SITUATION, THE CHOICE OF FEASIBLE SPACE AND THE CHOICE OF SOLUTION CONCEPT

by

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Abstract

This paper argues that two factors of the situation being modeled are critically important in predicting potential coalition formation and payoff distribution. These factors are: (1) whether coalitions form from the self-interest of their players or are pre-specified by the situation (this determines the appropriate choice of feasible space) and (2) the nature of bargaining which can be expected in the given situation (this determines the appropriate choice of solution concept on the given feasible space).

SECTION 1: INTRODUCTION

Characteristic function games have been used to model a wide variety of situations. Solution concepts for these games capture how the existence of alternative "profitable" coalitions for various players affects the payoffs they receive for their participation in a coalition. Examples include: the coalition production economy where the profit share an agent can bargain for in one firm depends on his ability to participate in other profitable firms; the distribution of marital benefits where the decision making power may depend on the relative abilities of the partners to form other desirable matches; and the allocation of cost of a university-wide watts line, where the "fair" allocation depends on the cost savings to various departments, faculties, etc.

In replacing the real world by its characteristic function representation, care must be taken to choose the feasible space and solution concept which are appropriate for the given situation.

To see this point we present two situations which are both represented by the same sidepayment game:

$N = \{1,2,3\}$ and v is given by:

$v([i]) = 0 \quad \forall i \in N$

$v([i,j]) = 6 \quad i \neq j$

$v(N) = 6$

Situation 1. Due to a recent change in the tax law, certain firms can save millions of tax dollars by merging. Only three firms meet the necessary requirements. A merger between any of them would result in a net savings of \$ 6 million and, if all three of them merged, the net savings would also be \$ 6 million.

Situation 2. A corporation is considering introducing special communications links among three of its divisional headquarters. It is estimated that linking (any) two of the divisions would result in a net annual savings of $ 6 million. Linking in the remaining division results in no additional net benefit or cost.

Although these situations have identical characteristic function representations we can intuitively expect different outcomes. In the first situation we expect two of the firms to merge and split the cost savings equally. Depending on which firm is left out, we expect an outcome of (3,3,0), (3,0,3), or (0,3,3). In the second situation we expect that, in absence of favoritism or malice towards the various divisions, all three of the divisions will be linked and the cost savings will be allocated, equally to the divisions. This results in the outcome of (2,2,2).

The difference in these outcomes can be attributed to two distinct factors. (1) In the first situation individual self-interest motivates players to form coalitions in the course of the game. In the second the corporation has authority which means that although it is in the self-interest of its players to form a 2-player coalition and exclude the third they can't break off to form the smaller coalition. In this circumstance we would say that the coalition of the whole had formed before the beginning of the game. This factor -- whether coalition formation is endogenous or given by the situation determines the appropriate feasible space in which to search for solutions. (2) There is also likely to be a different style of bargaining in the two situations and this difference in the style of bargaining can also lead to differences in the outcome. In the first situation the discussion over the relative payoffs in a two-player coalition is likely to focus on the prospects each would have with the third player. On the other hand, at the corporate headquarters arguments about fairness and equity are more likely to be heard. This second factor--the nature of the bargaining which can be expected in the given situation determines the appropriate choice of solution concept.

Although these factors are equally important for the analysis of games with non-transferable utility, this paper focuses on games with transferable utility (games with sidepayments). Section 2 discusses how the appropriate choice of feasible space (imputations, individually rational payoff configurations or aspirations) depends on whether at the time the game is begun there is already a natural coalition structure or, instead, coalitions form from the self-interest of their players. Section 3 shows how the appropriate choice of solution concepts depends on the bargaining notion which is appropriate for the situation being analyzed. We review the bargaining notions underlying the core, Shapley value and kernel. In Section 4 we show the significant impact on the predicted payoff distributions of alternative choices of feasible space. We fix the bargaining notion (that of the bargaining set) and compare the predicted payoff distributions of coalitions when the same bargaining notion is applied on two different feasible spaces--the space of aspirations and the space of individually rational payoff configurations. A series of examples are used to demonstrate that they are often strikingly different and why.

SECTION 2: SITUATION AND THE CHOICE OF FEASIBLE SPACE

In this section we examine three feasible spaces: the set of imputations, the set of individually rational payoff configurations (IRPCs) and the set of aspirations. These three are the spaces on which almost all solution concepts have been defined--including the core, nucleolus, Shapley value, the von Neumann Morgenstern solution, the bargaining set, kernel and all of the aspiration solution concepts.

We place the requirements for each of these feasible spaces in a common framework which makes clear the situations for which each is appropriate.

