THE SHAPLEY-VALUE COMPARED TO SOLUTIONS OF COOPERATIVE N-PERSON GAMES IN EXTENSIVE FORM AND TO ASPIRATIONS

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Preface

This paper was proposed by an article by A.Rapoport [9], where a new model of coalition formation and a corresponding payoff distribution is described. The model is treated in detail, whereby the solutions are compared to the Shapley-value, and the influence of indifferent players and dummy players is investigated. The model is extended to games with sidepayments and applied to 3- and 4-person games. Finally the attempt is made to relate the Shapley-value to the set of aspirations.

The question why Shapley-value and expected value differ for games with more than 3 players is answered satisfactory, as well as the question how dummy players and indifferent players influence the outcome of the game. The structures of 3- and 4-person games with sidepayments are described in detail. The comparison of Shapley-value and aspiratons shows the different nature of the two solution concepts. Only in very restricted cases some connexion can be seen.

I. Expected value and Shapley-value in games in extensive form

The Shapley-value of an n-person cooperative game implies that the value of the game to the grand coalition shall be so apportioned among the players that each player receives his average contribution to a coalition which he joins as an additional member.
Starting from the definition of the Shapley-value A.Rapoport suggests a game in extensive form for which he defines the rules [9]:

1. The player to make the first bid to another to join him in the growing coalition is chosen equiprobably.
2. This player invites another player (of his choice) to join him.
3. The player so invited can accept or decline.
4. If he accepts a coalition of two has formed. The second player, then, selects the next prospective partner from the remaining players and so on.
5. If the player invited to join declines, the player who invited him selects another prospective partner, and so on.
6. No invitation from the same player to the same player can be repeated. If all but one player have declined an invitation, the remaining player must accept it.
7. The play of the game terminates when the grand coalition has formed. The payoffs in any play of the game are awarded in accordance with the contributions of the players to the coalitions they have formed.

In his article A.Rapoport compares the expected value of the defined game in extensive form to the Shapley-value for

3-person games and concludes that the two solutions concepts coincide for 3-person games.
Rapoport expected the solutions to be different for games with more than 3 players. Moreover Rapoport raises some questions concerning the difference of expected value and Shapley-value.

In the first chapter we do a more formal approach in order to get some more insight in games in extensive form. We start with 3-person games and give a formal reason for the equality of expected value and Shapley value.
We then investigate 4-person games where already some typical features of n-person games in extensive form can be seen. For one special case of n-person games it is possible to give an explicit formula for the probability of coalition formation. The general n-person extensive game reveals some typical features influencing the difference of expected payoff and Shapley-value.

Furthermore we have answered the question how dummy players can influence the outcome of the game satisfactory. Some examples of 5-and 6-person games illustrate our results. We also give a reason why the expected value of a 7-person game may not be computed easily, even by a computer.

After the mentioned investigations we were able to answer the questions raised by Rapoport in his article[9].

1. Expected value and Shapley-value

1.1. Definitions [8]

Definition 1.: For an n-person game let $N=\{1,2,\ldots,n\}$ be the set of players. Any nonempty subset of N is called a coalition.

Definition 2.: By the characteristic function of an n-person game we mean a real-valued function, v, defined on the subsets of N, $\mathcal{P}(N)$:

$$v: \mathcal{P}(N) \longrightarrow R$$

satisfying the conditions:

2.1: $v(\emptyset) = 0$

2.2: $v(S \cup T) \geq v(S) + v(T) \quad \text{if } S \cap T = \emptyset$.

v(S) is the amount of utility that the members of S can obtain from the game.

Frequently the [0,1]-normalization of a characteristic function is used, that is:

$v: \mathcal{P}(N) \longrightarrow [0,1]$ satisfying

$v(\emptyset) = 0$ and

$v(N) = 1$.

Definitions 3.: By an n-person game in characteristic function form is meant a real-valued function v, defined on the subsets of N, satisfying conditions 2.1 and 2.2 .

Definition 4.: A game is said to be constant sum, if, for all $S \subseteq N$,

$$v(S) + v(N-S) = v(N) .$$

Definition 5.: An imputation for the n-person game, v, is a payoff vector $x = (x_1, \ldots, x_n)$ satisfying

(i) $$\sum_{i \in N} x_i = v(N)$$

(ii) $x_i \geq v(\{i\})$ for all i in N .

Definition 6.: A carrier for a game v is a coalition, T, such that, for any S,

$$v(S) = v(S \cap T) .$$

Heuristically, Definition 6. states that any player who does not belong to a carrier is a dummy, that is, can contribute nothing to any coalition.

Definition 7.: Let v be an n-person game, and let π be any permutation of the set N. Then, by πv, we mean the game u such that, for any $S = (i_1, i_2, \ldots, i_s)$

$$u(\{\pi(i_1), \pi(i_2), \ldots, \pi(i_s)\}) = v(S).$$

Definition 7 simply states, that the game πv is nothing other than the game v, with the roles of the players interchanged by the permutation .

Notice, as games are essentially real-valued functions, it is possible to talk of the sum of two or more games, or of a number times a game.

1.2. The Shapley-value

In his work "A Value for n-Person Games" [13], L.S.Shapley conceptually started from the von Neumann - Morgenstern theory. He thereby made some important underlying assumptions:

a) utility is objective and transferable
b) games are cooperative affairs
c) games granting a) and b) are adequately represented by their characteristic function.

Shapley did his value-approach axiomatically. By the value of a game v is meant an n-vector $\varphi(v) = (\varphi_1(v),\ldots,\varphi_n(v))$, satisfying

A1: If S is any carrier of S, then

$$\sum_S \varphi_i(v) = v(S)$$

A2: For any permutation Π, and i in N

$$\varphi_{\pi(i)}(\pi v) = \varphi_i(v) \; .$$

A3: If u and v are any games,

$$\varphi_i(u+v) = \varphi_i(u) + \varphi_i(v) \; .$$

These are Shapley's axioms.
It is a remarkable fact that these axioms determine a unique value, S, for all games. For the proof see [8] pp.180.
The following formula gives the Shapley-value explicitely:

$$S_i(v) = \sum_{\substack{T \subseteq N \\ i \in T}} (t-1)!(n-t)!/n![v(T) - v(T-\{i\}]$$

where $t = |T|$ and $n = |N|$.

The Shapley-value may be given the following heuristic interpretation:
Suppose the aim of the game is to form the all-player coalition. Suppose, players arrive randomly, join the coalition that has formed until now, and get their marginal contribution to that coalition, that is $v(T) - v(T-\{i\})$. Under this scheme every order of coalition formation is possible and occurs with equal probability $1/n!$. The Shapley-value assigns to each player i his expected value under this randomization scheme.

In the sequel a coalition in brackets [S] denotes an ordered coalition.

1.3. Probability of coalition formation and expected value in 3-person games

Let $N=\{1,2,3\}$ be the set of players.
Let $p_z(i)$ denote the probability that player i for $i=1,2,3$ is chosen at random to start the game. Obviously $p_z(i)=1/3$.

Let $p(j_{[i]})$ denote the probability that player i makes a bid to player j. If player i is indifferent with respect to which player will enter the coalition in the next place he will choose the next player at random. We will write $p_z(j_{[i]})$ instead of $p(j_{[i]})$. It is clear that

$p_z(j_{[i]})=1/2$ for $i=1,2,3$.

Let $p_+(j_{[i]})$ and $p_-(j_{[i]})$ denote the probabilities that player j accepts or refuses to join coalition [i] respectively. Of course

$$p_+(j_{[i]}) + p_-(j_{[i]}) = 1.$$

Let us assume that if players j and k for j,k in $\{1,2,3\} - \{i\}$ both accept (refuse) to join the coalition [i] for $i=1,2,3$, then $p_+(j_{[i]}) = p_+(k_{[i]})$.
Suppose now player 1 starts the game. Suppose player 3 prefers entering the coalition in the third rather than in the second place, that is

$$v(123) - v(12) > v(13) \quad \text{and therefore}$$
$$v(123) - v(13) > v(12).$$

This states, also player 2 prefers entering the coalition in the third rather than in the second place.
And therefore $p_-(2_{[1]}) = p_-(3_{[1]})$.
As $p_+(2_{[1]})+p_-(2_{[1]}) = 1$ also

$$p_+(2_{[1]})+p_-(3_{[1]}) = 1 .$$

Let [S] be any sequence of players in the all player coalition. Let P([S]) denote the probability that the sequence [S] will form. Then the expected value E for player 1 is defined as follows:

$$E(1) = v(1)\ (P([123]) + P([132]))$$
$$+ v(21).P([213]) + v(31).P([312])$$
$$+[v(123)-v(23)]\{P([231]) + P([321])\}$$

Consider as an example the sequence S=[123]. The probability for the sequence S is

$$P([123]) = p_z(1)\{p(2_{[1]})\cdot p_+(2_{[1]}) + p(3_{[1]})\ p_-(3_{[1]})\}$$

$$= 1/3\{p(2_{[1]})p_+(2_{[1]})$$

$$+ (1-p(2_{[1]})(1-p_+(2_{[1]})\}\ .$$

If moreover $p(i_{[1]}) = p_z(i_{[1]}) = 1/2$ for i=2,3 then

$$P([123]) = p_z(1)\{p_z(2_{[1]})p_+(2_{[1]})$$

$$+ p_z(3_{[1]})p_-(3_{[1]})\}$$

$$=1/3\ 1/2[p_+(2_{[1]} + p_-3_{[1]}] = 1/6\ .$$

To be clear by E_z we denote the expected value where $p(i_{[j]})=p_z(i_{[j]})$ for j in N, i in N-{i} .

For random choices $p_z(i_{[j]})$ the value P([123]) shows

that the probability that a coalition forms does not depend on that coalition. It only depends on the chance moves which are equal for all players. Therefore we note the following result:

> If in a 3-person game each player makes his bids with equal probability each sequence of players in the all-player coalition is equally likely.

Notice, the outcome of the game does not depend on players' decision whether to join the coalition or not.
In the random case we get for the expected value E_z

of player 1:

$$E_z(1) = v(1)1/3 + [v(21)+v(31)]\ 1/6 + [v(123)-v(23)]\ 1/3$$

which is exactly the Shapley-value for a 3-person game.

So we have found a formal proof for Rapoport's result:
For every 3-person game the Shapley value equals the expected value of the corresponding extensive form game.

In the general case where players do not make their bids at random, that is $p(i_{[j]}) \neq p_z(i_{[j]}) = 1/2$ for j in N, i in N-{i}

the expected value E for player 1 has the form:

$$\begin{aligned} E(1) = {} & v(12)\ 1/3[p(1_{[2]})p_+(1_{[2]}) \\ & + (1-p(1_{[2]}))(1-p_+(1_{[2]}))] \\ & + v(13)\ 1/3[p(1_{[3]})p_+(1_{[3]}) \\ & + (1-p(1_{[3]}))(1-p_+(1_{[3]}))] \\ & + [v(123)-v(23)]1/3\{(1-p(1_{[2]}))p_+(1_{[2]}) \\ & + (1-p_+(1_{[2]}))p(1_{[2]}) \\ & + (1-p(1_{[3]}))p_+(1_{[3]}) + (1-p_+(1_{[3]}))p(1_{[3]})\} \end{aligned}$$

using the formula P([S]) for the probability of coalition [S] and assuming that the payoff of 1-player coalitions equals zero.

