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Optimal Asset Allocation under Quadratic Loss Aversion

Ines Fortin
Jaroslava Hlouskova

Institut für Höhere Studien
Institute for Advanced Studies
Vienna

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Author(s):
Ines Fortin, Jaroslava Hlouskova

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Institute for Advanced Studies, Vienna

Contact:

Ines Fortin
Department of Economics and Finance
Institute for Advanced Studies
Stumpergasse 56
1060 Vienna, Austria
☎: +43/1/599 91-165
email: fortin@ihs.ac.at

Jaroslava Hlouskova
Department of Economics and Finance
Institute for Advanced Studies
Stumpbergasse 56
1060 Vienna, Austria
☎: +43/1/599 91-142
email: hlouskov@ihs.ac.at
and
Thompson Rivers University
School of Business and Economics
Kamloops, British Columbia, Canada

Abstract

We study the asset allocation of a quadratic loss-averse (QLA) investor and derive conditions under which the QLA problem is equivalent to the mean-variance (MV) and conditional value-at-risk (CVaR) problems. Then we solve analytically the two-asset problem of the QLA investor for a risk-free and a risky asset. We find that the optimal QLA investment in the risky asset is finite, strictly positive and is minimal with respect to the reference point for a value strictly larger than the risk-free rate. Finally, we implement the trading strategy of a QLA investor who reallocates her portfolio on a monthly basis using 13 EU and US assets. We find that QLA portfolios (mostly) outperform MV and CVaR portfolios and that incorporating a conservative dynamic update of the QLA parameters improves the performance of QLA portfolios. Compared with linear loss-averse portfolios, QLA portfolios display significantly less risk but they also yield lower returns.

Keywords

Quadratic loss aversion, prospect theory, portfolio optimization, MV and CVaR portfolios, investment strategy

JEL Classification

D03, D81, G11, G15, G24

Comments
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Contents

1 Introduction

Loss aversion, which is a central finding of Kahneman and Tversky's (1979) prospect theory,[1] describes the fact that people are more sensitive to losses than to gains, relative to a given reference point. More specifically returns are measured relative to a given reference value, and the decrease in utility implied by a marginal loss (relative to the reference point) is always greater than the increase in utility implied by a marginal gain (relative to the reference point).[2] The simplest form of such loss aversion is linear loss aversion, where the marginal utility of gains and losses is fixed. The optimal asset allocation decision under linear loss aversion has been extensively studied, see, for example, Gomes (2005), Siegmann and Lucas (2005), He and Zhou (2011), and Fortin and Hlouskova (2011a). It has been argued, however, that real investors may put an increasing rather than a fixed marginal penalty on losses, i.e., investors could be more averse to larger than to small losses. Thus a quadratic form of loss aversion, where the objective is linear in gains and quadratic in losses, may be more adequate. Quadratic loss aversion differs from the originally introduced (S-shaped) loss aversion in that it displays risk-aversion in both domains of gains and losses, while prospect theory (S-shaped) utility captures a risk-averse behavior in the domain of gains and a risk-seeking behavior in the domain of losses. Under quadratic loss aversion, investors face a trade-off between return on the one hand and quadratic shortfall below the reference point on the other hand. Interpreted differently, the utility function contains an asymmetric or downside risk measure, where losses are weighted differently from gains. Compared with linear loss aversion, large losses are punished more severely than small losses under quadratic loss aversion. The penalty on losses under quadratic loss aversion is also referred to as quadratic shortfall (see Siegmann and Lucas, 2005; Siegmann, 2007; and Lucas and Siegmann, 2008). Very recently, the analysis of optimal investment with capital income taxation under loss-averse preferences was conducted in Hlouskova and Tsigaris (2012). Some results indicate that it could be possible for a capital income tax increase not to stimulate risk taking even if the tax code provides the attractive full loss offset provisions. However, risk taking can be stimulated if the investor interprets part of the tax as a loss instead of as a reduced gain.

[1]Sometimes the different versions of prospect theory are classified as three generations of prospect theory. The first generation builds on the original model introduced in Kahneman and Tversky (1979), the second generation (cumulative prospective theory) features cumulative individual probabilities (see, e.g., Tversky and Kahneman, 1992), and the third generation treats the reference point as being uncertain (see Schmidt, Starmer and Sugden, 2008).

[2]This is also referred to as the first-order risk aversion (see Epstein and Zin, 1990).

Asymmetric – or rather downside – risk measures are extremely popular in applied finance, where their use has been promoted by banking supervisory regulations which specify the risk of proprietary trading books and its use in setting risk capital requirements. The measure of risk used in this framework is value-at-risk (VaR), which explicitly targets downside risk, see the Bank for International Settlements (2006, 2010). VaR has been developing into one of the industry standards for assessing the risk of financial losses in risk management and asset/liability management. Another risk measure, which is closely related to VaR but offers additional desirable properties like information on extreme events, coherence and computational ease, is conditional value-at-risk (CVaR).[3] Computational optimization of CVaR is readily accessible through the results in Rockafellar and Uryasev (2000).

The purpose of this paper is to investigate the asset allocation decision under quadratic loss aversion, both theoretically and empirically, and compare it to more traditional portfolio optimization methods like mean-variance and conditional value-at-risk as well as to the recent asset allocation problem under linear loss aversion. Our theoretical analysis of the problem under quadratic loss aversion is related to Siegmann and Lucas (2005) who mainly explore optimal portfolio selection under linear loss aversion and include a brief analysis on quadratic loss aversion.[4] Their setup, however, is in terms of wealth (while our analysis is based on returns) and they characterize the solution in a different way than we do. We contribute to the existing literature along different lines. First, we investigate theoretically how the optimization problems of quadratic loss aversion, mean-variance and CVaR relate to each other. Second, we analytically solve the portfolio selection problem of a quadratic loss-averse investor and compare the results to those implied by linear loss aversion. Third, we contribute to the empirical research involving loss-averse investors by investigating the portfolio performance under the optimal investment strategy, where the portfolio is reallocated on a monthly basis using 13 European and 13 US assets from 1985 to 2010. In addition to using fixed parameters in the loss-averse utility, we employ time-changing trading strategies which depend on previous gains and losses to better reflect the behavior of real investors. As opposed to a number of other authors, we do not consider a general equilibrium model but examine the portfolio selection problem from an investor's point of view.[5]

[3]See Artzner et al. (1999).

[4]Siegmann and Lucas (2005) refer to what we call linear and quadratic loss aversion as linear and quadratic shortfall, respectively.

[5]See, for example, De Giorgi and Hens (2006) who introduce a piecewise negative exponential function as the loss-averse utility to overcome infinite short-selling and to guarantee the existence of market equilibria and Berkelaar

The remaining paper is organized as follows. In Section 2 we first derive conditions under which the quadratic loss-averse (QLA) utility maximization problem is equivalent to the traditional mean-variance (MV) and conditional value-at-risk (CVaR) problems for the general n-asset case, under the assumption of normally distributed asset returns. Then we explore the two-asset case, where one asset is risk-free, and derive properties of the optimal weight of the risky asset under the assumption of binomially and (generally) continuously distributed returns, both for the case when the reference point is equal to the risk-free rate and for the case when it is not. We additionally contrast the derived results with those implied by linear loss aversion (LLA). In Section 3 we implement the trading strategy of a quadratic loss-averse investor, who reallocates her portfolio on a monthly basis, and study the performance of the resulting optimal portfolio. We also compare the optimal QLA portfolio to the optimal LLA portfolio and to the more traditional optimal MV and CVaR portfolios. Section 4 concludes.

2 Portfolio optimization under quadratic loss aversion

Under quadratic loss aversion investors are characterized by a utility of returns, $g(\cdot)$, which is linear in gains and quadratic in losses, where gains and losses are defined relative to a given reference point. Formally, $g(y) = y - \lambda([\hat{y} - y]^+)^2$, where y is the (portfolio/asset) return, $\lambda \geq 0$ is the loss aversion – or penalty – parameter, $\hat{y} \in \mathbb{R}$ is the reference point that defines gains and losses, and $[t]^+$ denotes the maximum of 0 and t. See Figure 1 for a graphical illustration of the quadratic loss-averse utility. Compared with linear loss aversion, large losses are punished more severely than small losses under quadratic loss aversion.

We start by studying the optimal asset allocation behavior of a quadratic loss-averse investor. This behavior depends on the reference return $\hat{y}$ and, in particular, on whether this reference return is below, equal to, or above the (requested lower bound on the) expected portfolio return or some threshold value that is larger than the risk-free interest rate. Investors maximize their expected utility of returns as

$$\max_x \left\{ \mathbb{E}\left(r'x - \lambda\left([\hat{y} - r'x]^+\right)^2\right) \,\middle|\, Ax \leq b \right\} \tag{2.1}$$

and Kouwenberg (2009) who analyze the impact of loss-averse investors on asset prices.

where $x = (x_1, \ldots, x_n)'$, with x_i denoting the proportion of wealth invested in asset i,[6] $i = 1, \ldots, n$, and r is the $n-$dimensional random vector of returns, subject to the usual asset constraints $Ax \leq b$, where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$. Note that in general the proportion invested in a given asset may be negative or larger than one due to short-selling.

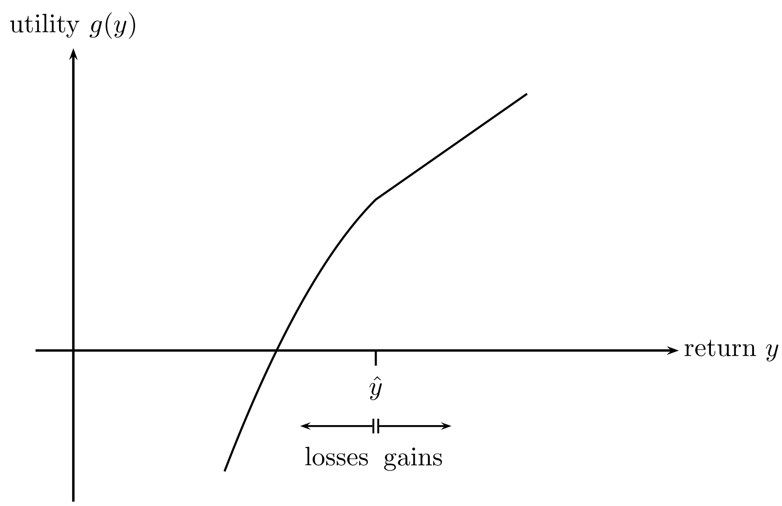

Figure 1: Quadratic loss-averse utility function

2.1 Quadratic loss-averse utility versus mean-variance and conditional value-at-risk

In this section we show the relationship between the quadratic loss-averse utility maximization problem (2.1) and both the MV and the CVaR problems, under the assumption of normally distributed asset returns.[7]

Let Z be a continuous random variable describing the stochastic portfolio return and $f_Z(\cdot)$ and $F_Z(\cdot)$ be its probability density and cumulative distribution functions. Then we define the expected quadratic loss-averse utility of return Z, given the penalty parameter $\lambda \geq 0$ and the reference point

[6]Throughout this paper, prime ($'$) is used to denote matrix transposition and any unprimed vector is a column vector.

[7]For presentations of the MV and CVaR optimizations, see Markowitz (1952) and Rockafellar and Uryasev (2000), respectively.

$\hat{y} \in \mathbb{R}$, as[8]

$$
\begin{aligned}
\mathrm{QLA}_{\lambda,\hat{y}}(Z) &= \mathbb{E}\left(Z - \lambda([\hat{y} - Z]^{+})^{2}\right) \\
&= \mathbb{E}(Z) - \lambda\mathbb{E}\left([\hat{y} - Z]^{+})^{2}\right) \\
&= \mathbb{E}(Z) - \lambda\mathbb{E}\left((\hat{y} - Z)^{2}\,|Z \leq \hat{y}\right) P(Z \leq \hat{y}) \\
&= \mathbb{E}(Z) - \lambda \int_{-\infty}^{\hat{y}} (\hat{y} - z)^{2} f_{Z}(z) dz \qquad (2.2) \\
&\leq \mathbb{E}(Z)
\end{aligned}
$$

As $\int_{-\infty}^{\hat{y}} (\hat{y} - z)^{2} f_{Z}(z) dz \geq 0$, the loss-averse utility of the random variable Z is its mean reduced by some positive quantity, where the size of the reduction depends positively on the values of the penalty parameter λ and the reference point $\hat{y}$. The expected quadratic loss-averse utility $\mathrm{QLA}_{\lambda,\hat{y}}(\cdot)$ is thus a decreasing function in both the penalty parameter and the reference point. Let the conditional value-at-risk $\mathrm{CVaR}_{F_z(\hat{y})}(Z)$ be the conditional expectation of Z below $\hat{y}$; i.e. $\mathrm{CVaR}_{F_z(\hat{y})}(Z) = \mathbb{E}(Z|Z \leq \hat{y})$. If Z is normally distributed such that $Z \sim N(\bar{z}, \sigma^{2})$ then it can be shown using (2.2) that

$$
\mathrm{QLA}_{\lambda,\hat{y}}(Z) = \bar{z} - \lambda\sigma^{2}\left[\left(\frac{\hat{y} - \bar{z}}{\sigma}\right)^{2} F\left(\frac{\hat{y} - \bar{z}}{\sigma}\right) + 2\frac{\hat{y} - \bar{z}}{\sigma} f\left(\frac{\hat{y} - \bar{z}}{\sigma}\right) + \int_{-\infty}^{\frac{\hat{y} - \bar{z}}{\sigma}} y^{2} f(y) dy\right] \qquad (2.3)
$$

$$
\mathrm{CVaR}_{F(\frac{\hat{y} - \bar{z}}{\sigma})}(Z) = \bar{z} - \sigma\frac{f(\frac{\hat{y} - \bar{z}}{\sigma})}{F(\frac{\hat{y} - \bar{z}}{\sigma})} \qquad (2.4)
$$

where $f(\cdot)$ and $F(\cdot)$ are the probability density and the cumulative probability functions of the standard normal distribution. Note that if $\hat{y} = \bar{z}$ then based on (2.3) and (2.4) $\mathrm{QLA}_{\lambda,\hat{y}}(Z) = \bar{z} - \lambda\sigma^{4}/2$ and $\mathrm{CVaR}_{F(\frac{\hat{y} - \bar{z}}{\sigma})}(Z) = \bar{z} - \sigma\sqrt{2/\pi}$.

If asset returns are normally distributed (i.e., $r \sim N(\mu, \Sigma)$, where μ, $r \in \mathbb{R}^{n}$ and $\Sigma \in \mathbb{R}^{n \times n}$ such that covariance matrix Σ is positive definite) then the portfolio return is also normally distributed (i.e., $r'x \sim N(\mu' x, x'\Sigma x)$, where $x \in \mathbb{R}^{n}$). Thus, using our formulation of quadratic loss aversion given normal returns, see (2.3), we introduce the following quadratic loss-averse utility maximization

[8] Note that $\mathrm{QLA}_{\lambda,\hat{y}}$ already accounts for the expectation of utility.

problem[9]

$$\left.\begin{aligned} \text{maximize}_x: \quad & \mathrm{QLA}_{\lambda,\hat{y}}(r'x) = \mu'x - \lambda x'\Sigma x \left[\frac{(\hat{y}-\mu'x)^2}{x'\Sigma x} F\left(\frac{\hat{y}-\mu'x}{\sqrt{x'\Sigma x}}\right) + 2\frac{\hat{y}-\mu'x}{\sqrt{x'\Sigma x}} f\left(\frac{\hat{y}-\mu'x}{\sqrt{x'\Sigma x}}\right) + \int_{-\infty}^{\frac{\hat{y}-\mu'x}{\sqrt{x'\Sigma x}}} y^2 f(y)dy\right] \\ \text{subject to}: \quad & Ax \leq b \\ & \mu'x = \bar{R} \end{aligned}\right\} \tag{2.5}$$

Under the same assumptions, the MV problem can be stated as

$$\min_x \left\{ \mathrm{var}(r'x) = x'\Sigma x \,\middle|\, Ax \leq b,\ \mu'x = \bar{R} \right\} \tag{2.6}$$

and, based on (2.4), the CVaR problem can be written as

$$\max_x \left\{ \mathrm{CVaR}_{F\left(\frac{\hat{y}-\mu'x}{\sqrt{x'\Sigma x}}\right)}(r'x) = \mu'x - \sqrt{x'\Sigma x}\frac{f\left(\frac{\hat{y}-\mu'x}{\sqrt{x'\Sigma x}}\right)}{F\left(\frac{\hat{y}-\mu'x}{\sqrt{x'\Sigma x}}\right)} \,\middle|\, Ax \leq b,\ \mu'x = \bar{R} \right\} \tag{2.7}$$

We can now state the two main theorems of equivalence, which describe how the QLA problem is related to the more traditional MV and CVaR problems.

Theorem 2.1 *Let $\left\{ x \mid Ax \leq b,\ \mu'x = \bar{R} \right\} \neq \emptyset$, $r \sim N(\mu, \Sigma)$ and $\lambda > 0$. Then the QLA problem (2.5) and the MV problem (2.6) are equivalent, i.e., they have the same optimal solution, if either (i) $\hat{y} = \bar{R}$ or (ii) $\lambda = 1/F\left(\frac{\hat{y}-\bar{R}}{\sqrt{(x^*)'\Sigma x^*}}\right)$ and $\hat{y} > \bar{R}$, where x^* is the optimal portfolio of (2.6).*

Proof. (i) If $\hat{y} = \bar{R}$ and $\lambda > 0$ then $\mathrm{QLA}_{\lambda,\hat{y}}(r'x) = \hat{y} - \lambda(x'\Sigma x)^2/2$. This and the fact that $x'\Sigma x > 0$ (for any $x \neq 0$) imply the equivalence between (2.5) and (2.6).