Imputations

The space of imputations is appropriate for situations where, like the cost allocation problem, the coalition of the whole, N, has formed at the beginning of play. The constraint to form the coalition of the whole means that coalition remains stable even though in some games some players could increase their payoffs by forming some smaller coalition.

For these situations we can restrict our search for solutions to those payoff vectors, $(x_1,...,x_n)$ which are distributions of v(N). The coalition N cannot allocate more than its value (we call this condition "payoff feasibility"). If N allocated a total of less than v(N), presumably some players would demand and obtain a higher payoff. We therefore expect no surplus of payoff for the coalition N. Lastly, imputations require individual rationality--that each player be given a payoff at least as much as he can earn alone.

Formally: the vector $x \in \mathbb{R}^n$

1. Payoff feasibility: $x(N) \leqq v(N)$
2. No surplus: $x(N) \geqq v(N)$
3. Individual Rationality: $x([i]) \geqq v([i]) \; \forall i \in N$.

IRCPs

Selecting the space of individually rational payoff configurations is appropriate for situations in which, before the beginning of play, coalitions have already formed. Examples include (1) the partition induced by country over game theorists (where the game may be to determine the relative salaries of the game theorists) and (2) the division of countries into alliances such as NATO and the Warsaw Pact (where the game may be the allocation of defense costs).

The set of IRPCs includes the set of imputations as a special case--the set of IRPCs for the coalition structure N. The bargaining set and kernel were defined on the space of IRPCs and, Aumann and Dreze (1975) extended the imputation solutions--core, Shapley value, and nucleolus to the space of IRPCs.

For applications where the coalition structure is situationally determined we can narrow the search for solutions in the following way. Each coalition can allocate no more (payoff feasibility) and no less (no surplus) than its value and no coalition can allocate to a player less than he can earn on his own (individual rationality).

The vector $x \in \mathbb{R}^n$, is an individually rational payoff configuration, IRPC for the coalition structure, T, if:

1. Payoff feasibility: $x(S) \leqq v(S)$ for all S in T
2. No surplus: $x(S) \geqq v(S)$ for all S in T
3. Individual rationality: $x_1 \geqq v([i])$ for all i in N

Aspirations

The space of aspirations is the appropriate feasible space for situations where individual self-interest is the force which motivates the formation of coalitions. In these situations the competition among the coalitions determines both the coalitions which are likely to form as well as the payoff distributions within these coalitions. Such situations include the tax-saving merger of firms in our introductory example, and markets for various commodities where the (negotiated) selling price of the item determines the payoff to both buyer and seller and the existence of alternative buyers and sellers affects the price.

In these situations each player sets a price for his participation in coalitions. Since each player can set his price in light of his "most profitable" opportunities, the resulting vector of players' prices is not an imputation or an IRPC except when the coalition of the whole on a given coalition structure is the "most profitable" alternative(s) for every player.

A player can regard his price to be feasible (given other players' prices) if there is some coalition which can afford to pay him and its other players their prices. The payoff feasibility condition for aspirations requires that each player be able to view his price as feasible. If, after paying its players their prices, there would be a surplus in some coalition, we would expect one or more of its players to raise their prices. The no surplus condition for aspirations therefore requires that there is no surplus in any coalition. Lastly, for aspirations we do not have to make a separate individual rationality requirement since the no surplus condition applied to the singelton coalitions quarantees individual rationality.

The payoff vector x is an <u>aspiration</u> if it satisfies:

1. Payoff feasibility: $\forall i \varepsilon N$, $\exists S$ with $i \varepsilon S$ such that $x(S) \leqq v(S)$
2. No surplus: $\forall X \subset N$, $x(S) \geqq v(S)$.

We use G(x) to denote the collection of coalitions which can afford the aspiration x, that is, $GC(x) = \{S \subset N \mid x(S) \leqq v(S)\}$. Since the aspiration action of feasibility requires only that each player have some coalition which can afford its players' prices, an aspiration may not be feasible for any coalition structure. Once players have picked an aspiration, one or more coalitions from GC(x) form. When as many coalitions as possible from GC(x) form, the remaining players are left out and "form" singleton coalitions. We call a coalition structure which could arise from an aspiration in this way a coalition structure consistent with the aspiration. When players form a coalition structure consistent with the aspiration x, the players who succeeded in forming coalitions from GC(x) obtain their components of x. Those who are left out presumably receive only what they could earn alone. Formally, the payoff distribution y for the coalition structure $\mathcal{T}$ is said be to consistent with the spiration x if:

(1) y is an IRPC for $\mathcal{T}$;

(2) $y_i = x_i$ for $S \varepsilon \mathcal{T} \cap GC(x)$;

(3) If $S \subset US'$ and $S \neq \emptyset$, then $S \notin GC(x)$.

