IHS Economics Series
Working Paper 290
September 2012

The Inefficiency of Price Taking Behavior in Multiperiod Production Economies with Incomplete Markets

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Vienna

Impressum

Author(s):

Egbert Dierker

Title:

The Inefficiency of Price Taking Behavior in Multiperiod Production Economies with Incomplete Markets

ISSN: Unspecified

2012 Institut für Höhere Studien - Institute for Advanced Studies (IHS)

Josefstädter Straße 39, A-1080 Wien

E-Mail: office@ihs.ac.at

Web: www.ihs.ac.at

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Abstract

The purpose of this paper is to explore how the concept of a Drèze equilibrium can be extended to multiperiod production economies with incomplete markets. Constrained efficiency cannot serve as a basis for such an extension because multiperiod models tend to violate even weak constrained efficiency requirements. We show by means of examples how the difficulties that arise in the case of sequential trade can be taken into account. Finally, we employ the concept of minimal efficiency, which has been introduced by Dierker et al. (2005) in a two-period model, to derive a natural extension of the Drèze rule. This is possible because minimal efficiency relies on a planner who can choose the production plan but who cannot interfere with future consumption otherwise.

Keywords

Incomplete markets with production, the objective of a firm, Drèze equilibria with sequential trade, efficiency and social welfare

JEL Classification

D21, D52, D61

Comments

I would like to thank the participants of the Micro Research Afternoon at the Institute for Advanced Studies in Vienna for their helpful comments. I am particularly grateful to Larry Blume, Hildegard Dierker, Klaus Ritzberger, and Leopold Sögner for their valuable suggestions.

Contents

1 Introduction.

Drèze (1974) extends General Equilibrium Theory to the realistic case of incomplete markets with production. When markets are incomplete missing prices tend to make it impossible to define profits in an objective manner and different shareholders of a firm rank its production plans differently. This leads to the question of how one can define the goal of a firm. We address this question from a purely normative perspective. The resulting model presents a benchmark case based on welfare and efficiency considerations. We do not aim to analyze the power struggle among the members of a firm's control group.

When markets are complete and every agent acts as a price taker equilibrium allocations are Pareto efficient according to the first theorem of welfare economics. Drèze equilibria aim at efficiency in two-period models with incomplete markets and price taking behavior. Consider a planner who can choose the production plan of every firm, allocate shareholdings, and redistribute consumption in the initial period 0. An allocation is *constrained efficient* if this planner is unable to achieve a Pareto improvement. Drèze equilibria can be characterized by the fact that they satisfy the first order condition for constrained efficiency.

The difficulty to extend Drèze's approach to multiperiod economies with production arises in the *consumption* sector because *price taking behavior fails to organize the consumption sector efficiently when trade is sequential.* More precisely, consider another planner who is weaker than the previous planner. The second planner takes the production plans as given and intervenes only at the initial date $t = 0$ by reallocating the shares ϑ_0^i carried over from $t = 0$ to $t = 1$ and by redistributing the initial consumption x_0^i. We say that an allocation is slightly (constrained) efficient, or *slightly efficient* for short, if the second, weaker planner cannot achieve a Pareto improvement.[1] We will provide simple examples in single good models without spot markets that show that the first order condition for slight efficiency is still too demanding to lend itself to a useful definition of the objective of a firm in multiperiod models.

The fundamental conclusion drawn in this paper is the following. In the multiperiod case, an appropriate efficiency concept must take into account that price taking behavior tends to entail a distribution of the shares ϑ_0^i acquired at $t = 0$ that prevents the efficient organization of the consumption sector. As a consequence, we consider a third planner who is *unable to affect future consumption other than by choosing production plans.* In particular, the third planner cannot redistribute any shares. This planner can only redistribute the total consumption at $t = 0$ among consumers to compensate them for changes in production. The compensation takes place after the stock market is closed so that the changes in initial consumption do not affect consumers' investments [cf. Section 5]. An allocation is called minimally (constrained) efficient, or *minimally efficient* for

[1]We speak of slight rather than weak efficiency because weakly constrained efficiency has another meaning in the literature.

short, if the third planner cannot find a Pareto improvement.[2]

Such a planner has been introduced before in a two-period model for another reason. Dierker et al. (2002) have presented an example of a finance economy in a two-period model with a unique, but constrained inefficient Drèze equilibrium. In the example, the inefficiency is caused by strong income effects. To cut the link between $t = 0$ and future consumption, Dierker et al. (2005) introduce the concept of minimal efficiency in a two-period model and show that the unique, constrained inefficient Drèze equilibrium in the example is minimally efficient. In Section 5 we present a definition of minimal efficiency in a multiperiod model in order to derive a generalized Drèze rule from efficiency considerations.

We are going to recall, in a brief and simplified fashion, the traditional theory for the two-period case. Throughout the paper, we aim at simplicity rather than generality in order to enhance the understanding of the fundamental issues and conceptual difficulties.

Suppose that there are only the time periods $t = 0$ and $t = 1$. The state $s = 0$ has been realized at $t = 0$, but it is unknown which of the states $s = 1, \ldots, S$ will occur at $t = 1$. We assume that there is only one (perishable) good at each state which is interpreted as state dependent income. Hence, there is no need for spot markets. Spot markets are known to entail constrained inefficiency generically [cf. Geanakopos et al. (1986) and Geanakopos et al. (1990)].

We follow §31 of Magill and Quinzii (1996), or MQ for short, that deals with partnership economies in a two-period model. In particular, consumers are not endowed with exogenously given initial shares. The only shares in the two-period partnership model are acquired at $t = 0$ on the stock market and carried over to $t = 1$ where they expire after the payment of dividends in the form of the firm's state dependent output. Furthermore, MQ assume that production in a partnership economy exhibits constant returns to scale.

Consider, for simplicity's sake, an economy with a single firm which produces $y = (y_0, y_+) = (y_0, y_1, \ldots, y_S) \in \mathbb{R}_- \times \mathbb{R}^S_+$ and let ϑ^i be $i's$ optimal share in the firm and π^i be the gradient of i's quasilinear utility function $x_0 + V^i(x_1, \ldots, x_S)$.[3] Throughout the paper, the index + refers to all states after $t = 0$. Then $\pi^i_s = \partial_s V^i(x^i_+)$ represents i's present value of one (infinitesimal) additional unit of income in state $s \geq 1$. That is to say, i is indifferent between receiving the additional unit at state s or π^i_s units of good 0. The vector $\pi^i = (1, \pi^i_1, \pi^i_2, \ldots, \pi^i_S)$ is called consumer i's state price system or vector of i's stochastic discount rates.

The firm maximizes profits with respect to a price system π which is linked to the individual state price systems π^i as follows. Suppose the production plan y is changed such that y_s is increased by one infinitesimal unit where $s \geq 1$. Then the benefit which i obtains from his dividend at $t = 1$ increases by π^i_s if i holds all

[2]The term minimally efficient is a bit misleading because the third planner can alter the production plan, the second cannot.

[3]If the utility function is not quasilinear one needs to normalize the utility gradient such that the marginal utility of good 0 equals 1.

shares. Since i's share of the firm is ϑ^i his marginal benefit equals $\pi^i_s\vartheta^i$. Here we use the fact that i's shares can be kept fixed according to the envelope theorem.[4] In total, the economy gains $\sum_i \pi^i_s\vartheta^i$ in terms of future benefits and has to cover the marginal cost π_s of the additional unit of good s. If the difference, the marginal social surplus, does not vanish the original production plan is not socially optimal. When shareholders can make transfers in units of good 0 then the winners of the change can compensate the losers so that a unanimous agreement can be reached to change the production of good s marginally. The marginal social surplus vanishes iff $\pi_s = \sum_i \pi^i_s\vartheta^i$ for any $s \geq 1$, that is to say, $\pi = \sum_i \pi^i\vartheta^i$.

This argument suggests that, in a model with several firms, each firm j should choose its production plan so as to maximize profits with respect to the price system $\pi_j = \sum_i \pi^i\vartheta^i_j$ where ϑ^i_j denotes consumer i's shares in firm j. We call this objective the *Drèze rule*. For a precise and more general statement and a proof not based on quasilinear utility functions see MQ, §31, 31.5 Proposition. A *Drèze equilibrium* obtains in a two-period framework if the stock market clears, i.e. $\sum_i \vartheta^i_j = 1$ for all firms j, and every firm acts according to the Drèze rule. This can be interpreted as follows: *At a Drèze equilibrium, the marginal cost of good $s \geq 1$ equals the marginal social benefit* $\sum_i \pi^i_s\vartheta^i_j$ for each firm j. Our goal is to extend this idea to the multiperiod case.

If there are only two periods then the shares cannot be sold after the dividends have been paid at $t = 1$ because there is no future. The step from two to three time periods makes a decisive difference. MQ point out on p. 423: "In a multiperiod setting, shares will always be traded after a production decision is made, so that the shareholders must take into account the influence of production decisions not only on the firm's subsequent dividends but also of the subsequent capital value of its shares."