1.4. Probability of coalition formation and expected value in 4-person games

Let $N=\{1,2,3,4\}$ be the set of players. The probability $p_z(i)$ that player i in N is chosen at random to start the game equals 1/4.

Let $p_0(j_{[i]})$ denote the probability that player i makes a bid to player j where no player before player j has been invited by player i.

Correspondingly $p_1(j_{[i]})$ denotes the probability that player i invites player j where already one player has been invited by player i.

Again for the random case we will write $p_{z_0}(j_{[i]})$ and $p_{z_1}(j_{[i]})$ correspondingly. It is clear that

$$p_{z_0}(j_{[i]})=1/3 \text{ and } p_{z_1}(j_{[i]})=1/2 \text{ for } i \text{ in } N \text{ and } j \text{ in } N-\{i\}.$$

In the case of $p_{z_1}(j_{[i]})$ player i chooses between two

players which gives the value 1/2. Of course there is no p because at this stage of the 4-person game there is only one player left who must join the coalition corresponding to the rules of the game.

More general $p_r(j_{[S]})$ denotes the probability with which the last player of coalition (sequence) [S] makes a bid to player j where already r players have rejected the invitation, for j in N-S, with the range of r going from 0 to n-s-2 where n denotes the number of players.

For a 4-person game there exists one more value $p_0(j_{[S]})$ where $|S|=2$ and j in N-S. If the only possible value for r=0 we simply write $p(j_{[S]})$.
If player j is chosen at random
$p_z(j_{[S]})=1/2$ for $|S|=2$ and j in N-S.

Again by $p_+(j_{[S]})$ and $p_-(j_{[S]})$ we denote the probabilities that player j in N-S accepts or declines the invitation of the last player of coalition [S].
Of course $p_+(j_{[S]})+p_-(j_{[S]}) = 1$.

Suppose $v(134)-v(13) > v(14)$,
therefore $v(134)-v(14) > v(13)$.

This states that if player 1 starts the game players 3 and 4 have competing interests in the third place. But this does not say anything about the interest of players 2 and 3 or players 2 and 4 in the second and third place.
One can only be sure that if there are 2 players left who have not yet joined the coalition, then these two players will be competing either in the third or in the forth place.
For example if $v(1234)-v(123)>v(124)-v(12)$ then $v(1234)-v(124)>v(123)-v(12)$, that is, both players 3 and 4 prefer entering the coalition in the forth place.
In other words: If in a 4-person game $|S|=2$, then
$p_+(i_{[S]}) = p_+(j_{[S]})$ and therefore
$p_+(i_{[S]}) + p_-(j_{[S]}) = 1$ for i,j in N-S .

Let $P(i_{[S]})$ denote the probability that player i joins coalition [S] with S in N, i in N-S.
The probability that coalition [1234] forms
$P([1234]) = p_z(1).P(2_{[1]}).P(3_{[12]})$
where $p_z(1)=1/4$.

$$P(2_{[1]}) = p_0(2_{[1]}) \cdot p_+(2_{[1]})$$

$$+ p_0(3_{[1]}) \cdot p_-(3_{[1]}) \cdot p_1(2_{[1]}) \cdot p_+(2_{[1]})$$

$$+ p_0(4_{[1]}) \cdot p_-(4_{[1]}) \cdot p_1(2_{[1]}) \cdot p_+(2_{[1]})$$

$$+ p_0(3_{[1]}) \cdot p_-(3_{[1]}) \cdot p_1(4_{[1]}) \cdot p_-(4_{[1]})$$

$$+ p_0(4_{[1]}) \cdot p_-(4_{[1]}) \cdot p_1(3_{[1]}) \cdot p_-(3_{[1]})$$

$$= p_+(2_{[1]})\{p_0(2_{[1]})+p_0(3_{[1]}) \cdot p_-(3_{[1]}) \cdot p_1(2_{[1]})$$

$$+ p_0(4_{[1]}) \cdot p_-(4_{[1]}) \cdot p_1(2_{[1]})\}$$

$$+ p_-(3_{[1]}) \cdot p_-(4_{[1]})\{p_0(3_{[1]})p_1(4_{[1]})+p_0(4_{[1]})p_1(3_{[1]})\}$$

It can be seen easily that in a 4-person game the probability that a player joins a 2-person coalition in a 4-person game is equal the probability that a player in a 3-person game joins a 1-person coalition. Therefore

$$P(3_{[12]}) = p(3_{[12]}) \cdot p_+(3_{[12]})$$

$$+ p(4_{[12]}) \cdot p_-(4_{[12]})$$

As player 1 as the starting player has 3 possibilities to choose the next player

$$p_0(2_{[1]}) + p_0(3_{[1]}) + p_0(4_{[1]}) = 1,$$

and for the same reason

$$p_1(i_{[1]}) + p_1(j_{[1]}) = 1 \quad \text{for } i,j \text{ in } N-\{1\},\ i \neq j$$

$$p(i_{[S]}) + p(j_{[S]}) = 1 \quad \text{for } i,j \text{ in } N-S,\ |S|=2\ ,\ i \neq j\ .$$

Assuming that 1-player coalitions get a zero payoff the expected value E for player 1 is:

$$\begin{aligned}
E(1) = {} & v(12)\{P([2134]) + P([2143])\} \\
& + v(31)\{P([3124]) + P([3142])\} \\
& + v(41)\{P([4123]) + P\ ([4132]) \\
& + [v(231)-v(23)]\{P([2314]) + P([3214])\} \\
& + [v(241)-v(24)]\{P([2413]) + P([4213])\} \\
& + [v(341)-v(34)]\{P([3412]) + P([4312])\} \\
& + [v(2341)-v(234)]\{P([2341]) + P([2431]) + P([3241]) \\
& \qquad + P([3421]) + P([4321]) + P([4231])\}.
\end{aligned}$$

4-person games with random choices

Now suppose that players choose their successors at random. Then

$$P(2_{[1]}) = p_+(2_{[1]})\{1/3+(1/3)(1/2)p_-(3_{[1]})$$
$$+ (1/3)(1/2)\cdot p_-(4_{[1]})\}$$
$$+ (1/3)\cdot p_-(3_{[1]})\cdot p_-(4_{[1]})$$
$$= p_+(2_{[1]})\{1/3 + 1/3!(p_-(3_{[1]}) + p_-(4_{[1]})\}$$
$$+ (2!/3!)p_-(3_{[1]})\cdot p_-(4_{[1]})$$
$$= p_+(2_{[1]})[1/3 + 1/3!\sum_{i=3}^{4} p_-(i_{[1]})] + 1/3\prod_{i=3}^{4} p_-(i_{[1]})$$

and

$$P(3_{[12]}) = 1/2[p_+(3_{[12]}) + p_-(4_{[12]})]=1/2 \quad .$$

The probability that coalition [1234] forms is

$$P([1234]) = p_z(1).P(2_{[1]}.P(3_{[12]}).1$$
$$= (1/4)P(2_{[1]})(1/2)$$
$$= 1/8\ P(2_{[1]})$$

stating that in a 4-person game with random choices the probability that a certain coalition forms only depends on the player in the second position. Therefore it is clear that $P([1234]) = P([1243])$.

Thus we get for the expected value of player 1:

$$
\begin{aligned}
E(1) = {} & v(21)\ 2P([2134]) \\
& + v(31)\ 2P([3124]) \\
& + v(41)\ 2P([4123]) \\
& + [v(231)-v(23)]\{P([2314]) + P([3214])\} \\
& + [v(241)-v(24)]\{P([2413]) + P([4213])\} \\
& + [v(341)-v(34)]\{P([3412]) + P([4312])\} \\
& + [v(N)-v(234)]\{P([2341]) + P([2431]) \\
& \qquad + P([3214]) + P([3421]) \\
& \qquad + P([4231]) + P([4321])\} \\
& = \sum_{\substack{S \subseteq N \\ 1 \in S}} [v(S) - v(S-\{1\})]\ P([S] \cup N-S)
\end{aligned}
$$

The probability that coalition [1234] forms can also be written as:

$$
\begin{aligned}
P([1234]) = {} & 1/4\ \{p_+(2_{[1]})[1/3 + 1/3!(p_-(3_{[1]}) + p_-(4_{[1]})] \\
& + 2!/3! \cdot p_-(3_{[1]}) \cdot p_-(4_{[1]})\ \}1/2 \\
& = 1/4!\ p_+(2_{[1]}) + (1/4!)(1/2)\ p_+(2_{[1]})(p_-(3_{[1]}) + p_-(4_{[1]})) \\
& + 1/4!\ p_-(3_{[1]})p_-(4_{[1]})\ .
\end{aligned}
$$

This expression shows that the probability for a certain sequence of players in the great coalition is determined by the probabilities of acceptance or rejection of the players, or in other words by the characteristic function which determines the payoffs players can hope for in certain positions.

Notice that therefore in general the probabilities for different sequences of players in the great coalition will not be equal.

However some restrictions can be made for the $p_+(i_{[j]})$ for j in N, i in N-{j}. In the following table we show all possible combinations for the values of $p_+(i_{[1]})$ for i=2,3,4.

$p_+(2_{[1]})$	$p_+(3_{[1]})$	$p_+(4_{[1]})$	
1	1	1	
1	0	0	
0	0	0	
w	w	w	0<=w<=1
w	0	0	0<=w<=1
0	0	w	0<=w<=1
w	1	1	0< w<=1
1	1	w	0< w<=1

Table I.1.4.1: probabilities of acceptance $p_+(i_{[1]})$ of player i for i=2,3,4

The table shows 8 different combinations. Of course the first three rows need not be mentioned separately.

Let $p_+(2,3,4)_{[1]}$ denote the vector of acceptance probabilities for players 2,3,4 invited by player 1. For each vector $p_+(2,3,4)_{[1]}$ it is possible to compute $P(2_{[1]})$ which determines

$$P([1234]) = P([1243]) = 1/8 \cdot P(2_{[1]}).$$

The expressions for $P(2_{[1]})$ are given in the next table.

$p_+(2,3,4)_{[1]}$.	$P(2_{[1]})$	
(w,w,w)	1/3	0<=w<=1
(w,0,0)	1/3 + w2/3	0<=w<=1
(0,0,w)	1/3(1-w)	0<=w<=1
(w,1,1)	1/3(w)	0< w<=1
(1,1,w)	1/3+1/3(1-w)	0< w<=1

table I.1.4.2: Probabilities $P(2_{[1]})$ that player 2 joins coalition [1] for the different values of $p_+(2,3,4)_{[1]}$.