(ii) If $\hat{y} > \bar{R}$, $\mu'x = \bar{R}$, and $\lambda = 1/F\left(\frac{\hat{y}-\bar{R}}{\sqrt{x'\Sigma x}}\right)$ then the objective functions of (2.5) can be stated as

$$\mathrm{QLA}_{1/F\left(\frac{\hat{y}-\bar{R}}{\sqrt{x'\Sigma x}}\right),\hat{y}}(r'x) = \bar{R} - (\hat{y}-\bar{R})^2 - \frac{2\sqrt{x'\Sigma x}(\hat{y}-\bar{R}) f\left(\frac{\hat{y}-\bar{R}}{\sqrt{x'\Sigma x}}\right) + x'\Sigma x \int_{-\infty}^{\frac{\hat{y}-\bar{R}}{\sqrt{x'\Sigma x}}} y^2 f(y)dy}{F\left(\frac{\hat{y}-\bar{R}}{\sqrt{x'\Sigma x}}\right)}$$

Maximizing this is equivalent to minimizing the variance $x'\Sigma x$ over the same set of feasible solutions,

[9] Note that as in Barberis and Xiong (2009) and Hwang and Satchell (2010), we use an objective probability density function rather than a subjective weight function to calculate the loss-averse utility function.

which follows from the fact that $F(\cdot)$ is an increasing function, $f(z)$ is decreasing for $z \geq 0$, $\hat{y} > \bar{R}$, and $u(z) \equiv z \int_{-\infty}^{\frac{\hat{y}-\bar{R}}{\sqrt{z}}} y^2 f(y)dy$ is increasing for $z > 0$. The latter follows from the fact that $\frac{du(z)}{dz} = u_1(z) - u_2(z) > 0$ for $z > 0$ and $\hat{y} > \bar{R}$, where[10]

$$
\begin{aligned}
u_1(z) &= \int_{-\infty}^{\frac{\hat{y}-\bar{R}}{\sqrt{z}}} y^2 f(y)dy \\
u_2(z) &= \frac{(\hat{y}-\bar{R})^3}{2\,z\sqrt{z}} f\left(\frac{\hat{y}-\bar{R}}{\sqrt{z}}\right)
\end{aligned}
$$

This concludes the proof. $\square$

Theorem 2.2 *Let* $\left\{ x \,|\, Ax \leq b,\ \mu' x = \bar{R} \right\} \neq \emptyset$, $r \sim N(\mu, \Sigma)$ *and* $\lambda > 0$. *Then the QLA problem (2.5) and the CVaR problem (2.7) are equivalent, i.e., they have the same optimal solution, if either (i)* $\hat{y} = \bar{R}$ *or (ii)* $\lambda = 1/F\left(\frac{\hat{y}-\bar{R}}{\sqrt{(x^*)'\Sigma x^*}}\right)$ *and* $\hat{y} > \bar{R}$, *where* x^* *is the optimal portfolio of (2.7).*

Proof. (i) If $\hat{y} = \bar{R}$ and $\lambda > 0$ then $\mathrm{QLA}_{\lambda,\hat{y}}(r'x) = \hat{y} - \lambda x'\Sigma x/2$ and $\mathrm{CVaR}_{F(0)}(r'x) = \bar{R} - \sqrt{x'\Sigma x}\ f(0)/F(0)$. This, the fact that $x'\Sigma x > 0$ (for any $x \neq 0$) and the fact that $\sqrt{z}$ is increasing for $z > 0$ imply the equivalence between (2.5) and (2.7).

(ii) If $\lambda = 1/F\left(\frac{\hat{y}-\bar{R}}{\sqrt{x'\Sigma x}}\right)$ then problems (2.5) and (2.7) can be written as

$$
\mathrm{QLA}_{1/F\left(\frac{\hat{y}-\bar{R}}{\sqrt{x'\Sigma x}}\right),\hat{y}}(r'x) = \bar{R} - (\hat{y}-\bar{R})^2 - \frac{2\sqrt{x'\Sigma x}(\hat{y}-\bar{R}) f\left(\frac{\hat{y}-\bar{R}}{\sqrt{x'\Sigma x}}\right) + x'\Sigma x \int_{-\infty}^{\frac{\hat{y}-\bar{R}}{\sqrt{x'\Sigma x}}} y^2 f(y)dy}{F\left(\frac{\hat{y}-\bar{R}}{\sqrt{x'\Sigma x}}\right)}
$$

and

$$
\mathrm{CVaR}_{F\left(\frac{\hat{y}-\mu' x}{\sqrt{x'\Sigma x}}\right)}(r'x) = \bar{R} - \sqrt{x'\Sigma x}\frac{f\left(\frac{\hat{y}-\bar{R}}{\sqrt{x'\Sigma x}}\right)}{F\left(\frac{\hat{y}-\bar{R}}{\sqrt{x'\Sigma x}}\right)}
$$

and the statement of the theorem can be shown in an analogous way as in Theorem 2.1. $\square$

Theorem 2.1 states the conditions under which the QLA and MV problems are equivalent provided returns are normally distributed: they are equivalent (i) when the reference point is equal to the mean of the portfolio return at the optimum, or (ii) when the reference point is strictly larger than the mean of the portfolio return and the loss aversion parameter is equal to some specific value (depending on the reference point and on the optimal solution). In the latter case, the loss aversion

[10]The statement $u_1(z) > u_2(z)$ for $z > 0$, $\hat{y} > \bar{R}$ can be verified by, first, showing that $u_1(z)$ is a decreasing function with values above 0.5, and, second, showing that a maximum of $u_2(z)$ is strictly smaller than 0.5.

parameter yielding equivalence is smaller for larger reference points. The equivalence of the QLA and CVaR problems, stated in Theorem 2.2, is established under the same conditions.[11]

2.2 Analytical solution for one risk-free and one risky asset

To better understand the attitude with respect to risk of quadratic loss-averse investors, we consider a simple two-asset world, where one asset is risk-free and the other is risky, and analyze what proportion of wealth is invested in the risky asset under quadratic loss aversion.

Let r^0 be a certain (deterministic) return of the risk-free asset and let r be the (stochastic) return of the risky asset. Then the portfolio return is $R(x) = xr + (1-x)r^0 = r^0 + (r - r^0)x$, where x is the proportion of wealth invested in the risky asset, and the maximization problem of the quadratic loss-averse investor is

$$\begin{aligned} \max_x \{ \, \mathrm{QLA}_{\lambda,\hat{y}}(R(x)) &= \mathbb{E}\left(R(x) - \lambda \left([\hat{y} - R(x)]^+ \right)^2 \right) \\ &= \mathbb{E}(r^0 + (r - r^0)x) - \lambda \mathbb{E}\left(\left([\hat{y} - r^0 - (r - r^0)x]^+ \right)^2 \right) \mid x \in \mathbb{R} \} \end{aligned} \tag{2.8}$$

where $\lambda \geq 0$, $\hat{y} \in \mathbb{R}$ and $[t]^+ = \max\{0, t\}$. The following two cases present characterizations of the optimal solution when the risky asset's return is binomially distributed (discrete distribution) and when it is (generally) continuously distributed. We shall see that the main properties of the optimal solution and its sensitivity with respect to the loss aversion parameter and the reference point do not depend on the distributional assumptions.

The risky asset is binomially distributed

First we assume for the sake of simplicity and because in this case we can show a number of results analytically, that the return of the risky asset follows a binomial distribution. We assume two states of nature: a good state of nature which yields return r_g such that $r_g > r^0$ and which occurs with probability p and a bad state of nature which yields return r_b such that $r_b < r^0$ and which occurs with probability $1 - p$. In the good state of nature the portfolio thus yields return $R_g(x) = r^0 + (r_g - r^0)x$ with probability p, in the bad state of nature it yields return

[11] The condition $\mu' x = \bar{R}$, which is required in both theorems, can be interpreted as setting a lower bound on the portfolio return, $\bar{R} \leq \mu' x$, which is binding at the optimum.

$R_b(x) = r^0 + (r_b - r^0)x$ with probability $1-p$. Note that

$$\begin{aligned}
\mathbb{E}(r) &= pr_g + (1-p)r_b \;=\; p(r_g - r_b) + r_b, && (2.9)\\
\mathbb{E}(R(x)) &= \mathbb{E}\left(r^0 + \left(r - r^0\right)x\right) = p\left(r^0 + \left(r_g - r^0\right)x\right) + (1-p)\left(r^0 + \left(r_b - r^0\right)x\right) &&\\
&= r^0 + \left[p(r_g - r_b) - r^0 + r_b\right]x = r^0 + \mathbb{E}(r - r^0)x, && (2.10)
\end{aligned}$$

$$[\hat{y} - R_g(x)]^+ = \begin{cases} \hat{y} - r^0 - \left(r_g - r^0\right)x, & \text{for } \; x \le \frac{\hat{y} - r^0}{r_g - r^0} \\ 0, & \text{for } \; x > \frac{\hat{y} - r^0}{r_g - r^0} \end{cases} \tag{2.11}$$

$$[\hat{y} - R_b(x)]^+ = \begin{cases} 0, & \text{for } \; x \le \frac{r^0 - \hat{y}}{r^0 - r_b} \\ \hat{y} - r^0 - \left(r_b - r^0\right)x, & \text{for } \; x > \frac{r^0 - \hat{y}}{r^0 - r_b} \end{cases} \tag{2.12}$$

Thus, based on (2.8), the loss-averse utility of the two-asset portfolio including the risk-free asset and the binomially distributed risky asset is

$$\mathrm{QLA}_{\lambda,\hat{y}}(R(x)) = r^0 + \mathbb{E}\left(r - r^0\right)x - \lambda\left(p\left([\hat{y} - R_g(x)]^+\right)^2 + (1-p)\left([\hat{y} - R_b(x)]^+\right)^2\right) \tag{2.13}$$

The next proposition presents the analytical solution of the loss-averse utility maximization problem (2.8) for the binomially distributed risky asset with respect to a certain threshold value of the loss aversion parameter λ.

Theorem 2.3 *Let $r_b < r^0 < r_g$, $\mathbb{E}(r - r^0) > 0$, $\lambda > 0$, x^* be the optimal solution of (2.8) and*

$$\hat{\lambda} \;\equiv\; \frac{(r_g - r^0)\mathbb{E}(r - r^0)}{2(1-p)(\hat{y} - r^0)(r^0 - r_b)(r_g - r_b)} \quad for \;\; \hat{y} > r^0 \tag{2.14}$$

where the risky asset's return r is binomially distributed with r_g (r_b) being the return in the good (bad) state of nature, which occurs with probability p $(1-p)$. Then the following holds:

(i) *If $\hat{y} \le r^0$ then $x^* = \frac{r^0 - \hat{y}}{r^0 - r_b} + \frac{\mathbb{E}(r - r^0)}{2\lambda(1-p)(r^0 - r_b)^2} > 0$*

(ii) *If $\hat{y} > r^0$ and $\lambda \le \hat{\lambda}$ then $x^* = \frac{r^0 - \hat{y}}{r^0 - r_b} + \frac{\mathbb{E}(r - r^0)}{2\lambda(1-p)(r^0 - r_b)^2} > 0$*

(iii) *If $\hat{y} > r^0$ and $\lambda > \hat{\lambda}$ then $x^* = \frac{\left(\frac{1}{2\lambda} + \hat{y} - r^0\right)\mathbb{E}(r - r^0)}{\mathbb{E}(r - r^0)^2} > 0$*

Proof. See Appendix A.

Under quadratic loss aversion the optimal investment in the risky asset is thus always positive and finite, for any given degree of loss aversion and any given reference point.[12] For the case when the reference point is larger than the risk-free rate, the analytical form of the solution depends on the investor's loss aversion, more precisely, it depends on the loss aversion parameter being below or above some threshold value. This threshold value is a function of the reference point, and thus the assumption with respect to the loss aversion parameter ($\lambda \leq \hat{\lambda}$, $\lambda > \hat{\lambda}$), for the case when $\hat{y} > r^0$, can be translated into an assumption with respect to the reference point: $\lambda \leq (>)\hat{\lambda} \Leftrightarrow \hat{y} \leq (>)\hat{y}_{min}$, where $\hat{y}_{min} = \frac{(r_g - r^0)(\mu - r^0)}{2\lambda(1-p)(r^0 - r_b)(r_g - r_b)} + r^0 > r^0$.[13] Using this latter assumption we can combine cases (i) and (ii) of Theorem 2.3 to require $\hat{y} \leq \hat{y}_{min}$. The next corollary describes the sensitivity of the optimal solution with respect to the penalty parameter and the reference point.

Corollary 2.1 *Let $r_b < r^0 < r_g$, $\mathbb{E}(r - r^0) > 0$ and $\lambda > 0$. Then the optimal solution of (2.8), x^*, has the following properties*

$$\frac{dx^*}{d\lambda} < 0 \tag{2.15}$$

and

$$\frac{dx^*}{d\hat{y}} = \begin{cases} < 0, & \text{if } \ \hat{y} < \hat{y}_{min} \\ > 0, & \text{if } \ \hat{y} > \hat{y}_{min} \end{cases} \tag{2.16}$$

where

$$\hat{y}_{min} = \frac{(r_g - r^0)(\mu - r^0)}{2\lambda(1-p)(r^0 - r_b)(r_g - r_b)} + r^0 > r^0 \tag{2.17}$$

Proof. Property (2.15) follows directly from Theorem 2.3 which implies also

$$\frac{dx^*}{d\hat{y}} = \begin{cases} < 0, & \text{if } \ \hat{y} < r^0 \\ & \text{if } \ \hat{y} > r^0 \ \text{ and } \ \lambda \leq \hat{\lambda} \\ & \\ > 0, & \text{if } \ \hat{y} > r^0 \ \text{ and } \ \lambda > \hat{\lambda} \end{cases}$$

[12] Note that under linear loss aversion the investor has to be sufficiently loss averse to yield a finite investment in the risky asset.

[13] The next corollary will explain why we call this threshold the *minimum reference point.*

The statement of the corollary follows then from this and the fact that

$$\lambda \leq \hat{\lambda} \Leftrightarrow \hat{y} \leq \hat{y}_{min}$$

where $\hat{y}_{min}$ is given by (2.17). $\square$

The corollary implies that the optimal solution as a function of the reference point is U-shaped, where the minimum (which is strictly positive) is attained for a reference point that is strictly larger than the risk-free rate. This reference point, which we call the *minimum reference point*, depends on the loss aversion parameter and can be stated explicitly, see equation (2.17).

Table 1 summarizes and contrasts the optimal investments into the risky asset for the linear and the quadratic loss-averse investor (for more details see Fortin and Hlouskova, 2011a). An analogous summary including the case for binomial and continuous returns as well as the sensitivities of the optimal solution with respect to the penalty parameter λ and the reference point $\hat{y}$, is presented in Table 2.

assumptions	solutions			
$\hat{y} \leq r^0, \quad \lambda > \lambda_{\mathrm{LLA}}$	$+\infty$	$>$	$x_1^* =$	$x^*_{\mathrm{QLA}} > x^*_{\mathrm{LLA}} = \frac{r^0-\hat{y}}{r^0-r_b} \geq 0$
$\hat{y} > r^0, \quad \lambda_{\mathrm{LLA}} < \lambda < \lambda_{\mathrm{QLA}}$	$+\infty$	$>$	$x_1^* =$	$x^*_{\mathrm{QLA}} > x^*_{\mathrm{LLA}} = \frac{\hat{y}-r^0}{r_g-r^0} > 0$
$\hat{y} > r^0, \quad \lambda > \lambda_{\mathrm{QLA}} \ \ (> \lambda_{\mathrm{LLA}})$	0	$<$	$x_2^* =$	$x^*_{\mathrm{QLA}} < x^*_{\mathrm{LLA}} = \frac{\hat{y}-r^0}{r_g-r^0}$
$\hat{y} \in \mathbb{R}, \quad \lambda < \lambda_{\mathrm{LLA}}$	0	$<$	$\{x_1^*, x_2^*\} \ni$	$x^*_{\mathrm{QLA}} < x^*_{\mathrm{LLA}} = +\infty$

Table 1: Overview of optimal solutions under linear and quadratic loss aversion.
We assume that $\mathbb{E}(r - r^0) > 0$ and $\lambda > 0$. The threshold values of the loss aversion parameter are $\lambda_{\mathrm{LLA}} = \frac{\mathbb{E}(r-r^0)}{(1-p)(r^0-r_b)}$ under linear loss aversion and $\lambda_{\mathrm{QLA}} = \lambda_{\mathrm{LLA}} \frac{r_g-r^0}{2(\hat{y}-r^0)(r_g-r_b)}$ for $\hat{y} > r^0$ under quadratic loss aversion. x_1^* and x_2^* correspond to the optimal solutions given in (i) and (iii) of Theorem 2.3. Note that $\lambda < \lambda_{\mathrm{QLA}} \Leftrightarrow \hat{y} < \hat{y}_{min}$, where $\hat{y}_{min}$ is given by (2.17).