Stochastic discount rates or state price systems have been used to form conjectures about market values of future income streams. A prominent example is Grossman and Hart (1979) where this approach is applied to a multiperiod setting in which a firm aims to act in the interest of its group $\mathcal{O}$ of original shareholders. In their model, the original owners of a firm are endowed with exogenously given initial shares δ^i, determine the production plan before shares are traded, and pay the production costs. They make the assumption of competitive price perceptions according to which each original shareholder i, who does not know how a change of the output vector impacts its price, feels that his induced utility change is exactly offset by the associated price change.[5] That is to say, i uses his own state price system $\pi^i = (\pi^i_s)_{s=0,\dots,S}$ for future benefits in order to assess changes of the firm's capital value. As a consequence, the firm maximizes profits

[4]The fact that the induced consumption changes can be disregarded entails a huge difference between two-period and multiperiod models where stocks are traded at intermediate dates so that the envelope theorem does no longer apply.

[5]Strictly speaking, Grossman and Hart consider a model in which the input and not the output is chosen.

with respect to the price system $\pi = \sum_{i \in \mathcal{O}} \pi^i \delta^i$.

Bonnisseau and Lachiri (2004) avoid the use of subjective perceptions and suggest to generalize the Drèze rule as follows. They consider a setting with a finite number of time periods, in which uncertainty evolves gradually in the form of a finite tree as described in chapter 4 of MQ, and derive first order conditions for constrained Pareto optimality in a multi-commodity framework with stock and spot markets. As pointed out by MQ, it is not obvious how one can define constrained Pareto optimality appropriately in a multiperiod model. Loosely speaking, Bonnisseau and Lachiri (2004) use several versions which entail the same conclusion. Bonnisseau and Lachiri (2006) is more closely related to our setting because it uses a finance model with a single good at each event. We refer to both papers as BL.

BL's pricing rule evolves sequentially in a piecewise fashion along the date-event tree. A node in the tree is a date-event $\xi = (t, s)$ where $t = 0, 1, \ldots, T$ and $s = 0, 1, \ldots, S$. Roughly speaking, the basic idea is that one proceeds, at each non-terminal node, in the same way as the Drèze rule does at node 0. Consider the simplest case with $T = 2$ future periods and suppose the initial node branches out into k immediate successor nodes at $t = 1$. Altogether there are $k + 1$ stock markets, one at each non-terminal node. The price π_0 of the initial good is normalized to 1. The prices associated with the immediate successors ξ_0^+ of the initial node ξ_0 are given by a convex combination of the corresponding coordinates of the individual state prices weighted with the shares ϑ_0^i carried over from ξ_0. That is to say, the firm's price of good s is given by $\pi_s = \sum_i \pi_s^i \vartheta_0^i$ where $s = 1, \ldots, k$. Thus, the first step of BL's inductive description coincides with the classical Drèze formula applied to the tree truncated at $t = 1$.

The same principle is used again starting from each of the k nodes at $t = 1$. Pick any such node $\xi = (1, s), s = 1, \ldots, k$, and assume that $\xi' = (2, s')$ is a terminal node that follows ξ. Then the firm's price of good s' is $\pi_{s'} = \sum_i \pi_{s'}^i \vartheta_s^i$, where ϑ_s^i denotes the amount of shares i carries over from ξ_s. To determine the firm's price system π for all goods in the case of $T = 2$, one uses the classical Drèze formula $k + 1$ times.[6]

For any finite tree, the price system π can be computed by the same principle. The price of good 0 is normalized to 1. The prices of the immediate successors ξ^+ of each non terminal node ξ equal the convex combination of the individual state prices associated with ξ^+ with weights equal to the shares carried from ξ to ξ^+.

Pricing rules such as the Drèze or the BL rule share the following property. Let ξ be a non-terminal node and denote the set of immediate successors of ξ by ξ^+. Then the amount of shares i carries over from ξ to ξ^+ serves simultaneously as weight for those coordinates of i's utility gradient which correspond to some node in ξ^+. In a multiperiod model, the nodes in ξ^+ need not be final and

[6] This procedure will be illustrated by means of numerical examples in subsequent sections.

can have active stock markets. At these stock markets, i adjusts his portfolio and faces capital gains or losses which affect i's utility typically in different, state dependent ways. Pricing rules which assign prices to the nodes in ξ^+ that are convex combinations of the corresponding coordinates of consumers' utility gradients are too rigid to capture these effects. A generalization of the Drèze rule to multiperiod models has to differ substantially from the traditional Drèze rule.

We show that constrained inefficiency prevents the use of the BL rule and entails that all individual stock transactions and the induced utility changes as well as the consequences of production plan variations must be taken into account. Although the examples analyzed in the following sections are chosen such that they can be solved numerically with modest effort, they show that the simplicity of the original Drèze formula and its intuitive appeal are lost as soon as one considers three instead of two time periods.

Drèze equilibria and their generalization to multiperiod models have a normative character. They deserve to be investigated as valuable benchmark cases. The theory is static and all decisions are made at the initial period $t = 0$. This is in line with Debreu (1959), p.50, who characterizes the role of a consumer as follows: "His role is to choose (and to carry out) a consumption plan made now for the whole future, i.e., a specification of the quantities of all his inputs and all his outputs."

Since the notion of constrained efficiency does not provide a sound basis for the analysis of multiperiod models we adopt a framework in which social welfare maximization is well defined. We begin with examples in which good 0 can be used to transfer utility, cf. Sections 2 and 3. In this case, the firm aims to maximize the social surplus measured in units of good 0. In Subsection 3.2, we modify the example from section 2 such that costs are paid at all non-final states. In the final example in Section 4, quasilinearity is violated and no good is additively separated from the others. Section 5 derives a generalized Drèze rule from the first order condition for minimal efficiency. Section 6 concludes.

2 Surplus maximization in an economy with additively separable, quasilinear utilities.

We consider a setting with three periods, $t = 0, 1, 2$, a single firm and a single good per state. There are no assets other than the firm's shares. Each non-terminal node has two immediate successors. The initial node ξ_0 is followed by the intermediate nodes ξ_1 and ξ_2. The successors of node ξ_1 are the final nodes ξ_3 and ξ_4, those of ξ_2 are the final nodes ξ_5 and ξ_6. Altogether the tree has seven nodes. The good at node ξ_j is called good j. There are two versions of the example. In the first, costs are only paid at the initial node. In the second, there is a cost at every non-terminal node. The second version will be presented in Subsection 3.2 because the model is no longer additively separable.

At $t = 0$, the production of the output $y_+ = (y_1, y_2, \ldots, y_6)$ requires the input

$$y_0 = -C(y_1, \ldots, y_6) = -\left(\sqrt{y_1^2 + y_2^2} + \sqrt{y_3^2 + y_4^2} + \sqrt{y_5^2 + y_6^2}\right). \tag{1}$$

No consumer is endowed with initial shares. Consumer i can obtain shares ϑ_0^i at the initial node ξ_0 by contributing to the cost $C = -y_0$ in proportion to the size of ϑ_0^i. At the intermediate nodes $\xi_s, s = 1, 2$, shares ϑ_s^i are traded against good s at the market clearing price q_s. At a *stock market equilibrium*, all three stock markets clear, that is to say, $\sum_i \vartheta_s^i = 1$ for $s = 0, 1, 2$.

There are two consumers, $i = A, B$, with the quasilinear and additively separable, concave utility functions

$$\begin{aligned} U^A(x_0, x_1, \ldots, x_6) &= x_0 + 1\log(x_1) + 2\log(x_2) + 3\log(x_3) \\ &\quad + 4\log(x_4) + 5\log(x_5) + 6\log(x_6) \\ U^B(x_0, x_1, \ldots, x_6) &= x_0 + \log(x_1) + \log(x_2) + \log(x_3) \\ &\quad + \log(x_4) + \log(x_5) + \log(x_6), \end{aligned} \tag{2}$$

respectively. For simplicity, consumer i has no initial endowment e_s^i except at node ξ_0 and consumes $e_0^i + \vartheta_0^i y_0$ at $t = 0$. To be specific, let $e_0^A = 25, e_0^B = 15$. The consumption at an intermediate node ξ_s is $x_s^i = q_s(\vartheta_{s\_}^i - \vartheta_s^i) + \vartheta_{s\_}^i y_s$ at $t = 1$, where $\xi_{s\_}$ is the immediate predecessor of ξ_s. If ξ_s is a terminal node then i consumes $x_s^i = \vartheta_{s\_}^i y_s$.

Let $y_+ \gg 0$ be given and assume that the consumers know the costs $C(y_+)$ and foresee the market clearing share prices $q_1(y_+)$ and $q_2(y_+)$. Consumer A maximizes his utility by choosing $\vartheta_0^A = 21/C, \vartheta_1^A = 147(q_1 + y_1)/(8q_1C), \vartheta_2^A = 231(q_2 + y_2)/(13q_2C)$ where we have dropped the variable y_+. For Consumer B we obtain $\vartheta_0^B = 6/C, \vartheta_1^B = 4(q_1 + q_2)/(q_1C), \vartheta_2^B = 4(q_2 + y_2)/(q_2C)$.

There are three market clearing equations. Solving those for markets 1 and 2 we obtain $q_1 = 179\, y_1/(-179 + 8\, C)$ and $q_2 = 283\, y_2/(-283 + 13\, C)$. The market clearing of the initial stock market entails that the costs are identically equal to 27 for all stock market equilibria.