Notice the second row of table 1.2.2. For w=1 $P(2_{[1]})$= 1/3+2/3 = 1, which states that for

$p_+(2,3,4)_{[1]}$ = (1,0,0) player 2 will certainly join coalition [1] in the second place. There is no possibility for players 3 and 4 to join coalition [1]. That is P([1324]) = P([1342]) = P([1423]) = P([1432]) = 0. Moreover, notice the first row of the table where $p_+(i_{[1]})$ is equal for i=2,3,4. In this special case the value $P(2_{[1]})$ is independent of $p_+(i_{[1]})$ for i=2,3,4, and therefore the probabilities P(S) for each sequence S in the great coalition are equiprobable, thus suggesting that the expected value might be equal the Shapley value. Notice that in the general case where the vector $p_+(2,3,4)_{[1]}$ has different values for players 2,3 and 4, $P(2_{[1]})$ will depend on the acceptance probabilities, which states that for different sequences [S] of players in the great coalitions P([S]) will have different values.

1.5. A special case for the probability of coalition formation and the expected value in n-person games

As in the case of 3- and 4-person games we denote by [S] the ordered coalition S. P([S]) denotes the probability that coalition [S] forms.

$p_+(i_{[S]})$ denotes the probability that player i in N-S accepts to join coalition [S], correspondingly $p_-(i_{[S]})$.
Moreover s=|S| and n=|N|.
For the general case suppose N={1,2,3, ... ,n}.
Suppose all choices are made at random.
Then the probability $P(k_{[S]})$ that player k joins coalition [S] can be written in the following way:

$$P(k_{[S]}) = p_+(k_{[S]})[1/(n-s) + 1/((n-s)(n-s-1))\sum_{\substack{i\notin S\\ i\neq k}} p_-(i_{[S]})$$

$$+ 1/((n-s)(n-s-1)(n-s-2))\sum_{\substack{i\notin S\\ i\neq k}} p_-(i_{[S]})\sum_{\substack{j\notin S\\ j\neq k\\ j\neq i}} p_-(j_{[S]})$$

$$+ 1/((n-s)(n-s-1)(n-s-2)(n-s-3))\sum_{\substack{i\notin S\\ i\neq k}} p_-(i_{[S]})\sum_{\substack{j\notin S\\ j\neq k\\ j\neq i}} p_-(j_{[S]})$$

$$\cdot\sum_{\substack{m\notin S\\ m\neq k\\ m\neq i\\ m\neq j}} p_-(m_{[S]})$$

$$+ \; . \; . \; . \; +$$

$$+ 1/(n-s)! \sum_{\substack{i \notin S \\ i \neq k}} p_-(i_{[S]}) \sum_{\substack{j \notin S \\ j \neq k \\ j \neq i}} p_-(j_{[S]}) \ \ldots \sum_{\substack{m \notin S \\ m \neq k \\ m \neq i \\ m \neq j \\ m \neq \ldots}} p_-(m_{[S]})]$$

$$+ (n-s-1)!/(n-s)! \prod_{l \notin N-S-\{k\}} p\,(l_{[S]}) \qquad \text{where } k \text{ in } N-S .$$

In case $p_+(i_{[S]})$ and $p_-(i_{[S]})$ are equal w and 1-w with w in [0,1] respectively for all i in N-S the expression for $P(k_{[S]})$ can be written as:

$$\begin{aligned} P(k_{[S]}) = {} & w[1/(n-s) + (1/((n-s)(n-s-1)))(n-s-1)(1-w) \\ & + (1/((n-s)(n-s-1)(n-s-2)))(n-s-1)(1-w)(n-s-2)(1-w) \\ & + (1/((n-s)(n-s-1)(n-s-2)(n-s-3))) \\ & \qquad (n-s-1)(1-w)(n-s-2)(1-w)(n-s-3)(1-w) \\ & + \ldots . \\ & + (1/(n-s)!)(n-s-1)(n-s-2) \ \ldots \ 3.2(1-w)^{n-s-2} \ ] \\ & + (n-s-1)!/(n-s)! \ (1-w)^{n-s-1} \end{aligned}$$

$$= 1/(n-s)[w \sum_{i=0}^{n-s-2} (1-w)^i + (1-w)^{n-s-1}]$$

For 4-person games we have seen that the expression in brackets equals 1. We now show the general result by induction.
Denote by $m-1:=n-s-2$.

Suppose

$$B_m := w\sum_{i=0}^{m-1}(1-w)^i + (1-w)^m = 1 \quad \text{does hold}$$

If m becomes m+1 we get

$$B_{m+1} = w\sum_{i=0}^{m}(1-w)^i + (1-w)^{m+1} .$$

This expression can also be written as

$$B_{m+1} = w\sum_{i=0}^{m-1}(1-w)^i + w(1-w)^m + (1-w)^m - w(1-w)^m ,$$

by our assumption this term obsiously equals 1.

We have thus shown:
If in an n-person game in extensive form at some stage of the game all players behave equally, any player in N-S may enter coalition [S] in place s+1 with equal probability.

Therefore: If at every stage of the game, all players behave equally, then every order of coalition formation is possible and equally likely.

But this is exactly the assumption that is made by the solution concept of the Shapley-value.
We notice the following result for n-person games that are played corresponding to the rules proposed by A.Rapoport:

If in an n-person game at every stage of the game all players behave equally, then the expected value equals the Shapley-value.

Some examples (section 3.) will show that situations where all players behave equally, occur easily.

Interpretation

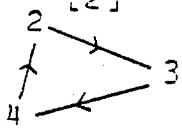
The assumption that in a certain game-situation all players behave equally with respect to invitations is certainly true for symmetric games. But notice that a game need not necessarily be symmetric for the players to behave all equally. As an example consider game (6) of the 6-person games in section 3.4.

1.6. On the equiprobability of orders of coalitions

Consider once more the expression for the expected value E(1) in a four-person game. For the expected value to be equal the Shapley value, P(S) need not necessarily be equal for all sequences S in the great coalition. Assuming that the probability for a coalition S where player 1 is in the second place equals 1/24, there seems to be some kind of cyclic relation between the coalitions where player 1 is in the third or fourth place. For example, equality of expected value and Shapley-value is assured by P([3214]) = P([4312]) = P([2413])
or equivalently

$P(2_{[3]}) = P(3_{[4]}) = P(4_{[2]})$.

Consider the circle



which may be read as
probability of players (2 behind 3) equals (3 behind 4) equals (4 behind 2) .
If in a 4-person game the cyclic relation is true for any permutation of players 2,3 and 4 then expected value and Shapley-value are equal.

We cannot be sure about the relations between the values P(S) for some coalitions S in an n-person game. However our considerations suggest that for the expected value to be equal the Shapley value P([S]) need not necessarily be equal for all sequences [S] in the great coalition. Or in other words, we would expect equality of expected value and Shapley-value also in some games where $p(i_{[S]})$ is not equal for all i in N-S, for some S in N.

Before going on to some examples illustrating the difference or equality of Shapley-value and expected value let us do some considerations concerning dummy players and indifferent players.

2. Dummy players and indifferent players

2.1. Dummy players

Consider a game as it was proposed by A.Rapoport in [9]. Remember one player is chosen to start the game. At any stage of the game a player makes a bid to another player. The invited player may accept or decline. Each time a coalition has formed the last player is payed his marginal contribution to that coalition. A player i is called a dummy player if

$v(S \cup \{i\}) = v(S)$ for all S in N.

Our question is: Can a dummy player ever influence the outcome of the game?

One might expect that a dummy player by accepting or declining an invitation could influence the payoff for other players.

Let $N=\{a_1,a_2,...,a_n\}$ be the set of players.
Let $G(a_i)/n!$ be the payoff for player a_i.
Suppose coalition $K = [a_1,a_2,...,a_n]$ has

formed. Corresponding to the definition a dummy player d may join the sequence in any place without changing the payoff $G(a_i)/n!$ of player a_i in N.
Add one dummy player d to the game. Then starting from the given sequence player d has the following possibilities to enter coalition K:

$$
\begin{array}{llllll}
d & a_1 & a_2 & \dots & & a_n \\
 & a_1 & d & a_2 & \dots & a_n \\
 & a_1 & a_2 & d & \dots & a_n \\
 & \dots & & & & \\
 & a_1 & a_2 & \dots & d & a_n \\
 & a_1 & a_2 & \dots & & a_n \quad d
\end{array}
$$

We get (n+1) possibilities out of one given sequence.
Denote by $G_n(a_i)$ the new gain of player a_i if dummy player d is added to the game. Then

$$G_n(a_i) = G(a_i)(n+1)/(n!(n+1)) = G(a_i)/n!$$

This shows for general n-person games that dummy players do not have any influence on the outcome of the game.

2.2. Indifferent players

Let c denote some constant real number.
A player i is called indifferent with respect to a coalition S, if

$$v(S'\cup\{i\}) - v(S') = c \quad \text{for all } S' \supseteq S \text{ and } i \notin S' .$$

In simple games c equals 1 or 0.
If $v(S'\cup\{i\}) = v(S')$ for all coalitions S' then i is a dummy player.
A player may become indifferent up from a certain stage of the game. If for example in a simple game a winning coalition has formed, there is no gain left and all remaining players will be indifferent.

Or think of a game where one coalition [T] has formed.
Let $N-T = \{a_{t+1}, a_{t+2}, \ldots, a_n\}$.
Suppose there is only one prospective winner. So all players not yet in the coalition will be indifferent.
In this case the expected value $E(a_i)$ for any a_i in N-T depends on

$$P([Ta_{t+1}\ a_{t+2}\ \ldots\ a_n]) = P([T])\ P(a_{t+1} \ldots a_n\ {}_T)$$

As all players a_i in $\{a_{t+1}, a_{t+2}, \ldots, a_n\}$ behave equally we know that

$$P(a_{t+1}\ {}_{[T]}) = 1/(n-t-1)$$

$$P(a_{t+2}\ {}_{[Ta_{t+1}]}) = 1/(n-t-2) \quad \text{etc.}$$

Therefore, if coalition [T] has formed all sequences in the great coalition are equally likely, that is the probabilities of coalition formation are independent of the probabilities of acceptance of players in N-T.
In other words, players in N-T do not have the power to influence the payoff of any other player.

<u>More general:</u>

Indifferent players cannot influence the outcome of the game.

