3.8702	4.2976	5.3584	7.0277	6.2427	4.8671	3.7879	3.1985	2.9851	3.1622	3.7319	4.6634	6.0038
	10.0743	7.6397	5.8352	4.5859	3.9367	3.8702	4.2976	5.3584	7.0277	6.2427	4.8671	3.7879	3.1985	2.9851	3.1622	3.7319	4.6634	6.0038
-0.6	8.9851	6.6541	4.8046	3.6222	2.9043	2.7659	3.203	4.3312	5.8104	5.8972	4.2988	3.286	2.5416	2.252	2.3516	2.8001	3.6678	4.9696
	8.9851	6.6541	4.8046	3.6222	2.9043	2.7659	3.203	4.3312	5.8104	5.8972	4.2988	3.286	2.5416	2.252	2.3516	2.8001	3.6678	4.9696
-0.4	8.3924	5.9053	4.0761	2.823	2.1	1.9249	2.3477	3.2835	4.86	5.6203	4.0181	2.8768	2.0803	1.6885	1.698	2.0815	2.8729	4.0755
	8.3924	5.9053	4.0761	2.823	2.1	1.9249	2.3477	3.2835	4.86	5.6203	4.0181	2.8768	2.0803	1.6885	1.698	2.0815	2.8729	4.0755
-0.2	7.9899	5.4592	3.5909	2.2938	1.5024	1.3235	1.735	2.7043	4.2155	5.5776	3.9012	2.6338	1.7837	1.3402	1.2624	1.5651	2.2785	3.3882
	7.9899	5.4592	3.5909	2.2938	1.5024	1.3235	1.735	2.7043	4.2155	5.5776	3.9012	2.6338	1.7837	1.3402	1.2624	1.5651	2.2785	3.3882
0	7.7329	5.2817	3.3769	1.9918	1.1978	0.9881	1.3642	2.2661	3.7311	5.7564	3.9875	2.6393	1.7098	1.1361	1.0001	1.2521	1.8717	2.9246
	7.7329	5.2817	3.3769	1.9918	1.1978	0.9881	1.3642	2.2661	3.7311	5.7564	3.9875	2.6393	1.7098	1.1361	1.0001	1.2521	1.8717	2.9246
0.2	7.8282	5.2633	3.3298	1.9454	1.1444	0.8833	1.2538	2.1234	3.5862	6.0438	4.2392	2.856	1.801	1.2355	0.9679	1.1217	1.6973	2.6603
	7.8282	5.2633	3.3298	1.9454	1.1444	0.8833	1.2538	2.1234	3.5862	6.0438	4.2392	2.856	1.801	1.2355	0.9679	1.1217	1.6973	2.6603
0.4	8.1497	5.6167	3.6033	2.199	1.3172	1.0491	1.3626	2.1751	3.6241	6.609	4.7185	3.2148	2.1299	1.4333	1.1031	1.2064	1.6852	2.5486
	8.1497	5.6167	3.6033	2.199	1.3172	1.0491	1.3626	2.1751	3.6241	6.609	4.7185	3.2148	2.1299	1.4333	1.1031	1.2064	1.6852	2.5486
0.6	8.7117	6.022	4.0747	2.6505	1.7484	1.4508	1.7234	2.5658	3.9843	7.3801	5.3954	3.8279	2.6811	1.8592	1.4743	1.4516	1.8862	2.6894
	8.7117	6.022	4.0747	2.6505	1.7484	1.4508	1.7234	2.5658	3.9843	7.3801	5.3954	3.8279	2.6811	1.8592	1.4743	1.4516	1.8862	2.6894
0.8	9.503	6.8604	4.7947	3.3562	2.4577	2.0675	2.3362	3.117	4.4886	8.2596	6.2569	4.6032	3.3294	2.4715	2.0182	1.9594	2.2348	2.9717
	9.503	6.8604	4.7947	3.3562	2.4577	2.0675	2.3362	3.117	4.4886	8.2596	6.2569	4.6032	3.3294	2.4715	2.0182	1.9594	2.2348	2.9717
	Σ_2, W_1									Σ_2, W_2								
-0.8	9.298	6.9665	5.3372	4.0177	3.3965	3.2209	3.3793	4.2145	5.5004	13.8989	10.3168	7.6098	5.7599	4.5508	4.2295	4.7183	5.9087	7.9978
	9.298	6.9665	5.3372	4.0177	3.3965	3.2209	3.3793	4.2145	5.5004	13.8989	10.3168	7.6098	5.7599	4.5508	4.2295	4.7183	5.9087	7.9978