Let $\hat{y} = (y_1, \ldots, y_5)$. The last component y_6 can be written as $g(\hat{y})$ where g is implicitly defined by

$$\sqrt{y_1^2 + y_2^2} + \sqrt{y_3^2 + y_4^2} + \sqrt{y_5^2 + (g(\hat{y}))^2} = 27. \tag{3}$$

When we put $y(\hat{y}) = (\hat{y}, g(\hat{y}))$ then every function depending on y becomes indirectly a function of $\hat{y}$. To simplify the notation, we shall often write $\vartheta_s^i(\hat{y})$ instead of $\vartheta_s^i(y(\hat{y}))$ as long as there is no danger of a confusion. The same slight abuse of language is also used for other functions such as the stock prices q_j.

In this notation, the *stock market equilibria* are parameterized by $\hat{y}$ and lie in

$$\mathcal{E} = \{y_+ \in \mathbb{R}_{++}^6 \mid \sum_i \vartheta_s^i(\hat{y}) = 1 \text{ for } s = 0, 1, 2\}.$$

The choice of the parametrization is arbitrary and has no significance.

As in the two-period case, $\mathcal{E}$ has one dimension less than the output vector. This reflects the fact that the firm operates at the output level desired by the consumers. In the Walrasian tradition, the firm is instructed to sell its production plan at $t = 0$ at marginal (=unit) costs.

We write, dropping the variable $\hat{y}$, the consumption of $i = A, B$ as

$$x^i = (e^i_0 - \vartheta^i_0 C, q_1(\vartheta^i_1 - \vartheta^i_0) + \vartheta^i_0 y_1, q_2(\vartheta^i_2 - \vartheta^i_0) + \vartheta^i_0 y_1, \vartheta^i_1 y_3, \vartheta^i_1 y_4, \vartheta^i_2 y_5, \vartheta^i_2 g). \quad (4)$$

Let $u^i(\hat{y}) = U^i(x^i(\hat{y}))$ be the utility i obtains if $\hat{y}$ is chosen. Due to quasilinearity, we can write

$$u^i(\hat{y}) = x^i_0(\hat{y}) + v^i(\hat{y}) = U^i(x^i(\hat{y})) = x^i_0(\hat{y}) + V^i(x^i(\hat{y})). \quad (5)$$

Social welfare is given by

$$\mathcal{W}(\hat{y}) = x^A_0(\hat{y}) + v^A(\hat{y}) + x^B_0(\hat{y}) + v^B(\hat{y}). \quad (6)$$

To find a generalized Drèze equilibrium in the example, we solve the first order condition $D\mathcal{W}(\hat{y}) = 0$ for welfare maximization numerically and obtain the solution $\hat{y}^* \approx (3.162, 3.873, 6, 6.708, 8.832)$. The last coordinate of the production plan y^* is $y^*_6 = g(\hat{y}^*) \approx 9.539$. A consumes $x^A(y^*) \approx (9, 1.79, 2.39, 4.93, 5.51, 7.21, 7.79)$ and B consumes $x^B(y^*) \approx (24, 1.37, 1.48, 1.07, 1.20, 1.62, 1.75)$.

The firm sells its output y^*_+ at marginal costs. That is to say, the firm acts as if it maximizes profits with respect to the price system $(1, DC(\hat{y}^*)) = \pi(\hat{y}^*)$. We ask the question of whether coordinates 1 and 2 of $\pi(\hat{y}^*) = (1, \pi_+)(\hat{y}^*)$, which correspond to the two intermediate nodes, are a convex combination of the corresponding coordinates of A's and B's utility gradients.

The utility gradients, the firm price system $\pi(\hat{y}^*)$, and the Bonnisseau-Lachiri or BL price system π_{BL} at the generalized Drèze equilibrium are, respectively,

$$\begin{aligned} DU^A(\hat{y}^*) = \pi^A(\hat{y}^*) &\approx (1, 0.557, 0.836, 0.609, 0.726, 0.694, 0.771) \\ DU^B(\hat{y}^*) = \pi^B(\hat{y}^*) &\approx (1, 0.731, 0.675, 0.932, 0.834, 0.616, 0.571) \\ (1, DC(\hat{y}^*)) = \pi(\hat{y}^*) &\approx (1, 0.632, 0.775, 0.667, 0.745, 0.679, 0.734) \\ \pi_{BL}(\hat{y}^*) &\approx (1, 0.596, 0.800, 0.667, 0.745, 0.679, 0.734). \end{aligned} \quad (7)$$

The BL prices $(\pi_1, \pi_2)_{BL} = \vartheta^A_0(\hat{y}^*)(0.557, 0.836) + \vartheta^B_0(\hat{y}^*)(0.731, 0.675) \approx (0.596, 0.8)$ differ from the correct values $(0.632, 0.775)$. Therefore, the equilibrium at $\hat{y}^*$ does not satisfy the first order condition for constrained efficiency in the sense of BL. However, the BL prices of goods 3 to 6, which are associated with the terminal nodes, coincide with the corresponding coordinates of $\pi(\hat{y}^*)$. One might conjecture that this is a general property. We shall explain in the following section why this conjecture is false.

Concerning the two intermediate nodes ξ_1 and ξ_2, we search for a γ such that $\gamma(0.557, 0.836) + (1 - \gamma)(0.731, 0.675) = (0.632, 0.775)$. However, it is easy to check that the equation has no solution.

Proposition 1. *Suppose the firm maximizes profits with respect to some price system π such that (π_1, π_2) is a convex combination of the corresponding coordinates of A's and B's utility gradient. Then the firm fails to maximize social welfare.*

Proposition 1 shows that, in spite of the simplicity of the example, public knowledge of every consumer's stochastic discount factors and shareholdings does not suffice to define a pricing rule that prevents the firm to waste resources. To clarify the nature of the efficiency failure that underlies Proposition 1 we pose the following definition.

Definition 1. *Consider a stock market equilibrium with production $y^* = (y_0^*, y_+^*)$. The stock market equilibrium is slightly constrained efficient, or* slightly efficient *for short, iff a planner P cannot achieve a Pareto improvement in the following way. Nobody including P can change the production plan. First, P assigns the shares $\bar{\vartheta}_0^i$ to every consumer i under the condition that $\sum_i \bar{\vartheta}_0^i = 1$. Then, given their consumption $e_0^i + \bar{\vartheta}_0^i y_0^*$, consumers choose their future shares optimally on every stock market held after $t = 0$ at market clearing prices. Finally, P redistributes the total consumption $\sum_i e_0^i + y_0^*$ at $t = 0$ among the consumers while their consumption at $t \geq 1$ remains unaltered.*

Proposition 2. *The stock market equilibrium at $\hat{y}^*$ in the example does not satisfy the first order condition for slight efficiency.*

We describe the *feedback effect* underlying the inefficiency . Since both consumers have quasilinear utility functions, income effects at $t = 0$ cannot play any role. Instead, the feedback effect works through the impact of $\bar{\vartheta}_0^i$ on future share prices.

The planner P maximizes social welfare under the condition that consumers take the shares $\bar{\vartheta}_0^A = \bar{\vartheta}_0$ and $\bar{\vartheta}_0^B = 1 - \bar{\vartheta}_0$ at $t = 0$ as given and react optimally. Consumers foresee the market clearing prices q_1 and q_2 associated with $\bar{\vartheta}_0$ and choose their future shares optimally. Given $\bar{\vartheta}_0$, markets clear if $q_1 = (16 + 5\bar{\vartheta}_0)y_1/(8 - 5\bar{\vartheta}_0)$ and $q_2 = (26 + 7\bar{\vartheta}_0)y_2/(13 - 7\bar{\vartheta}_0)$. A's demand for shares is $\vartheta_1^A = 7\bar{\vartheta}_0^A(q_1 + y_1)/(8q_1)$ and $\vartheta_2^A = 11\bar{\vartheta}_0^A(q_2 + y_2)/(13q_2)$ and B's demand is $\vartheta_1^B = 21\bar{\vartheta}_0^B(q_1 + y_1)/(3q_1)$ and $\vartheta_2^B = 2\bar{\vartheta}_0^B(q_2 + y_2)/(3q_2)$.

Observe that planner P can manipulate the equilibrium prices via $\bar{\vartheta}_0$. Therefore, P's decision has a long lasting impact on the equilibrium allocation. *This feedback effect is ignored by the consumers because they act as price takers.*

P's welfare $\mathcal{W}^P(\bar{\vartheta}_0)$ is the total utility resulting from the responses of the consumption sector to his choice $\bar{\vartheta}_0$. Because y^* is fixed and transfers leave $\mathcal{W}^P$ invariant, $\mathcal{W}^P$ depends only on $\bar{\vartheta}_0$. $\mathcal{W}^P$ takes its maximum at $\bar{\vartheta}_0 \approx 0.77815$. However, if consumers can choose their shares freely on the initial stock market, A chooses $\vartheta_0^A(\hat{y}^*) = 7/9 = 0.777....$ The derivative of $\mathcal{W}^P$ at $\vartheta_0^A(\hat{y}^*) = 7/9$ is about $0.055 > 0$. Therefore, P achieves a first order Pareto improvement by raising ϑ_0^A above the level of $7/9$.[7]

The underlying intuition is as follows. Consider an infinitesimal increase of ϑ_0^A at $\vartheta_0^A = 7/9$. Then consumer A's infinitesimal utility loss of about 1.139 is more than compensated by B's utility gain of about 1.195. It is a rare coincidence that the marginal utility gains and losses caused by infinitesimal changes $\Delta\vartheta_0^i$ with $\sum_i \Delta\vartheta_0^i = 0$ happen to cancel out. In general, these changes or the changes in the opposite direction entail a Pareto improvement.