This is a workable definition of consistency for the examples used in this paper. However, for cases where either we would like to predict the payoff distribution of players who are "left out" when the coalition structure forms or when initial coalition formation does not absorb all the gains from coalition formation, a more complicated notion of consistency is required (see "Predicting Partitions", Bennett (1983a)).

SECTION 3: MODELING BARGAINING - THE CHOICE OF SOLUTION CONCEPT

In this section we turn our attention to the second factor which leads to substantially different outcomes for situations having identical characteristic functions. In the play of the game, bargaining among the players determines the payoffs the players receive. Different situations may lead to different bargaining strategies and cause differences in the resulting payoffs. Solution concepts capture alternative notions of how bargaining might proceed. In this section we review the bargaining notions of three solution concepts--the core, Shapley value, and kernel and their aspiration space counterparts. (The bargaining set is treated in detail in the following section.)

The Core (C)

The Core (Gillies (1953)) was defined on the space of imputations. The underlying bargaining notion is based on the idea that, when a payoff distribution is proposed, if some group of players is assigned a total of less than they could earn by breaking off and forming a coalition on their own, that group can overturn payoff distributions. When no group of players would earn more by forming a subcoalition, we say the imputation is coalitionally rational. This consideration seems relevant in two types of situations: first, when the coalitions do have the power to veto such proposals and second, when the coalition of the whole is very affluent. In this case coalitional rationality is arguably a minimum basic requirement of a "fair" distribution of payoff. As we see in the following examples, when the coalition of the whole is not affluent, the core is empty and when the coalition of the whole is affluent but not very affluent, the core is not empty but it picks intuitively implausible imputations.

Example 3.1: Left Shoes and Right Shoes

In this game there are 5 shoes - 2 left and 3 right. The value of any coalition is \$ 10 for each pair of shoes it contains (unpaired shoes have no value). The left shoes are labeled 1 and 5 and the right shoes, 2, 3, and 4. The coalition of the whole in this game is worth 20. The core of this game is not empty, $C = \{(10,0,0,0,10)\}$. In this case the only way to satisfy coalitional rationality is to give the left shoes the entire payoff. I doubt that in such a situation, the left shoes can make the argument of coalitional rationality sufficiently pursuasive to convince the right shoes all of the payoff.

Formally, the core, C, is the set of imputations that satisfies $x(S) \geq v(S)$ for every coalition S.

Consider first the introductory game where the coalition of the whole earns 6 and every two players can earn 6 also. The coalition of the whole is not affluent - it cannot afford to pay it's players any coalitionally rational payoff distribution so $C = \emptyset$. For the merger problem, this suggests that the coalition of the whole will not form. For the corporate cost allocation, emptiness of the core suggests that the divisions' arguments for coalitionality should not be heeded.

The Aspiration Core (AC)

If coalitional rationality were the only requirement placed on aspirations in order to be in the aspiration core then AC would be the entire set of aspirations the coalitional rationality condition is precisely the no-surplus condition. So every aspiration is coalitionally rational. However, there is an alternative characterization of the core in the space of imputations due to Bondareva (1962, 1963). When extended to the space of aspirations, this solution concept picks the aspirations with least sum. In Bennett (1983a), the author defined the aspiration core to be the set of least sum aspirations. Formally, the aspiration x is in the aspiration core, AC, if $x(N) \leq y(N)$ for every aspiration y. The aspiration core can be thought of as an extension of the core solution concept because when the core is not empty, C = AC and AC is always nonempty. Aspiration solution concepts capture how players bargain over the prices they charge for their coalitional participation. Cross (1967) gives the following description:

> "The first (hypothesis) states that each individual will select from the whole set of feasible coalitions that one which will give him the highest possible return given the demands or payoff expectations of the necessary allies. The coalitions which would have to form in order to yield these maximal returns to all individuals, however, may be incompatible with one another. The association which would yield the highest return to Albert may require the membership of Sidney, and Sidney may also be essential to Bill's alliance; hence the satisfaction of the expectations of both Albert and Bill is impossible. The second hypotheses deals with the response of individuals to such in compatible expectations, suggesting that competition will arise for "scarce" members (e.g., Sidney)."

> "If the value of a potential exceeds the payoff which he is currently receiving elsewhere, it is possible to make a bid which will induce him to join, other things being equal. Conversely, the members of any coalition which finds that a crucial member is likely to transfer to some other alliance can attempt to retain his membership, if possible, by increasing his share of the value of the agreement."