-0.6	8.5698	6.4376	4.755	3.549	2.805	2.6566	2.9491	3.6286	4.8654	12.821	9.1024	6.5137	4.58	3.3795	3.0965	3.4942	4.7206	6.8361
	8.5698	6.4376	4.755	3.549	2.805	2.6566	2.9491	3.6286	4.8654	12.821	9.1024	6.5137	4.58	3.3795	3.0965	3.4942	4.7206	6.8361
-0.4	8.2349	6.019	4.378	3.116	2.4691	2.2123	2.4966	3.2223	4.4947	11.8942	8.2672	5.5351	3.626	2.4974	2.1455	2.5437	3.8201	5.8688
	8.2349	6.019	4.378	3.116	2.4691	2.2123	2.4966	3.2223	4.4947	11.8942	8.2672	5.5351	3.626	2.4974	2.1455	2.5437	3.8201	5.8688
-0.2	7.9495	5.791	4.0123	2.7728	2.173	1.961	2.2289	2.9392	4.2044	11.1565	7.6967	4.8963	3.0373	1.8473	1.5064	1.9407	3.2015	5.2729
	7.9495	5.791	4.0123	2.7728	2.173	1.961	2.2289	2.9392	4.2044	11.1565	7.6967	4.8963	3.0373	1.8473	1.5064	1.9407	3.2015	5.2729
0	7.8406	5.6054	3.9477	2.7577	1.9988	1.8459	2.1016	2.8387	4.0823	10.8647	7.2033	4.535	2.604	1.4442	1.0966	1.5336	2.7723	4.8536
	7.8406	5.6054	3.9477	2.7577	1.9988	1.8459	2.1016	2.8387	4.0823	10.8647	7.2033	4.535	2.604	1.4442	1.0966	1.5336	2.7723	4.8536
0.2	7.849	5.6278	3.9574	2.7056	2.044	1.8298	2.1226	2.8781	4.1542	10.6351	7.1733	4.4261	2.4663	1.301	0.9819	1.4414	2.6705	4.7382
	7.849	5.6278	3.9574	2.7056	2.044	1.8298	2.1226	2.8781	4.1542	10.6351	7.1733	4.4261	2.4663	1.301	0.9819	1.4414	2.6705	4.7382
0.4	8.0089	5.7993	4.1268	2.9359	2.1949	2.0427	2.2679	3.0087	4.2708	10.7256	7.2823	4.529	2.5889	1.4435	1.1172	1.5487	2.791	4.8369
	8.0089	5.7993	4.1268	2.9359	2.1949	2.0427	2.2679	3.0087	4.2708	10.7256	7.2823	4.529	2.5889	1.4435	1.1172	1.5487	2.791	4.8369
0.6	8.2731	6.0675	4.4757	3.2584	2.5628	2.2948	2.6278	3.3315	4.6286	11.19	7.5635	4.9208	2.995	1.8278	1.4857	1.9249	3.1436	5.254
	8.2731	6.0675	4.4757	3.2584	2.5628	2.2948	2.6278	3.3315	4.6286	11.19	7.5635	4.9208	2.995	1.8278	1.4857	1.9249	3.1436	5.254
0.8	8.7209	6.5416	4.8768	3.6965	2.9353	2.7174	3.0543	3.7825	5.0247	11.9381	8.2392	5.4718	3.6221	2.489	2.1332	2.5927	3.8209	5.8864
	8.7209	6.5416	4.8768	3.6965	2.9353	2.7174	3.0543	3.7825	5.0247	11.9381	8.2392	5.4718	3.6221	2.489	2.1332	2.5927	3.8209	5.8864

Table 14. The MSE of estimates and approximates for Theorem 4 (N=10)

ρ_2 \ ρ_1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8
	Σ_1, W_1									Σ_1, W_2								
-0.8	2.1185	1.2084	2.1342	2.1878	1.6684	1.3213	1.4477	2.1492	3.526	3.4365	3.8385	3.4095	2.675	2.1215	1.8867	2.0152	2.52	3.4219
	4.3764	3.3497	2.7399	2.5716	2.8919	3.65	4.8926	6.4795	8.6162	4.4455	3.3064	2.5988	2.2925	2.5259	3.203	4.3125	5.9037	7.9606