We conclude that the concept of constrained efficiency should be combined with the requirement that the planner cannot exploit feedback effects to overrule future consequences of decisions made by consumers on the basis of their characteristics including their initial endowments before transfers. That is to say, the planner should only be allowed to choose the output vector and to redistribute good 0 *after* all market transactions have been determined.

This efficiency concept has been introduced in Dierker et al. (2005) in a two-period model under the name of minimal (constrained) efficiency.[8] In a two-period model, the first order conditions for constrained efficiency and for minimal efficiency coincide. This ceases to be true if there are more than two periods because of the feedback effect between the shares ϑ_0^i and future shareholdings. The concept of minimal efficiency takes all market transactions explicitly into account. For the multiperiod case, see Section 5.

We are now going to compute the firm's state price vector $\pi(\hat{y}^*)$ with the aid of the state prices of the consumers. First, we focus on the intermediate node ξ_1 with an active stock market. An infinitesimal change of y_1 changes the stock price q_1. The resulting utility changes need to be taken into account on the individual level because of market incompleteness.

We consider infinitesimal output changes tangent to $\mathcal{E}$ and insert the equilibrium value of $C \equiv 27$ into the definitions of q_1 and q_2 so that we have $q_1 = 179q_1/37$ and $q_2 = 283y_2/68$.

When we differentiate $x_+^A(\hat{y}, g(\hat{y}))$ at $\hat{y}^*$ with respect to y_1 we obtain the vector $(21/37, 0, 0, 0, 0, -0.704)$. The first coordinate measures the direct effect, the last coordinate represents the indirect effect through $\partial_1 g(\hat{y}^*)$. We evaluate the consumption changes coordinatewise with A's stochastic discount factors $DV^A(\hat{y}^*) \approx (0.557, 0.836, 0.609, 0.726, 0.694, 0.771)$ and obtain $(0.316, 0, 0, 0, 0, -0.542)$.

Similarly, we use $DV^B(\hat{y}^*) \approx (0.731, 0.675, 0.932, 0.834, 0.616, 0.571)$ to evaluate B's consumption changes $(16/37, 0, 0, 0, 0, -0.158)$ and obtain B's evaluation $(0.316, 0, 0, 0, 0, -0.09)$. The sum 0.632 of the first coordinates of A's and of B's evaluations is the firm's state price π_1^*. The last coordinate -0.632 is $\partial_1 g(\hat{y}^*)\pi_6(\hat{y}^*)$. Thus $\pi_6(\hat{y}^*) \approx 0.632/0.862 \approx 0.734$. The sum 0.632–0.632 of all six coordinates vanishes because $\partial_1 \mathcal{W}\hat{y}^* = 0$ and the cost is constant on $\mathcal{E}$.

[7] I am particularly thankful to Larry Blume and Klaus Ritzberger for valuable discussions about this feedback effect.

[8] A formal definition for the multiperiod case will be given in Section 5.

In the example, the cost C stays constant on $\mathcal{E}$. Hence, marginal costs vanish and the first order condition for welfare maximization reduces to

$$\pi_s(\hat{y}^*) + \pi_6(\hat{y}^*)\partial_s g(\hat{y}^*) = 0 \text{ for } s = 1, \ldots, 5. \tag{8}$$

Because $\pi_6 \approx 0.734$ and $Dg(\hat{y}^*) \approx (-0.862, -1.056, -0.909, -1.016, -0.926)$, the firm's stochastic discount factors are given by

$$\pi(y^*) \approx (1, 0.632, 0.775, 0.667, 0.745, 0.679, 0.734),$$

which is in accordance with (7).

In the two-period model, the firm's state prices are given by $\pi_s = \sum_i \pi_s^i \vartheta^i$ and first order changes in the consumption of good s can be ignored. This is no longer correct if there is sequential trade. In the example, one has to replace $\pi_s = \sum_i \pi_s^i \vartheta^i$ by

$$\pi_s(\hat{y}^*) = \sum_{i=A,B} \pi_s^i(\hat{y}^*)\ \partial_s x_s^i(\hat{y}^*) \tag{9}$$

for $s = 1, 2$ where portfolios are readjusted. However, as in the two-period model, $\pi_s(\hat{y})$ does not involve changes in the consumption of goods other than s.

For comparison, we determine the *Walrasian equilibrium* in the example under the assumption of complete markets as in Chapter 7 of Debreu (1959).[9] At a Walrasian equilibrium, the firm maximizes profits with respect to marginal cost prices and consumers maximize utility with respect to the same price system. In this example, the Walrasian equilibrium price system coincides with the firm's state price system $\pi(\hat{y}^*)$ as listed in (7). Therefore, the production plans are equal but the consumption plans are different. A's Walrasian equilibrium consumption is (approximatively) equal to $(4, 1.581, 2.582, 4.5, 5.366, 7.360, 8.177)$ and B's equilibrium consumption equals $(9, 1.581, 1.291, 1.5, 1.342, 1.472, 1.363)$. The associated utility levels are 40.176 for A and 11.109 for B so that social welfare becomes 51.285.

Recall that the initial endowments of good 0 are $e_0^A = 25, e_0^B = 15$. If one evaluates the consumption plans in the incomplete market case with the Walrasian price system one obtains the value $24.9904 < e_0^A = 25$ for A and $15.0096 > e_0^B = 15$ for B which amounts to a redistribution from A to B.

The redistribution in favor of B entails that B *prefers incomplete over complete markets* although social welfare increases when markets become complete. In the incomplete market model, A reaches a utility level of 40.130 and B of 11.003 so that social welfare becomes 51.133.

[9] I am grateful to Leopold Sögner for the suggestion to illustrate the difference between the complete and the incomplete market case with the aid of numerical examples.

3 Departures from additive separability.

The state price formula (8) has the property that the price for state s does only involve i's marginal utility of good s and i's changes in consumption of good s. When one gives up additive separability, cross effects between goods come into play. The goal of this section is to illustrate how the analysis changes. This will be done with the aid of two examples. The first and simpler one connects the goods at ξ_1 and its successor ξ_3 in A's utility function in a non additively separable way. The second example leaves both utility functions of Section 2 unaltered but assumes that costs are not only paid at the initial node but also at the intermediate nodes 1 and 2.

3.1 A not additively separable utility function.

We use the following utility functions

$$\begin{aligned} U^A(x_0, x_1, \ldots, x_6) &= x_0 + \log(x_1 + x_3) + \log(x_2) \\ &\quad + \log(x_4) + \log(x_5) + \log(x_6) \\ U^B(x_0, x_1, \ldots, x_6) &= x_0 + \log(x_1) + \log(x_2) + \log(x_3) \\ &\quad + \log(x_4) + \log(x_5) + \log(x_6), \end{aligned} \tag{10}$$

where U^A is no longer additively separable. The initial endowments are $e^A = e^B = (30, 0, \ldots, 0)$. For simplicity's sake, the cost of producing $y_+ = (y_1, \ldots, y_6)$ is defined as

$$C = -y_0 = y_1 + \cdots + y_6. \tag{11}$$

Observe that the vector $(1, 1, \ldots, 1)$ is perpendicular to the technology at every $y_+ \gg 0$ and cannot help to locate the optimal production plan.

As before, consumers can obtain shares ϑ_0^i at $t = 0$ by contributing $\vartheta_0^i C$ to the firm's cost. Furthermore, they anticipate the market clearing prices q_1 and q_2 prevailing at the intermediate nodes. Consumer A maximizes his utility by choosing $\vartheta_0^A = 5/C, \vartheta_1^A = 5(q_1+y_1)/(2C(q_1-y_3))$, and $\vartheta_2^A = 10(q_2+y_2)/(3Cq_2)$.[10] Consumer B chooses $\vartheta_0^B = 6/C, \vartheta_1^B = 4(q_1+y_1)/(Cq_1)$, and $\vartheta_2^B = 4(q_2+y_2)/(Cq_2)$.

As in Section 2, we use the equilibrium values of the production cost in the calculations. When we insert $C = 5 + 6 = 11$ we obtain

$$q_1 = \frac{1}{18}(13y_1 + 14y_3 + \sqrt{169y_1^2 + 76y_1y_3 + 196y_3^2}) \quad \text{and} \quad q_2 = 2y_2. \tag{12}$$

Observe that y_1 and y_3 play similar roles in the determination of q_1.