Think of a game where a coalition [T] has formed. Up from this stage there is one player i who now is indifferent. Suppose players in N-T-{i} are not indifferent but want all to join coalition T next.
Let a_{t+1} be in N-T-{i}.
If coalition [T] has formed the possibilities for player i to join the coalition are:

$$\begin{array}{l} T\ i\ a_{t+1}\ \dots\ a_n \\ T\ a_{t+1}\ i\ \dots\ a_n \\ \dots \\ T\ a_{t+1}\ \dots i\ a_n \\ T\ a_{t+1}\ \dots\ a_n\ i \end{array}$$

Leaving out player i the payoff $G(a_{t+1})$ for player a_{t+1} is

$$G(a_{t+1}) = [v(T \cup \{a_{t+1}\}) - v(T)]P([T\ a_{t+1}\ a_{t+2}\ \dots\ a_n])$$

where $|\{T, a_{t+1}, \dots, a_n\}| = n-1$.
Reintroducing player i to the game

$$|\{T, a_{t+1}, \dots, i, \dots a_n\}| = n \quad \text{and}$$

$$G(a_{t+1}) = \{v([Tia_{t+1}]) - v([Ti])\}P([Tia_{t+1}\ \dots\ a_n])$$

$$+ \{v([Ta_{t+1}]) - v([T])\}P([Ta_{t+1}\ ..\ i\ ..\ a_n])(n-t-1).$$

In the sequence $[Tia_{t+1}\ \dots\ a_n]$ all players behave equally after player i has joined the coalition. In $[Ta_{t+1}\ ..i..\ a_n]$ all players behave equally after player a_{t+1} has joined coalition [T], that is

$$P(\{a_{t+1}, \dots, i, \dots, a_n\}_{[Ta_{t+1}]}) =$$

$$P(\{a_{t+1}, \dots, a_n\}_{[T_i]})$$

Therefore
the probabilities $P([Tia_{t+1}\ \dots\ a_n])$ and

$P([Ta_{t+1} \ldots i \ldots a_n])$ only differ in

$P(a_{t+1[T]})$ and $P(i_{[T]})$.

From the formula in section 1.5. and assuming that all players in N-T-{i} behave equally and player i accepts an invitation with probability w we get:

$$P(a_{t+1[T]}) = 1[1/(n-t) + 1/((n-t)(n-t-1))\ (1-w) + 0] + 0$$
$$= 1/(n-t) + 1/((n-t)(n-t-1))(1-w)$$

and

$$P(i_{[T]}) = w(1/(n-t)) .$$

For the gain of player a_{t+1}, $G(a_{t+1})$, we get:

$$G(a_{t+1}) = P([Tia_{t+1} \ldots a_n]) +$$
$$+ (n-t-1).P([Ta_{t+1}ia_{t+2} \ldots a_n])$$

as i may enter the coalition after player a_{t+1} in any place.

$$G(a_{t+1}) = P([T]).P(i_{[T]}).P(\{a_{t+1},..,a_n\})_{[Ti]})$$
$$+ (n-t-1).P\ ([T]).P(a_{t+1[T]}).P(\{i,a_{t+2},..,a_n\}_{[Ta_{t+1}]})$$
$$= P([T]).P(\{a_{t+1},..,a_n\}_{[Ti]}[P(i_{[T]} +$$
$$+ (n-t-1).P(a_{t+1[T]})] .$$

The expression in brackets equals

$$P(i_{[T]}) + (n-t-1)P(a_{t+1[T]})$$

$$= w/(n-t) + (n-t-1)(1/(n-t) + 1/((n-t)(n-t-1))(1-w))$$

$$= w/(n-t) + (n-t)/(n-t) - 1/(n-t) + (1-w)/(n-t) = 1 ,$$

and therefore

$$G(a_{t+1}) = P([T]).1/(n-t-1)! ,$$

stating that the payoff of player a_{t+1} cannot be influenced by an indifferent player.
If there are more than one indifferent player, apply the described procedure as often as there are still indifferent players, and the payoff for some prospective winner remains unchanged.

<u>So we have shown:</u>
If there are only interested players, that is, players who want to join the coalition in the next step, indifferent players cannot influence the payoff for the prospective winners.
If in a certain situation of the game there are only indifferent players and players who do not want to join the coalition in the next step, it is even more simple. As for any of the 'non interested' players the payoff is zero if he must enter the coalition the outcome does not depend on the $P(k_{[S]})$ for all indifferent players k in N-S.

As a dummy player is also an indifferent player, dummy players too cannot influence the expected value for any player in N.

As an example consider the game with characteristic function $v(1245) = v(1246) = v(1235) = 1$ and $v(S) = 0$ otherwise.
Whenever the coalition [124] has formed players 5 and 6 are prospective winners. But player 3 will be indifferent as there is no chance for him to get any positive payoff.
Consider the 5-person game:
$v(1245) = v(1246) = 1$
$v(S) = 0$ otherwise.
If coalition [124] has formed a possible gain for player 5 will be:

$$G(5) = (v(1245) - v(124))P([12456])$$
$$= P([124])\ 1/2$$

Reintroducing player 3 to the game results in

$$G_n(5) = (v(1245) - v(124))(P([124536]) + P([124563]))$$
$$+ (v(12435) - v(1243))P([124356])$$

$$= P([124])P(5_{[124]})(1/2)2 + P([124])P(3_{[124]})1/2$$

$$= 1/2\ P([124])(2P(5_{[124]}) + P(3_{[124]})))$$

$$= 1/2\ P([124])\{2p_+(5_{[124]})(1/3 + 1/3!(p_-(3_{[124]})+p_-(6_{[124]})))$$

$$+ (2/3)p_-(3_{[124]})p_-(6_{[124]})$$

$$+ p_+(3_{[124]})(1/3 + 1/3!(p_-(5_{[124]}) + p_-(6_{[124]})))$$

$$+ (1/3)\cdot p_-(5_{[124]})p_-(6_{[124]})\}$$

$$= (1/2)P([124])[(2/3) + (1/3)p\_(3_{[124]}) + (1-p\_(3_{[124]}))/3)]$$

$$= (1/2)P([124]) .$$

This example shows that the indifferent player 3 cannot influence the payoff of player 5.

Until now we have only considered games where all but the indifferent players behave equally. Now let us consider the more general case where some players want to join the coalition next and some do not.

Let us consider a game with dummy players, interested and non interested players.
We start with a simple n-person game that has only interested and non interested players. We add one dummy player to the game and show that the expected payoff for a prospective winner remains unchainged.

Let n be the number of players.
Assume some coalition [T] with $|T|=t$ has formed.
Let $\{a_{t+1},\dots,a_{n-2},b_1,b_2\}$ be the players in N-T.
Let the vector $p\ (a_{t+1},\dots,a_{n-2},b_1,b_2)_{[T]}$

have the values $(1,\dots,1,0,0)$, that is, there are two players b_1,b_2 who do not want to join coalition [T] in place t+1.
Of course $P(b_{i_{[T]}})=0$ for i=1,2.
Denote by $P_1(a_{i_{[T]}})$ the probability that player a_i in N-T joins coalition [T] in the n-person game.
Denote by $P_2(a_{i_{[T]}})$ the corresponding value in the (n+1)-person game. Then

$P_1(a_{i_{[T]}}) = 1/(n-t) + 2/((n-t)(n-t-1)) + 2/((n-t)(n-t-1)(n-t-2))$.

Add one dummy player i to the game who accepts an invitation with probability w in [0,1]. Then

$P_2(i_{[T]}) = w(1/(n+1-t) + 2/((n+1-t)(n-t)) + 2/((n+1-t)(n-t)(n-t-1)))$

The probability that any interested player a_i joins coalition [T] is

$$\begin{aligned}P_2(a_{i_{[T]}}) &= 1/(n+1-t) + (2+1-w)/((n+1-t)(n-t)) \\ &+ 1/((n+1-t)(n-t)(n-t-1))\cdot(1(1+1-w) + 1(1+1-w) + (1-w)2) \\ &+ 1/((n+1-t)(n-t)(n-t-1)(n-t-2))\cdot(6(1-w)) \\ &= 1/(n+1-t) + 2/((n+1-t)(n-t)) + 2/((n+1-t)(n-t)(n-t-1)) \\ &+ 1/((n+1-t)(n-t))(1-w) + 4(1-w)/((n+1-t)(n-t)(n-t-1)) \\ &+ 6(1-w)/((n+t-1)\ldots(n-t-2)) .\end{aligned}$$

Correspondingly denote by $G_1(a_i)$ and $G_2(a_i)$ the gain for player a_i in the n-person and (n+1)-person game respectively.
Denote by a_{t+1} the interested player that joins coalition [T] in place (t+1). He gets a positive payoff in the n-person game only if he joins T in position (t+1) . Therefore

$$G_1(a_{t+1}) = P_1(a_{t+1\ [T]}).P([T])(1/(n-t-1)!) ,$$

as all other players are indifferent after player a_{t+1} has joined [T].

In the (n+1)-person game player a_{t+1} gets a positve payoff if he joins [T] in position (t+1). Then there are (n-t) possible positions for player i. And player a_{t+1} gets a positive payoff if he joins T in position (t+2) after player i. Therefore

$$G_2(a_{t+1}) = (n-t)P_2([Ta_{t+1}])\cdot 1/(n-t)!$$

$$+ P_2([Tia_{t+1}])\cdot 1/(n-t-1)!$$

$$= 1/(n-t-1)!\{P_2([Ta_{t+1}]) + P_2([Tia_{t+1}])\}$$

$$= 1/(n-t-1)!\ P([T])\{(P_2(a_{t+1[T]})$$

$$+ P_2(i_{[T]}).P_2(a_{t+1[Ti]})\ .$$

We still need the expression

$$P_2(a_{t+1[Ti]}) = 1/(n-t) + 2/((n-t)(n-t-1)) + 2/((n-t)(n-t-1)(n-t-2))$$

We want to show that

$$G_1(a_{t+1}) = G_2(a_{t+1})\ .$$

So we have to verify that

$$P_1(a_{t+1[T]}) = P_2(a_{t+1[T]})$$

$$+ P_2(i_{[T]}).P_2(a_{t+1[Ti]}) \qquad \text{or}$$

$$1/(n-t) + 2/((n-t)(n-t-1)) + 2/((n-t)(n-t-1)(n-t-2))$$
$$= 1/(n+1-t) + 2/((n+1-t)(n-t)) + 2/((n+1-t)(n-t)(n-t-1))$$
$$+ (1-w)/((n+1-t)(n-t)) + 4(1-w)/((n+1-t)(n-t)(n-t-1))$$
$$+ 6(1-w)/((n+1-t)(n-t)(n-1-t)(n-2-t))$$
$$+ w[1/(n+1-t) + 2/((n+1-t)(n-t)) + 2/((n+1-t)(n-t)((n-t-1))].$$
$$[1/(n-t) + 2/((n-t)(n-t-1)) + 2/((n-t)(n-t-1)(n-t-2))]$$

The terms with w cancel out. Multiplying each side with the term $(n+1-t)(n-t)(n-t-1)(n-t-2)$ we get

$$(n+1-t)(n-t-1)(n-t-2) + 2(n+1-t)(n-t-2) + 2(n+1-t) =$$
$$(n-t)(n-t-1)(n-t-2) + 3(n-t-1)(n-t-2) + 6(n-t-2) + 6\ .$$

Transfer all terms including variables n and t on the left-hand-side. Let the right-hand-side consist of the constant. Then

$$(n-t-1)(n-t-2)1 + (n-t-2)(-n+t+5) - 4n + 14t + 4t = 6$$

and after some transformations

$$(n-t-2)4 - 4n + 14 + 4t = 6$$

which obviously does hold.

So we have shown that $G_1(a_{t+1}) = G_2(a_{t+1})$,
which states that also if not all players behave equally, dummy players do not have the power to influence the payoff of other players.