> "The competitive process will tend to increase the payoff offers to any scarce player who is essential to two or more coalitions, and simultaneously it will tend to reduce the payoff expectations of other members of those coalitions. It is also true that competition will tend to reduce the arithmetic sum of all payoff demands." [1,2]

For the introductory example AC = (3,3,3) . For the merger problem selection of (3,3,3) is the prediction that some two-player coalitions will form and in the coalition that does form the payoff distribution will be (3,3). AC is not appropriate for the corporate cost allocation problem since coalitions do not form endogenously.

For Left Shoes and Right Shoes AC = C = {(10,0,0,0,10)}.

The Shapley Value

The Shapley value (Shapley, 1953) is defined for superadditive games on the space of imputations. The Shapley value assigns to each player the "average" increase in the value of the coalitions due to his participation. The increase in value of the coalition S due to i's participation is $v(S) - v(S - \{i\})$, and the "average" is taken over coalitions which are formed sequentially by players who "arrive" in random order. The Shapley value of player i can be calculated using the following formula:

$$\phi_i = \frac{1}{n} \sum_{s=1}^{n} \frac{1}{c(s)} \sum_{\substack{S \ni i \\ |S| = s}} \left[v(S) - v(S - \{i\}) \right],$$

where c(s) is the number of coalitions of size s containing player i:

$$c(s) = \binom{n-1}{s-1}$$

Although Harsanyi (1977) has provided a bargaining procedure which does result in the Shapley value, payoff distributions like those given by the Shapley value more often arise from arguments over fairness and equity of the payoff distribution. In Harsanyi's procedure every coalition is imagined to form a "syndicate" that declares a divided (positive or negative) and distributes to each of its members an equal share of the difference between its coalitional value and the total of the dividends of all of its subsyndicates. For each player the sum of his dividends is precisely his Shapley value.

Since the payoffs assigned by the Shapley value can be interpreted as the average worth of each player, the Shapley value seems appropriate for those situations where the bargaining centers on what is fair and equitable - situations such as the corporate cost allocation problem.

So far no extension of the Shapley value to the space of aspirations has been proposed.

The Shapley value for the introductory example is (2,2,2). Within the context of the corporate cost allocation problem, the Shapley value is the "fair" allocation which reflects the cost savings to each division. Since the Shapley value is an imputation solution, it is only applicable to the merger problem if the three firms merge into a single entity. If they did merge, the Shapley value would be a "fair" way to distribute the tax savings among the three firms.

The Kernel (K)

The Kernel was defined on the space of IRPCs by Davis and Maschler (1965). The kernel solution concept is appropriate for situations where, as we describe below, the bargaining between pairs of players in each coalition in the given coalition structure leads to "equal surpluses".

Suppose the IRPC x is proposed. At s, define the surplus (or excess) of the coalition S to be $e(x,S) = v(S) - x(S)$. For a fixed pair of players i and j, let $s_{ij} = \max\{e(x,S) \mid i \in S, j \notin S\}$. The amount s_{ij} is the maximum surplus of any coalition containing i but not j. When $s_{ij} > 0$, then s_{ij} can be interpreted as the amount i can hope to earn by forming his best alternative coalition without j (assuming that the other players will accept as little as their current components of x). When $s_{ij} < 0$, it has the interpretation as the loss i would sustain by forming his best alternative coalition (assuming that he must pay the other players at least their current components of x).

When, for a pair of players i and j in a coalition in the given coalition structure, $s_{ij} > s_{ji}$ player i can argue that he deserves a larger payoff at j's expense on the grounds such as: (1) i would gain more (or lose less) by forming an alternative coalition and only when $s_{ij} = s_{ji}$ do i and j have equal bargaining strength or (2) the condition $s_{ij} = s_{ji}$ represents the equal gains (or loss) position and, therefore, is the fair distribution of payoff between i and j. If j is making more in his coalition with i than he could make on his own, then these arguments may induce j to give i some of his payoff. The IRPC, x, is in the kernel, K, for the coalition structure T if, for every coalition S in T and every i and j in S, either $s_{ij} = s_{ji}$, or else $s_{ij} > s_{ji}$ and $x_j = v([j])$.

K predicts a payoff distribution for every coalition structure. For the introductory example:

T	K(T)
$\{[1],[2],[3]$	$K = \{(0,0,0)\}$
$\{[1,2],[3]\}$	$K = \{(3,3,0)\}$
$\{[1],[2,3]\}$	$K = \{(0,3,3)\}$
$\{[1,3],[2]\}$	$K = \{(3,0,3)\}$
$\{[1,2,3]\}$	$K = \{(2,2,2)\}$

In the corporate cost allocation problem the coalition of the whole has formed so K predicts (2,2,2). For the merger problem if a two player coalition formed, K predicts (3,3) for their payoff distribution.