First of all, x^*_{QLA}, which is the optimal investment in the risky asset under quadratic loss-averse preferences, is always strictly positive, while the optimal investment in the risky asset of a sufficiently loss-averse investor under linear loss-averse preferences, x^*_{LLA}, is zero when the reference point coincides with the risk-free rate. Second, the optimal investment in the risky asset of a QLA investor never explodes, while this can be the case ($x^*_{\text{LLA}} = +\infty$) for an LLA investor who is not sufficiently loss-averse ($\lambda < \lambda_{\text{LLA}}$). This then is also referred to as an ill-posed problem. In addition, if the investor is sufficiently loss averse to guarantee a finite solution under linear loss aversion ($\lambda > \lambda_{\text{LLA}}$), then the optimal investment in the risky asset of a QLA investor is strictly larger than the optimal investment of an LLA investor for all reference points below the minimum reference point, and it is strictly smaller for all reference points above the minimum reference point.

When comparing the sensitivity analysis of the optimal investment in the risky asset with respect to changes of the loss aversion parameter and the reference point under QLA and LLA preferences (see Table 2) then one can see the following: while the investment in the risky asset decreases with an increasing degree of loss aversion under QLA preferences, it remains unchanged under LLA preferences. On the other hand, the sensitivity of the optimal investment in the risky asset with respect to the reference point is similar for both types of investors when they are sufficiently loss-averse ($\lambda > \lambda_{\text{LLA}}$), i.e., the optimal investment in the risky asset decreases when the reference point is below some threshold value and increases when it is above the same threshold. Under linear loss aversion this threshold is equal to the risk-free interest rate, while under quadratic loss aversion this threshold (which depends on the loss aversion parameter) is strictly larger than the risk-free rate. The situation is different for investors who are less loss-averse ($\lambda < \lambda_{\text{LLA}}$): while under quadratic loss aversion it is identical to the one just described, the LLA investment in the risky asset is not affected by the reference point. However, in this case the optimal investment is always infinite.

The risky asset is continuously distributed

Let us now assume that the risky asset's return is continuously distributed with probability density function $f_r(\cdot)$ and expected return $\mathbb{E}(r) = \mu$ such that the expected excess return (risk premium) is positive, i.e., $\mathbb{E}(r - r^0) > 0$ (or $\mu > r^0$). Then the expected loss-averse utility can be formulated as

$$\text{QLA}_{\lambda,\hat{y}}(R(x)) = \left\{ \begin{array}{rl} r^0 + (\mu - r^0)x - \lambda \int_{\frac{\hat{y}-r^0}{x}+r^0}^{\infty} \left(\hat{y} - r^0 - \left(r - r^0\right)x\right)^2 f_r(r)dr, & x < 0 \\ r^0 - \lambda \left(\left[\hat{y} - r^0\right]^+\right)^2, & x = 0 \\ r^0 + (\mu - r^0)x - \lambda \int_{-\infty}^{\frac{\hat{y}-r^0}{x}+r^0} \left(\hat{y} - r^0 - \left(r - r^0\right)x\right)^2 f_r(r)dr, & x > 0 \end{array} \right\} \quad (2.18)$$

We consider two cases, first the case when the reference point does not coincide with the risk-free rate ($\hat{y} \neq r^0$) and second the case when it does ($\hat{y} = r^0$). The latter is the case more often considered in the literature. One reason for investigating $\hat{y} = r^0$ is that the risk-free rate seems to be a natural choice for the reference point. Another reason may be that the corresponding analysis is often more straightforward. Let us use the term *zero excess* reference return to describe the case $\hat{y} = r^0$ and *positive (negative) excess* reference return to describe the case $\hat{y} > r^0$ ($\hat{y} < r^0$).[14] For the latter we also use the term *non-zero excess* reference return. Another interpretation of the negative and positive excess reference returns can be seen from writing down the portfolio return net of the reference point for the case when the investor stays out of the market ($x = 0$)

$$R(x) - \hat{y}|_{x=0} = r^0 + (r - r^0)x - \hat{y}|_{x=0} = r^0 - \hat{y}$$

Thus, if the residual of the relative portfolio return with respect to the reference point $\hat{y}$ with zero risky investment is positive, $\hat{y} < r^0$, i.e., the investor is modest in setting her return goals, then even when she stays out of the market she will be in her *comfort zone.* On the other hand if the investor is more ambitious in setting her goals, $\hat{y} > r^0$, then the residual of the relative portfolio return with respect to the reference point with zero risky investment is negative and thus if she stays out of the market she will be not that well off and be in her *discomfort zone.*

The following theorem characterizes the solution to the asset allocation decision under quadratic loss aversion, see (2.8). For the special case when $\hat{y} = r^0$, the solution can be stated explicitly, which is shown in the subsequent corollary.

Theorem 2.4 *Let $\mathbb{E}(r - r_0) > 0$ and $\lambda > 0$. Then problem (2.8) has a unique solution $x^* > 0$ which satisfies*

$$\mu - r^0 + 2\lambda \int_{-\infty}^{\frac{\hat{y}-r^0}{x^*}+r^0} \left(\hat{y} - r^0 - (r - r^0)x^*\right)(r - r^0) f_r(r)dr = 0 \tag{2.19}$$

Proof. See Appendix A.

Corollary 2.2 *Let $\mathbb{E}(r - r_0) > 0$, $\lambda > 0$ and $\hat{y} = r^0$. Then the solution to problem (2.8), as*

[14] In the wealth setup the case corresponding to $\hat{y} > r^0$ ($\hat{y} < r^0$, $\hat{y} = r^0$) is called the negative (positive, zero) surplus case.

characterized by Theorem 2.4, can be stated explicitly as

$$x^* = \frac{\mu - r^0}{2\lambda \int_{-\infty}^{r^0} (r - r^0)^2 f_r(r) dr} \tag{2.20}$$

Proof. Note that for $\hat{y} = r^0$ the first order condition (2.21) simplifies to

$$\mu - r^0 - 2\lambda x^* \int_{-\infty}^{r^0} (r - r^0)^2 f_r(r) dr = 0 \tag{2.21}$$

which immediately yields (2.20). □

Note that both for the non-zero excess reference point ($\hat{y} \neq r^0$) and the zero excess reference point ($\hat{y} = r^0$) the existence of a positive bounded solution does not depend on the degree of loss aversion. This is in contrast to linear loss aversion, where the investor needs to be sufficiently loss-averse to guarantee a bounded solution (see Table 2 or, e.g., Fortin and Hlouskova, 2011a; and Siegmann and Lucas, 2005).[15] If the linear loss-averse investor displays a low degree of loss aversion (i.e., a small penalty parameter) then she would invest an infinite amount in the risky asset ($x^* = +\infty$). He and Zhou (2011) refer to this as an ill-posed problem. In that sense, the investment problem under quadratic loss aversion is always well-posed: for both the non-zero and the zero excess reference point, a unique positive solution exists for any given penalty parameter. Another fundamental difference between linear and quadratic loss aversion is that for a zero excess reference point the LLA investor stays out of the market ($x^* = 0$) while the QLA investor always buys a strictly positive amount of the risky asset ($x^* > 0$). This difference is a direct consequence of the quadratic penalty. A large penalty parameter drives the risky investment to zero, however. From a normative point of view it might be undesirable to see positive investments in the risky asset if the reference point is equal to the risk-free rate. This has been found especially concerning given the use of quadratic down-side risk measures in financial planning (see Siegmann and Lucas, 2005).

The following two corollaries summarize properties of the optimal solution with respect to the degree of loss aversion and the level of the reference point, for the non-zero and the zero excess reference points.

[15] Also for S-shaped loss aversion, a bounded solution depends on the degree of loss aversion, see Fortin and Hlouskova (2011b).

Corollary 2.3 *Let $\mathbb{E}(r - r_0) > 0$, $\hat{y} \neq r^0$ and $\lambda > 0$. Then the solution of problem (2.8) has the following properties*

$$\begin{aligned} \frac{dx^*}{d\lambda} &= -\frac{\mu - r^0}{2\lambda^2 \int_{-\infty}^{\frac{\hat{y}-r^0}{x^*}+r^0} (r-r^0)^2 f_r(r) dr} < 0 \\ \frac{dx^*}{d\hat{y}} &= \frac{\int_{-\infty}^{\frac{\hat{y}-r^0}{x^*}+r^0} (r-r^0) f_r(r) dr}{\int_{-\infty}^{\frac{\hat{y}-r^0}{x^*}+r^0} (r-r^0)^2 f_r(r) dr} \begin{cases} < 0, & \text{if } \hat{y} < \hat{y}_{min} \\ > 0, & \text{if } \hat{y} > \hat{y}_{min} \end{cases} \end{aligned} \tag{2.22}$$

where $\hat{y}_{min} = \operatorname{argmin}\{x^*(\lambda, \hat{y}) \,|\, \hat{y}\}$ such that $r^0 < \hat{y}_{min} < +\infty$ and $\hat{y}_{min}$ solves $\int_{-\infty}^{\frac{\hat{y}_{min}-r^0}{x^*}+r^0} (r - r^0)^2 f_r(r) dr = \frac{\mu - r^0}{2\lambda x^*}$.

Proof. The proof is based on implicit function differentiation and Theorem 2.4. Let

$$G(\lambda, \hat{y}, x) \equiv \mu - r^0 + 2\lambda \int_{-\infty}^{\frac{\hat{y}-r^0}{x}+r^0} \left(\hat{y} - r^0 - (r - r^0)x\right)(r - r^0) f_r(r) dr = 0$$

then

$$\frac{dx}{d\lambda} = -\frac{\partial G/\partial \lambda}{\partial G/\partial x} \quad \text{and} \quad \frac{dx}{d\hat{y}} = -\frac{\partial G/\partial \hat{y}}{\partial G/\partial x} \tag{2.23}$$

where $\hat{y}$ is fixed in the first case and λ is fixed in the second case, and

$$\begin{aligned} (\partial G/\partial \lambda)_{x=x^*} &= 2\int_{-\infty}^{\frac{\hat{y}-r^0}{x^*}+r^0} \left(\hat{y} - r^0 - (r-r^0)x^*\right)(r-r^0) f_r(r) dr \\ &= -\frac{\mu - r^0}{\lambda} < 0 \\ (\partial G/\partial \hat{y})_{x=x^*} &= 2\lambda \int_{-\infty}^{\frac{\hat{y}-r^0}{x^*}+r^0} (r-r^0) f_r(r) dr \\ (\partial G/\partial x)_{x=x^*} &= -2\lambda \int_{-\infty}^{\frac{\hat{y}-r^0}{x^*}+r^0} (r-r^0)^2 f_r(r) dr \end{aligned}$$

This and (2.23) imply expressions for $\frac{dx^*}{d\lambda}$ and $\frac{dx^*}{d\hat{y}}$ as stated in (2.22). Positive equity premium,

$\mathbb{E}(r-r^0)>0$, and the expression for $\frac{dx^*}{d\lambda}$ imply that $\frac{dx^*}{d\lambda}<0$. Regarding $\frac{dx^*}{d\hat{y}}$, (2.21) gives

$$\int_{-\infty}^{\frac{\hat{y}-r^0}{x^*}+r^0}(r-r^0)f_r(r)dr=\frac{1}{\hat{y}-r^0}\left(x^*\int_{-\infty}^{\frac{\hat{y}-r^0}{x^*}+r^0}(r-r^0)^2f_r(r)dr-\frac{\mu-r^0}{2\lambda}\right)$$

which implies, in addition to (2.22), that

$$\frac{dx^*}{d\hat{y}}=\frac{\int_{-\infty}^{\frac{\hat{y}-r^0}{x^*}+r^0}(r-r^0)f_r(r)dr}{\int_{-\infty}^{\frac{\hat{y}-r^0}{x^*}+r^0}(r-r^0)^2f_r(r)dr}\left\{\begin{array}{ll}<0, & \text{if } \hat{y}<r^0 \\ & \text{if } \hat{y}>r^0 \text{ and } \lambda<\hat{\lambda} \\ =0, & \text{if } \hat{y}>r^0 \text{ and } \lambda=\hat{\lambda} \\ >0, & \text{if } \hat{y}>r^0 \text{ and } \lambda>\hat{\lambda}\end{array}\right\} \tag{2.24}$$

where $\hat{\lambda}=\frac{\mu-r^0}{2x^*\int_{-\infty}^{\frac{\hat{y}-r^0}{x^*}+r^0}(r-r^0)^2f_r(r)dr}$. As

$$\begin{aligned}\lim_{\hat{y}\to+\infty}\frac{dx^*}{d\hat{y}} &= \frac{\int_{-\infty}^{+\infty}(r-r^0)f_r(r)dr}{\int_{-\infty}^{+\infty}(r-r^0)^2f_r(r)dr}=\frac{\mu-r^0}{\int_{-\infty}^{+\infty}(r-r^0)^2f_r(r)dr}>0 \\ \lim_{\hat{y}\to(r^0)^+}\frac{dx^*}{d\hat{y}} &= \frac{\int_{-\infty}^{r^0}(r-r^0)f_r(r)dr}{\int_{-\infty}^{r^0}(r-r^0)^2f_r(r)dr}<0 \\ \frac{d\hat{\lambda}}{d\hat{y}} &= -\frac{(\mu-r^0)(\hat{y}-r^0)^2}{2(x^*)^4\left(\int_{-\infty}^{\frac{\hat{y}-r^0}{x^*}+r^0}(r-r^0)^2f_r(r)dr\right)^2}f_r\left(\frac{\hat{y}-r^0}{x^*}+r^0\right)<0\end{aligned}$$

then this and (2.24) imply the U-shape of the optimal solution x^* with respect to the reference point $\hat{y}$ as stated by (2.22). This concludes the proof. □

The corollary implies that the optimal solution as a function of the reference point is U-shaped, displaying its minimum for a reference point that is strictly larger than the risk-free interest rate (minimum reference point). This threshold value depends on the loss aversion parameter and cannot be stated explicitly.[16] There will thus be investors with positive excess reference points who take on less risk than an investor with a zero excess reference point. This is also interesting from a normative point of view and is clearly different from the case of linear loss aversion, where the minimum degree of risk (in fact zero risk) is always attained for a zero excess reference point. The behavior of the optimal solution with respect to the reference point is similar for sufficiently loss-

[16] It can, however, be computed numerically for a given loss aversion parameter and a given distribution.

averse investors under linear loss aversion, except for the level of the threshold (minimum reference point). Under linear loss aversion, the threshold that yields the minimum investment in the risky asset, is equal to the risk-free interest rate.

Corollary 2.4 *Let $\mathbb{E}(r - r_0) > 0$, $\hat{y} = r^0$ and $\lambda > 0$. Then the solution of problem (2.8) has the following properties*

$$\frac{dx^*}{d\lambda} = -\frac{x^*}{\lambda} < 0 \tag{2.25}$$

Proof. The statement can be shown by differentiating x^*, as given in (2.20), with respect to λ. □

Thus, the optimal fraction invested in the risky asset decreases with an an increasing degree of the penalty parameter λ, both for non-zero and zero excess reference points. This behavior coincides with the one of a linear loss-averse investor for a non-zero excess reference point. For the case of a linear loss-averse investor with a zero excess reference point the optimal fraction invested in the risky asset does not depend on the penalty parameter. Table 2 summarizes the properties of the optimal solution under linear and quadratic loss aversion and lists the corresponding sensitivities of the optimal risky asset's weight with respect to the loss aversion parameter and with respect to the reference point. For continuously distributed returns, the comparison of optimal investments under linear and quadratic loss aversion is not so straightforward as for binomial returns. What we can say, however, is that the optimal investment of an QLA investor exceeds that of an LLA investor in a neighborhood of the zero excess reference point (since the former is strictly positive and the latter is equal to zero).

	binomial	continuous
linear loss aversion (LLA)		
$\hat{y} \neq r^0$, $\lambda > \lambda_{\text{LLA}}$	$x^* > 0$ (expl.)	$x^* > 0$
$\hat{y} = r^0$, $\lambda > \lambda_{\text{LLA}}$	$x^* = 0$	$x^* = 0$
$\lambda < \lambda_{\text{LLA}}$	$x^* = +\infty$	$x^* = +\infty$
$dx^*/d\lambda$, $\hat{y} \neq r^0$, $\lambda > \lambda_{\text{LLA}}$	$= 0$	< 0
$\hat{y} = r^0$ or $\lambda < \lambda_{\text{LLA}}$	$= 0$	$= 0$
$dx^*/d\hat{y}$, $\hat{y} < r^0$, $\lambda > \lambda_{\text{LLA}}$	< 0	< 0
$\hat{y} > r^0$, $\lambda > \lambda_{\text{LLA}}$	> 0	> 0
$\lambda < \lambda_{\text{LLA}}$	=0	=0
quadratic loss aversion (QLA)		
$\hat{y} \neq r^0$	$x^* > 0$ (expl.)	$x^* > 0$
$\hat{y} = r^0$	$x^* > 0$ (expl.)	$x^* > 0$ (expl.)
$dx^*/d\lambda$	< 0	< 0
$dx^*/d\hat{y}$, $\hat{y} < \hat{y}_{min}$	< 0	< 0
$\hat{y} > \hat{y}_{min}$	> 0	> 0

Table 2: Overview of optimal solutions under linear and quadratic loss aversion.
We assume that $\mathbb{E}(r - r^0) > 0$ and $\lambda > 0$. The threshold values of the loss aversion parameter are $\lambda_{\text{LLA}} = \frac{\mathbb{E}(r-r^0)}{(1-p)(r^0-r_b)}$ for the binomial case under linear loss aversion, $\lambda_{\text{QLA}} = \lambda_{\text{LLA}} \frac{r_g - r^0}{2(\hat{y}-r^0)(r_g-r_b)}$ for the binomial case under quadratic loss aversion with $\hat{y} > r^0$, $\lambda_{\text{LLA}} = \frac{\mathbb{E}(r-r^0)}{\int_{\infty}^{r^0}(r^0-r)f_r(r)dr}$ for the continuous case under linear loss aversion, and $\lambda_{\text{QLA}} = \frac{\mathbb{E}(r-r^0)}{2x^* \int_{-\infty}^{\frac{\hat{y}-r^0}{x^*}+r^0}(r-r^0)^2 f_r(r)dr}$ for the continuous case under quadratic loss aversion with $\hat{y} > r^0$. $\hat{y}_{min} = \frac{(r_g-r^0)(\mu-r^0)}{2\lambda(1-p)(r^0-r_b)(r_g-r_b)} + r^0$ for the binomial case and $\hat{y}_{min}$ solves $\int_{-\infty}^{\frac{\hat{y}_{min}-r^0}{x^*}+r^0}(r-r^0)^2 f_r(r)dr = \frac{\mu-r^0}{2\lambda x^*}$ for the continuous case.