3. Examples

To verify our results let us now consider some examples. The Shapley-value may be computed by hand even for games with more than 4 players. The computation of the expected value makes some difficulties. So we let a computer programm do the job.
In this section we compare the solutions (expected value and Shaplye-value) for 5-person and 6-person games. The computer did not terminate the computations of a 7-person game in a 2-days-time. The reason for these long-time-runs will be given in a later section.

As we consider only simple games we restrict to specify minimal winning coalitions (MWC).
Let us denote by E the vector of the expected values and by S the vector of the Shapley-values.

3.1. Some 5-person games

(1) v(12) = v(13) = v(145) = 1
    v(S) = 0 otherwise
    E = (7/10, 4/30,4/30,1/60,1/60)
    S = (7/10,7/60,7/60,1/30,1/30)

(2) v(1234) = v(1245) = v(1345) = 1
    v(S) = 0 otherwise
    E = (7/20,1/10,1/10,7/20,1/10)
    S = (7/20,1/10,1/10,7/20,1/10)

(3) v(1234) = v(145) = v(245) = 1
    v(S) = 0 otherwise
    E = (7/60,7/60,1/60,14/30,17/60)
    S = (11/60,11/60,1/20,11/30,13/60)

(4) v(1234) = v(245) = 1
    v(S) = 0 otherwise
    E = (1/30,23/60,1/30,23/60,1/6)
    S = (1/20,23/60,1/20,23/60,2/15)

(5) v(123) = v(124) = v(134) = v(145) = 1
    v(S) = 0 otherwise
    E = (8/15,1/10,1/10,7/30,1/30)
    S = (8/15,7/60,7/60,1/5,1/30)

(6) v(123) = v(124) = v(125) = v(345) = 1
    v(S) = 0 otherwise
    E = (3/10,3/10,4/30,4/30,4/30)
    S = (1/4,1/4,1/6,1/6,1/6)

(7) v(12) = v(13) = v(14) = v(15) = 1
    v(S) = 0   otherwise
    E = (4/5,1/20,1/20,1/20,1/20)
    S = (4/5,1/20,1/20,1/20,1/20)

(8) v(12) = v(13) =v(145) = v(235) = 1
    v(S) = 0   otherwise
    E = (13/20,19/120,19/120,1/60,1/60)   !!
    S = (9/20,1/5,1/5,1/30,7/60)

(9) v(12) = v(134) = v(135) = v(145) = 1
    v(S) = 0   otherwise
    E = (13/20,1/4,1/30,1/30,1/30)
    S = (13/20,3/20,1/15,1/15,1/15)

(10) v(12) = v(1345) = v(245) = v(345) = 1
     v(S) = 0   otherwise
     E = (3/10,2/5,1/20,1/8,1/8)
     S = (1/4,1/3,1/12,1/6,1/6)

3.2. Comments on the solutions of 5-person games

The values for E and S of the investigated 5-person games show that for almost all games E≠S. Notice that in some games [(1),(4),(5),(9)] E and S give equal payoffs to some players, but differ in the other players.
Notice that for games (2) and (7) E and S are equal.
Denote by E(i) and S(i) the expected value and the Shapley-value for player i in N respectively.
Consider as an example game (1) where E(1) = S(1).
If players 2 or 3 start the game player 1 will be the only prospective winner. Therefore all players are indifferent in the above defined sense and behave equally. Or in other words, if players 2 or 3 start the game any coalition formation will be possible and moreover equally likely.
If players 4 or 5 start the game the rest of the players will not be indifferent but however have all the same interest, that is, not joining the coalition in the next place.
If any other player but 1 joins the coalition in the second place, player 1 again is the only prospective winner. Again all players behave therefore equally and all coalition formations are equally likely.
If player 1 joins the coalition in the second place each of the remaining players wants to join the coalition in the third place as each one has the chance to win. Once again all players have equal interests, which says, each sequence in the grand coalition with players 4 or 5 as starting players is equally likely.
From these considerations it is clear that player 1

has as well in the Shapley-value as in the expected value exactly the same chances to win which explains the equality of E(1) and S(1).

If player 1 starts the game players 2 and 3 will be competing players in the second place. Players 4 and 5 have different interests in the second place than players 2 and 3, that is, they do not want to join the coalition in the second place, as they can only hope for a positive payoff in the third place. Therefore coalitions of the form [14...] or [15...] will not form. In other words, the situations where players 4 and 5 are prospective winners differ in the Shapley-value and in the expected value. Therefore $E(4)\neq S(4)$ and $E(5)\neq S(5)$ and by consequence $E(2)\neq S(2)$ and $E(3)\neq S(3)$.

It seems that game (2) is in some sense a symmetric game. A reason for the equality of E and S might be that each two coalitions only differ in one player. Let us consider this game more closely to see whether in any game situation all players behave equally.
All players certainly behave equally if the formed coalition consists of only one or two players. The only difficulty may arise when a three-person coalition has formed. But it is no difficulty to check the $\binom{5}{3}=10$ possible 3-person coalitions.
If for example the coalition [124] has formed, players 3 and 5 are competing and therefore behave equally.
If the coalition [123] has formed player 4 is the only prospective winner. Therefore players 4 and 5 are indifferent and behave equally. Each time the result is the equality of the probability for the corresponding sequences in the great coalition.

<u>More general</u>: If in game (2) of the 5-person games
a 3-person coalition has formed then the two remaining players are both

1.- either competing in the forth place, that is want to join the coalition or
2.- indifferent with respect to an invitation.

It is not possible that both or even one player decline in the forth place, because

1.- then only the great coalition would be winning,
2.- if one player declines in the forth place there must be at least one more player who who declines as well in the forth place. So case 2. corresponds to case 1.

As in our example the minimal winning coalitions have less than 5 players this case may not occur.

<u>The given examples show:</u>
At any stage of the game all players behave equally. Therefore the probability of any sequence in the great coalition is equal which is a sufficient condition for the expected value to be equal the Shapley-value.

For game (7) the equality can be seen even more easily. By a similar reasoning it is clear that the expected value for player 1, $E(1)$, equals his Shapley-value $S(1)$. As the four remaining players have the same 'power' the remaining payoff is shared equally among them.

Notice game 3!
In this game $E(4)=E(5)$ but $S(4)\neq S(5)$.

Considering the charcteristic function it does not seem that players 4 and 5 have the same power. This phenomenon results from the fact that for the expected value not all sequences in the great coalition are possible.
It is easy to find the situations where players 4 and 5 are prospective winners for the expected value. If player 4 starts the game noone wants to be in the second place. So it may happen that any player must enter the coalition in the second place. It is possible that coalition [41] forms. Then players 2,3 and 5 are all prospective winners in the third place. The coalition [415] is the only possible chance for player 5 to win. As players 2 and 3 may be arranged in any order in the forth and fifth place the expected value for player 5 equals 2/120.

If player 1 starts the game then player 2 or 3 will certainly enter the coalition in the second place. The coalitions [145] or [154] cannot form. Correspondingly, if players 2 or 3 start the game coalition [21] or [31] will certainly form, but not coalitions [235] and [325]. The winning coalition {235} is therefore of no importance for player 5. In any game player 4 can hope for the same payoff as player 5 does. In other words, the expected value treats players 4 and 5 as if they had the same power.

The Shapley-value considers all sequences in the great coalition. This is exactly the reason why player 5 is more powerful corresponding to the Shapley-value than player 4 is.

3.3. Expected value for 5-person games

Consider as an example the coalition [12345].
The probability that this coalition forms is

$$P([12345]) = p_z(1).P(2_{[1]}).P(3_{[12]}).P(4_{[123]}).1$$

We already know from our consideration of 4-person games that

$$P(4_{[123]}) = p_+(4_{[123]})(1/3 + 1/3!(p_-(5_{[123]}) + p_-(6_{[123]}))) + 1/3\, p_-(5_{[123]})\, p_-(6_{[123]})$$

Using the formula for $P(i_{[S]})$ we get for the probability $P(2_{[1]})$ that player 2 joins coalition [1]:

$$P(2_{[1]}) = p_+(2_{[1]})[1/4 + 1/(3.4)\sum_{i=3}^{5} p_-(i_{[1]}) + 1/4![\sum_{i=3}^{5} p_-(i_{[1]})\sum_{\substack{j=3\\ j\neq i}}^{5} p_-(j_{[1]})]] + 3!/4!\cdot p_-(3_{[1]})\cdot p_-(4_{[1]})\cdot p_-(5_{[1]}) \quad .$$

Now assume for the vector $p_+(2,3,4,5)_{[1]}$ - that is the probabilities that players 2,3,4 and 5 accept to join coalition [1] - the values $p_+(2,3,4,5)_{[1]} = (w,w,0,0)$.
Therefore

$$\begin{aligned} P(2_{[1]}) &= w\{1/4 + 1/12(3-w) + 1/4!((1-w)2 + (2-w)2)\} + 1/4\cdot(1-w) \\ &= w\,\{1/4 + 1/4 - w/12 + 1/12\,(3-2w)\} + 1/4\cdot(1-w) \\ &= w\{9/12 - 3w/12\} + 1/4(1-w) \end{aligned}$$

$= 3w/4 - w^2/4 + 1/4 - w/4$
$= w/2 - w^2/4 + 1/4$
$= 1/4(1 + 2w - w^2)$

For w=0, $P(2_{[1]}) = 1/4$, indicating that any of players 2,3,4,5 may enter coalition [1].
For w=1, $P(2_{[1]}) = 1/2$, indicating that either coalition [12...] or [13...] forms, but not coalitions [14...] and [15...] .
This is exactly the situation which orrurs in game (1).
Thus we have shown the intuitive result that in game (1) coalitions of the form [14...] and [15...] with any permutations of players {2,3,5} or {2,3,4} in the remaining positions are not possible.