For situations where the coalition structure is not given a priori and it is not obvious by looking at the characteristic function which coalition structures are likely to form, then the apparent flexibility of K in providing a payoff prediction for every coalition structure, is a distinct liability. Further, even in cases when the relative payoffs specified in the characteristic function make predicting coalition formation easy, K can make unintuitive predictions - predictions which are appropriate only if the coalitions have formed <u>a priori</u>. The following example makes this point clearer. See also Aumann and Dreze (1975) section 12, for further discussion.

Consider the example which Davis and Maschler called "My Aunt and I".

Example 3.1: My Aunt and I $N = \{1,2,3,4,5\}$

$v(1,j) = 100$ $j = 2,3,4,5$

$v(2,3,4,5) = 100$ $v(S) = 0$ otherwise

In this game coalition formation is clear. Either the Aunt (player 1) forms a coalition with "I" (player 2) or 3 or 4 or 5 or else [2,3,4,5] forms.

If [2,3,4,5] did form, we would expect (by symmetry) a payoff distribution of (25,25,25,25). Indeed K ({[1], [2,3,4,5]}) = (0,25,25,25,25). The difficulty arises if a coalition of the form [1,j] j = 2,3,4,5 formed instead. Suppose [1,2] formed. K([{1,2], [3], [4], [5]}) = (50,50,0,0,0). Now if indeed [1,2] fomed a prior, players 2,3,4 and 5 would expect to receive 0 and, therefore, would presumably settle for any positive payoff in forming [2,3,4,5]. In this case $s_{21} = 50$ does reflect 2's potential gain in forming an alternative coalition without 1. Player 1's best alternative coalitions are to form [1,3], [1,4] or [1,5]. Each of these fives a potential gain, $s_{12} = 50$. For situation where players 1 and 2 don't have a prior commitment to [1,2] players 3,4 and 5 are unlikely to settle for a miniscule payoff. Indeed, 3,4 and 5 would otherwise be likely to demand 25 each. In this case s_{12} and s_{21} are unlikely to reflect the bargaining positions of the players and, therefore, the kernel will pick less defensible payoff distributions.

Aspiration Kernel (AK)

The formal extension of the kernel to the space of aspirations is straightforward. The aspiration x is in AK if either $s_{ij} = s_{ji}$ or else $s_{ij} > s_{ji}$ and $x_j = v(j)$.

Notice that when x is an aspiration, $s_{ij} \leq 0$ for every i and j. Therefore, s_{ij} has the interpretation as the loss incurred by i by forming an alternative coalition without j. Suppose x_j = v(j). Then s_{jk} = 0 for all k and, from these facts we observe that the second condition can actually never hold the space of aspirations.

AK for the introductory example selects the aspiration (3,3,3). Selection of this aspiration for the merger problem is the prediction that two firms will merge and split the tax savings (3,3). AK is not appropriate for the corporate cost allocation problem since coalition formation is not endogenous.

For "My Aunt and I" (Example 3.1) AK = (75,25,25,25,25). Since the components of an aspiration are the prices players would charge to form a coalition s_{12} and s_{21} do reflect 1 and 2's potential "gain" from forming alternative coalitions when coalition formation is endogenous. The payoff distribution (75, 25,0,0,0) for the coalition structure = {[1,2], [3], [4], [5]} is consistent with AK. (By comparison K(T) = (50,50,0,0,0)).

SECTION 4: IMPORTANCE OF THE CHOICE OF FEASIBLE SPACE - B vs AB

This section studies the bargaining set of Aumann and Maschler (1964) and its aspiration space counterpart. We explore the bargaining notion captured in this solution concept and then use them to demonstrate, through a series of examples, how strongly the choice of feasible space affects the predicted payoff distributions.

The Bargaining Set (B)

The bargaining set was originally defined on the space of IRPC. Consider, therefore, the bargaining that takes place over the payoff distributions within each coalition of a pre-existing coalition structure. Player 1 has an objection against 2 if there is another coalition where player 1 can earn more than x_1 while paying his partners at least as much as they earn in x. A player's objection is basically a threat to disrupt the given coalition structure to form an alternative coalition. Unless player 2 can maintain his payoff in an alternative coalition without 1's cooperation, player 2 is well advised to give 1 a higher payoff at his own expense. If, on the other hand, player 2 has an alternative coalition where he can maintain his payoff while paying his partners at least as much as they can otherwise earn, then player 2 has a counterobjection to player 1. When player 2 has a counterobjection, then player 1's objection presumably does not bring about a payoff redistribution. An IRPC, x, for the coalition structure, T, is in the bargaining set if there are no objections without counter-objections.