Let us now look at a concrete example of optimal investment under quadratic and linear loss aversion. Table 3 presents the optimal investments in the risky assets for investors under both linear and quadratic loss-averse preference, for different values of the reference point, $\hat{y} \in \{3\%, 5\%, 7\%\}$, and the penalty parameter, $\lambda \in \{1, 2, 3\}$. We assume that the risky asset is normally distributed with a mean of 10% and a sigma of 20%, and that the risk-free rate is 5%. First, under QLA the optimal investment into the risky asset is always positive, even when the reference point is equal to the risk-free rate, while the risky investment under LLA is zero for the case when $\hat{y} = r^0$ (for a sufficiently high degree of loss aversion). This reflects our theoretical results.[17] Second, the optimal investment in the risky asset is smaller under QLA than under LLA if the reference point is sufficiently far away from the risk-free rate,[18] which is a consequence of the quadratic shortfall. We thus expect QLA optimal portfolios to exhibit a clearly smaller risk than LLA optimal portfolios in empirical applications. This conjecture will be confirmed in our empirical study, see Section 3. Third, the investment in the risky asset decreases with an increasing value of λ, for a given reference point, under both QLA and LLA preferences. On the other hand, the investment in the risky asset decreases (increases) with an increasing reference point, for a given penalty parameter, provided the reference point is below (above) the threshold value. This threshold is equal to r^0 for the LLA investor and equal to $\hat{y}_{min}^{\text{QLA}}$ for the QLA investor. This again reflects our theoretical results.[19]

2.3 Numerical solution

In empirical applications or simulation experiments, the quadratic loss-averse utility maximization problem (2.1) has to be solved numerically. We thus reformulate the original problem as the following parametric problem of $n-$variables

$$\max_{x} \left\{ \frac{1}{S} \sum_{s=1}^{S} \left(r_s' x - \lambda \left([\hat{y} - r_s' x]^+ \right)^2 \right) \;\middle|\; Ax \leq b \right\} \tag{2.26}$$

[17]See Theorem 2.2 and Proposition 5 in Fortin and Hlouskova (2011a), and the summary of results in Table 2.

[18]What "sufficiently far away" means, depends on the specific distribution assumed as well as on the loss aversion parameter. In our example, the neighborhoods around r^0, in which $x^*_{\text{QLA}} > x^*_{\text{LLA}}$, are $(4.72, 5.11)$, $(3.93, 5.10)$ and $(3.51, 5.08)$ for $\lambda = 1, 2, 3$. The neighborhood is thus not symmetric around the risk-free rate, it includes a larger interval below the risk-free rate.

[19]See Corollary 2.3 and Proposition 6 in Fortin and Hlouskova (2011a), and the summary of results in Table 2.

	$\lambda=1$				$\lambda=2$				$\lambda=3$			
$\hat{y}/\hat{y}^{\text{QLA}}_{min}$	3	$5=r^0$	5.30	7	3	$5=r^0$	5.15	7	3	$5=r^0$	5.10	7
x^*_{LLA}	0.2162	0.0000	0.0350	0.2336	0.0850	0.0000	0.0080	0.1061	0.0691	0.0000	0.0046	0.0925
x^*_{QLA}	0.0871	0.0190	0.0109	0.0294	0.0712	0.0095	0.0055	0.0265	0.0645	0.0063	0.0036	0.0255

Table 3: Optimal share in the risky asset.
The table reports the optimal risky asset's weight of a linear loss-averse (LLA) and a quadratic loss-averse (QLA) investor. The risky asset's return is assumed to be normally distributed, $r \sim N(10, 20^2)$, and $r^0 = 5$ (units in percent or percentage points). The LLA investor is sufficiently loss averse ($\lambda > \hat{\lambda} = 0.8731$) in order to show a bounded investment in the risky asset. $\hat{y}^{\text{QLA}}_{min}$ (the value to the right of the risk-free rate) is the reference point yielding the minimum QLA investment in the risky asset.

where λ, x, $\hat{y}$, A and b are defined as in (2.1), and r_s is the $n-$vector of observed returns, $s = 1, \ldots, S$.

It can be shown that (2.26) is equivalent to the following $(n+S)-$dimensional quadratic programming problem

$$\max_{x,y^-} \left\{ \hat{\mu}'x - \frac{\lambda}{S}(y^-)'y^- \middle| Ax \le b,\ Bx + y^- \ge \hat{y}e,\ y^- \ge 0 \right\} \tag{2.27}$$

where $\hat{\mu} = (\hat{\mu}_1, \ldots, \hat{\mu}_n)'$ is the vector of estimated expected returns, i.e., $\hat{\mu}_i = \frac{1}{S}\sum_{s=1}^S r_{si}$, e is an $S-$vector of ones, $B' = [r_1, r_2, \ldots, r_S]$ and $y^- \in \mathbb{R}^S$ is an auxiliary variable. The equivalence should be understood in the sense that if x^* is the x portion of an optimal solution for (2.27), then x^* is optimal for (2.26). On the other hand, if x^* is optimal for (2.26) then $((x^*)', (y^-)')'$ is optimal for (2.27) where $y_s^- = [\hat{y} - (r_s)'x^*]^+$, $s = 1, \ldots, S$. Thus, the utility function of problem (2.27) maximizes the expected return of the portfolio penalized for cases when its return drops below the reference value $\hat{y}$.

3 Empirical application

In this section we investigate the performance of an optimal asset portfolio constructed by a quadratic loss-averse investor. We study the *benchmark scenario*, where the penalty parameter is constant and the reference point is equal to zero percent, as well as five modified versions of the benchmark scenario. The first modification uses the risk-free interest rate as the reference point (*risk-free scenario*), the remaining four modifications employ time-changing versions of the penalty parameter and the reference point, which depend on previous gains and losses. The second and third modifications of the risk-free scenario describe the usual conservative loss-averse investor who becomes even more loss-averse after losses (*dynamic scenarios*), while the forth and fifth modifications describe a risk-seeking investor who becomes less loss-averse after losses and accepts further risk and gambles which offer a chance to break even (*break-even scenarios*). The setup of the dynamic scenarios follow Barberis and Huang (2001) while that of the break-even scenarios follow Zhang and Semmler (2009).

Let $d_t = r_B/r_t$ be a state variable describing the investor's sentiment with respect to prior gains or losses, which depends on the prior benchmark return $r_B = 1/T\sum_{i=1}^{T} r_{t-i}$ and the current portfolio return r_t. The benchmark return, which is the average value of the latest T realized portfolio returns, is compared with the current portfolio return. If $d_t \leq 1$, then the current portfolio return is greater than or equal to the benchmark return, making the investor feel that her portfolio has performed well and that she has accumulated gains. If $d_t > 1$, then the current portfolio return is lower than the benchmark return, making the investor feel she has experienced losses. We take $T = 1$ because in general investors seem to be most sensitive to the most recent loss. The current portfolio return is thus compared to the previous period's portfolio return.

The *dynamic scenario 1* is modeled as follows. If the investor has experienced gains, then her penalty parameter is equal to the pre-specified λ while her reference point is lower than the risk-free interest rate due to the investor's decreasing loss aversion. On the other hand, if the investor has experienced losses, then her loss aversion and thus her penalty parameter increases. At the same time her reference point is equal to the risk-free interest rate. The quadratic loss-averse utility function adjusted for a time-changing penalty parameter and reference point is

$$g(r_t) = \begin{cases} r_t, & r_t \geq \hat{y}_t \\ r_t - \lambda_t(\hat{y}_t - r_t)^2, & r_t < \hat{y}_t \end{cases}$$

where

$$\lambda_t = \begin{cases} \lambda, & r_t \geq r_{t-1} \text{ (gain)} \\ \lambda + \left(\frac{r_{t-1}}{r_t} - 1\right), & r_t < r_{t-1} \text{ (loss)} \end{cases} \qquad \hat{y}_t = \begin{cases} \frac{r_{t-1}}{r_t} r_t^0, & r_t \geq r_{t-1} \text{ (gain)} \\ r_t^0, & r_t < r_{t-1} \text{ (loss)} \end{cases}$$

and r_t^0 is the risk-free interest rate at time t. Note that $\lambda_t \geq \lambda$ and $\hat{y}_t \leq r_t^0$, where higher values of the loss aversion parameter and the reference point reflect a higher degree of loss aversion.

The *dynamic scenario 2* is again designed for a conservative investor and in that sense is similar to the dynamic scenario 1. If the investor has experienced recent gains, her loss aversion and thus the penalty parameter decreases while her reference point is equal to the risk-free interest rate. On the other hand, if the investor has experienced recent losses, her penalty parameter is equal to the pre-specified λ while her reference point is bigger than the risk-free interest rate due to the investor's increasing loss aversion. Thus, the time-changing penalty parameter and reference point are

$$\lambda_t = \begin{cases} \lambda + \left(\frac{r_{t-1}}{r_t} - 1\right), & r_t \geq r_{t-1} \text{ (gain)} \\ \lambda, & r_t < r_{t-1} \text{ (loss)} \end{cases} \qquad \hat{y}_t = \begin{cases} r_t^0, & r_t \geq r_{t-1} \text{ (gain)} \\ \frac{r_{t-1}}{r_t} r_t^0, & r_t < r_{t-1} \text{ (loss)} \end{cases}$$

The forth and fifth modifications are based on the "break-even" effect as described in Zhang and Semmler (2009) and we refer to it as the *break-even scenario 1* and the *break-even scenario 2*. The main idea is that sometimes people become more risk-seeking after losses in order to make up for previous losses. In other words, even if they have experienced losses in the previous period, investors may be ready to incur further risks and accept gambles which offer them a chance to break even. In both scenarios the case of the losses is modeled in the same way, namely, the penalty parameter decreases and the reference point becomes smaller than the risk-free interest rate due to the investor's increased risk-seeking. The gains in the first break-even scenario are modeled as in the risk-free scenario while in the second break-even scenario they are modeled as if the investor was risk-averse, namely, the penalty parameter increases and the reference point becomes larger than the risk-free interest rate. The time-changing penalty parameter and reference point are then

$$\lambda_t = \begin{cases} \lambda, & r_t \geq r_{t-1} \text{ (gain)} \\ \lambda + \left(\frac{r_t}{r_{t-1}} - 1\right), & r_t < r_{t-1} \text{ (loss)} \end{cases} \qquad \hat{y}_t = \begin{cases} r_t^0, & r_t \geq r_{t-1} \text{ (gain)} \\ \frac{r_t}{r_{t-1}} r_t^0, & r_t < r_{t-1} \text{ (loss)} \end{cases}$$

for the first break-even scenario and

$$\lambda_t = \lambda + \left(\frac{r_t}{r_{t-1}} - 1\right), \qquad \hat{y}_t = \frac{r_t}{r_{t-1}} r_t^0,$$

for both prior gains and losses for the second break-even scenario.

To summarize the different investors Figure 2 shows plots of the four different types of time-changing quadratic loss-averse utility we consider. In particular the different utility functions for gains and losses in the two dynamic and the two break-even scenarios are shown. As a reference the utility for the risk-free scenario is also plotted. In the dynamic scenarios (top row) the investor's loss aversion increases after losses, while it decreases after gains. The dotted line (losses) thus reflects a higher penalty and the dashed line (gains) reflects a lower penalty than in the risk-free scenario. In the break-even scenarios, on the other hand, the investor's loss aversion decreases after losses, since the investor feels she has to make up for the recent losses. We will see that the specific form of the utility affects the performance of the optimal portfolio.

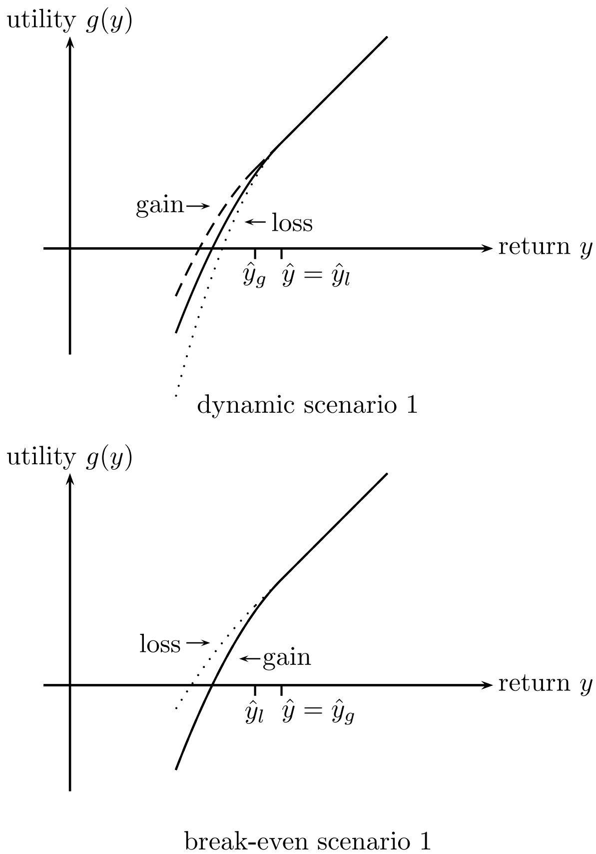

Figure 2: Dynamic and break-even scenarios
The plot shows quadratic loss-averse utility for gains (dashed line) and losses (dotted line) in the modified scenarios and the loss-averse utility in the risk-free scenario (solid line). $\hat{y}$ denotes the reference point in the risk-free scenario, $\hat{y}_g$ and $\hat{y}_l$ denote the reference point in the modified scenarios for the case of gains and losses, respectively.

In the empirical analysis we consider two geographical markets, the European and the US markets, each including different types of financial assets among which the investor may select. These assets include sectoral stock indices, government bonds and the two commodities gold and crude oil, yielding a total of 13 assets. Tables 7 and 8 in Appendix B report the summary statistics of the considered European and US financial assets. In general, the stock indices exhibit comparatively high risk and return, the government bonds show a low risk and return, and gold exhibits moderate risk and a low return while crude oil shows high risk and a moderate return. For additional information we report the overall stock market index as a benchmark portfolio. Returns are computed as $r_t = p_t/p_{t-1} - 1$, where p_t is the monthly closing price at time t. All prices are extracted from Thomson Reuters Datastream from January 1982 to December 2010. The overall stock market indices for the EMU and the US are as calculated by Datastream. The sectoral stock indices follow the Datastream classification for EMU and US stock markets and cover the following 10 sectors: oil and gas, basic materials, industrials, consumer goods, health care, consumer services, telecom, utilities, financials, and technology. We use Brent and WTI crude oil quotations for the European and US markets, respectively. Prices in the European markets are quoted in, or transformed to, Euro; prices in the US markets are quoted in US dollar, hence we consider European and US investors who completely hedge their respective currency risk.[20]

The investor is assumed to re-optimize her portfolio once a month using monthly closing prices and an optimization sample of 36 months, i.e., three years. This yields an out-of-sample evaluation period from February 1985 until December 2010. We have experimented with other, longer optimization samples, e.g., five years, but the performance of the resulting optimal QLA portfolio is generally better for shorter periods indicating that changing market conditions should be taken into immediate account.

We use different values of λ in all scenarios to allow for different degrees of loss aversion. Specifically, we let the penalty parameter be equal to $0.5, 1, 1.25, 1.5$ and 2. The value $\lambda = 1.25$ is the one estimated by Kahneman and Tversky ($\lambda = 2.25$ in their set-up) when dealing with the prospect theory (S-shaped) utility function.[21] For the European and US quadratic loss-averse investors we report optimization results for different scenarios, as described above. In particular, we present descriptive statistics including mean, standard deviation, downside volatility, CVaR, and

[20]The gold price, which is quoted in US dollar, is transformed to Euro for the European investor. Differences between the descriptive statistics of the US and the European gold price are thus entirely due to fluctuations in the USD/EUR exchange rate.

[21]It makes sense to use this value, as we compare the performance of QLA investment with that of LLA investment which is a reasonable approximation of prospect theory investment.

various risk-adjusted performance measures of the optimal quadratic loss-averse portfolio return as well as the average optimal portfolio weights. Risk-adjusted performance measures include the Sharpe and Sortino ratios and the Omega measure.[22] The downside volatility, the Sortino ratio and the Omega measure are calculated with respect to two targets, the risk-free interest rate and the overall stock market index. To be able to compare the new quadratic loss-averse portfolio optimization to more standard approaches, we also report optimization results for the MV and CVaR methods, and for the linear loss-averse (LLA) investor.