3.4. Some 6-person games

First we consider 6-person games with only 2 minimal winning coalitions:

(1) v(12) = v(13456) = 1

v(S) = 0 otherwise
E = (8/15,13/30,1/120,1/120,1/120,1/120)
S = (8/15,1/3,1/30,1/30,1/30,1/30)

(2) v(123) = v(1456) = 1
v(S) = 0 otherwise
E = (5/12,23/120,23/120,1/15,1/15,1/15)
S = (5/12,1/6,1/6,1/12,1/12,1/12)

(3) v(123) = v(13456) = 1
v(S) = 0 otherwise
E = (11/30,13/60,11/30,1/60,1/60,1/60)
S = (11/30,1/6,11/30,1/30,1/30,1/30)

(4) v(1234) = v(12456) = 1
v(S) = 0 otherwise
E = (17/60,17/60,1/10,17/60,1/40,1/40)
S = (17/60,17/60,1/12,17/60,1/30,1/30)

(5) v(1234) = v(1256) = 1
v(S) = 0 otherwise
E = (1/3,1/3,1/12,1/12,1/12,1/12)
S = (1/3,1/3,1/12,1/12,1/12,1/12)

(6) v(12345) = v(12356) = 1
v(S) = 0 otherwise
E = (7/30,7/30,7/30,1/30,7/30,1/30)
S = (7/30,7/30,7/30,1/30,7/30,1/30)

Games with more than two winning coalitions:

(7) v(123) = v(124) = v(1456) = v(2456) = 1
v(S) = 0 otherwise
E = (13/40,13/40,1/15,11/60,1/20,1/20)
S = (4/15,4/15,1/12,11/60,1/10,1/10)

(8) v(1234) = v(2456) = v(3456) = v(1256) = 1
v(S) = 0 otherwise
E = (13/120,23/120,13/120,23/120,1/5,1/5)
S = (2/15,11/60,2/15,11/60,11/60,11/60)

(9) v(12)= v(23) = v(134) = v(1356) = 1
v(S) = 0 otherwise
E = (13/60,17/30,13/60,0,0,0)
S = (4/15,23/60,4/15,1/20,1/60,1/60)

(10) v(12) = v(234) = v(256) = v(13456) = 1

v(S) = 0 otherwise

E = (23/90,19/30,1/36,1/36,1/36,1/36)

S = (1/5,7/15,1/12,1/12,1/12,1/12)

(11) v(123) = v(145) = v(2345) = v(1346) = 1

v(S) = 0 otherwise

E = (11/30,47/360,11/60,11/60,47/360,1/180)

S = (17/60,3/20,1/5,1/5,3/20,1/60)

3.5. Comments on the results of 6-person games

Games (5) and (6) give equal values for the expected value and the Shapley-value. One might argue for the result of game (5) in the following way:
All coalitions where players 1 and 2 are prospective winners are possible. Moreover, players 1 and 2 do not profit of an impossible coalition. Therefore E and S give the same amount to players 1 and 2.
As players 3,4,5 and 6 have all the same power, the remaining payoff is shared equally among them.
It can also be seen easily that the situations where a player wins or looses as a consequence of an impossible coalition are exactly the same for players 3,4,5 and 6.
One might say that game (5) is in some sense symmetric with respect to certain players.
As long as no 3-person coalition has formed, all players behave equally, that is, decline to join a coalition. The possible situations that may occur in this game are the following:

1.- A 3-person coalition may form which is included in one minimal winning coalition. In the next step the player who is needed to form the minimal winning coalition will certainly enter the coalition. For example, if coalition [123] has formed player 4 will enter the coalition next. Players 5 and 6 would reject an invitation and so a coalition of the form [1235..] or [1236..] is not possible.

Let us denote the coalition that has formed by K. Denote by A={1234} and B={1256}.

2.- If $|K|=3$ it may happen that $|K \cap A| = |K \cap B| = 2$ and $|A \cap B| = 1$. That is

$$|A - K \cap A| = |B - K \cap B| = 2$$

stating that in each coalition A and B two players are needed to form a winning coalition.
In this situation still all players behave equally, that is, reject an invitation.
Only as soon as a 4-player coalition has formed one MWC will have formed up to one player. So we will have the analogue situation to case 1.

An example for such a situation is that coalition [235] has formed.

3.- The last situation that may occur is that

$$|A \cap K| = |B \cap K| = 2$$

and $A \cap B \cap K = \emptyset$.
Of course $K = \{3,4,5,6\}$.
In this case the only prospective winners are players 1 and 2.

The result for game (6) seems to be reasonable too. Here, as long as no 4-player coalition has formed all players behave equally, that is reject an invitation adressed to them. After a 4-player coalition has formed there are two situations that may occur in this game:

Denote by $C = \{1,2,3,4,5\}$ and
$D = \{1,2,3,5,6\}$.

1.- One 5-person MWC has formed up to one player, for exaple, $|K \cap C| = 4$. Then either

a) $|D - D \cap K| = 2$ or

b) $|D - D \cap K| = 1$.

In case a) there is only 1 prospective winner. Therefore the remaining players behave equally and all further coalition formations are equally likely.

An example for case a) is the coalition [1234].

In case b) $C \cap K = D \cap K$.

The players that are needed to form coalition C or D are competing and therefore behave equally. The resulting sequences in the great coalition are all equally likely.

The only situation where case b) applies is when coalition [1235] has formed and players 4 and 6 are competing.

2.- A 4-person coalition K has formed such that

$|K \cap C| = |K \cap D| = 3$.

It is clear that $C - K = D - K$

and $|C - K| = |D - K| = 2$.

Moreover both players that are not yet in K are needed to form a minimal winning coalition.

The two players are competing and behave equally. Each resulting sequence in the great coalition is equally likely.

In game (6) only situations occur where all players that are not yet in the coalition behave equally. We know that if $p_+(i_{[T]})$ for a coalition T in N and i in N-T is equal for all i in N-T then $P(i_{[T]}) = 1/(n-t)$ where $t=|T|$.

Therefore all sequences in the great coalition are possible and moreover even equally likely, which is a sufficient condition for the expected value to be equal the Shapley value and the explanation for the obtained result.

Consider game (2). Player 1 has the same chances to win either for the expected value or for the Shapley-value. Moreover he does not profit from impossible coalitions. Therefore E(1) = S(1) . Suppose coalition [13] has formed. Of course $p_+(4,5,6)_{[13]} = (0,0,0)$ as players 4,5,6 cannot hope for any positive payoff by joining the coalition [13]. But player 2 can hope for a positive payoff in this third position.
But notice that also player 2 may reject an invitation to join coalition [13]. There is no risk for player 2 if coalition [13i] with i in {4,5,6} forms. Only if his competetors have formed their winning coalition up to two players then it is necessary for player 2 to join the coalition [13i] where i in {4,5,6} with the probability $p_+(2_{[13i]}) = 1$ in order to get a positive payoff.

However it is players 2 and 3 that profit and players 4,5 and 6 that loose as a consequence of impossible coalitions. Therefore their expected values and Shapley-values are different.

Similar reasoning holds for games (1),(3),(4) and (7).

Game (9) shows that if a player is pivot only in situations which are not possible for the expected value then his expected value becomes 0.

Notice game (8) where the Shapley-value gives equal payoff to players 2,4,5 and 6. The expected value gives the same payoff to players 2 and 4 but somewhat more than the Shapley-value does. The expected value gives the same payoff to players 5 and 6 but gives more to players 5 and 6 than to players 2 and 4. The loosing players are of course players 1 and 3. Their expected values are equal as their Shapley values are. But $S(i) > E(i)$ for $i=1,3$.

Considering all 3-person coalitions it turns out that players 1 and 3 each has once the chance to profit as a consequence of an impossible coalition. Players 2 and 4 each twice the chance and player 5 and 6 each have 3 times the chance to win more. This might yield an explanation at least for the fact that players 5 and 6 earn more in the expected value than players 2 and 4 do.

If W_i and W_j are two different MWC in game (8) then notice that $2 <= |W_i \cap W_j| <= 3$ always. In this game one might expect some nicer result than the one obtained as there are situations similar to those already discussed.

However, one may consider game (3) as a representative of more general games. And then it shows that expected value and Shapley-value may behave very differently.

For games (10) and (11) all values $E(i) \neq S(i)$ for all $i = \{1,2,..,6\}$. Only the ordinal power of players is the same for the Shapley-value and for the expected value.

4. On the difference between expected value and Shapley-value

4.1. Predicting coalition formation in games in extensive form

We have seen several examples where certain orders of coalition formation were not possible. In this section we treat this phenomenon from a general point of view.

Let N be the set of players with $|N|=n$.
Let T be some subcoalition that has formed, where $t=|T|$.
Notice, whenever there is one player i in N-T with $p_+(i_{[T]})=0$ there is at least one more player j in N, $j \neq i$, such that $p_+(j_{[T]})=0$.
Player i to win in some position $k>t+1$ needs some other player, say j. Of course the needed player j has the same interest as player i.

Let $N-T=\{a_{t+1},\dots,a_{n-2},b_1,b_2\}$ and

suppose $p_+(a_{t+1},\dots,a_{n-2},b_1,b_2)_{[T]} = (1,1,\dots,1,0,0)$

that is players b_1, b_2 do not want to join [T] in position t+1. Using the formula of section 1.5. we get for the probability that a player b_i with $i=1,2$ joins coalition [T]:

$$P(b_{i_{[T]}}) = 0[1/(n-t)+ \ldots] + 1/(n-s)\ 0.0.\ \ldots\ 0.1.1 = 0\ .$$

This states: Whenever at some stage of the game - when some coalition [T] has formed - there are n-2 players who want to join [T] in place t+1 - call them 'interested' players - and 2 players who do not want to join [T] in place t+1 - call them 'non interested' players -, then only interested players can enter coalition [T] in place t+1. Similarily it can be verified that the assertion is also true for more than 2 non interested players. In other words, if the vector

$$p_+(\{N-T\})_{[T]} = (1,1,\ldots,1,0,0,\ldots,0)\ ,$$

that is, contains zero and 1, only some player i with $p_+(i_{[T]}) = 1$ will enter [T] in place t+1.

4.2. A Remark on the computation of Shapley-value and expected value

Consider as an example a simple 4-person game with characteristic function v(12) = v(134) = 1 and v(S) = 0 otherwise.
Orders of coalitions in this game are:

$1\underline{2}34$	$2\underline{1}34$	$31\underline{2}4$	$41\underline{2}3$
$1\underline{2}43$	$2\underline{1}43$	$31\underline{4}2$	$41\underline{3}2$
$[13\underline{2}4$	$23\underline{1}4$	$32\underline{1}4$	$42\underline{1}3$
$13\underline{4}2$	$234\underline{1}$	$324\underline{1}$	$423\underline{1}$
$14\underline{2}3$	$24\underline{1}3$	$34\underline{1}2$	$43\underline{1}2$
$14\underline{3}2]$	$243\underline{1}$	$342\underline{1}$	$432\underline{1}$

Table I.4.2.1: Orders of coalition formation in the game v(12) = v(134) = 1

Underscored players in a sequence are the corresponding winners. The coalitions in brackets are not possible in the extensive form game.
The Shapley-value of this game is S = (7/12,1/4,1/12,1/12), the expected value gives E = (7/12,1/3,1/24,1/24).
Notice that both Shapley value and expected value average over all n! permutations. The Shapley-value, per definitionem, admits every permutation with equal probability. The expected value does not admit all orders of players in the great coalition. There are some players who win as a consequence of 'impossible' coalitions and some players who loose as a consequence of impossible coalitions.
In the cited example player 2 wins what is lost by players 3 and 4.

As in games with more than 4 players impossible coalitions occur very easily, it is reasonable to expect the expected value to be different from the Shapley-value.

Notice also what the outcome would be if the solution only did average over possible coalitions. Then we would get:

$$S'(1) = 14/20 = 7/10$$
$$S'(2) = 4/20 = 1/5$$
$$S'(3) = S(4) = 1/20$$

where $S'(i)$ denotes the solution for player i. These values obviously differ from the Shapley-values. Such a solution concept reminds of the a-priori-unions considered by Hart and Kurz [6].