-0.6	2.7607	1.8352	2.8082	2.8368	2.2723	1.9308	2.079	2.7902	4.1791	3.618	4.0804	3.5347	2.9215	2.3089	2.1143	2.2539	2.7601	3.6209
	3.5942	2.617	1.9863	1.8681	2.1236	2.9084	4.1006	5.7903	7.9033	4.0117	2.872	2.1579	1.9108	2.0839	2.8051	3.88	5.3787	7.585
-0.4	2.5724	1.6782	2.6117	2.6715	2.1391	1.7837	1.8999	2.6016	3.9959	3.4964	3.9088	3.4826	2.788	2.1794	1.9791	2.1184	2.6409	3.4713
	3.1834	2.1306	1.52	1.3639	1.6796	2.4086	3.6203	5.2498	7.4384	3.7923	2.593	1.9396	1.644	1.8221	2.541	3.7239	5.2713	7.2279
-0.2	2.2208	1.2798	2.2633	2.3084	1.7731	1.3694	1.5525	2.2235	3.6227	3.2726	3.6792	3.287	2.6008	2.0007	1.7136	1.8949	2.4267	3.2658
	2.9642	1.9118	1.2601	1.1447	1.4594	2.1592	3.435	5.0709	7.2161	3.6951	2.4291	1.7752	1.5359	1.7705	2.3769	3.5434	5.1548	7.0957
0	1.9366	1.0356	1.9863	2.0423	1.4857	1.1463	1.2498	1.9922	3.3649	3.0608	3.4753	3.0506	2.4364	1.8088	1.5409	1.687	2.2354	3.0771
	2.9717	1.9487	1.3304	1.1784	1.4857	2.2794	3.4213	5.1125	7.2019	3.7385	2.5106	1.8655	1.6379	1.8088	2.4663	3.5529	5.2047	7.104
0.2	1.8737	0.9656	1.9054	1.9382	1.4117	1.0779	1.1789	1.9226	3.3101	3.0656	3.4616	3.0271	2.3524	1.7197	1.5018	1.633	2.1122	3.001
	3.2528	2.2215	1.6442	1.4731	1.7561	2.5321	3.7062	5.4241	7.5659	3.938	2.8015	2.0789	1.8353	1.9957	2.673	3.8052	5.3925	7.3909
0.4	2.0503	1.1416	2.1009	2.1334	1.6496	1.2301	1.3621	2.0922	3.4751	3.0752	3.502	3.1354	2.412	1.829	1.5792	1.7223	2.2708	3.089
	3.8748	2.8184	2.2232	2.0433	2.4227	3.1117	4.3268	5.9264	8.0858	4.2416	3.2348	2.4834	2.1943	2.3971	3.0231	4.14	5.7002	7.7573
0.6	2.4351	1.5247	2.49	2.5217	2.0106	1.6348	1.7468	2.4827	3.859	3.3023	3.7147	3.347	2.6768	2.067	1.8246	1.9821	2.5098	3.3573
	4.6615	3.6206	3.0669	2.8687	3.2099	4.002	5.2326	6.8291	8.9236	4.8437	3.639	2.9155	2.7322	2.9239	3.5463	4.6634	6.228	8.2249
0.8	3.0964	2.1958	3.1379	3.1882	2.616	2.3107	2.4255	3.152	4.5173	3.7852	4.214	3.7535	3.0549	2.5361	2.2483	2.3946	2.9618	3.7809
	5.779	4.7245	4.0597	4.0167	4.2214	5.0124	6.2102	7.9358	10.0105	5.5412	4.3594	3.6331	3.3596	3.5867	4.2296	5.4474	6.9695	9.0659
	Σ_2, W_1									Σ_2, W_2								
-0.8	2.3427	3.2779	3.5012	3.1945	2.6842	2.4467	2.6007	3.216	4.3631	2.7058	2.0111	2.2721	2.0268	1.6193	1.3699	1.6078	2.3611	3.7145
	4.3066	3.3755	2.8456	2.6255	2.7526	3.3862	4.3009	5.6919	7.4566	3.6098	2.6244	2.0256	1.8217	2.0215	2.6022	3.5551	4.9391	6.6117
-0.6	2.2827	3.2087	3.4766	3.0981	2.7081	2.4043	2.5149	3.1621	4.3033	2.9443	2.2368	2.5311	2.32	1.87	1.6598	1.8545	2.6142	4.0017
	3.7588	2.7457	2.2272	2.0069	2.2448	2.8173	3.7682	5.0749	6.7805	3.3726	2.3157	1.7463	1.5611	1.7279	2.3001	3.2365	4.6556	6.3547