Formally, Player i has an <u>objection</u> to j in the coalition structure T for the IRPC x if:

(a) $k, j \in S \in T$

(b) $\exists S', y \in \mathbb{R}^{S'}$ which satisfy:

$y(S') = v(S')$

$i \in S', \quad j \notin S'$

$y_k \geqq x_k \quad \forall k \in S'$

and $y_i > x_i$.

Player j has a <u>counter objection</u> against i if $\exists S", z \in \mathbb{R}^{S"}$ which satisfies:

$z(S") = v(S")$

$j \in S", \ i \notin S"$

$z_k \geqq x_k \quad \forall k \in S"$

$z_k \geqq y_k \quad \forall k \in S" \cap S'$

The IRPC x is in <u>B</u> for the coalition structure T if for every objection there is a counterobjection.

<u>The Aspiration Bargaining Set (AB)</u>

To transfer the spirit of the bargaining set to the space of aspiration requires not only notational substitution but also a substantive change in the definition of objection. (1) Notice in the definition of objection (for B) that for i to have objection he must be made strictly better off in S' than in S. Since y is feasible for S' and since the other players are given at least their components in x we know that $x(S') < v(S')$ - that there is a positive surplus in S' given x. When we consider payoff vectors drawn from the space of aspirations, by definition, there can be no surplus in any coalition. If we strictly adhered to the definition of objection, no player could ever have an objection against an aspiration and therefore AB would be the entire set of aspirations. However, consider a fixed aspiration x. Even though player i can only be made as well off in a coalition without j, i's threat to form an alternative coalition is still a viable threat if j cannot maintain his aspiration in any coalition without i. It seems closer to the spirit of the bargaining set to only require in the space of aspirations that i be made at least as well off in his alternative coalition in order to make an (aspiration) objection. (2) In the definition of objection and counterobjection players are allowed to conjecture <u>any</u> payoff distribution in S' and S" which meets the given criteria. Since an aspiration is a vector of the prices the players will demand from any coalition, when the aspiration x is proposed, the appropriate payoff distributions to consider for S' and S" are simply $x|_{S'}$ and $x|_{S''}$. (This difference between B and AB greatly simplifies the task of determining whether or not players have justified (aspiration) objections.)

With these changes AB can be formally defined

Player i has an <u>aspiration objection</u> against j for the aspiration x if:

(1) $\exists S$ with $i,j \in S$ and $x(S) = v(S)$

(2) $\exists S'$ with $i \in S$, $j \notin s$ and $x(S') = v(S')$

Player j has an <u>aspiration counter objection</u> against i if:

S'', $i \notin S''$, $j \in S''$ and $x(S'') = v(S'')$

The aspiration $x \in$ <u>AB</u> if there are no aspiration objections without aspiration counterobjections.

B vs AB

Since for every coalition structure B is nonempty, B predicts a payoff distribution for each coalition in each coalition structure. For each game AB is nonempty; selection of an aspiration in AB is the prediction that certain coalitions are likely to form and a payoff distribution for each of these coalitions which does form. Intuitively since both B and AB express the same bargaining notion, one might expect these solution concepts to agree on the payoff distributions of the coalitions for which they both predict payoff distributions. We provide examples where (1) AB "is larger" than B, (2) B is larger" than AB, and lastly (3) B and AB are "disjoint". These examples show clearly the force of each solution concept and the fundamental reason for the divergence in their predictions.

Consider the coalition structure {[1,2], [3], [4,5]}. The unique IRPC in B for this coalition structure is (10,0,0,0,10). Further, for any coalition containing at least one right and one left shoe, in any coalition structure whatsoever, every payoff distribution in B assigns 10 to the left shoe and 0 to the right. To see why, consider (7,3,0,3,7) for {[1,2], [3], [4,5]}. 1L can object against 2R with the coalition 1,3,4,5 and payoff distribution (8,0,3,9). Since the only other left shoe, 4L, is well paid in the objection, player 2R has no counter objection.

For this game the set of aspirations in AB is given by AB = $\{(10-\lambda,\lambda,\lambda,\lambda,10-\lambda)$ for $0 \leqq \lambda \leqq 10\}$. Any payoff distribution (including (10,0)) between left and right shoes is consistent with AB. Consider the aspiration (7,3,3,3,7). Selection of this aspiration is the prediction that coalitions with exactly 1 or 2 pairs of shoes (no leftovers) will form with a payoff of 7 to left and 3 to right shoes. The leftover shoe presumably gets zero; so a final outcome consistent with AB is (7,3, 0,3,7) for {[1,2], [3], [4,5]}.