We use $y_6 = g(\hat{y}) = 11 - (y_1 + \cdots + y_5)$ and (4), (5), and (6) to define the welfare function $\mathcal{W}(\hat{y})$, solve the first order condition $D\mathcal{W}(\hat{y}) = 0$ for welfare maximization numerically, and obtain $y_+^* \approx (0.989, 2, 2.011, 2, 2, 2)$.

[10] Note that y_3 enters into the denominator of ϑ_1^A due to the term $\log(x_1 + x_3)$ in A's utility function.

Proposition 3. *The stock market equilibrium at y^* in the example does not satisfy the first order condition for slight efficiency.*

The proof is an adaptation of the proof of Proposition 2 to the present example. The assignment of $\vartheta^A = \bar{\vartheta}_0$ and $\vartheta^B = 1 - \bar{\vartheta}_0$ at $t = 0$ causes a feedback effect at $t \geq 1$ which entails $\vartheta_1^A = \bar{\vartheta}_0(q_1 + y_1)/(2(q_1 - y_3))$ for A and $\vartheta_2^A = 2\bar{\vartheta}_0(q_2 + y_2)/(3q_2)$ and $\vartheta_1^B = 2(1-\bar{\vartheta}_0)(q_1+y_1)/(3q_1)$ and $\vartheta_2^B = 2(1-\bar{\vartheta}_0)(q_2+y_2)/(3q_2)$ for B. The market clearing prices $q_1 = (\bar{\vartheta}_0(y_1 - 2y_3) + 2y_3)/(2 - 3\bar{\vartheta}_0)$ and $q_2 = 2y_2$ depend on planner P's choice of $\bar{\vartheta}_0$. The derivative of $\mathcal{W}^P$ at $\vartheta_0^A(\hat{y}^*) = 5/11$ equals 0.357. That is to say, P wants to move shares ϑ_0^i from B to A.

The equilibrium utility gradients are

$$\begin{aligned} DU^A(\hat{y}^*) &= \pi^A(\hat{y}^*) \approx (1, 0.860, 11/10, 0.860, 0.910, 11/10, 11/10) \\ DU^B(\hat{y}^*) &= \pi^B(\hat{y}^*) \approx (1, 1.074, 11/12, 1.103, 1.109, 11/12, 11/12). \end{aligned} \tag{13}$$

The BL price system $(1, 0.977, 1, 0.970, 1, 1, 1)$ differs from $\pi(\hat{y}^*) = (1, 1, \ldots, 1)$ not only at the intermediate node ξ_1 but also at the final node ξ_3 because q_1 depends on y_1 and y_3 due to the term $\log(x_1 + x_3)$ in A's utility function. The BL price of node ξ_2 equals 1 because there is no trade at ξ_2, that is to say, $\vartheta_0^i = \vartheta_2^i$ for $i = A, B$ due to the special nature of the utility functions.

In the context of the previous, additively separable example, we have obtained equation (8) according to which the firm's state price $\pi_s(\hat{y}^*)$ involves only the consumption changes $\partial_s x_s^i(\hat{y}^*)$ occurring at ξ_s. When additive separability is violated as in the present example, however, the state prices $\pi_s(\hat{y}^*)$ tend to involve consumption changes of goods other than s.

It is instructive to compute the state price $\pi_1(\hat{y}^*)$ step by step. A's consumption changes of goods 1 to 6 at $\hat{y}^*$ that are induced by an infinitesimal increase of y_1 (without the induced change of y_6) are about $(0.613, 0, -0.129, -0.129, 0, 0)$, those of B are $(0.387, 0, 0.129, 0.129, 0, 0)$. If one evaluates these changes coordinatewise with the corresponding state prices of the consumers, one obtains for A the utility changes $(0.527, 0, -0.111, -0.117, 0, 0)$. The values for B are $(0.416, 0, 0.143, 0.143, 0, 0)$. Therefore A's marginal benefit of an infinitesimal increase of y_1 amounts to $0.527 - 0.111 - 0.117 = 0.299$ and B's to $0.416 - 2 \cdot 0.1425 = 0.701$ when one ignores the indirect effect via $y_6 = g(\hat{y})$. The firm's state price is $\pi_1(\hat{y}^*) = \sum_{i=A,B} \sum_{\sigma=1}^{6} \pi_\sigma^i \partial_1 x_\sigma^i = 0.299 + 0.701 = 1$.

More generally, consider the production plan $y_+(\hat{y}^*) = (\hat{y}, g(\hat{y}))$ and let $x^{i*} = x^i(y_+(\hat{y}^*))$. Consumer i's marginal benefit of a change of y_s, $s = 1, \ldots, 5$, depends on his utility gradient $DU^i(x^{i*})) = (1, \pi_1^{i*}, \ldots, \pi_6^{i*})$ and the induced consumption changes. When we differentiate $x_\sigma^i(\hat{y}, g(\hat{y}))$ with respect to y_s, we obtain $\partial_s x_\sigma^i + \partial_6 x_\sigma^i\, \partial_s g$, where $\sigma = 1, \ldots, 6$ and $s = 1, \ldots, 5$. Therefore, i's marginal benefit equals $\sum_{\sigma=1}^{6} \pi_\sigma^i (\partial_s x_\sigma^i + \partial_6 x_\sigma^i\, \partial_s g)$ and the marginal social benefit of a change of y_s combined with the corresponding change of y_6 is $\sum_i \sum_{\sigma=1}^{6} \pi_\sigma^i (\partial_s x_\sigma^i + \partial_6 x_\sigma^i\, \partial_s g)$ where $s = 1, \ldots, 5$. The direct effect of an infinitesimal increase of y_s, $s = 1, \ldots, 5$, on social benefit is

$$\pi_s = \sum_i \partial_s v^i = \sum_i \sum_{\sigma=1}^{6} \pi_\sigma^i \partial_s x_\sigma^i \tag{14}$$

and the indirect effect via $g(\hat{y})$ is $\pi_6\ \partial_s g = \sum_i \sum_{\sigma=1}^{6} \pi^i_\sigma \partial_6 x^i_\sigma\ \partial_s g$. Altogether, the total marginal social benefit is given by $\pi_s + \pi_6\ \partial_s g$.

In the example, the first order condition for welfare maximization requires that $\pi_s + \pi_6\ \partial_s g = 0$ for $s = 1, \ldots, 5$, because the cost is constant on $\mathcal{E}$.

This leads us to the question of how much information is required in the multiperiod case. In the two-period case, the Drèze rule is formulated without reference to $\mathcal{E}$. Its generalization to multiperiod models requires at least local information about $\mathcal{E}$ at $\hat{y}^*$ and this is encapsulated in $Dg(\hat{y}^*)$ as in equation (3).

3.2 Costs at intermediate nodes cause a wedge between the evaluations of the consumers and the producer.

In the complete market case, costs can occur at different time periods. When markets are incomplete there is also no need to assume that all costs are borne in the initial period. Therefore, we modify the example from Section 2 such that costs are paid at all non-final nodes. We denote the costs paid additionally at the intermediate nodes $\xi_s, s = 1, 2$, by C_s.

We rename the initial cost as $C_0 = -y_0$. Then $i = A, B$ consumes $(e^i_0 - \vartheta^i_0 C_0, \vartheta^i_0 y_1 + (\vartheta^i_0 - \vartheta^i_1)q_1 - \vartheta^i_1 C_1, \vartheta^i_0 y_2 + (\vartheta^i_0 - \vartheta^i_2)q_2 - \vartheta^i_2 C_2, \vartheta^i_1 y_3, \vartheta^i_1 y_4, \vartheta^i_2 y_5, \vartheta^i_2 y_6)$. Consumer A's shares are given by

$$\vartheta^A_0 = \frac{21}{C_0}, \quad \vartheta^A_1 = \frac{147(q_1 + y_1)}{8C_0(q_1 + C_1)}, \quad \vartheta^A_2 = \frac{231(q_2 + y_2)}{13C_0(q_2 + C_2)}.$$

Consumer B's shares are

$$\vartheta^B_0 = \frac{6}{C_0}, \quad \vartheta^B_1 = \frac{4(q_1 + y_1)}{C_0(q_1 + C_1)}, \quad \vartheta^B_2 = \frac{4(q_2 + y_2)}{C_0(q_2 + C_2)}.$$

The market clearing share prices are

$$q_1 = \frac{179y_1 - 8C_0C_1}{8C_0 - 179}, \qquad q_2 = \frac{283y_2 - 13C_0C_2}{13C_0 - 283}.$$

To be specific, we assume

$$C_1 = \sqrt{y_3^2 + y_4^2} \quad \text{and} \quad C_2 = \frac{1}{2}\sqrt{y_5^2 + y_6^2}.$$

and choose $e^A_0 = 25, e^B_0 = 15$ in the numerical example. As before, social welfare $\mathcal{W}$ is defined by equation (6). We solve the first order condition $D\mathcal{W}(\hat{y}) = 0$ for welfare maximization and obtain $\hat{y}^* \approx (8.279, 8.865, 3.566, 3.987, 6.468)$. The last coordinate of the production plan y^* is $y^*_6 = g(\hat{y}^*) \approx 6.986$. The consumption plans are $x^A(\hat{y}^*) \approx (4, 1.663, 2.535, 2.929, 3.274, 5.280, 5.703)$ and $x^B(y^*) \approx$

$(9, 1.267, 1.569, 0.638, 0.713, 1.188, 1.284)$. We distinguish between the *gross production* plan and the *net production* plan which are, respectively,

$$\begin{aligned} y^*_{gross} =& (y_0, y_1, y_2, \ldots, y_6)(\hat{y}^*) \\ \approx& (-27, 8.279, 8.865, 3.566, 3.987, 6.468, 6.986) \\ y^*_{net} =& (y_0, y_1 - C_1, y_2 - C_2, \ldots, y_6)(\hat{y}^*) \\ \approx& (-27, 2.930, 4.105, 3.566, 3.987, 6.468, 6.986). \end{aligned} \tag{15}$$

The stochastic discount factors of the consumers are given by

$$\begin{aligned} \pi^A(\hat{y}^*) &\approx (1, 0.601, 0.789, 1.024, 1.222, 0.947, 1.052) \\ \pi^B(\hat{y}^*) &\approx (1, 0.789, 0.637, 1.569, 1.403, 0.841, 0.779). \end{aligned} \tag{16}$$

We do not intend to replicate the analysis of the feedback effect in the previous subsection. Instead we want to shed light on the novelty of this subsection, that is to say, the difference between gross and net production.