For a more formal reasoning consider the order [1234] . We know

$$P([1234]) = 1/8 \cdot P(2_{[1]})$$

$$= 1/8\{p_+(2_{[1]})(1/3 + 1/3!(p_-(3_{[1]}) + p_-(4_{[1]})))$$

$$+ 1/3 \cdot p_-(3_{[1]}) . p_-(4_{[1]})\}$$

$$= 1/4!\{p_+(2_{[1]})[1 + 1/2(p_-(3_{[1]}) + p_-(4_{[1]}))]$$

$$+ p_-(3_{[1]})\ p_-(4_{[1]})\}$$

$$= 1/4!\ f(p_+(2_{[1]}), p_+(3_{[1]}), p_+(4_{[1]}))$$

where f is some function depending on $p_+(2_{[1]}), p_+(3_{[1]}), p_+(4_{[1]})$.

Denote by $\{a_{s+1}, \ldots, a_n\}$ the players in N

that are not yet members of coalition [S].
Then the expected value for some player i in
a 4-person game is given by:

$$E(i) = \sum_{\substack{S \subseteq N \\ i \in S}} [v(S)-v(S-\{i\})]P([S] \cup N-S) .$$

$$= 1/4! \sum_{\substack{S \subseteq N \\ i \in S}} [v(S)-v(S-\{i\})]\ f(p_+(a_{s+1_{[S]}}),..,p_+(a_{n_{[S]}}))$$

It is clear that in the general case

$$E(i) = 1/n! \sum_{\substack{S \subseteq N \\ i \in S}} [v(S)-v(S-\{i\})]f(p_+(a_{s+1_{[S]}}),..,p_+(a_{n_{[S]}})).$$

Thus the expected value E(i) averages over n!. The expression under the sum may be changed (increased or decreased) only

by $f(p_+(a_{s+1_{[S]}}),..,p_+(a_{n_{[S]}}))$. It is clear

from the definition for the expected value that the only difference between E(i) for i in N may arise from
$p_+(k_{[S]})$ with k in N-S, that is the willingness of player k to join coalition [S].

4.3. Characteristic function and difference between Shapley-value and expected value

In his article A.Rapoport [9] raised some questions concerning the difference between expected payoff and Shapley-value. We now are able to answer these questions.

The first of Rapoport's questions is:

Will the expected payoffs be more or less equally distributed than in the Shapley-value?

The answer to this question is not unique. We have seen an example (game (8) of 5-person games) where the expected values are equal for more players than the Shapley-values are. But we have also seen an example (game (8) of 6-person games) with more equal Shapley-values than expected values. With this fact in mind one might answer: Sometimes the expected payoffs are more equally distributed than the Shapley-values are, and sometimes they are less equally distributed.

But considering the fact that the expected value may not take into account all sequences in the great coalition - what in the extreme case may result in a zero-payoff for some players whereas the Shapley-value is strictly positive (game (9) of the 6-person games) - it seems reasonable to say that the expected values are less equally distributed than the Shapley values are.

The second of Rapoport's questions is:

Does the answer depend on the characteristic function, if so, how?

Of course does the answer depend on the characteristic function. The characteristic function defines which positions are profitable for each player. As in the extensive-form-game not every position is a profitable one for a player and as a player has the choice to join the coalition or not, it may easily happen that there is some sequence in the all-player-coalition that is not possible.

Remember: If the characteristic function defines a game where in every situation of the game all players behave equally, all sequences in the great coalition are equally likely.

The following two lemmata will help to get an answer to the question how the difference of E and S depends on the characteristic function. Denote by K the coalition that has formed until now, where $k = |K|$. Denote by A and B two different winning coalitions.

<u>Lemma 1:</u> If a) $|A - K \cap A| = 1$
and b) $|B - K \cap B| >= 2$
then

1. coalition A will certainly form, and
2. if equality in b) holds, not all sequences in the all-player-coalition are possible.

<u>proof:</u> Let $|A - K \cap A| = 1$ and $|B - K \cap B| = 2$, then in the next step all the players in B-K will reject an invitation on the (k+1)st place as they cannot hope for a positive payoff in this position. As a consequence the player in A-K will certainly be invited to join the coalition in the (k+1)st position, and moreover he can be sure to get the maximum payoff, that is 1, in this position.

One might expect the player in A-K to be the only prosective winner and therefore to be indifferent. If the player in A-K rejects he runs the risk that coalition B will form and get the whole payoff.
Assume that the player in A-K is not the last player to be asked and that he rejects, then $|A - K \cap A| = |B - K \cap B| = 1$ as soon as one player of B-K had to join. In this situation the players that are needed to form coalition A or B are competing. Whether coalition A or B forms depends on which player is invited. So coalition A need not necessarily form.
The player in A-K being rational will certainly join coalition K in the (k+1)st place.
Obviously no player in B-K can join coalition K in the (k+1)st place.
The sequences [Ki...] where i in B-K are exactly those that are not possible. So we have shown the assertion to hold for $|B - B \cap K| = 2$.

If $|B - B \cap K| > 2$ then the player in A-K may be indifferent with respect to an invitation as long as $|B - B \cap K| > 2$. If he joins coalition K while $|B - B \cap K| > 2$ he wins. As players in B-K will reject any invitation as long as $|B - B \cap K| >= 2$ the player in A-K can be sure

1.- to be invited before coalition B has formed and therefore
2.- to win.

It is only when $|B - B \cap K| = 2$ that the player in A-K will join coalition K for the above discussed reasons.
As long as $|B - B \cap K| > 2$ any player may join coalition K in the (k+1)st place. In other words,

all sequences of players are possible while $|B - B\cap K| > 2$. Even if the player in A-K joins coalition K while $|B - B\cap K| > 2$ all players in B-K will become indifferent and again all sequences of players are possible.
If $|B - B\cap K| = 2$ then the only possible player to join coalition K in the $(k+1)$st place is the player in A-K. And this is the only situation where not all sequences of players are possible.

<u>Lemma 2:</u> Let A and B be two different MWC with $|A| = |B|$.
Whenever $|A - A\cap K| = |B - B\cap K| = 1$
then 1. $A\cap K = B\cap K$ or
2. $A - (A\cap K) = B - (B\cap K)$.

ad 1): assume $A\cap K \neq B\cap K$ and $A - (A\cap K) \neq B - (B\cap K)$.
As $|A| = |B|$ it is clear that $|A\cap K| = |B\cap K|$.
<u>One step before</u> it must have been true that either

$|A - A\cap K| = 1$ or (exclusive)

$|B - B\cap K| = 1$.

Assume that $|A - A\cap K| = 1$ and therefore

$|B - B\cap K| > 1$ say 2 .

From Lemma 1 we know that A will certainly form, and then

$|A - A\cap K| = 0$ and $|B - B\cap K| = 2$,

which contradicts the assumption that
$|A - A\cap K| = |B - B\cap K| = 1$.
Therefore, one possible situation is that
$A\cap K = B\cap K$.

ad 2): Whenever $A\cap K \neq B\cap K$
remains the possibility that

$A - (A \cap K) = B - (B \cap K)$

In this case it may well happen that

$|A - A \cap K| = |B - B \cap K| = 1$.

as the player not yet in coalition K is needed in both coalitions A and B to form a winning coalition and therefore is indifferent.

<u>Remark:</u> If $|A| \neq |B|$ and $|A - A \cap K| = |B - B \cap K| = 1$, then
$A - A \cap K = B - B \cap K$
is the only possibility.

As $|A| \neq |B|$ also $|A \cap K| \neq |B \cap K|$ and therefore $A \cap K \neq B \cap K$.

Lemma 2 states that whenever two different minimal winning coalitions of the same size have formed up to one player then either they differ only in this one player, or the player needed is a player the two coalitions have in common.

Lemma 1 describes the situations, where the expected value does not admit all coalition formations as possible.
Whether such a situation may occur or not is determined by the characteristic function. More precisely, the characteristic function defines for every situation of the game the probability of acceptance for the players that are not yet in the formed coalition K, that is, the vector $p_+(a_{k+1},\dots,a_n)_{[K]}$.
If the situation described in Lemma 1 occurs, then there is some player a in A-K, such that $p_+(a_{[K]}) = 1$, and there are at least 2 players b_1 and b_2 in B-K,

such that $p_+(b_{i[K]}) = 0$ for i=1,2.
We already know from section 4.1., that [Ka] will certainly form.
Lemma 2 describes a special situation where all coalition formations are possible.

Our conclusion is: The characteristic function determines which situations may occur in a game. For two such situations we have shown the impact on the probability of coalition formation and thus on the expected value. In fact, it is the situation described in Lemma 1 that accounts for the difference of expected value and Shapley-value.

Of course, in more general games there will be more than two MWC. However, the situation of Lemma 1 may still occur. There may be one or more MWC $\overline{B}_i$ with $|\overline{B}_i - K| \geq 2$ (see game(10) in section 3.4.), but this is of no influence on the fact that coalition A will form. There are only more players who loose as a consequence of impossible coalitions.
Or there may be a coalition $\overline{A}$, such that $|\overline{A}-K| = 1$. Lemma 2 shows under which cicumstances this situation may occur. But still there will be some impossible coalitions (see game (8) of section 3.1.).
Note the important fact: The characteristic function determines the probability of acceptance for every game situation, thus the probability of coalition formation and by consequence the equality or difference of E and S.

The two Lemmata already give an answer to Rapoport's third question, namely:
What is the relation between the characteristic function

and the orders of coalition formation excluded from the outcome?
Whenever a characteristic function is defined to which Lemma 1 applies, it is clear which coalitions are excluded from the outcome.

If case 1) of Lemma 2 applies it can be seen easily that all orders of coalition formation are possible. Also in case 2) of Lemma 2 all orders of coalition formation may arise, and moreover are equally likely. As long as no winning coalition has formed, all players reject an invitation and thus behave equally. As soon as there is only one prospective winner, all players are indifferent.

Notice: If more than two MWC are defined, it may easily happen, that case 2) of Lemma 2 applies but that there are more than 2 prospective winners. Therefore not all players are indifferent.
As an example consider the game
$v(1245) = v(1246) = v(1235) = 1$
in section 2.2.

5. The number of coalition formations in a game in extensive form

Think of an extensive game that is played according to the rules proposed by Rapoport [9]. As we have seen, it may happen that some orders of coalition formation are excluded from the outcome. However, the computer has to go through every order of coalition formation in order to decide whether it's probability is positive or not.
The purpose of this section is to answer the following question: How much time will it take the computer to find the expected value for some n-person game. Of course the time needed depends on the number of coalitions that may form. This number is not just n! but much bigger, as every order of coalition formation may occur several times.