To see why (7,3,0,3,7) is consistent with AB and not B recall the objection of 2 against 1. In the IRPC framework the coalition structure {[1,2], [3], [4,5]} is given a priori. In it player 3R receives 0 no matter what the payoff distributions are in the other coalitions and therefore is presumably to accept as little in [1,3,4,5]. This makes it possible for 1 to increase his payoff while giving 5 so large a payoff that there can be no counterobjection. In the aspiration context no coalitions given a priori so there is no reason to expect that 3 accept so little. When left shoes demand 7 and the other right shoes demand 3, a payoff demand of 3 for player 3 is reasonable. While 1 still has an objection - he can threaten to form [1,3,5,6] with payoff of (7,3,3,7), player 1 can maintain his aspiration in [2,5] with a payoff of (3,7).

<u>Example 4.2</u>: A 13-player Simple Majority Voting Game

A winning coalition is worth 21 and takes a simple majority of the players to form a coalition to win. $N = \{1,...,13\}$ v is given by:

$$v(s) = \begin{cases} 21, & |S| \geq 7 \\ 0 & \text{otherwise.} \end{cases}$$

The aspiration (3,3,...,3) is the unique vector in AB. Since the players are identical and since any 7 players can earn 21, a payoff demand of 3 each is quite plausible. Selection of (3,3,...,3) is the prediction that some 7-player coalition will form and in it each player recieves 3. One final outcome consistent with this aspiration is (3,3,3,3,3,3,3,0,0,...0) for the coalition structure {[1-7], [8], [9],...,[13]}.

The payoff distribution (3,3,3,3,3,3,3,0,0,...,0) is also in B for the same coalition structure. B for this coalition structure is quite large including, in particular (7,7,7,0,0,...,0). To see that this (unlikely) vector is in B suppose 4 made an objection against 3. The objection most difficult to counter results from paying 3's potential partners as much as possible.

For small $\varepsilon > 0$, the objection most difficult to counter is to threaten to form [4,5,...,13] with the payoff distribution (9ε, $7/3-\varepsilon$, $7/3-\varepsilon$,...,$7/3-\varepsilon$). Player 3 does have a counter objection, for example, [3,5,6,7,8,9,10] with the payoff distribution (7,7/3,7/3,...,7/3).

In this type of game, as the number of players increases it becomes more difficult to have a justified objection because there are more potential partners who must, in an objection, be paid sufficiently well to remove the possibility of a counterobjection.

The reason why (7,7,7,0,0,0) is a payoff distribution for [1-7] which is consistent with B but not AB is that if the coalition structure {[1-7], [8], [9],...,[13]} is given a priori then players 8-13 who would expect 0 would presumably be willing to 7/3 in an objection or counterobjection. On the other hand, when coalitions are not given a priori, threats which involve paying their partners less than 3 are unlikely to be credible.

For the sake of completeness we present an example where B and AB do not agree at all.

Example 4.3: $N = \{1,2,3,4,5\}$ and v is given by:

$$v([1,2,3]) = v([1,4,5]) = 30$$
$$v([2,4]) = v([3,4]) = v([3,5]) = 20$$
$$v(S) = 0 \text{ otherwise.}$$

The aspiration bargaining set for this example is:
$AB = \{(10,\alpha,20-\alpha,\alpha),\ 0 \leqq \alpha \leqq 10\}$. As α varies over its range, all the positive valued coalitions are predicted by AB. If [1,2,3] forms, none of the other coalitions can form. The payoff distributions for [1,2,3] which are consistent with AB are $(10,\alpha,20-\alpha)$ $0 \leqq \alpha \leqq 10$.

For the coalition structure {[1,2,3], [4], [5] , B = { (14+β, 4+β, 12-2β, 0, 0) for $0 \leq \beta \leq 3$. For example, the IRPC (10,5,15) is consistent with AB but is not in B. To see why notice that player 1 can object with the coalition [1,4,5] and payoff distribution (15,15,0). Player 2 has no counterobjection. In the space of aspirations 2's objection using the coalition [1,4,5] would have to be (10,15,5) for which 2 does have a counterobjection.

Appendix: The Aspiration Approach

The aspiration approach corresponds to the following intuition on how characteristic function games are played.

The values of coalitions become known to the players (either before or during coalitional discussions). The players of various coalitions meet together to decide how the coalition's value would be distributed within the coalition if it were to form. These "discussions" may range from a fierce bargaining process to immediate agreement on the "fair" and "equitable" distribution of payoff. After a series of such negotiations each agent selects his payoff demand (the price the player will charge for his participation in a coalition). He selects his price in light of the prices he expects other players to demand and in light of his opportunities in various coalitions. After perhaps further bargaining over whether these prices are reasonable, a "stable" vector of prices evolves and, given this price vector, the coalitions which can form are the coalitions that can "afford" their members' prices and in each of these coalitions which actually forms each player receives his price.