A short calculation shows that A's as well as B's evaluation of the net production plan vanish. That is to say,

$$\pi^A(\hat{y}^*)\ y^*_{net} = \pi^B(\hat{y}^*)\ y^*_{net} = 0. \tag{17}$$

We use formula (14) to compute the firm's state prices. When one differentiates the social benefit $v^A + v^B$ with respect to $y_1, \ldots, y_6$ one obtains the vector $(0.683, 0.731, 0.667, 0.745, 0.679, 0.734)$. The social benefit takes the costs C_1 and C_2 into account. Optimality requires that the social benefit accruing at $t \geq 1$ is equal to the marginal cost $DC_0(y^*_+)$ paid at $t = 0$. A simple calculation shows that this is the case. The firm's state price system is

$$\pi(\hat{y}^*) \approx (1, 0.683, 0.731, 0.667, 0.745, 0.679, 0.734). \tag{18}$$

Concerning the two intermediate nodes ξ_1 and ξ_2 we search for a γ such that $\gamma(0.601, 0.789) + (1-\gamma)(0.789, 0.637) = (0.683, 0.731)$. One checks easily that the equation has no solution. Therefore, Proposition 1 applies also to the present case.

The firm's evaluation of the *gross* production plan vanishes, that is to say

$$\pi(\hat{y}^*)\ y^*_{gross} = 0, \tag{19}$$

whereas $\pi(\hat{y}^*)\ y^*_{net}$ is negative. Equations (17) and (19) show:

Proposition 4. *The costs C_1 and C_2 at the intermediate nodes act like a tax that introduces a wedge between the evaluations of the consumers and the producer.*

The intuition behind Proposition 4 can be described as follows. We have interpreted $C_1(y^*_+)$ and $C_2(y^*_+))$ as technologically unavoidable costs. Suppose, however, that only $C_0(y^*_+)$ is actually needed to produce the output y^*_+ and that

$C_1(y_+^*)$ and $C_2(y_+^*)$ are the amounts of the intermediate goods that a foreign owner of the firm extracts from the economy for personal consumption in his own country. Both interpretations explain a loss of social benefit. The consumers are left with the net production plan whereas the firm has to provide the gross production plan. Because the costs C_1 and C_2 are not available for consumption, they are accounted for in the social benefit and should be disregarded by the firm.

As at the end of Section 2, we look at a *Walras equilibrium* in the case of complete markets. The firm's technology is given by all net production plans $\eta = (\eta_0, \eta_1, \ldots, \eta_6) \in \mathbb{R}_- \times \mathbb{R}_+^6$ such that $\eta_0 \leq C_0(y_+), \eta_1 \leq y_1 - C_1(y_+),\ \eta_2 \leq y_2 - C_2(y_+)$, and $\eta_s = y_s$ for $s \geq 3$ for some gross output $y_+ \geq 0$. The technology is convex because $-C_0, -C_1$ and $-C_2$ are concave functions.

The firm aims to maximize profits with respect to a price system which is equal to the utility gradient of each consumer in equilibrium. Using A's utility gradient, we define $C = C_0 + (1/x_1^A)C_1 + (2/x_2^A)C_2$. Because consumers' utility gradients are equal in a Walras equilibrium, that is to say $x_1^B = x_1^A, x_2^B = x_2^A/2, \ldots, x_6^B = x_1^A/6$, we can eliminate every x_s^B from the market clearing conditions for goods 1 to 6 and solve for the output vector y_+. This leads to

$$y_1 = 2x_1^A + \sqrt{\frac{16(x_3^A)^2}{9} + \frac{25(x_4^A)^2}{16}},\ y_2 = \frac{1}{2}\left(3x_2^A + \sqrt{\frac{36(x_5^A)^2}{25} + \frac{49(x_6^A)^2}{36}}\right)$$

and $y_3 = 4x_3^A/3,\ y_4 = 5x_4^A/4,\ y_5 = 6x_5^A/5,\ y_6 = 7x_6^A/6$. When this substitution is used the derivative $DC(y_+)$ of the total cost function becomes a function of $x_1^A, \ldots, x_6^A$.

In a Walras equilibrium, $(1, DC(y_+))$ equals A's utility gradient. Solving this condition numerically we obtain that A's equilibrium consumption is given by $(1, 1.465, 2.737, 2.675, 3.190, 5.390, 5.988)$. The associated price system is

$$p^* = (1, 0.683, 0.731, 1.122, 1.254, 0.928, 1.002) \tag{20}$$

and *the profit maximizing production plan* η^* *equals* y_{net}^* in (15). It is easy to check that $p^* y_{net}^* = 0$ so that y_{net}^* maximizes profits given p^*.

For comparison, in the incomplete market case the firm's price system in (18) is $(1, 0.683, 0.731, 0.667, 0.745, 0.679, 0.734)$. Observe that prices of goods 1 and 2 are the same in (18) and (20). However, when markets are complete the equilibrium prices of goods 3 and 4 are 68.3% and those of goods 5 and 6 are 36.5% higher than the corresponding prices in (18).

The underlying reason is as follows. The firm's prices in (18) are the marginal costs C_0 paid at $t = 0$ because C_1 and C_2 are taken into account by the marginal benefit. When markets are complete equilibrium prices can be used to aggregate all cost components into one total cost. Therefore, the Walrasian prices in (20) are the derivative of the total cost $C = C_0 + p_1^*\ C_1 + p_2^*\ C_2$ at $\eta^* = y_{net}^*$. The price ratios of the final goods in (20) and in (18) correspond to

$$\left(\frac{\partial_3 C}{\partial_3 C_0}, \frac{\partial_4 C}{\partial_4 C_0}, \frac{\partial_5 C}{\partial_5 C_0}, \frac{\partial_6 C}{\partial_6 C_0}\right)(y_{net}^*) \approx (1.683, 1.683, 1.365, 1.365).$$

4 Welfare maximization without additive separability and quasilinearity.

In the quasilinear examples considered above, it is natural to maximize social surplus measured in units of good 0. Social surplus maximization presents an example of the maximization of a utilitarian welfare function that is based on the principle that one additional unit of good 0 raises the welfare by an amount that is independent of the person that receives this unit.

In this section, we analyze an example in which consumer A does not possess a quasilinear utility function. This can be taken into account in the same way as in the two-period model. To convert i's utility gradient into i's present value vector we divide U^i by the marginal utility of good 0.

The utility functions are as follows:

$$\begin{aligned} U^A(x_0, x_1, \ldots, x_6) &= 10\log(x_0) + 2\log(3x_1 + x_3) + \log(2x_2 + x_6) \\ &\quad + \log(x_4) + \log(x_5) \\ U^B(x_0, x_1, \ldots, x_6) &= x_0 + \log(2x_1 + x_4) + \log(2x_2 + x_5) \\ &\quad + \log(x_3) + \log(x_6), \end{aligned} \tag{21}$$

Each consumer has an initial endowment of 30 at ξ_0 and no initial endowments elsewhere. The cost function is given by equation (11).

Consumer A's demand for shares is $\vartheta_0^A = 10/C$, $\vartheta_1^A = \vartheta_0^A(q_1 + y_1)/(3q_1 - y_3)$, and $\vartheta_2^A = \vartheta_0^A(q_2 + y_2)/(2q_2 - y_6)$. Consumer B demands $\vartheta_0^B = 4/C$, $\vartheta_1^B = \vartheta_0^B(q_1 + y_1)/(2q_1 - y_4)$, and $\vartheta_2^B = \vartheta_0^B(q_2 + y_2)/(2q_2 - y_5)$. The scale of production is given by the market clearing condition $\vartheta_0^A + \vartheta_0^B = 10/C + 4/C = 1$ which entails that the costs are identically equal to 14. Thus, any $\hat{y} = (y_1 \ldots y_5)$ determines the amount $y_6 = g(\hat{y}) = 14 - y_1 \cdots - y_5$.