As the empirical results are very similar for the two constant scenarios, for the two dynamic scenarios and for the two break-even scenarios, respectively, we only report results for one at a time. We thus report results for the benchmark scenario, where the loss aversion parameters are constant and the reference point is equal to zero, for the dynamic scenario 2, where loss aversion increases after losses, and for the break-even scenario 2, where loss aversion decreases after losses. We first discuss the results for the EU investor.

Considering the benchmark scenario (see Table 4), we note that the optimal QLA portfolios generally display a higher expected return and a higher median, but also a higher risk (in terms of standard deviation, downside volatility with respect to the risk-free rate and conditional value-at-risk, except for the downside volatility with respect to the benchmark portfolio) than the optimal MV and CVaR portfolios. The reported risk-adjusted performance measures (Sharpe and Sortino ratios as well as the Omega measure) of most QLA portfolios are significantly larger than those of the MV or CVaR portfolios, suggesting a clear outperformance of QLA portfolios over the MV and CVaR portfolios. In addition, also the downside volatilities (with respect to the market index) of QLA portfolios are significantly smaller than those of the MV and CVaR portfolios. When comparing the performance of QLA portfolios to LLA portfolios, the risk (standard deviation and downside volatility) is significantly smaller for QLA portfolios while the return is only a bit smaller. In total, however, the reported risk-adjusted performance measures are slightly smaller for QLA portfolios. Still, QLA investment seems to be an acceptable compromise between relatively safe (but less profitable) MV and CVaR investment and relatively risky (but also more profitable) LLA investment. Over the past 10 years (2001-2010), QLA investment would have produced the highest realized returns (compared to MV, CVaR and LLA investment). For an investor with $\lambda = 1.25$,

[22] The Sortino ratio is a modified version of the Sharpe ratio which uses downside volatility with respect to a target return (instead of standard deviation) as the denominator. The Omega measure is a ratio of upside potential of portfolio return relative to its downside potential with respect to a target return (see Shadwick and Keating, 2002).

QLA investment would have led to an average yearly return of 6.95%, while MV, CVaR and LLA investment would have lead to 5.99%, 4.60% and 5.55%, respectively.

Turning to the discussion of the results for the dynamic scenario 2 (see Table 5), the main observations of QLA investment as compared to MV, CVaR and LLA investment in the benchmark scenario are still true. I.e., QLA investment is much more profitable, yet also more risky (except for the downside volatility with respect to the market index) than MV and CVaR investment, but clearly outperforms MV and CVaR investment with respect to risk-adjusted performance measures; and QLA investment is considerably less profitable, yet also less risky than LLA investment, but is slightly dominated by LLA investment with respect to risk-adjusted performance measures. Over the past 10 years (2001-2010), QLA investment would have yielded a higher realized return than MV and CVaR investment, but a slightly smaller return than LLA investment (for most values of the loss aversion parameter). For an investor with $\lambda = 1.25$, QLA investment would have lead to an average yearly return of 6.96%, while MV, CVaR and LLA investment would have led to 5.99%, 4.60% and 4.65%, respectively. Compared to the benchmark scenario, time-changing QLA investment seems to be more profitable and slightly more risky in the dynamic scenario, and it clearly outperforms constant QLA investment (benchmark scenario) in terms of the risk-adjusted performance measures. Thus it seems to be important that the investment behavior takes recent market developments into account. This is also in line with the observation that shorter optimization samples tend to yield better QLA portfolios.

Even though the investment behavior of the risk-seeking investor in the break-even scenario (loss aversion *de*creases after losses) is inherently different from that of the conservative investor in the dynamic scenario (loss aversion *in*creases after losses), the two sets of empirical results are very similar (see Table 6 for the results in the break-even scenario 2). In general, risk-seeking QLA investment (break-even scenario) seems to perform marginally worse than conservative QLA investment (dynamic scenario), in terms of return, risk and risk-adjusted performance measures. Also, the risk-seeking QLA investor (break-even scenario) would have realized a slightly lower return than the conservative QLA investor (dynamic scenario) over the past 10 years (2001-2010). For an investor with $\lambda = 1.25$, QLA investment would have led to an average yearly return of 4.09%, while MV, CVaR and LLA investment would have lead to 5.99%, 4.60% and 4.04%, respectively. Comparing different types of investors, the results are similar as in the risk-free and the dynamic scenarios: QLA investment is more profitable and more risky (except for the downside volatility with respect to the market index) than MV and CVaR investment but in total yields a higher risk-adjusted performance. In addition, QLA investment is considerably less profitable and less risky than LLA investment, and it shows a slightly lower risk-adjusted performance.

The results for the US markets are roughly similar to those for the European markets apart from two main issues. First, QLA portfolios do not outperform MV portfolios in terms of risk-adjusted performance measures. Second, risk-seeking QLA investment (break-even scenario 2) does not outperform constant QLA investment (benchmark scenario), while conservative QLA investment (dynamic scenario 2) still does. See Tables 9, 10, and 11 in Appendix B.

For both the EU and the US markets, we verify the robustness of our empirical results in the presence of transaction costs, namely 0.3% of turnover. Obviously, the absolute performance in all scenarios is reduced due to the transaction costs. Apart from that, however, the results remain qualitatively the same.

	Risk-free	Market Index	MV	CVaR	Linear loss-averse (LLA), λ 0	0.50	1.00	1.25	1.50	2.00	Quadratic loss-averse (QLA), λ 0.50	1.00	1.25	1.50	2.00
Performance of 1-month returns (in percent p.a.)															
Mean	4.15	11.78	6.49	5.40	20.09	19.11	15.33	14.43	13.02	12.95	9.91	8.82	8.61	8.42	7.93
Median	3.64	18.30	7.35	6.43	15.88	16.37	15.96	14.72	13.27	12.59	10.88	9.84	9.33	9.45	9.02
Std.Dev.	0.68	17.46	5.07	6.13	29.56	22.26	19.30	17.91	16.63	15.28	11.98	9.41	8.85	8.49	7.94
Std.Dev.riskfree		11.01	2.97	3.83	16.12	11.88	10.60	9.59	9.10	8.36	7.29	5.97	5.62	5.40	5.09
Std.Dev.market index	9.86		9.24	9.12	15.50	10.21	9.03	8.18	7.96	7.79	7.91	7.90	8.01	8.10	8.24
CVaR	0.10	-79.39	-28.31	-36.13	-90.61	-80.84	-77.30	-73.65	-71.75	-67.59	-59.97	-50.14	-47.89	-46.41	-44.30
Minimum	0.00	-93.99	-43.53	-58.80	-97.67	-96.20	-94.97	-94.97	-94.91	-94.53	-94.59	-92.85	-91.59	-90.64	-89.02
Sharpe's (in percent)		41.88	43.73	19.43	51.80	64.48	55.62	55.05	51.15	55.21	45.98	47.45	48.09	47.99	45.48
Sortino (risk-free)		66.69	75.70	31.39	95.24	121.30	101.60	103.27	93.90	101.40	76.04	75.37	76.28	75.98	71.53
Sortino (market index)	-69.80		-51.68	-63.08	48.44	64.83	35.60	29.28	14.16	13.64	-21.33	-33.76	-35.76	-37.44	-42.12
Omega (risk-free)		11.26	4.13	2.05	36.20	64.59	36.41	33.81	25.19	29.93	11.55	9.48	9.07	8.59	7.18
Omega (market index)	0.09		0.17	0.11	7.33	8.99	3.33	2.53	1.57	1.53	0.50	0.34	0.32	0.30	0.25
Total realized return (in percent p.a.)															
Last 10 Years	0.22	0.68	5.99	4.60	2.91	5.49	3.28	5.55	6.50	8.93	7.06	6.85	6.95	6.94	6.66
Last 5 Years	0.21	0.19	4.49	2.92	8.79	12.43	7.40	9.39	10.99	12.74	9.75	8.27	8.30	8.12	7.39
Last 3 Years	0.15	-8.74	5.93	3.67	6.60	5.20	-2.62	-1.21	-0.96	1.29	-0.03	1.53	1.81	1.87	2.29
Last Year	0.04	5.60	7.60	5.77	36.61	37.60	10.97	11.57	10.40	11.52	10.21	7.97	7.43	7.03	6.79
Percentiles															
5	0.39	-63.87	-21.70	-25.43	-79.18	-66.70	-58.53	-58.43	-55.35	-51.03	-45.05	-35.47	-33.81	-32.29	-30.01
10	0.93	-43.94	-15.30	-18.43	-61.00	-50.07	-42.01	-39.17	-38.50	-35.26	-29.94	-25.10	-23.10	-22.35	-20.70
25	2.64	-19.11	-3.26	-6.36	-25.68	-18.91	-13.79	-12.07	-10.55	-7.37	-9.62	-7.56	-6.11	-5.96	-6.12
50	3.64	18.30	7.35	6.43	15.88	16.37	15.96	14.72	13.27	12.59	10.88	9.84	9.33	9.45	9.02
75	4.99	58.00	17.36	21.20	99.36	64.74	47.88	42.91	38.74	39.01	33.50	27.00	26.36	25.63	23.42
90	8.33	107.03	30.19	32.92	252.69	160.83	116.78	99.83	89.21	81.37	69.23	50.99	50.07	47.79	45.66
95	9.13	153.49	40.26	46.34	431.78	283.52	190.21	190.21	136.90	124.76	95.47	77.40	74.11	69.70	65.71
Mean allocation (in percent)															
OIL			0.70	3.65	7.40	8.44	7.80	6.24	5.04	4.30	3.43	3.18	3.17	3.19	3.13
BASICMAT			1.52	1.06	5.47	5.86	4.78	3.86	3.27	2.87	3.11	2.57	2.12	1.84	1.46
INDUS			0.62	0.14	1.93	0.83	0.12	0.15	0.27	0.44	0.21	0.18	0.19	0.21	0.24
CONSGDS			1.14	1.46	0.32	0.98	1.20	1.14	1.22	1.18	0.61	0.47	0.42	0.39	0.37
HEALTH			3.64	3.99	3.54	3.87	5.81	5.82	5.48	5.21	3.29	3.46	3.49	3.44	3.80
CONSSVS			0.91	0.49	0.00	0.36	0.54	0.73	1.09	1.56	2.01	2.14	2.25	2.35	2.40
TELE			0.54	1.79	5.79	4.88	5.49	6.28	6.62	6.44	5.17	3.51	3.17	2.90	2.59
UTIL			3.23	6.21	4.82	13.20	18.22	19.40	19.64	18.56	17.88	14.41	13.36	12.53	10.86
FIN			0.20	1.91	0.64	2.25	3.23	3.42	3.50	3.45	2.54	1.15	0.92	0.82	0.83
TECH			0.45	0.67	36.66	31.14	21.51	17.77	13.77	9.52	7.09	4.39	3.82	3.43	2.87
BOND			74.52	63.91	4.82	12.85	21.74	26.55	31.65	37.93	44.03	52.11	53.99	55.37	57.34
GOLD			9.61	10.85	9.00	8.15	5.21	4.61	4.57	4.78	6.42	8.77	9.57	10.06	10.68
CRUDEOIL			2.92	3.86	19.61	7.19	4.34	4.01	3.87	3.75	4.21	3.65	3.54	3.49	3.44

Table 4: Out-of-sample evaluation of EU portfolios: Benchmark scenario

The table reports statistics of a monthly reallocated optimal linear loss-averse portfolio based on an optimization period of 36 months as well as the average of the optimal asset weights. The benchmark scenario assumes a constant loss-averse parameter λ and a zero reference point. The evaluation period covers February 1985 to December 2010. Statistics are calculated on the basis of monthly returns and then annualized assuming discrete compounding. The annual standard deviation is computed as $\sigma_{pa} = \sqrt{12}\sigma_{pm}$.

	Risk-free	Market Index	MV	CVaR	Linear loss-averse (LLA), λ 0	0.50	1.00	1.25	1.50	2.00	Quadratic loss-averse (QLA), λ 0.50	1.00	1.25	1.50	2.00
Performance of 1-month returns (in percent p.a.)															
Mean	4.15	11.78	6.49	5.40	20.09	19.17	20.67	15.56	17.06	15.46	9.74	9.92	9.56	9.16	8.96
Median	3.64	18.30	7.35	6.43	15.88	19.75	18.33	14.95	15.51	14.82	11.17	9.56	9.35	9.54	9.21
Std.Dev.	0.68	17.46	5.07	6.13	29.56	19.72	19.55	19.09	18.37	17.13	11.27	9.59	9.25	9.18	8.45
Std.Dev.riskfree		11.01	2.97	3.83	16.12	10.36	9.72	10.18	9.47	9.02	6.96	5.70	5.54	5.63	4.91
Std.Dev.market index	9.86		9.24	9.12	15.50	8.15	8.70	8.72	7.84	8.11	7.89	7.92	8.03	8.05	8.21
CVaR	0.10	-79.39	-28.31	-36.13	-90.61	-75.29	-73.56	-75.85	-72.29	-71.21	-57.91	-48.35	-48.01	-48.56	-42.53
Minimum	0.00	-93.99	-43.53	-58.80	-97.67	-95.19	-95.04	-94.97	-94.88	-94.57	-94.49	-92.48	-91.27	-90.34	-88.64
Sharpe's (in percent)		41.88	43.73	19.43	51.80	73.04	81.19	57.38	67.40	63.34	47.42	57.51	55.97	52.18	54.40
Sortino (risk-free)		66.69	75.70	31.39	95.24	139.66	163.78	107.99	131.26	120.84	77.32	97.38	94.12	85.77	94.44
Sortino (market index)	-69.80		-51.68	-63.08	48.44	81.93	92.41	39.25	60.83	41.05	-23.33	-21.20	-24.86	-29.31	-30.94
Omega (risk-free)		11.26	4.13	2.05	36.20	94.35	159.45	39.81	64.87	50.49	11.60	16.10	14.33	12.09	12.11
Omega (market index)	0.09		0.17	0.11	7.33	14.15	21.31	3.87	6.58	3.70	0.47	0.51	0.45	0.39	0.37
Total realized return (in percent p.a.)															
Last 10 Years	0.22	0.68	5.99	4.60	2.91	10.55	9.77	4.65	8.40	7.91	7.36	6.99	6.96	7.58	6.45
Last 5 Years	0.21	0.19	4.49	2.92	8.79	10.81	11.41	13.99	14.36	11.46	10.17	8.34	7.77	8.35	6.58
Last 3 Years	0.15	-8.74	5.93	3.67	6.60	7.39	3.34	8.16	8.63	0.57	0.53	1.36	0.53	2.30	2.78
Last Year	0.04	5.60	7.60	5.77	36.61	20.55	27.69	17.04	33.68	23.87	9.89	7.95	7.58	7.31	8.18
Percentiles															
5	0.39	-63.87	-21.70	-25.43	-79.18	-59.86	-58.51	-59.40	-58.55	-54.92	-42.89	-33.18	-32.51	-30.67	-27.60
10	0.93	-43.94	-15.30	-18.43	-61.00	-44.08	-42.66	-45.01	-42.10	-36.57	-28.29	-22.86	-21.15	-21.14	-20.33
25	2.64	-19.11	-3.26	-6.36	-25.68	-16.94	-12.54	-14.62	-15.20	-12.75	-9.18	-6.54	-5.41	-5.51	-5.17
50	3.64	18.30	7.35	6.43	15.88	19.75	18.33	14.95	15.51	14.82	11.17	9.56	9.35	9.54	9.21
75	4.99	58.00	17.36	21.20	99.36	61.07	63.24	49.41	51.70	45.85	33.64	26.50	25.59	24.87	23.15
90	8.33	107.03	30.19	32.92	252.69	123.97	115.18	107.15	124.88	89.78	62.13	52.98	52.72	50.71	44.16
95	9.13	153.49	40.26	46.34	431.78	227.74	241.84	160.28	194.39	158.64	86.08	76.14	70.69	68.06	64.95
Mean allocation (in percent)															
OIL			0.70	3.65	7.40	7.66	7.05	7.33	7.07	6.11	3.15	2.91	3.14	2.97	2.97
BASICMAT			1.52	1.06	5.47	7.17	5.77	6.27	5.74	4.88	3.14	2.59	2.19	1.85	1.42
INDUS			0.62	0.14	1.93	4.33	2.84	2.42	2.09	1.66	0.22	0.21	0.20	0.21	0.25
CONSGDS			1.14	1.46	0.32	3.86	3.00	2.78	2.35	2.28	0.55	0.42	0.45	0.37	0.37
HEALTH			3.64	3.99	3.54	5.72	6.56	6.39	6.67	7.16	3.33	3.77	3.70	3.96	4.28
CONSSVS			0.91	0.49	0.00	3.65	2.70	2.59	2.55	2.16	2.12	1.93	2.07	2.17	2.16
TELE			0.54	1.79	5.79	6.52	7.08	7.39	7.04	6.57	4.87	3.38	2.93	2.95	2.38
UTIL			3.23	6.21	4.82	9.16	11.95	13.21	14.40	15.85	17.19	13.58	12.60	11.76	10.26
FIN			0.20	1.91	0.64	4.15	3.77	3.64	3.82	3.43	2.26	0.93	0.75	0.72	0.74
TECH			0.45	0.67	36.66	22.03	20.03	19.46	17.60	15.50	6.04	4.30	3.87	3.44	3.34
BOND			74.52	63.91	4.82	9.43	12.52	14.97	17.00	22.73	47.11	54.09	55.71	56.87	59.01
GOLD			9.61	10.85	9.00	8.15	7.36	6.35	6.43	5.30	6.21	8.42	8.90	9.32	9.55
CRUDEOIL			2.92	3.86	19.61	8.55	9.37	7.20	7.24	6.38	3.82	3.46	3.50	3.42	3.27

Table 5: Out-of-sample evaluation of EU portfolios: Dynamic scenario 2

The table reports statistics of a monthly reallocated optimal linear loss-averse portfolio based on an optimization period of 36 months as well as the average of the optimal asset weights. The dynamic scenario 2 assumes a smaller λ and a reference point equal to the risk-free rate for prior gains, and a constant λ and a higher reference point for prior losses. The evaluation period covers February 1985 to December 2010. Statistics are calculated on the basis of monthly returns and then annualized assuming discrete compounding. The annual standard deviation is computed as $\sigma_{pa} = \sqrt{12}\sigma_{pm}$.