Two assumptions about the way players set their prices are captured in the definition of aspiration. The first is that players will not price themselves out of the market, that is, no player will set his price so high that given the other players' prices no coalition will be able to afford him. This condition is called the "feasibility" condition. The second condition corresponds to the notion the players' prices will rise whenever there is a surplus in any coalition. This means that whenever there is a coalition where the value of the coalition exceeds the sum of its members' prices, then one or more of its members will raise their prices. This condition is called the "no-surplus" condition.

<u>Definition</u> The payoff vector x is an <u>aspiration</u> if it satisfies:

1. the feasibility condition: $\forall i \in N, \exists S \ni i$ such that $x(S) \leq v(S)$.

2. the no-surplus condition: $\forall S \subseteq N, x(S) \geq v(S)$.

Once an aspiration is selected, the coalitions which can afford the aspiration are the coalitions which are likely to form in the game. The collection of coalitions which can afford the payoff vector x is called the generating collection of x.

<u>Definition</u> The <u>generating collection</u> of x, GC(x), is given by:

$$GC(x) = \{S \subseteq N \mid x(S) \geq v(S)\}.$$

Using these terms we restate our view of the play of characteristic function games. After the characteristic function is known, each player, in the course of many discussions over "appropriate" payoffs, selects a price for his participation within any coalition. We expect this vector of players' prices to be an aspiration. Solution concepts model the bargaining over prices and select aspirations. Selection of a particular aspiration is the prediction that the coalitions in its generating collection are the coalitions that are likely to form and in each of the coalitions that does form, its members receive their prices, that is, their components of the selected aspiration.

Example: A Weighted Voting Game

There are 5 players: N = {1, 2, 3, 4, 5}. Let the number of votes each agent controls be given by w = (2, 2, 1, 1, 1). Any coalition that controls five votes or more can pass the bill. If we assign 1 to winning coalitions and 0 to losing coalitions, we have the following characteristic function:

$$v(S) = \begin{cases} 1, & \sum_{i \varepsilon S} w_i \geq 5 \\ 0, & \text{otherwise.} \end{cases}$$

Aspirations for this example include: (1, 1, 0, 0, 0), (1/2, 1/2, 1/6, 1/6, 1/6) and (0, 0, 1, 1, 1). The aspiration core and the aspiration bargaining set of this game select the aspiration (.4, .4, .2, .2, .2). Given that the players have selected these prices, only the coalitions that can afford these prices can actually form. In this example all of the minimal winning coalitions and only these coalitions can afford these prices:

$$GC(.4, .4, .2, .2, .2) =$$
$$\{[1, 2, 3], [1, 2, 4], [1, 2, 5], [1, 3, 4, 5], [2, 3, 4, 5]\}.$$

Selection of this aspiration is, therefore, the prediction that the minimal winning coalitions are the coalitions which are likely to form. In any of these coalitions which does form, each player will receive his price. Suppose the coalition [1, 2, 3] forms. Players 1, 2, and 3 receive .4, .4, and .2, respectively. Since the agreement to form a coalition is viewed as a binding agreement, after [1, 2, 3] has formed, player 4 and 5 can no longer be part of a winning coalition, the highest payoff players 4 and 5 can receive is 0. The payoff distribution (.4, .4, .2, 0, 0) for the coalition structure {[1,2, 3], [4], [5]} is an outcome in the game which is consistent with the given aspiration. If instead [2,3,4,5] formed, a consistent outcome would be (0,.4,.2,.2,.2) for the coalition structure {[1], [2,3,4,5]}.

A Brief History of the Aspiration Approach

Several solution concepts were invented (and reinvented) on the space of aspiration before the general approach was suggested. The solution concept I call the aspiration core was first proposed in a paper by John Cross (1967). It was reinvented by Wulf Albers (1974) and carefully studied in Gabriel Turbay's (1977) dissertation. It was also reinvented and applied to coalition economies in Myrna Wooders' (1978) working paper. The aspiration bargaining set was first proposed in a paper by Wulf Albers (1974) and was reinvented by this author in her dissertation (Bennett (1980)). The author also proposed the equal gains aspiration solution concept (Bennett (1983a)) and extended several of the "traditional" solution concepts: von Neumann-Morgenstern solution, core, bargaining set, kernel and nucleolus to the space of aspiration (see Bennett (1980) and (1983b)). This general approach to characteristic function games was proposed in Bennett (1983a) for games with sidepayments and in Bennett (1982a) for games without sidepayments.

FOOTNOTES

1. Cross (1967) pp. 184, 185 and 187.
2. See the critique of this "least sum" aspiration solution concept in Bennett (1983a).

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