Consumer A's consumption at the initial node equals $x_0^A = 30 - \vartheta_0^A C = 30 - 10 = 20$. Therefore, A's marginal utility $\partial_0 U^A$ is identically equal to $10/x_0^A = 1/2$. If we multiply U^A by 2 then the marginal utility of good 1 is normalized to 1. Let $u^i(\hat{y}) = U^i(x^i(\hat{y}))$ be i's indirect utility function and define:

Definition 2. *The* social welfare function $\mathcal{W}$ *is given by*

$$\mathcal{W}(\hat{y}) = 2u^A(\hat{y}) + u^B(\hat{y}).$$

In the previous examples, social welfare is determined independently of the location of the welfare maximum. In general, this is not the case because $\partial_0 U^i(x)$ depends on x. Then one can proceed as in the two-period case. That is to say, one rules out production plans at which first-order Pareto improvements can be achieved when the production plan is varied and infinitesimal transfers of good 0 are made.

When one solves the first order condition $D\mathcal{W}(\hat{y}) = 0$ numerically one obtains $\hat{y}^* \approx (4.248, 2.085, 1.245, 2.503, 2.459)$. Furthermore, $y_6^* = g(\hat{y}^*) \approx 1.456$.

The shareholdings $(\vartheta_0^A, \vartheta_0^B) = (5/7, 2/7)$ at the initial node ξ_0 are not optimal for planner P. This planner, who cannot change the production plan, can improve social welfare by giving some of A's shares to B. Thus, one expects that the BL rule is violated.

A's state price system is $\pi^A \approx (1, 1.033, 0.941, 0.344, 1.431, 1.201, 0.471)$ and B's is $\pi^B \approx (1, 0.861, 1.177, 1.814, 0.430, 0.588, 2.129)$. It is easy to check that the system of equations $\lambda_s \pi_s^A + (1 - \lambda_s)\pi_s^B = 1$ for $s = 1, \ldots, 6$ has the following property: For each pair of nodes with the same immediate predecessor the solutions are different. That is to say, $\lambda_1 \neq \lambda_2$, $\lambda_3 \neq \lambda_4$, and $\lambda_5 \neq \lambda_6$, so that there cannot be any weighted gradient formula where the immediate successors of at least one node have the same weight.

The BL price system $(1, 0.984, 1.009, 0.993, 0.989, 1.003, 1.006)$ differs from the technology gradient at every coordinate $s \geq 1$. This reflects the complete lack of additive separability of the pair of utility functions.

The example illustrates the difficulty to find intuitively plausible price perceptions that can be used to specify the objective of a firm in multiperiod models with incomplete markets in a satisfactory manner. Local information about $\mathcal{E}$ can hardly be detected by individual introspection. This fact is a consequence of a failure of markets with price taking behavior to coordinate consumers' decisions in a constrained efficient way when sequential trade is allowed. Otherwise one could use the BL rule which is far less complex because it is as simple as the original Drèze rule applied to each non-terminal node.

Although this conclusion appears disappointing, it does not mean that the solutions that we have obtained in our examples cannot be derived from an efficiency goal. This will be the topic of the next section.

5 How can one derive a generalized Drèze rule from efficiency considerations?

Drèze (1974) and Magill and Quinzii (1996) emphasize the importance of constrained efficiency for two period models of incomplete markets with production. On the other hand, we have analyzed examples in which firms are unable to achieve a slightly efficient allocation. This leads to the question of how one can reconcile our approach with the original intention to satisfy an appropriate efficiency requirement.

For this reason, we adapt the approach taken by E. and H. Dierker (2010) in a two period setting to the multiperiod framework. As before, we consider a single firm with constant returns to scale. There is a finite date-event tree with dates $t = 0, 1, \ldots, T$, states $s = 0, 1, \ldots, S$ and date-event pairs or nodes (t, s) denoted by $\xi_0, \xi_1, \ldots, \xi_S$. The node ξ_0 is the root of the tree. At each intermediate node $\xi_s \neq \xi_0$, shares are traded at the market clearing price q_s. For simplicity, we assume that no costs occur at $t \geq 1$ so that the price at ξ_0 equals the production cost C.

The tree structure can be used to impose separability restrictions on the utility functions under consideration. If the tree structure is such that the consumption of good s precludes the consumption of s' then one would like that s and s' are additively separated in the utility function. Although it appears natural, we do not make this assumption because it does not impact our arguments.

Consider the production plan $y = (y_0, y_+) = (y_0, y_1, y_2, \dots, y_S)$. Consumer i consumes $x_0^i = e_0^i - \vartheta_0^i C$ at the initial node ξ_0. At each intermediate node ξ_s, the consumption of i equals $x_s^i = e_s^i + q_s(\vartheta_{s\_}^i - \vartheta_s^i) + \vartheta_{s\_}^i y_s$, where $\xi_{s\_}$ is the immediate predecessor of ξ_s. If ξ_s is a terminal node then i consumes $x_s^i = e_s^i + \vartheta_{s\_}^i y_s$.

At a stock market equilibrium, all stock markets clear. *Stock market equilibria* are elements of

$$\mathcal{E} = \{y_+ \in \mathbb{R}_{++}^S \mid \sum_i \vartheta_s^i(y_+) = 1 \text{ for all non-terminal states } s\}.$$

Because the stock price at ξ_0 equals the production cost C we have one equation more than is needed to determine all stock market prices. Under the rank assumption of the implicit function theorem, one can express one coordinate of y_+ locally as a function of the others. For the ease of notation, we assume that $y_S = g(\hat{y})$ where $\hat{y} = (y_1, \dots, y_{S-1})$. We shall consider stock market equilibria in a neighborhood of a reference stock market equilibrium with output vector $y_+^* = (\hat{y}^*, g(\hat{y}^*)) \gg 0$.

As before, all functions have $\hat{y}$ directly or indirectly as their argument so that the restriction to the stock market equilibria is incorporated in the notation. At a stock market equilibrium, consumer $i = 1, \dots, I$ consumes the vector $x^i(y_+(\hat{y})) \in \mathbb{R}^{S+1}$ with coordinates $x_s^i(y_1, \dots, y_{S-1}, g(y_1, \dots, y_{S-1}))$, where $s = 0, \dots, S$.

Assumption . *There is an open neighborhood O of $\hat{y} = (\hat{y}_1, \dots, \hat{y}_{S-1})$ and a C^1-function $g : O \to \mathbb{R}$ such that $y_+(\hat{y}) = (\hat{y}, g(\hat{y}))$ is an equilibrium output plan. All functions used in the description of an equilibrium are C^1.*

Consider a planner who can choose the stock market equilibrium and redistribute the total consumption $\sum_i x_0^i(y_+^*)$ at $t = 0$ by assigning to i the amount c_0^i. A minimally efficient stock market equilibrium must not be Pareto dominated by an allocation resulting from some other stock market equilibrium followed by a redistribution of consumption at $t = 0$.

Definition 3. *The stock market equilibrium associated with y_+^* is minimally constrained efficient, or* minimally efficient *for short, iff there is no y_+ with associated equilibrium consumption plans $(x_0^i(y_+), x_+^i(y_+))_{i=1,\dots,I}$ such that the allocation $(c_0^i, x_+^i(y_+))_{i=1,\dots,I}$ is Pareto preferred to $(x_0^i(y_+^*), x_+^i(y_+^*))_{i=1,\dots,I}$ where $\sum_i c_0^i = \sum_i x_0^i(y_+^*)$.*

Next we introduce a way to measure deviations from minimal efficiency that falls into the tradition of the four Hicksian surplus concepts, cf. Hicks (1956) and

E. and H. Dierker (2010). Consumer i's *compensating surplus* $CS^i_{\hat{y}^*}(y_+(\hat{y}))$ is the amount of good 0 which i has to lose after the move from the reference stock market equilibrium associated with $\hat{y}^*$ to the alternative equilibrium associated with $\hat{y}$. That is to say,

$$U^i(x^i_0(y_+(\hat{y})) - CS^i_{\hat{y}^*}(y_+(\hat{y})), x^i_+(y_+(\hat{y}))) = U^i(x^i(y_+(\hat{y}^*))). \tag{22}$$

Definition 4. *The* total compensating surplus *associated with the change from $\hat{y}^*$ to $\hat{y}$ is*

$$CS_{\hat{y}^*}(y_+(\hat{y})) = \sum_{i=1}^{I} CS^i_{\hat{y}^*}(y_+(\hat{y})).$$

The total compensating surplus $CS_{\hat{y}^*}(y_+(\hat{y}))$ can be interpreted as the amount of good 0 that can be taken out of the economy at $\hat{y}$ without making any consumer worse off than at $\hat{y}^*$. Minimal efficiency prevails if this amount is nowhere positive.

Remark . *The equilibrium associated with $\hat{y}^*$ is minimally efficient iff*

$$CS_{\hat{y}^*}(y_+(\hat{y})) \leq 0 = CS_{\hat{y}^*}(y_+(\hat{y}^*)).$$

Thus, minimal efficiency of y^* is based on the maximization of the total compensating surplus $CS_{\hat{y}^*}$. In the presence of income effects, $CS^i_{\hat{y}^*}$ is typically not a utility function because different indifference classes must be disjoint. Hence, $CS_{\hat{y}^*}$ is typically not a social welfare function.