-0.4	2.1079	3.0501	3.264	2.911	2.4797	2.2049	2.3431	2.9468	4.1096	3.0272	2.3239	2.5313	2.3415	1.8803	1.7056	1.9008	2.6523	4.0027
	3.452	2.48	1.8802	1.7002	1.8722	2.4432	3.4234	4.8381	6.4769	3.1595	2.1533	1.5671	1.3616	1.5374	2.1652	3.1066	4.4943	6.1762
-0.2	1.791	2.7542	3.003	2.6008	2.1775	1.9078	2.0596	2.6317	3.8252	2.8353	2.2142	2.4159	2.2027	1.7902	1.5789	1.7923	2.5606	3.9175
	3.2587	2.3209	1.7935	1.547	1.7658	2.3482	3.2922	4.6498	6.435	3.1483	2.114	1.5652	1.3361	1.5316	2.1178	3.0744	4.4362	6.2167
0	1.5373	2.4408	2.67	2.3118	1.9176	1.6402	1.7739	2.3532	3.5338	2.6964	2.091	2.2537	2.027	1.6367	1.3904	1.6106	2.3772	3.7517
	3.4247	2.4841	1.8664	1.659	1.9176	2.4634	3.4708	4.7582	6.5017	3.1441	2.2669	1.601	1.4054	1.6367	2.2043	3.1367	4.4832	6.2644
0.2	1.3471	2.2896	2.5049	2.1796	1.7288	1.5071	1.6423	2.2012	3.3876	2.5991	1.9128	2.133	1.9713	1.5085	1.2705	1.5051	2.2783	3.6099
	3.7624	2.8771	2.248	2.0836	2.2504	2.8549	3.8466	5.152	6.8793	3.374	2.4846	1.8593	1.6884	1.8229	2.4002	3.4117	4.7327	6.5319
0.4	1.418	2.3475	2.5758	2.2344	1.7851	1.5296	1.6292	2.2729	3.4483	2.6427	1.9537	2.1961	1.9944	1.5393	1.3338	1.5414	2.3181	3.7012
	4.39	3.4347	2.8195	2.6725	2.8137	3.438	4.3345	5.7091	7.4841	3.767	2.7723	2.2161	1.9881	2.1563	2.7406	3.7449	5.099	6.7496
0.6	1.7401	2.6932	2.9263	2.5774	2.1144	1.8645	1.9693	2.5775	3.7674	2.9494	2.2844	2.4984	2.3059	1.8654	1.6289	1.8822	2.637	3.9849
	5.2541	4.2377	3.6628	3.4664	3.6785	4.3122	5.2009	6.6427	8.3206	4.2007	3.2641	2.6509	2.4818	2.6556	3.2006	4.174	5.4867	7.3294
0.8	2.3574	3.3118	3.4676	3.1644	2.7266	2.4721	2.5829	3.177	4.3603	3.5291	2.8718	3.0862	2.8856	2.462	2.2491	2.4552	3.2264	4.5703
	6.2489	5.29	4.7876	4.6181	4.7609	5.3904	6.2971	7.7821	9.3949	4.7892	3.8358	3.2613	3.0257	3.2329	3.8159	4.8226	6.1307	7.8946

Table 15. The MSE of Estimates and approximates for Theorem 7 (N=10)

$\rho_2 \backslash \rho_1$	−0.8	−0.6	−0.4	−0.2	0	0.2	0.4	0.6	0.8	−0.8	−0.6	−0.4	−0.2	0	0.2	0.4	0.6	0.8
	Σ_1, W_1									Σ_1, W_2								
-0.8	4.9046	3.8771	2.8562	2.2997	2.1605	2.17	2.2027	2.5853	3.429	3.7068	3.2028	2.4969	2.0439	1.8408	1.8136	1.8469	2.2152	2.6185
	1.5025	1.5409	1.5611	1.6375	1.7752	1.9179	2.059	2.2941	2.5334	1.3549	1.4376	1.5175	1.6207	1.7859	2.0157	2.1412	2.4326	2.6358
-0.6	4.5138	3.4316	2.5699	2.124	1.9277	1.9124	2.0067	2.3538	3.1571	3.667	2.9917	2.4162	1.9343	1.6485	1.5956	1.6567	1.9378	2.4031
	1.4793	1.4582	1.494	1.5957	1.6811	1.8115	1.9129	2.1464	2.3366	1.3107	1.3609	1.4215	1.494	1.6333	1.7663	1.9213	2.1379	2.4099