	Risk-free	Market Index	MV	CVaR	Linear loss-averse (LLA), λ 0	0.50	1.00	1.25	1.50	2.00	Quadratic loss-averse (QLA), λ 0.50	1.00	1.25	1.50	2.00
Performance of 1-month returns (in percent p.a.)															
Mean	4.15	11.78	6.49	5.40	20.09	17.68	17.76	15.78	16.37	15.10	9.60	9.56	8.77	8.25	8.37
Median	3.64	18.30	7.35	6.43	15.88	17.34	16.13	16.71	16.80	12.99	11.35	9.78	9.93	9.56	9.24
Std.Dev.	0.68	17.46	5.07	6.13	29.56	19.11	19.43	19.24	19.22	17.67	11.77	9.91	9.85	10.12	9.07
Std.Dev.riskfree		11.01	2.97	3.83	16.12	10.44	10.82	11.07	10.33	9.30	7.23	5.93	6.45	6.83	5.71
Std.Dev.market index	9.86		9.24	9.12	15.50	10.07	8.73	8.95	8.23	8.56	8.11	7.97	8.09	8.11	8.17
CVaR	0.10	-79.39	-28.31	-36.13	-90.61	-77.28	-77.84	-78.77	-75.70	-70.92	-58.68	-50.33	-53.91	-55.09	-49.72
Minimum	0.00	-93.99	-43.53	-58.80	-97.67	-95.19	-97.34	-97.22	-94.37	-93.90	-94.48	-91.60	-89.68	-96.20	-90.64
Sharpe's (in percent)		41.88	43.73	19.43	51.80	67.85	67.15	57.96	61.01	59.44	44.29	52.26	44.95	38.84	44.61
Sortino (risk-free)		66.69	75.70	31.39	95.24	124.86	121.19	101.24	113.96	113.33	72.49	87.81	68.94	57.86	71.13
Sortino (market index)	-69.80		-51.68	-63.08	48.44	52.95	61.96	40.41	50.42	35.01	-24.27	-25.15	-33.52	-39.23	-37.62
Omega (risk-free)		11.26	4.13	2.05	36.20	66.89	71.08	40.14	50.78	41.83	9.94	12.56	9.00	7.25	8.25
Omega (market index)	0.09		0.17	0.11	7.33	8.48	8.46	4.05	5.03	3.27	0.45	0.45	0.34	0.28	0.29
Total realized return (in percent p.a.)															
Last 10 Years	0.22	0.68	5.99	4.60	2.91	8.73	7.94	4.04	4.28	4.41	6.06	6.52	4.09	4.43	4.46
Last 5 Years	0.21	0.19	4.49	2.92	8.79	13.22	11.73	7.65	2.37	7.34	9.55	8.18	7.14	6.11	4.71
Last 3 Years	0.15	-8.74	5.93	3.67	6.60	3.62	5.26	-2.31	-5.94	-4.19	0.05	0.12	0.06	-0.97	-0.35
Last Year	0.04	5.60	7.60	5.77	36.61	23.08	7.76	10.28	2.99	12.87	10.34	8.23	7.39	7.16	6.82
Percentiles															
5	0.39	-63.87	-21.70	-25.43	-79.18	-56.48	-59.28	-58.52	-57.05	-57.60	-44.77	-36.40	-35.86	-33.96	-31.36
10	0.93	-43.94	-15.30	-18.43	-61.00	-42.13	-39.53	-44.73	-44.60	-44.34	-32.98	-24.80	-21.51	-21.41	-22.48
25	2.64	-19.11	-3.26	-6.36	-25.68	-14.61	-12.56	-13.43	-13.67	-12.62	-9.88	-7.92	-6.76	-6.36	-6.30
50	3.64	18.30	7.35	6.43	15.88	17.34	16.13	16.71	16.80	12.99	11.35	9.78	9.93	9.56	9.24
75	4.99	58.00	17.36	21.20	99.36	61.15	57.36	52.82	52.58	43.10	34.28	29.20	28.25	26.76	24.78
90	8.33	107.03	30.19	32.92	252.69	112.41	115.25	121.30	113.91	99.73	71.93	53.27	53.80	50.36	46.34
95	9.13	153.49	40.26	46.34	431.78	204.05	252.49	226.02	179.91	169.12	92.54	83.16	79.76	70.29	67.53
Mean allocation (in percent)															
OIL			0.70	3.65	7.40	7.03	7.24	6.80	7.82	5.90	3.53	3.50	3.20	3.02	3.00
BASICMAT			1.52	1.06	5.47	5.90	5.54	5.76	4.85	3.71	3.57	2.85	2.20	1.97	1.65
INDUS			0.62	0.14	1.93	4.25	2.59	2.14	1.73	1.58	0.26	0.19	0.16	0.29	0.29
CONSGDS			1.14	1.46	0.32	4.00	3.38	2.90	2.52	2.15	0.53	0.36	0.35	0.50	0.50
HEALTH			3.64	3.99	3.54	5.63	6.20	5.66	5.43	6.19	3.31	3.81	3.85	3.52	3.95
CONSSVS			0.91	0.49	0.00	3.29	2.45	2.38	1.88	2.02	1.79	2.12	2.17	2.39	2.46
TELE			0.54	1.79	5.79	7.33	6.30	7.06	5.92	6.36	5.71	3.92	3.46	3.44	2.72
UTIL			3.23	6.21	4.82	10.55	10.35	11.26	13.69	15.46	17.48	14.17	13.41	12.66	11.04
FIN			0.20	1.91	0.64	4.15	3.26	3.16	3.02	3.21	2.47	1.16	0.91	0.85	0.85
TECH			0.45	0.67	36.66	17.47	20.32	18.56	18.00	16.32	7.63	4.73	4.42	4.02	3.70
BOND			74.52	63.91	4.82	13.67	17.55	19.40	21.92	25.97	43.22	51.22	53.00	54.25	56.77
GOLD			9.61	10.85	9.00	7.05	6.47	7.35	6.14	5.89	5.93	8.22	8.81	9.24	9.52
CRUDEOIL			2.92	3.86	19.61	9.68	8.03	7.57	7.08	5.24	4.58	3.73	4.05	3.86	3.54

Table 6: Out-of-sample evaluation of EU portfolios: Break-even scenario 2

The table reports statistics of a monthly reallocated optimal linear loss-averse portfolio based on an optimization period of 36 months as well as the average of the optimal asset weights. The break even scenario assumes a risk-seeking behavior after prior losses. The evaluation period covers February 1985 to December 2010. Statistics are calculated on the basis of monthly returns and then annualized assuming discrete compounding. The annual standard deviation is computed as $\sigma_{pa} = \sqrt{12}\sigma_{pm}$.

4 Conclusion

A large body of experimental evidence suggests that loss aversion plays an important role in the asset allocation decision. In this paper we investigate the quadratic loss-averse utility maximization problem along different dimensions. First we examine the theoretical relationship between the optimal asset allocation under quadratic loss aversion and more traditional asset allocation methods; i.e., the MV and CVaR methods. We formulate assumptions under which the QLA, MV and CVaR problems are equivalent, provided that portfolio returns are normally distributed. Then we investigate the two-asset case, involving one risky and one risk-free asset, and analytically derive the optimal risky asset's weight, under the assumption of binomially and (generally) continuously distributed returns of the risky asset. We consider both the zero excess reference point case, where the reference point is equal to the risk-free rate, and the non-zero excess reference point case, where the reference point is different from the risk-free rate. One reason for investigating the zero excess reference point case is that the risk-free rate seems to be a natural candidate for the reference point and it is also mostly used in the literature; another reason may be that the corresponding analysis is more straightforward and analytical solutions can typically be provided in an explicit form. In both cases, the optimal QLA investment in the risky asset is always finite and strictly positive. This is different from investment under linear loss aversion, where, first, the investor would invest an infinite amount in the risky asset if she displayed a low degree of loss aversion (small penalty parameter) and, second, she would completely stay out of the market for a zero excess reference point. We find that under QLA the minimum risk allocation with respect to the reference point is attained for some value strictly larger than the risk-free rate, while under LLA the portfolio risk is minimal (actually zero) for the zero excess reference point.

Then we implement the trading strategy of a quadratic loss-averse investor (as well as of a linear loss-averse investor) who reallocates her portfolio on a monthly basis in the period 1985 to 2010. In addition to the benchmark QLA scenario, which uses a constant penalty parameter and a constant reference point, we introduce time-changing QLA scenarios, where we update both the penalty parameter and the reference point conditional on previous gains and losses. The considered trading strategies/scenarios are either conservative, where loss aversion *in*creases after losses, or risk-seeking, where loss aversion *de*creases after losses. The assets available for portfolio selection include sectoral stock indices, government bonds as well as the two commodities gold and crude oil, yielding a total of 13 assets, and we consider a European and a US investor.

Our empirical results suggest that – independent of the loss aversion parameter's value – the optimal QLA portfolio outperforms the optimal MV and CVaR portfolios, when we use the Sharpe ratio, the Sortino ratio or the Omega measure as performance measures. Among the different QLA scenarios, the conservative time-changing method achieves the highest performance measures, which indicates that investors reacting to changing market conditions perform better than investors behaving the same all the time. In this context, it seems to be important, however, in which form investors update their investment strategy. *In*creasing loss aversion after losses (conservative QLA investment) usually seems to be a wiser choice than *de*creasing loss aversion after losses (risk-seeking QLA investment).

When comparing QLA and LLA portfolios, we find that the risk (standard deviation and downside volatility) is significantly smaller for QLA portfolios while the return is only a bit smaller. In total, however, risk-adjusted performance measures are slightly smaller for QLA portfolios. Still, QLA investment seems to be an acceptable compromise between relatively safe (but less profitable) MV and CVaR investment and relatively risky (but also more profitable) LLA investment.

An interesting topic for further research would be to consider the S-shaped form of loss-averse (prospect theory type) utility, and to investigate the properties and performance of the corresponding optimal portfolios with respect to the loss aversion parameter and the reference point. Another interesting topic would be to examine more closely the effect of investment under (quadratic) loss aversion for different market climates, e.g., to answer the question whether QLA investment performs fundamentally different in bearish and bullish markets.

Appendix A

Proof of Theorem 2.3. We show that $\mathrm{QLA}_{\lambda,\hat{y}}(R(x))$ is (a) increasing in $I_1 \equiv \left(-\infty, \min\left\{\frac{\hat{y}-r^0}{r_g-r^0}, \frac{r^0-\hat{y}}{r^0-r_b}\right\}\right]$; (b) increasing in $I_2 \equiv \left[\min\left\{\frac{\hat{y}-r^0}{r_g-r^0}, \frac{r^0-\hat{y}}{r^0-r_b}\right\}, \max\left\{\frac{\hat{y}-r^0}{r_g-r^0}, \frac{r^0-\hat{y}}{r^0-r_b}\right\}\right]$ for $\hat{y} < r^0$ and increasing in I_2 also for $\hat{y} > r^0$ and $\lambda \leq \hat{\lambda}$, but having a global maximum $x^* = \frac{\left(\frac{1}{2\lambda}+\hat{y}-r^0\right)\mathbb{E}(r-r^0)}{\mathbb{E}(r-r^0)^2} > 0$ in I_2 for $\hat{y} > r^0$ and $\lambda > \hat{\lambda}$; (c) having a global maximum $x^* = \frac{r^0-\hat{y}}{r^0-r_b} + \frac{\mathbb{E}(r-r^0)}{2\lambda(1-p)(r^0-r_b)^2} > 0$ in $I_3 \equiv \left[\max\left\{\frac{\hat{y}-r^0}{r_g-r^0}, \frac{r^0-\hat{y}}{r^0-r_b}\right\}, +\infty\right)$ for $\hat{y} \leq r^0$ and also for $\hat{y} > r^0$ and $\lambda \leq \hat{\lambda}$ but decreasing in I_3 for $\hat{y} > r^0$ and $\lambda > \hat{\lambda}$. The statement of the theorem then follows from (a)-(c).

It follows from (2.11), (2.12) and (2.13) that for $x \in I_1$

$$\mathrm{QLA}_{\lambda,\hat{y}}(R(x)) = r^0 + \mathbb{E}(r-r^0)x - \lambda p\left[\hat{y} - r^0 - (r_g - r^0)x\right]^2$$

thus

$$\frac{d\mathrm{QLA}_{\lambda,\hat{y}}(R(x))}{dx} = \mathbb{E}(r-r^0) + 2\lambda p\left[\hat{y} - r^0 - (r_g - r^0)x\right](r_g - r^0) > 0$$

as $x \leq \frac{\hat{y}-r^0}{r_g-r^0}$, $\mathbb{E}(r-r^0) > 0$, $\lambda > 0$ and $\mathrm{QLA}_{\lambda,\hat{y}}(R(x))$ is thus increasing in I_1. This finishes the proof of part (a).

For $\hat{y} < r^0$ is $I_2 = \left[\frac{\hat{y}-r^0}{r_g-r^0}, \frac{r^0-\hat{y}}{r^0-r_b}\right]$ and thus for $x \in I_2$, is $\mathrm{QLA}_{\lambda,\hat{y}}(R(x)) = r^0 + \mathbb{E}(r-r^0)x$, which implies that $\frac{d\mathrm{QLA}_{\lambda,\hat{y}}(R(x))}{dx} = \mathbb{E}(r-r^0) > 0$ and thus $\mathrm{QLA}_{\lambda,\hat{y}}(R(x))$ is increasing in I_2.

For $\hat{y} > r^0$ is $I_2 = \left[\frac{r^0-\hat{y}}{r^0-r_b}, \frac{\hat{y}-r^0}{r_g-r^0}\right]$ and thus for $x \in I_2$

$$\mathrm{QLA}_{\lambda,\hat{y}}(R(x)) = r^0 + \mathbb{E}(r-r^0)x - \lambda\left[p\left(\hat{y} - r^0 - (r_g - r^0)x\right)^2 + (1-p)\left(\hat{y} - r^0 - (r_b - r^0)x\right)^2\right]$$

implying

$$\begin{aligned}
\frac{d\mathrm{QLA}_{\lambda,\hat{y}}(R(x))}{dx} &= \\
= \mathbb{E}(r-r^0) &+ 2\lambda\left[p\left(\hat{y} - r^0 - (r_g - r^0)x\right)(r_g - r^0) - (1-p)\left(\hat{y} - r^0 + (r^0 - r_b)x\right)(r^0 - r_b)\right] \\
= \mathbb{E}(r-r^0) &+ 2\lambda\left[p(r_g - r^0) - (1-p)(r^0 - r_b)\right](\hat{y} - r^0) - 2\lambda\left[p\left(r_g - r^0\right)^2 + (1-p)\left(r^0 - r_b\right)^2\right]x \\
&= [1 + 2\lambda(\hat{y} - r^0)]\mathbb{E}(r-r^0) - 2\lambda\mathbb{E}\left(\left(r - r^0\right)^2\right)x
\end{aligned}$$

$\mathrm{QLA}_{\lambda,\hat{y}}(R(x))$ is concave in I_2 for $\hat{y} > r^0$ as

$$\frac{d^2\mathrm{QLA}_{\lambda,\hat{y}}(R(x))}{dx^2} = -2\lambda\mathbb{E}\left(\left(r-r^0\right)^2\right) < 0$$

As $\frac{d\mathrm{QLA}_{\lambda,\hat{y}}(R(x))}{dx} = 0$ for $x = x_1^* \equiv \frac{\left(\frac{1}{2\lambda}+\hat{y}-r^0\right)\mathbb{E}(r-r^0)}{\mathbb{E}((r-r^0)^2)} > 0$, then based on this and concavity $\mathrm{QLA}_{\lambda,\hat{y}}(R(x))$ reaches its global maximum in I_2 at x_1^* if $x_1^* < \frac{\hat{y}-r^0}{r_g-r^0}$. However, on the other hand, if $x_1^* \geq \frac{\hat{y}-r^0}{r_g-r^0}$ then $\mathrm{QLA}_{\lambda,\hat{y}}(R(x))$ reaches its maximum in I_2 for $x = \frac{\hat{y}-r^0}{r_g-r^0}$. Note that

$$x_1^* \geq \frac{\hat{y}-r^0}{r_g-r^0} \Leftrightarrow -\left(r_g-r^0\right)\mathbb{E}\left(r-r^0\right) \leq 2\lambda\left(\hat{y}-r^0\right)\left[\left(r_g-r^0\right)\mathbb{E}\left(r-r^0\right)-\mathbb{E}\left(\left(r-r^0\right)^2\right)\right] \tag{4.28}$$

As

$$\left(r_g-r^0\right)\mathbb{E}\left(r-r^0\right)-\mathbb{E}\left(\left(r-r^0\right)^2\right) = -(1-p)(r_g-r_b)\left(r^0-r_b\right)$$

then based on this (4.28) boils down to

$$x_1^* \geq \frac{\hat{y}-r^0}{r_g-r^0} \Leftrightarrow \lambda \leq \hat{\lambda} \tag{4.29}$$

Based on concavity of $\mathrm{QLA}_{\lambda,\hat{y}}(R(x))$ in I_2 when $r^0 < \hat{y}$ and (4.29) it follows that $\mathrm{QLA}_{\lambda,\hat{y}}(R(x))$ is increasing in I_2 if $\lambda \leq \hat{\lambda}$ and it has a global maximum in I_2 if $\lambda > \hat{\lambda}$ (as then $x_1^* < \frac{\hat{y}-r^0}{r_g-r^0}$). This finishes the proof of part (b).