When we differentiate $x^i_\sigma(\hat{y}, g(\hat{y}))$ with respect to $y_s, s = 1, \ldots, S-1$, we obtain $\partial_s x^i_\sigma + \partial_S x^i_\sigma \; \partial_s g$. Similarly, we have $\partial_s CS^i_{\hat{y}^*} + \partial_S CS^i_{\hat{y}^*} \; \partial_s g$. We normalize all utility gradients such that the marginal utility of good 0 at $\hat{y}^*$ equals 1 in order to obtain the consumers' present value vectors $\pi^i(x^i(y_+(\hat{y}^*)))$. We differentiate (22) and obtain, for $s = 1, \ldots, S-1$,

$$\partial_s(x^i_0 - CS^i_{\hat{y}^*}) + \partial_S(x^i_0 - CS^i_{\hat{y}^*}) \; \partial_s g + \sum_{\sigma=1}^{S} \pi^i_\sigma(\partial_s x^i_\sigma + \partial_S x^i_\sigma \; \partial_s g) = 0.$$

When we sum over all consumers and use the definition of x^i_0 we obtain

$$\begin{aligned} &\sum_{i=1}^{I} \left(\partial_s CS^i_{\hat{y}^*} + \partial_S CS^i_{\hat{y}^*} \; \partial_s g\right) = \\ &\partial_s y_0 + \partial_S y_0 \; \partial_s g + \sum_{i=1}^{I} \sum_{\sigma=1}^{S} (\pi^i_\sigma \; \partial_s x^i_\sigma + \pi^i_\sigma \; \partial_S x^i_\sigma \; \partial_s g). \end{aligned} \tag{23}$$

As in (14), state prices are defined by $\pi_s = \sum_i \partial_s v^i = \sum_i \sum_\sigma \pi^i_\sigma \; \partial_s x^i_\sigma$. The first order condition for minimal efficiency states that (23) vanishes for $s = 1, \ldots, S-1$, that is to say,

$$-(\partial_s y_0 + \partial_S y_0 \; \partial_s g) = \pi_s + \pi_S \; \partial_s g. \tag{24}$$

Equation (24) has the following interpretation. Consider an infinitesimal change at $y^* = y_+(\hat{y}^*)$ in a direction tangent to $\mathcal{E}$. This change induces a change of the cost, which is represented by the left hand side of (24), and a change of the social benefit, which is captured by the right hand side. Optimality requires that the two effects cancel out. Equation (24) states that *the marginal cost equals the marginal social benefit in every direction tangent to* $\mathcal{E}$.

We want to compare (24) with the first order condition for welfare maximization and normalize all utility functions so as to obtain individual present values at y^*_+. Define

$$\mathcal{W}_{y^*} = \sum_i \frac{U^i(x^i((y_+(\hat{y})))}{\partial_0 U^i(x^i((y_+(\hat{y}^*)))}. \tag{25}$$

When we differentiate (25) with respect to $y_s, s = 1, \ldots, S-1$, we obtain, dropping the argument $y_+(\hat{y}^*)$,

$$\partial_s y_0 + \partial_S y_0 \, \partial_s g + \sum_{i=1}^{I} \sum_{\sigma=1}^{S} \left(\pi^i_\sigma \, \partial_s x^i_\sigma + \pi^i_\sigma \, \partial_S x^i_\sigma \, \partial_s g \right) = 0 \text{ for } s = 1, \ldots, S-1,$$

which is equivalent to (24).

Theorem . *1) The first order condition for minimal efficiency requires that the directional derivatives of the cost function and the social benefit function in any direction tangent to* $\mathcal{E}$ *coincide.*
2) The same condition results from the first order condition for welfare maximization if every utility function U^i *is normalized such that* $\partial_0 U^i(y_+(\hat{y}^*)) = 1$.

6 Conclusions

The economic intuition provided by traditional two-period models of production economies with incomplete markets needs to be modified substantially when stock markets operate sequentially. For simplicity's sake, we have ruled out spot markets by assuming that there is only one good per state.

First, even in the quasilinear, additively separable case, consumer i's present value of an intermediate state ξ_s with an active stock market must take i's transactions on that market into account. There is no envelope type argument that allows us to ignore how i adjusts his shares. This fact has severe implications for the price perceptions because perceptions about individual transactions are needed. That is to say, in multiperiod models *portfolio adjustments replace the fixed shareholdings* in the two-period Drèze rule.

Second, if utility functions are not additively separable then i's state price of ξ_s is typically impacted by other states due to repercussions across different stock markets. For instance, the stock market price q_s can depend on changes in the production of good $s' \neq s$. This is *similar to oligopolistic competition* such as Cournot competition, where the price of good s depends on the output of the competing good s'.

These difficulties are caused by the inefficiency of the consumption sector. If this sector would not violate BL's constrained efficiency assumption then the transition from two to more periods would have far less severe consequences. The pricing rule derived by BL is in the spirit of the original Drèze rule although it incorporates all finite date-event trees.

Third, in the two-period case, the assumption that production takes time implies that investments are made at $t = 0$ and dividends are received at $t = 1$. In multiperiod models, costs can occur at intermediate states. Consumers do not only disagree because of different evaluations of dividend streams. They also hold different opinions about cost streams unless there are enough market prices for inputs at different times so that one can determine the total cost objectively.

This point is illustrated in Subsection 3.2 by comparing the generalized Drèze equilibrium in the case of incomplete markets with the Walrasian equilibrium when markets are complete. *The firm and the consumers evaluate the firm's production plan differently* in the former case and equally in the latter.

Fourth, the Drèze rule does not refer to $\mathcal{E}$, but our generalization does. Our analysis involves *local knowledge of* $\mathcal{E}$ in the form of tangent directions at the equilibrium. Again, it is difficult to imagine how an individual person develops reasonable perceptions about this tangent space by introspection.

To summarize, the informational requirements rise enormously when one leaves the two-period framework. This may appear regrettable if one aims to develop a more or less plausible, positive model of a realistic world with incomplete markets and production. However, a generalized Drèze equilibrium can be viewed as a suitable normative concept that lends itself to the analysis of benchmark models.

Finally, although the consumption sector does not satisfy the first order condition for slight efficiency in the sense of Definition 1, efficiency considerations can be used to derive a generalization of the Drèze rule. However, it would be far-fetched to interpret this rule as maximization of a profit function.

To generalize the Drèze rule, one has to weaken the efficiency requirements so much that the feedback effect that entails slight inefficiency is fully taken account. This is done by the use of minimal efficiency in Definition 3. The first order condition for minimal efficiency coincides with the first order condition for welfare maximization and serves as a basis for the concept of a generalized Drèze rule in the same way as the first order condition for constrained efficiency provides a foundation for the original Drèze rule. In the two-period case, the first order condition for constrained and for minimal efficiency coincide. This is no longer true in multiperiod models because of the efficiency loss caused by the feedback effect in the consumption sector.

References

Bonnisseau, J.-M. and O. Lachiri: On the objective of firms under uncertainty with stock markets. *J. Math. Econ.* **40**, 493–513 (2004)

Bonnisseau, J.-M. and O. Lachiri: About the second theorem of welfare economics with stock markets. *Pacific J. Optim.* **2**, 469–485 (2006)

Debreu, G.: Theory of Value. New York: Wiley (1959)

Dierker, E., H. Dierker, and B. Grodal: Nonexistence of constrained efficient equilibria when markets are incomplete. *Econometrica* **70**, 1245–1251 (2002)

Dierker, E., H. Dierker, and B. Grodal: Are incomplete markets able to achieve minimal efficiency? *Economic Theory* **25**, 75–87 (2005)

Dierker, E. and H. Dierker: Welfare and efficiency in incomplete markets with a single firm. *J. Math. Econ.* **46**, 652–665 (2010)

Drèze, J.H., Investment under private ownership: Optimality, equilibrium and stability. In: Drèze J.H. (ed.) Allocation Under Uncertainty; Equilibrium and Optimality. New York: Wiley, 129–165 (1974)

Geanakopos, J. and H. Polemarchakis: Existence, regularity, and constrained suboptimality of competitive allocations when markets are incomplete. In: Uncertainty, Information, and Communication: Essays in Honor of Kenneth Arrow, vol. 3, W.P. Heller, R.M. Ross, and D.A. Starett (eds.). Cambridge: Cambridge University Press (1986)

Geanakopos, J., M. Magill, M. Quinzii, and J. Drèze: Generic inefficiency of stock market equilibrium when markets are incomplete. *Journal of Mathematical Economics* **19**, 113–151 (1990)

Grossman, S. and O. Hart, A Theory of Competitive Equilibrium in Stock Market Economies, *Econometrica* **47**, 293-330 (1979)

Hicks, J.: A Revision of Demand Theory, Oxford: Oxford University Press (1956)

Magill, M. and M. Quinzii: Theory of Incomplete Markets. Cambridge, London: MIT Press (1996)

Author: Egbert Dierker

Title: The Inefficiency of Price Taking Behavior in Multiperiod Production Economies with Incomplete Markets

Reihe Ökonomie / Economics Series 290

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ISSN: 1605-7996

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ISSN: 1605-7996