-0.4	4.2434	3.1823	2.3735	1.9068	1.7049	1.6767	1.7908	2.1338	2.9154	3.4934	2.9167	2.3595	1.8475	1.5087	1.4405	1.5011	1.6565	2.1246
	1.5102	1.4721	1.4894	1.5166	1.5513	1.6458	1.8366	2.004	2.1596	1.3155	1.3186	1.3472	1.425	1.4938	1.5819	1.7782	1.9429	2.1545
-0.2	3.9435	2.9999	2.1925	1.7616	1.5657	1.5071	1.5815	1.9231	2.7046	3.4553	2.8036	2.3052	1.7128	1.4195	1.2845	1.2868	1.5003	1.8726
	1.5843	1.5193	1.4743	1.4672	1.5188	1.5832	1.6892	1.8896	2.05	1.3171	1.278	1.3266	1.3281	1.4139	1.5188	1.6397	1.7902	1.9171
0	3.721	2.8408	2.1713	1.7371	1.4931	1.3905	1.4633	1.7167	2.498	3.4094	2.8114	2.2498	1.6526	1.3318	1.1643	1.1644	1.3104	1.705
	1.6454	1.5702	1.4904	1.4832	1.4931	1.53	1.6665	1.7261	1.8719	1.3494	1.2771	1.2904	1.2817	1.3318	1.3976	1.4952	1.6003	1.8107
0.2	3.7337	2.9001	2.2199	1.7756	1.4894	1.3553	1.3491	1.6085	2.3265	3.4625	2.8681	2.233	1.706	1.3268	1.1602	1.1135	1.2144	1.6182
	1.7724	1.6636	1.5647	1.5275	1.5075	1.5136	1.551	1.6549	1.7783	1.4171	1.3138	1.2974	1.2798	1.2958	1.3635	1.4138	1.4802	1.6242
0.4	3.814	3.0764	2.3778	1.8431	1.5647	1.3543	1.3577	1.5328	2.2445	3.5538	2.9378	2.3601	1.7567	1.3667	1.1011	1.0957	1.2119	1.5748
	1.9212	1.7868	1.6698	1.5681	1.5682	1.4912	1.5295	1.5912	1.69	1.4592	1.4012	1.3377	1.2866	1.2766	1.24	1.3206	1.4122	1.5225
0.6	3.9383	3.338	2.5946	2.0829	1.678	1.4844	1.4426	1.5607	2.342	3.8062	3.2071	2.4722	1.8404	1.4178	1.1939	1.1457	1.232	1.5701
	2.0628	1.8898	1.7841	1.6482	1.603	1.5569	1.5784	1.5919	1.6369	1.5983	1.4685	1.3815	1.3286	1.2924	1.2851	1.2877	1.3546	1.4226
0.8	4.2057	3.6882	2.8252	2.2412	1.9324	1.6624	1.6138	1.712	2.3848	4.1227	3.5214	2.7667	2.0759	1.6259	1.3912	1.2675	1.3641	1.6913
	2.2789	2.0965	1.9095	1.7787	1.7061	1.6206	1.5914	1.6113	1.6214	1.784	1.6086	1.4509	1.3884	1.3615	1.312	1.2793	1.3115	1.3671
	Σ_2, W_1									Σ_2, W_2								
-0.8	8.5912	6.7932	5.0758	3.9895	3.4717	3.391	3.4114	3.4554	3.8353	5.6578	5.5539	5.3419	4.9666	5.0265	4.9188	5.0078	5.2473	5.5995
	1.922	1.8095	1.7974	1.776	1.7004	1.7111	1.6623	1.6401	1.6618	3.3436	3.3466	3.2873	3.2066	3.3094	3.3064	3.3694	3.4058	3.3638
-0.6	7.8652	6.0636	4.3378	3.3038	2.7884	2.6587	2.7529	2.7784	3.2461	5.2199	5.1986	4.9152	4.8514	4.5249	4.5696	4.6562	4.6909	5.0461
	1.9406	1.8442	1.7569	1.7346	1.6722	1.628	1.6711	1.6535	1.6748	3.13	3.2164	3.0956	3.1906	3.1569	3.1685	3.1859	3.2022	3.1284
-0.4	7.1128	5.5418	3.8875	2.7288	2.3274	2.134	2.2004	2.2867	2.5651	4.9313	4.6608	4.1956	4.3074	4.2037	4.0609	3.9879	4.2776	4.5648
	1.9155	1.8409	1.8164	1.7471	1.7391	1.6649	1.6693	1.6575	1.6268	3.1726	3.11	2.9057	3.1028	3.1336	3.083	3.0526	3.1033	3.0729