For $x \in I_3$ is

$$\mathrm{QLA}_{\lambda,\hat{y}}(R(x)) = r^0 + \mathbb{E}\left(r-r^0\right)x - \lambda(1-p)\left[\hat{y}-r^0+\left(r^0-r_b\right)x\right]^2$$

and thus

$$\begin{aligned}
\frac{d\mathrm{QLA}_{\lambda,\hat{y}}(R(x))}{dx} &= \mathbb{E}\left(r-r^0\right) - 2\lambda(1-p)\left[\hat{y}-r^0+\left(r^0-r_b\right)x\right]\left(r^0-r_b\right) \\
\frac{d^2\mathrm{QLA}_{\lambda,\hat{y}}(R(x))}{dx^2} &= -2\lambda(1-p)\left(r^0-r_b\right)^2 < 0
\end{aligned}$$

which implies the concavity for $\mathrm{QLA}_{\lambda,\hat{y}}(R(x))$ in I_3. As $\frac{d\mathrm{QLA}_{\lambda,\hat{y}}(R(x))}{dx} = 0$ for $x = x_2^* \equiv \frac{r^0-\hat{y}}{r^0-r_b} + \frac{\mathbb{E}(r-r^0)}{2\lambda(1-p)(r^0-r_b)^2}$ then based on concavity, $\mathrm{QLA}_{\lambda,\hat{y}}(R(x))$ reaches its global maximum in I_3 at x_2^*

if $x_2^* > \max\left\{\frac{\hat{y}-r^0}{r_g-r^0}, \frac{r^0-\hat{y}}{r^0-r_b}\right\}$. On the other hand, if $x_2^* \leq \max\left\{\frac{\hat{y}-r^0}{r_g-r^0}, \frac{r^0-\hat{y}}{r^0-r_b}\right\}$ then $\mathrm{QLA}_{\lambda,\hat{y}}(R(x))$ reaches its glomal maximum in I_3 for $x = \max\left\{\frac{\hat{y}-r^0}{r_g-r^0}, \frac{r^0-\hat{y}}{r^0-r_b}\right\}$. Note that for $\hat{y} \leq r^0$ is $x_2^* > \frac{r^0-\hat{y}}{r^0-r_b} = \max\left\{\frac{\hat{y}-r^0}{r_g-r^0}, \frac{r^0-\hat{y}}{r^0-r_b}\right\}$ and thus in this case the global maximum in I_3 is reached for x_2^*. Finally, as in case (b), it can be shown that for $\hat{y} > r^0$ $\mathrm{QLA}_{\lambda,\hat{y}}(R(x))$ reaches its global maximum in I_3; i.e., $x_2^* > \frac{\hat{y}-r^0}{r_g-r^0}$, if $\lambda \leq \hat{\lambda}$ and is decreasing in I_3; i.e., $x_2^* \leq \frac{\hat{y}-r^0}{r_g-r^0}$, if $\lambda > \hat{\lambda}$. This finishes the proof of part (c) and thus of the whole theorem. $\square$

Proof of Theorem 2.4. The expected loss-averse utility (2.18) is continuous as

$$\lim_{x\to 0^+} \mathrm{QLA}_{\lambda,\hat{y}}(R(x)) = \lim_{x\to 0^-} \mathrm{QLA}_{\lambda,\hat{y}}(R(x)) = r^0 - \lambda\left([\hat{y}-r^0]^+\right)^2 \tag{4.30}$$

The derivative of $\mathrm{QLA}_{\lambda,\hat{y}}(R(x))$ with respect to x is

$$\frac{d}{dx}\mathrm{QLA}_{\lambda,\hat{y}}(R(x)) = \left\{\begin{array}{ll} \mu - r^0 + 2\lambda \int_{\frac{\hat{y}-r^0}{x}+r^0}^{\infty} \left(\hat{y}-r^0-(r-r^0)x\right)(r-r^0)f_r(r)dr, & x<0 \\ \mu - r^0 + 2\lambda \int_{-\infty}^{\frac{\hat{y}-r^0}{x}+r^0} \left(\hat{y}-r^0-(r-r^0)x\right)(r-r^0)f_r(r)dr, & x>0 \end{array}\right\} \tag{4.31}$$

Thus, the expected loss-averse utility function $\mathrm{QLA}_{\lambda,\hat{y}}(R(x))$ is increasing for $x<0$, $\mu - r^0 > 0$, and $\hat{y} > r^0$ since

$$\begin{aligned} \frac{d}{dx}\mathrm{QLA}_{\lambda,\hat{y}}(R(x)) &= \mu - r^0 + 2\lambda \int_{\frac{\hat{y}-r^0}{x}+r^0}^{\infty} \left(\hat{y}-r^0-(r-r^0)x\right)(r-r^0)f_r(r)dr \\ &\geq \mu - r^0 + 2\lambda(\hat{y}-r^0)\int_{-\infty}^{\infty}(r-r^0)f_r(r)dr + 2\lambda(-x)\int_{\frac{\hat{y}-r^0}{x}+r^0}^{\infty}(r-r^0)^2 f_r(r)dr \\ &= (1+2\lambda(\hat{y}-r^0))(\mu-r^0) + 2\lambda(-x)\int_{\frac{\hat{y}-r^0}{x}+r^0}^{\infty}(r-r^0)^2 f_r(r)dr \\ &> 0 \end{aligned} \tag{4.32}$$

The expected loss-averse utility function $\mathrm{QLA}_{\lambda,\hat{y}}(R(x))$ is also increasing for $x<0$, $\mu - r^0 > 0$, and $\hat{y} < r^0$ since

$$\begin{aligned} \frac{d}{dx}\mathrm{QLA}_{\lambda,\hat{y}}(R(x)) &= \mu - r^0 + 2\lambda \int_{\frac{\hat{y}-r^0}{x}+r^0}^{\infty} \left(\hat{y}-r^0-(r-r^0)x\right)(r-r^0)f_r(r)dr \\ &> 0 \end{aligned} \tag{4.33}$$

as $\hat{y} - r^0 - (r-r^0)x \geq 0$ and $r - r^0 \geq 0$ for $r \geq \frac{\hat{y}-r^0}{x} + r^0$.

Finally, the expected loss-averse utility function $\mathrm{QLA}_{\lambda,\hat{y}}(R(x))$ is also increasing for $x < 0$, $\mu - r^0 > 0$, and $\hat{y} = r^0$ since

$$\begin{aligned} \frac{d}{dx}\mathrm{QLA}_{\lambda,\hat{y}}(R(x)) &= \mu - r^0 - 2\lambda x \int_{r^0}^{\infty} (r - r^0)^2 f_r(r) dr \\ &> 0 \end{aligned} \tag{4.34}$$

as $x < 0$ and the integrand is non-negative for each r.

For $x > 0$ it holds that

$$\begin{aligned} \lim_{x\to 0^+} \frac{d}{dx}\mathrm{QLA}_{\lambda,\hat{y}}(R(x)) &= \mu - r^0 + 2\lambda \int_{-\infty}^{\infty} (\hat{y} - r^0)(r - r^0) f_r(r) dr \\ &= \mu - r^0 + 2\lambda(\hat{y} - r^0)(\mu - r^0) \\ &= \left(1 + 2\lambda(\hat{y} - r^0)\right)(\mu - r^0) \\ &> 0, \qquad \text{if } \ \mu - r^0 > 0, \ \hat{y} > r^0, \ \lambda > 0 \qquad (4.35) \\ \lim_{x\to 0^+} \frac{d}{dx}\mathrm{QLA}_{\lambda,\hat{y}}(R(x)) &= \mu - r^0 + 2\lambda \int_{-\infty}^{-\infty} (\hat{y} - r^0)(r - r^0) f_r(r) dr \\ &= \mu - r^0 \\ &> 0, \qquad \text{for } \ \mu - r^0 > 0, \ \hat{y} < r^0 \qquad (4.36) \\ \lim_{x\to 0^+} \frac{d}{dx}\mathrm{QLA}_{\lambda,\hat{y}}(R(x)) &= \mu - r^0 - 2\lambda \int_{r^0}^{\infty} (r - r^0)^2 f_r(r) dr \times \lim_{x\to 0^+} x \\ &= \mu - r^0 \\ &> 0, \qquad \text{for } \ \mu - r^0 > 0, \ \hat{y} = r^0 \qquad (4.37) \end{aligned}$$

and

$$\begin{aligned} &\lim_{x\to +\infty} \frac{d}{dx}\mathrm{QLA}_{\lambda,\hat{y}}(R(x)) \\ &= \mu - r^0 + 2\lambda \left((\hat{y} - r^0) \int_{-\infty}^{r^0} (r - r^0) f_r(r) dr - \int_{-\infty}^{r^0} (r - r^0)^2 f_r(r) dr \times \lim_{x\to +\infty} x \right) \\ &= -\infty < 0, \qquad \text{if } \lambda > 0 \end{aligned} \tag{4.38}$$

Finally, $\mathrm{QLA}_{\lambda,\hat{y}}(R(x))$ is strictly concave for $x > 0$ and $\lambda > 0$ since

$$\frac{d^2}{dx^2}\mathrm{QLA}_{\lambda,\hat{y}}(R(x)) = -2\lambda \int_{-\infty}^{\frac{\hat{y}-r^0}{x}+r^0} (r - r^0)^2 f_r(r) dr < 0 \tag{4.39}$$

It follows then from (4.30)-(4.39) that for $\lambda > 0$ there exists a unique positive solution $x^* > 0$ of (2.8) such that (2.21) is satisfied. This concludes the proof of the theorem. $\square$

Appendix B

	OIL	BASICMAT	INDUS	CONSGDS	HEALTH	CONSSVS	TELE	UTIL	FIN	TECH	BOND	GOLD	CRUDEOIL
Performance of 1-Month Returns (in percent p.a.)													
Mean	16.99	16.10	14.10	13.95	15.12	13.92	15.64	14.19	12.14	20.42	7.73	5.58	9.90
Std.dev.	18.34	19.15	20.07	21.12	14.50	18.31	23.56	14.44	20.30	29.96	5.73	16.70	36.59
VaR	-61.25	-60.92	-63.52	-64.40	-51.81	-61.00	-67.22	-51.95	-64.39	-81.36	-22.57	-56.91	-86.37
CVaR	-73.54	-80.50	-83.10	-82.48	-66.39	-78.02	-82.41	-66.43	-84.22	-90.63	-30.70	-69.12	-94.48
Percentiles (in percent p.a.)													
5	-61.25	-60.92	-63.52	-64.40	-51.81	-61.00	-67.22	-51.95	-64.39	-81.36	-22.57	-56.91	-86.37
10	-44.53	-47.74	-47.70	-52.92	-39.46	-44.79	-58.93	-40.39	-47.43	-65.28	-16.89	-47.51	-76.89
25	-21.80	-18.79	-22.26	-22.94	-10.15	-19.54	-28.48	-14.50	-19.27	-27.39	-5.43	-24.36	-45.80
50	20.74	20.97	18.41	14.49	19.87	19.00	18.57	17.85	16.72	17.21	9.98	1.83	8.61
75	70.64	70.17	74.50	72.08	54.81	62.02	82.00	58.62	58.76	107.56	22.13	43.41	116.30
90	137.34	141.05	145.36	161.02	101.93	117.00	183.15	99.82	137.34	263.20	37.48	103.16	272.38
95	194.08	205.18	192.01	249.85	139.30	171.95	241.39	138.00	212.31	524.24	48.22	167.75	466.34

Table 7: Summary statistics for European data (February 1982 - December 2010)
Statistics are calculated on the basis of monthly returns and then annualized using discrete compounding. The annualized standard deviation is calculated by multiplying the monthly standard deviation with $\sqrt{12}$.

	OIL	BASICMAT	INDUS	CONSGDS	HEALTH	CONSSVS	TELE	UTIL	FIN	TECH	BOND	GOLD	CRUDEOIL
Performance of 1-month returns (in percent p.a.)													
Mean	15.05	15.45	15.11	13.18	15.49	14.00	12.33	12.33	14.48	16.11	9.58	6.17	9.76
Std.dev.	18.65	21.54	18.91	19.30	15.34	18.73	19.61	14.81	20.24	25.82	7.03	16.13	33.77
VaR	-60.56	-63.49	-57.62	-63.88	-53.64	-63.75	-68.30	-52.28	-61.30	-78.11	-28.09	-52.19	-82.37
CVaR	-75.51	-81.37	-78.95	-78.83	-68.84	-76.19	-78.14	-67.29	-81.66	-87.82	-34.56	-67.20	-93.47
Percentiles (in percent p.a.)													
5	-60.56	-63.49	-57.62	-63.88	-53.64	-63.75	-68.30	-52.28	-61.30	-78.11	-28.09	-52.19	-82.37
10	-42.45	-49.15	-47.76	-49.88	-37.18	-48.01	-55.64	-44.15	-46.91	-59.84	-19.63	-43.29	-72.90
25	-21.88	-27.61	-19.22	-19.36	-14.27	-21.05	-24.36	-16.54	-22.46	-31.94	-5.61	-24.99	-45.66
50	12.68	16.98	19.19	11.98	17.70	15.98	17.71	15.93	17.98	19.97	8.76	-0.39	11.34
75	70.92	78.19	66.28	69.88	57.54	66.05	65.19	52.97	68.40	102.25	27.22	42.22	109.95
90	136.61	161.49	145.75	139.79	110.88	143.09	129.73	99.58	149.24	231.65	46.78	104.88	248.88
95	204.10	252.49	192.91	195.17	158.45	206.43	183.27	126.29	217.12	347.20	63.25	158.53	402.57

Table 8: Summary statistics for US data (February 1982 - December 2010)
Statistics are calculated on the basis of monthly returns and then annualized using discrete compounding. The annualized standard deviation is calculated by multiplying the monthly standard deviation with $\sqrt{12}$.