-0.2	6.7911	5.0648	3.4569	2.3899	1.9417	1.7591	1.8088	1.8907	2.3205	3.9785	3.8735	3.8312	3.5877	3.4403	3.4722	3.376	3.5724	3.9762
	1.9385	1.8259	1.7623	1.7252	1.7359	1.6524	1.6933	1.6681	1.705	2.8725	2.9145	3.0044	2.9483	2.9314	3.0403	2.9583	3.0015	3.0173
0	6.7298	4.8481	3.2447	2.1531	1.7112	1.5783	1.5978	1.6826	2.1016	3.5103	3.3219	3.1669	3.0587	2.8206	2.8591	2.745	2.8995	3.3137
	1.9343	1.8189	1.7783	1.7323	1.7112	1.679	1.6736	1.665	1.6867	2.8743	2.8416	2.8373	2.9178	2.8206	2.9425	2.8357	2.8503	2.8783
0.2	6.4467	4.8435	3.189	2.1225	1.6333	1.5129	1.4885	1.5789	1.9923	3.1469	3.0804	2.8065	2.6997	2.5794	2.4555	2.4954	2.6965	3.0917
	1.9473	1.9029	1.8283	1.7767	1.7227	1.7085	1.6752	1.6631	1.7043	2.7417	2.8548	2.7771	2.8178	2.8328	2.8047	2.8734	2.9104	2.9227
0.4	6.715	4.7566	3.1704	2.0995	1.6185	1.4942	1.4868	1.5779	2.0127	3.2199	3.0213	2.8146	2.634	2.5746	2.4371	2.4832	2.6403	3.0829
	1.9947	1.8286	1.8249	1.772	1.7372	1.6908	1.6801	1.6709	1.7423	2.7913	2.719	2.7546	2.7727	2.7995	2.7324	2.8156	2.8343	2.8529
0.6	6.7342	5.0427	3.2452	2.2227	1.7091	1.6269	1.6247	1.7324	2.0998	3.416	3.349	3.1089	2.8864	2.8407	2.7475	2.7171	2.8606	3.2677
	1.9801	1.8991	1.803	1.7902	1.719	1.7554	1.7385	1.7392	1.7369	2.7077	2.8216	2.7574	2.7299	2.8342	2.7686	2.7967	2.7482	2.7702
0.8	6.8283	5.2517	3.5741	2.5247	2.0169	1.9137	1.9052	2.0055	2.3868	4.0433	3.9305	3.7379	3.5647	3.4923	3.2939	3.233	3.5284	3.9248
	1.9631	1.883	1.8607	1.808	1.7708	1.7579	1.7412	1.7533	1.7442	2.7667	2.794	2.747	2.7562	2.8274	2.8671	2.7934	2.8435	2.7755

Table 16. The MSE of estimates and approximations for Theorem 9 (N=10)

θ	−0.8	−0.6	−0.4	−0.2	0	0.2	0.4	0.6	0.8	−0.8	−0.6	−0.4	−0.2	0	0.2	0.4	0.6	0.8
	Σ_1, W_1									Σ_1, W_2								
	2.927	2.2429	1.7015	1.3355	1.1071	1.0238	1.0042	1.145	2.0089	4.751	3.0467	2.0164	1.4563	1.1653	1.0674	1.0693	1.2235	1.9837
	2.6424	2.044	1.6229	1.3221	1.1071	1.0232	1.023	1.1442	1.4115	3.1711	2.4239	1.8434	1.4322	1.1653	1.0677	1.0969	1.2449	1.5358
	Σ_2, W_1									Σ_2, W_2								
	6.5262	5.5707	3.9714	2.7039	1.8453	1.563	1.5455	1.6901	2.8742	4.3525	3.4851	2.6916	2.1424	1.7572	1.567	1.6113	1.868	2.6745
	7.7677	5.7412	3.8483	2.6501	1.8453	1.5514	1.666	2.2036	3.2617	4.1534	3.2772	2.602	2.1245	1.7572	1.568	1.6268	1.8654	2.2302

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Reihe Ökonomie / Economics Series 294

Editor: Robert M. Kunst (Econometrics)
Associate Editors: Michael Reiter (Macroeconomics), Klaus Ritzberger (Microeconomics)

ISSN: 1605-7996

ISSN: 1605-7996