	Risk-free	Market Index	MV	CVaR	Linear loss-averse (LLA), λ						Quadratic loss-averse (QLA), λ				
					0	0.50	1.00	1.25	1.50	2.00	0.50	1.00	1.25	1.50	2.00
Performance of 1-month returns (in percent p.a.)															
Mean	4.47	12.40	7.59	6.24	15.92	14.07	10.82	10.32	10.75	9.81	7.79	8.04	7.79	7.64	7.38
Median	5.19	18.54	6.94	6.73	21.31	18.57	12.33	11.76	10.69	10.83	10.60	10.73	9.93	10.52	10.34
Std.Dev.	0.71	15.86	5.92	7.12	27.36	20.03	16.59	15.15	14.37	12.62	11.88	9.58	9.07	8.64	7.99
Std.Dev.riskfree		9.98	3.22	4.36	15.74	11.77	10.37	9.54	9.03	7.84	7.67	5.80	5.58	5.36	4.99
Std.Dev.market index	8.83		8.26	8.66	15.12	9.64	8.37	8.00	7.72	7.50	7.93	7.82	7.82	7.84	7.82
CVaR	0.08	-74.31	-32.15	-40.09	-89.82	-81.07	-76.78	-72.86	-69.79	-64.20	-62.80	-53.00	-51.65	-50.68	-47.93
Minimum	0.00	-93.86	-55.82	-78.07	-97.93	-94.05	-94.05	-94.05	-94.20	-91.61	-91.23	-75.38	-71.89	-70.44	-69.10
Sharpe's (in percent)		48.08	50.02	23.63	40.15	46.04	36.81	37.05	41.98	40.64	26.83	35.79	35.02	35.22	34.90
Sortino (risk-free)		76.37	92.92	38.88	69.90	78.34	58.83	58.86	66.86	65.41	41.55	59.12	57.08	56.83	56.03
Sortino (market index)	-80.70		-52.33	-63.94	20.90	15.51	-16.95	-23.44	-19.21	-31.03	-52.27	-50.12	-53.04	-54.56	-57.64
Omega (risk-free)		14.36	6.08	2.64	13.86	15.58	8.54	8.13	10.35	8.52	4.05	5.26	4.88	4.73	4.39
Omega (market index)	0.07		0.19	0.11	2.41	1.70	0.56	0.46	0.54	0.37	0.17	0.19	0.18	0.17	0.15
Total realized return (in percent p.a.)															
Last 10 Years	0.21	2.07	6.55	5.26	8.93	6.83	3.28	4.26	5.28	5.98	4.13	4.63	4.92	5.09	5.29
Last 5 Years	0.22	3.04	6.38	4.47	15.12	10.54	5.16	5.40	6.84	7.19	3.99	4.15	4.48	4.63	4.82
Last 3 Years	0.09	-2.15	4.18	1.12	15.07	3.25	-1.39	-1.15	1.20	0.76	-2.08	-0.31	0.38	0.51	1.10
Last Year	0.02	16.54	10.46	10.09	27.74	24.44	12.30	11.94	12.74	12.11	10.68	10.18	10.08	10.02	10.02
Percentiles															
5	0.25	-62.13	-22.22	-26.12	-76.76	-64.21	-54.13	-54.03	-47.86	-41.59	-41.84	-36.33	-35.30	-34.86	-33.35
10	0.42	-42.51	-14.72	-17.95	-63.15	-51.47	-42.01	-39.17	-35.05	-34.01	-32.00	-26.85	-25.22	-23.51	-21.32
25	2.46	-17.84	-4.18	-7.11	-33.06	-23.18	-18.56	-16.20	-13.76	-12.10	-12.69	-10.01	-9.36	-8.06	-6.83
50	5.19	18.54	6.94	6.73	21.31	18.57	12.33	11.76	10.69	10.83	10.60	10.73	9.93	10.52	10.34
75	6.06	61.30	20.22	21.78	102.98	70.90	54.05	48.57	43.78	37.83	35.26	32.30	29.08	27.95	26.63
90	7.94	112.89	33.46	36.16	213.89	150.36	110.27	94.90	95.76	79.23	71.81	55.94	52.18	49.75	44.46
95	8.50	135.33	42.68	45.68	335.25	214.54	158.05	144.82	130.74	111.14	95.07	82.10	73.62	70.40	63.55
Mean allocation (in percent)															
OIL			0.60	2.09	7.42	10.31	5.20	3.95	3.83	3.44	5.33	4.34	4.10	3.94	3.73
BASICMAT			0.30	1.86	1.29	1.90	2.24	2.01	1.86	1.63	1.84	1.77	1.74	1.74	1.68
INDUS			2.22	0.75	0.00	0.02	1.11	1.40	1.64	2.01	1.87	1.93	1.73	1.64	1.52
CONSGDS			1.71	1.32	4.84	2.28	2.01	1.91	1.86	1.68	1.00	1.00	1.16	1.20	1.22
HEALTH			4.17	5.69	14.19	17.37	15.72	15.04	14.03	12.86	10.89	9.22	8.91	8.62	8.13
CONSSVS			3.63	4.48	0.00	0.00	0.23	0.25	0.38	0.70	0.53	0.65	0.73	0.86	1.11
TELE			0.82	2.80	4.19	5.18	7.74	7.83	8.21	6.88	6.12	4.27	3.69	3.31	2.87
UTIL			1.15	4.24	0.32	4.77	8.56	9.27	9.09	8.88	7.66	6.06	5.54	5.15	4.62
FIN			0.80	3.41	9.68	10.65	8.64	7.80	6.81	5.34	5.55	4.73	4.63	4.59	4.44
TECH			2.03	2.22	24.19	20.13	15.52	13.77	12.41	10.52	9.08	7.09	6.44	5.97	5.36
BOND			62.42	47.06	1.29	7.62	18.40	22.89	25.95	31.39	34.52	42.73	44.84	46.27	48.14
GOLD			14.80	18.14	12.90	11.77	8.39	7.94	8.17	8.90	9.50	10.02	10.29	10.55	11.06
CRUDEOIL			5.35	5.94	19.68	8.00	6.24	5.92	5.76	5.76	6.09	6.18	6.19	6.15	6.12

Table 9: Out-of-sample evaluation of US portfolios: Benchmark scenario

The table reports statistics of a monthly reallocated optimal linear loss-averse portfolio based on an optimization period of 36 months as well as the average of the optimal asset weights. The benchmark scenario assumes a constant loss-averse parameter λ and a zero reference point. The evaluation period covers February 1985 to December 2010. Statistics are calculated on the basis of monthly returns and then annualized assuming discrete compounding. The annual standard deviation is computed as $\sigma_{pa} = \sqrt{12}\sigma_{pm}$.

	Risk-free	Market Index	MV	CVaR	Linear loss-averse (LLA), λ 0	0.50	1.00	1.25	1.50	2.00	Quadratic loss-averse (QLA), λ 0.50	1.00	1.25	1.50	2.00
Performance of 1-month returns (in percent p.a.)															
Mean	4.47	12.40	7.59	6.24	15.92	13.28	14.18	12.21	12.60	12.03	8.50	8.32	7.92	8.25	7.32
Median	5.19	18.54	6.94	6.73	21.31	14.22	17.66	14.37	16.53	14.61	10.56	8.56	9.26	9.44	9.39
Std.Dev.	0.71	15.86	5.92	7.12	27.36	15.54	17.27	18.44	16.09	15.14	10.94	8.99	8.85	8.40	8.82
Std.Dev.riskfree		9.98	3.22	4.36	15.74	8.54	9.95	10.14	9.46	9.19	6.75	5.38	5.33	4.99	5.86
Std.Dev.market index	8.83		8.26	8.66	15.12	7.45	8.17	8.73	8.04	7.33	7.81	7.78	7.90	7.86	9.06
CVaR	0.08	-74.31	-32.15	-40.09	-89.82	-68.35	-73.64	-75.44	-71.39	-71.34	-58.33	-50.99	-50.50	-48.49	-52.33
Minimum	0.00	-93.86	-55.82	-78.07	-97.93	-91.77	-94.14	-91.35	-87.34	-87.29	-82.99	-70.82	-69.15	-68.15	-90.58
Sharpe's (in percent)		48.08	50.02	23.63	40.15	54.50	53.93	40.27	48.38	47.85	35.40	41.07	37.40	43.09	30.87
Sortino (risk-free)		76.37	92.92	38.88	69.90	99.01	93.73	73.25	82.55	79.02	57.36	68.66	62.17	72.71	46.63
Sortino (market index)	-80.70		-52.33	-63.94	20.90	10.53	19.49	-2.04	2.18	-4.57	-44.87	-47.14	-50.98	-47.50	-50.42
Omega (risk-free)		14.36	6.08	2.64	13.86	21.52	23.62	12.14	15.28	13.95	5.75	6.29	5.33	6.53	4.19
Omega (market index)	0.07		0.19	0.11	2.41	1.42	1.93	0.93	1.08	0.86	0.23	0.22	0.19	0.22	0.16
Total realized return (in percent p.a.)															
Last 10 Years	0.21	2.07	6.55	5.26	8.93	5.96	7.70	5.89	9.92	9.99	4.84	5.36	5.50	5.51	5.54
Last 5 Years	0.22	3.04	6.38	4.47	15.12	6.14	4.95	5.49	11.73	10.87	4.88	4.91	4.95	4.79	4.68
Last 3 Years	0.09	-2.15	4.18	1.12	15.07	-1.95	0.50	1.61	11.33	8.63	-0.09	0.52	0.57	0.28	0.22
Last Year	0.02	16.54	10.46	10.09	27.74	38.81	11.01	20.36	27.56	13.84	11.85	9.93	9.99	9.96	9.95
Percentiles															
5	0.25	-62.13	-22.22	-26.12	-76.76	-51.37	-53.73	-63.40	-56.63	-54.86	-38.49	-35.16	-37.52	-34.58	-34.27
10	0.42	-42.51	-14.72	-17.95	-63.15	-36.28	-42.02	-43.85	-44.04	-42.35	-31.47	-23.31	-24.00	-22.31	-21.52
25	2.46	-17.84	-4.18	-7.11	-33.06	-19.25	-19.03	-21.11	-18.77	-13.85	-10.88	-8.12	-7.79	-6.58	-6.27
50	5.19	18.54	6.94	6.73	21.31	14.22	17.66	14.37	16.53	14.61	10.56	8.56	9.26	9.44	9.39
75	6.06	61.30	20.22	21.78	102.98	52.01	56.77	53.57	57.22	50.78	35.03	30.59	28.13	27.25	26.55
90	7.94	112.89	33.46	36.16	213.89	98.66	118.27	114.23	113.67	107.03	66.58	52.79	48.69	47.18	41.80
95	8.50	135.33	42.68	45.68	335.25	150.41	190.23	168.11	158.97	150.26	91.09	79.97	73.77	69.22	63.04
Mean allocation (in percent)															
OIL			0.60	2.09	7.42	7.18	7.41	6.83	6.56	5.58	4.71	3.88	3.67	3.59	3.37
BASICMAT			0.30	1.86	1.29	4.46	4.25	3.94	3.60	2.86	1.43	1.48	1.45	1.46	1.42
INDUS			2.22	0.75	0.00	4.18	3.12	2.92	2.52	2.51	1.96	2.08	1.91	1.80	1.63
CONSGDS			1.71	1.32	4.84	6.00	4.16	3.91	3.28	3.37	1.18	1.09	1.05	1.06	1.05
HEALTH			4.17	5.69	14.19	9.83	12.11	12.05	12.89	13.79	10.52	9.65	9.25	8.75	8.25
CONSSVS			3.63	4.48	0.00	4.27	3.19	2.57	2.37	2.23	0.66	0.78	0.89	0.97	1.20
TELE			0.82	2.80	4.19	6.29	5.99	6.80	7.98	8.03	5.88	3.66	3.32	2.90	2.72
UTIL			1.15	4.24	0.32	5.33	6.21	7.12	6.56	7.86	7.50	5.71	5.28	5.13	4.66
FIN			0.80	3.41	9.68	8.63	7.74	8.03	8.62	7.25	5.09	4.06	4.16	3.93	3.67
TECH			2.03	2.22	24.19	13.88	14.66	13.68	14.19	13.15	8.13	6.31	6.18	5.58	5.34
BOND			62.42	47.06	1.29	7.93	10.76	11.59	13.32	16.48	37.46	45.18	46.74	48.31	49.17
GOLD			14.80	18.14	12.90	12.14	10.55	10.31	8.81	8.78	9.46	10.04	10.14	10.55	11.12
CRUDEOIL			5.35	5.94	19.68	9.86	9.84	10.24	9.30	8.10	6.02	6.07	5.96	5.97	6.42

Table 10: Out-of-sample evaluation of US portfolios: Dynamic scenario 2

The table reports statistics of a monthly reallocated optimal linear loss-averse portfolio based on an optimization period of 36 months as well as the average of the optimal asset weights. The dynamic scenario 2 assumes a smaller λ and a reference point equal to the risk-free rate for prior gains, and a constant λ and a higher reference point for prior losses. The evaluation period covers February 1985 to December 2010. Statistics are calculated on the basis of monthly returns and then annualized assuming discrete compounding. The annual standard deviation is computed as $\sigma_{pa} = \sqrt{12}\sigma_{pm}$.

	Risk-free	Market Index	MV	CVaR	Linear loss-averse (LLA), λ 0	0.50	1.00	1.25	1.50	2.00	Quadratic loss-averse (QLA), λ 0.50	1.00	1.25	1.50	2.00
Performance of 1-month returns (in percent p.a.)															
Mean	4.47	12.40	7.59	6.24	15.92	11.07	12.33	9.54	12.74	11.93	7.50	7.98	7.55	7.41	6.76
Median	5.19	18.54	6.94	6.73	21.31	16.41	14.13	13.04	15.19	12.50	11.57	10.44	10.40	10.19	10.56
Std.Dev.	0.71	15.86	5.92	7.12	27.36	15.67	17.78	17.24	16.52	15.04	12.09	10.44	9.94	9.50	8.99
Std.Dev.riskfree		9.98	3.22	4.36	15.74	10.08	9.81	11.32	9.88	9.19	7.92	6.75	6.55	6.26	6.10
Std.Dev.market index	8.83		8.26	8.66	15.12	7.91	8.49	9.52	8.97	8.24	8.06	7.91	7.96	8.00	8.20
CVaR	0.08	-74.31	-32.15	-40.09	-89.82	-74.71	-73.10	-79.29	-74.94	-72.30	-64.47	-57.90	-56.66	-55.31	-55.16
Minimum	0.00	-93.86	-55.82	-78.07	-97.93	-94.42	-92.20	-97.93	-94.42	-94.13	-92.68	-89.36	-88.41	-86.98	-83.05
Sharpe's (in percent)		48.08	50.02	23.63	40.15	40.45	42.38	28.21	48.03	47.76	24.06	32.21	29.68	29.71	24.35
Sortino (risk-free)		76.37	92.92	38.88	69.90	62.86	77.01	43.04	80.43	78.01	36.71	49.88	45.13	45.16	36.01
Sortino (market index)	-80.70		-52.33	-63.94	20.90	-15.16	-0.74	-26.97	3.38	-5.12	-54.64	-50.29	-54.77	-55.98	-61.86
Omega (risk-free)		14.36	6.08	2.64	13.86	10.74	12.87	5.46	16.58	15.04	3.56	4.86	4.19	4.07	3.08
Omega (market index)	0.07		0.19	0.11	2.41	0.58	0.97	0.35	1.13	0.84	0.16	0.19	0.16	0.15	0.12
Total realized return (in percent p.a.)															
Last 10 Years	0.21	2.07	6.55	5.26	8.93	5.24	7.64	5.39	7.63	4.83	3.73	5.32	4.80	5.06	5.20
Last 5 Years	0.22	3.04	6.38	4.47	15.12	3.99	3.65	8.82	13.42	11.79	3.70	4.94	5.19	6.25	6.36
Last 3 Years	0.09	-2.15	4.18	1.12	15.07	-4.81	-5.30	4.16	8.64	5.14	-1.63	0.28	0.99	2.37	2.18
Last Year	0.02	16.54	10.46	10.09	27.74	17.57	10.22	25.99	20.79	11.86	10.69	10.33	10.21	10.09	10.12
Percentiles															
5	0.25	-62.13	-22.22	-26.12	-76.76	-49.25	-54.23	-57.92	-56.70	-47.78	-43.05	-36.58	-39.31	-34.81	-33.35
10	0.42	-42.51	-14.72	-17.95	-63.15	-37.29	-42.51	-42.42	-37.66	-34.20	-31.52	-28.15	-27.25	-26.57	-24.59
25	2.46	-17.84	-4.18	-7.11	-33.06	-14.10	-20.92	-18.51	-17.89	-15.81	-13.08	-10.56	-8.17	-7.88	-7.68
50	5.19	18.54	6.94	6.73	21.31	16.41	14.13	13.04	15.19	12.50	11.57	10.44	10.40	10.19	10.56
75	6.06	61.30	20.22	21.78	102.98	46.19	49.41	56.31	46.29	44.08	35.40	31.94	29.58	27.97	26.90
90	7.94	112.89	33.46	36.16	213.89	92.37	129.05	109.17	121.80	108.88	66.85	61.10	55.21	52.56	48.13
95	8.50	135.33	42.68	45.68	335.25	130.10	190.97	167.86	185.55	150.91	91.52	82.81	74.41	72.15	68.93
Mean allocation (in percent)															
OIL			0.60	2.09	7.42	7.43	6.77	7.79	6.98	4.75	5.08	4.44	4.23	4.22	3.65
BASICMAT			0.30	1.86	1.29	5.01	4.19	3.84	3.02	3.30	1.61	1.67	1.75	1.76	1.87
INDUS			2.22	0.75	0.00	4.35	3.64	2.63	2.71	2.48	1.73	1.94	1.81	1.71	1.50
CONSGDS			1.71	1.32	4.84	5.39	4.76	4.40	3.42	2.44	1.00	0.97	1.18	1.04	1.09
HEALTH			4.17	5.69	14.19	9.60	11.34	11.62	12.81	14.46	11.79	10.77	10.46	9.88	8.94
CONSSVS			3.63	4.48	0.00	4.24	3.38	2.79	2.44	1.68	0.61	0.79	0.83	0.84	1.13
TELE			0.82	2.80	4.19	7.03	7.11	6.07	6.56	7.04	6.17	4.18	3.76	3.51	3.24
UTIL			1.15	4.24	0.32	6.65	6.18	5.68	6.19	8.19	7.83	6.52	5.88	5.36	4.78
FIN			0.80	3.41	9.68	8.33	6.80	7.95	6.63	7.05	5.53	5.03	4.88	4.87	4.58
TECH			2.03	2.22	24.19	11.65	14.74	14.80	14.02	13.47	9.24	7.15	6.67	6.46	6.50
BOND			62.42	47.06	1.29	11.71	10.84	13.17	16.48	19.66	33.88	40.28	42.62	44.04	46.04
GOLD			14.80	18.14	12.90	9.48	9.52	9.59	10.01	8.54	9.14	9.91	9.75	10.15	10.57
CRUDEOIL			5.35	5.94	19.68	9.12	10.74	9.65	8.71	6.95	6.39	6.35	6.17	6.17	6.12

Table 11: Out-of-sample evaluation of US portfolios: Break-even scenario 2

The table reports statistics of a monthly reallocated optimal linear loss-averse portfolio based on an optimization period of 36 months as well as the average of the optimal asset weights. The break even scenario assumes a risk-seeking behavior after prior losses. The evaluation period covers February 1985 to December 2010. Statistics are calculated on the basis of monthly returns and then annualized assuming discrete compounding. The annual standard deviation is computed as $\sigma_{pa} = \sqrt{12}\sigma_{pm}$.

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