To fix ideas, suppose some coalition K has formed. Then denote by S(f,m) the number of orders of players in the all-player coalition where each order is counted.
f denotes the number of players that are not yet members of coalition K.
m denotes the number of players who may enter coalition K in the next step.
Some of the f players that are not yet in K may enter the coalition in the next step. But some of the f players may already have rejected an invitation and therefore cannot any more enter coalition K in the next step. That is, m<=f must hold. The number S(f,m) is determined by the following recursion:

$$S(f,m) = m.S(f-1,f-1) + m.S(f,m-1)$$

and

$$S(f,1) = S(f-1,f-1) \qquad \text{where } f \geq m \geq 1 \quad .$$

The interpretation of the formula is quite simple. Remember how the game is played. Some coalition K may form. In the next step each of the m players may join the coalition and moreover, there are (f-1) players that are not yet in the newly formed coalition and that are simultaneously all possible, therefore the term m.S(f-1,f-1).
On the other hand, each of the m players may decline, resulting in f players not in coalition K, but only (m-1) possible players, therefore the term m.S(f,m-1).
If there is only one possible player out of f players that are not in coalition K, he joins the coalition and the (f-1) players not yet in coalition K are all possible in the next step.

The formula also can be written as

$$S(f,m) = S(f-1,f-1).m![\sum_{k=0}^{m-1} 1/k!]$$

This equation can be easily proved by induction to m. For f=3 and m=2 we have

$$S(3,2) = S(2,2).2![1/1! + 1] = 16 ,$$

which can be easily verified by considering the game tree of a 4-person game. Now suppose the expression for S(f,m) does hold. Then

$$S(f,m+1) = (m+1).S(f-1,f-1) + (m+1).S(f,m)$$

and by our assumption

$$S(f,m+1) = ((m+1)!/m!).S(f-1,f-1) + (m+1).m!.S(f-1,f-1)[\sum_{k=0}^{m-1} 1/k!]$$

$$= (m+1)!.S(f-1,f-1)[\sum_{k=0}^{m} 1/k!] \quad .$$

We mention a second very similar formula for the number of coalition formations.
Denote by Z(e,s) the number of coalition formations where e is the number of players in the coalition and s is the number of players not yet in the coalition. Let n denote the number of all players.
Then

$$Z(e,s) = s.Z(e+1,n-e-1) + s.Z(e,s-1)$$

and

$$Z(e,1) = Z(e+1,n-e-1)$$

or

$$Z(e,s) = s!.Z(e+1,n-e-1)[\sum_{k=0}^{s-1} 1/k!]$$

where $e+s<=n$, $e>=0$, $s>=1$.
This result too can be shown by induction.

In the sequel we refer to the first given formula S(f,m).

Let us now consider the sequence resulting for f=m:

$S(1,1) = 1$
$S(2,2) = 4$
$S(3,3) = 60$
$S(4,4) = 3840$
$S(5,5) = 1\ 248\ 000$
$S(6,6) = 2\ 441\ 088\ 000$
. . .

The sequence grows very rapidly. For reasons of simplicity denote by $S_n = S(n,n)$.

Denote by $\overline{S}_n = (S_{n+1}/(n+1)!)/(S_n/n!)$

Define $S_0 := 1$.
Consider the following table:

n	0	1	2	3	4	5	6
S_n	0	1	4	60	3840	1 248 000	2 441 088 000
$S_n/n!$	0	1	2	10	160	10 400	3 390 400

$\overline{S}_n$	1	2	5	16	65	326
n	0	1	2	3	4	5

The sequence $(S_{n+1}/(n+1)!)/(S_n/n!)$ corresponds exactly to the sequence

$$P_n = n!\sum_{r=0}^{n} 1/r! = \sum_{r=0}^{n} n!/(n-r)!$$

(compare [12], p.16. [14]).

We know

$$S_n = S_{n-1} . n! \sum_{k=0}^{n-1} 1/k!$$

$$= a_n . S_{n-1} \qquad \text{where } a_n = n! \sum_{k=0}^{n-1} 1/k!$$

It is clear that $a_n = P_n - 1$

and also $a_n = n.P_{n-1}$.

Therefore results the well known recurrence

$$P_n = n.P_{n-1} + 1 \quad .$$

In terms of S_n, P_n can be written as

$$P_n = S_{n+1}/((n+1).S_n)$$

and

$$S_{n+1} = (n+1).S_n.P_n \quad .$$

This relation shows how many coalition formations more have to be considered when moving from n over to (n+1) persons.

In a 6-person game the computer has to go through $6{*}S_5$ coalition formations. Correspondigly, in a 7-person game there are $7{*}S_6$ coalitions to go through. Using the recurrence gives

$$S_6 = 6.S_5.P_5 \quad \text{where } P_5 = 326$$

and therefore

$$S_6 = 326*6*S_5 \quad ,$$

$$7*S_6 = 7*326*6*S_5$$

$$= 2282*(6*S_5) \quad .$$

Suppose the computer needs half an hour for solving a 6-person game, then it would need 2282 times as much for solving a 7-person game, that is 1141 hours. Depending on how many hours the computer is working a day, it may take up to about 3 months to solve a 7-person game.

As $S_n = a_n.S_{n-1} = a_n.a_{n-1}.S_{n-2}$

it follows by recurrence

$$S_n = \prod_{i=1}^{n} a_i \qquad \text{or}$$

$$\ln S_n = \sum_{i=1}^{n} \ln a_i \qquad \text{where}$$

$$a_i = i!\sum_{j=0}^{i-1} 1/j!$$

It is a well known result that

$$\sum_{n=0}^{\infty} P_n . t^n / n! = e^t / (1-t) \qquad \text{and therefore}$$

$$\sum_{n=0}^{\infty} S_{n+1} / ((n+1) . S_n) . t^n / n! = e^t / (1-t) \quad \text{or}$$

$$\sum_{n=0}^{\infty} S_{n+1} / S_n . t^n / (n+1)! = e^t / (1-t) \quad .$$

So we found a generating function for the quotient S_{n+1}/S_n.

II. Simple n-person games with Sidepayments

In this chapter we consider simple 3-person and 4-person games where sidepayments are allowed. We give some rules for how the game should be played and compare the obtained solutions to the Shapley value and the expected payoff of the corresponding extensive form game discussed in chapter I. The rules for the game are similar to those proposed by A.Rapoport in [9].
A game is played until the all-player-coalition has formed. At each stage of the game, the players have to decide how to share the payoff. Here, each time a player makes a bid to another player to join the growing coalition, he makes a proposal to the invited player how to share the payoff.

The rules for the game are:

1. The first player is chosen at random.
2. This player invites another player to join him in the growing coalition. Together with his invitation the first player makes a proposal to the invited player how to share the payoff.
3. The invited player may accept or decline.
4. If the invited player accepts, a coalition of two has formed and the last player makes a bid to one of the remaining players, and so on.
5. If the invited player rejects, again the first player makes another bid to one of the remaining players.
6. A player can make only one bid to each of

the others players.

7. If all players but one have been invited to join the coalition - they may have accepted or declined - the remaining player must join the coalition and the obtained payoff - probably zero - is shared equally.
8. The play of the game terminates when the grand coalition has formed.

Two assumptions are made:

1. Players that are indifferent with respect to invitations join the coalition.
2. Players that are indifferent with respect to their successors choose the next player at random.

1. 3-person games

Let us first consider the simple 3-person superadditive game with characteristic function

$v(13) = 1$

$v(S) = 0$ otherwise .

The possible coalitions resulting from this game as well as the corresponding payoffs are shown in table II.1.1.

notation:

- Each vertex in the tree is denoted by the player whose move it is.
- +(-) indicates that a player accepts (declines)

the invitation.

- The numbers beneath the branches indicate the probabilitis with which
  1. prospective partners are invited,
  2. invited players accept or decline.
- Each final vertex is denoted by the order, in which players have joined the growing coalition.
- The vector beneath each coalition denotes the corresponding payoff vector.

Suppose player 1 is chosen to start the game. He can make a bid either to player 2 or to player 3. Suppose he invites player 2. As [12] is not a winning coalition, player 1 cannot offer player 2 any positive payoff. However, as the payoff must be shared equally among the last two players, player 2 will get a payoff of 1/2, no matter whether he accepts or refuses the invitation. In this case player 1 will get a zero payoff.
If player 1 invites player 3 and player 3 refuses the invitation, player 2 must join the coalition in the second place and player 3 must join the coalition in the third place. Each one of players 2 and 3 will get 1/2 and player 1 will get 0.
To make the invitation for player 3 more attractive, player 1 will offer player 3 more than 1/2, say $(1+\varepsilon)/2$, which leaves $(1-\varepsilon)/2$ for himself. Player 3 being a rational player will accept the proposal and the coalition [13] forms. The remaining player 2 must join the coalition and so the coalition [123] with the payoff vector $((1-\varepsilon)/2, 0, (1+\varepsilon)/2)$ has formed.

If player 3 starts the game, the game is the same up to a permutation of players 1 and 3.

If player 2 starts the game, there is no chance for him to get any positive payoff. So player 2 is indifferent with respect to the invitation of the next player. In any case, players 1 and 3 will get 1/2 each. So they are indifferent with respect to the invitation and will accept corresponding to the rules of the game.

Think of the expected value E(i) of player i as the sum over all coalition formations, each time taking player i's payoff times the probabiliity of this coalition. The probability of one coalition (one end point in the tree) is of course the product of the probabilities which lead from the root to this end point.
Notice that in the game v(13) = 1 and 0 otherwise, the expected payoff results of 4 possible coalition formations: E = (1/2,0,1/2).
This vector equals the Shapley-values of the corresponding game in extensive form.

Now consider the simple 3-person game

v(12) = v(13) = 1

v(S) = 0 otherwise

(see table II.1.2).
If players 2 or 3 start the game, we have the same situation as just demonstrated.
If player 1 starts, he can guarantee himself a payoff of at least 1/2. For each of the players 2 and 3 it is preferable to accept player 1's invitation because otherwise he would get a zero payoff. To make his invitation attractive, player 1 will make an offer of somewhat more than zero, say ε_1. The invited player will accept and player 1 gets a payoff of $(1-\varepsilon_1)$.

This game shows that player 1 is in a very strong position. Notice that also in this game the expected value is computed on the basis of 4 possible coalition formations.

For the expected value we get:

$$E(1) = 1/3(2 + \varepsilon - \varepsilon_1)$$

$$E(2) = E(3) = 1/6(\varepsilon_1 + 1 - \varepsilon) .$$

ε and ε_1 being arbitrary small it is convenient to consider $\lim_{\varepsilon, \varepsilon_1 \to 0} E(i)$,

which gives the result: $E = (2/3, 1/6, 1/6)$.

Notice that these values are exactly the Shapley-values for the corresponding 3-person game.

If all two-player coalitions are winning coalitions, all coalition formations are possible. As one might expect, the expected value gives equal payoff to each player and is equal to the Shapley-value (see table II.1.3.).

If the all player coalition is the only possible winning coalition, the players behave indifferent as with respect to their successors as with respect to an invitation. The starting player always gets a zero payoff. The other players get a payoff of 1/2 each. Here all 6 coalitions are possible, each forming twice. The expected value is $E = (1/3, 1/3, 1/3)$ and equals the Shapley-value (see table II.1.4.).

Thus we notice the following result: In a simple 3-person game with the newly defined rules the expected value always equals the Shapley-value.

This result does not seem surprising for symmetric games. It might be by chance that the solutions coincide for the non-symmetric case.

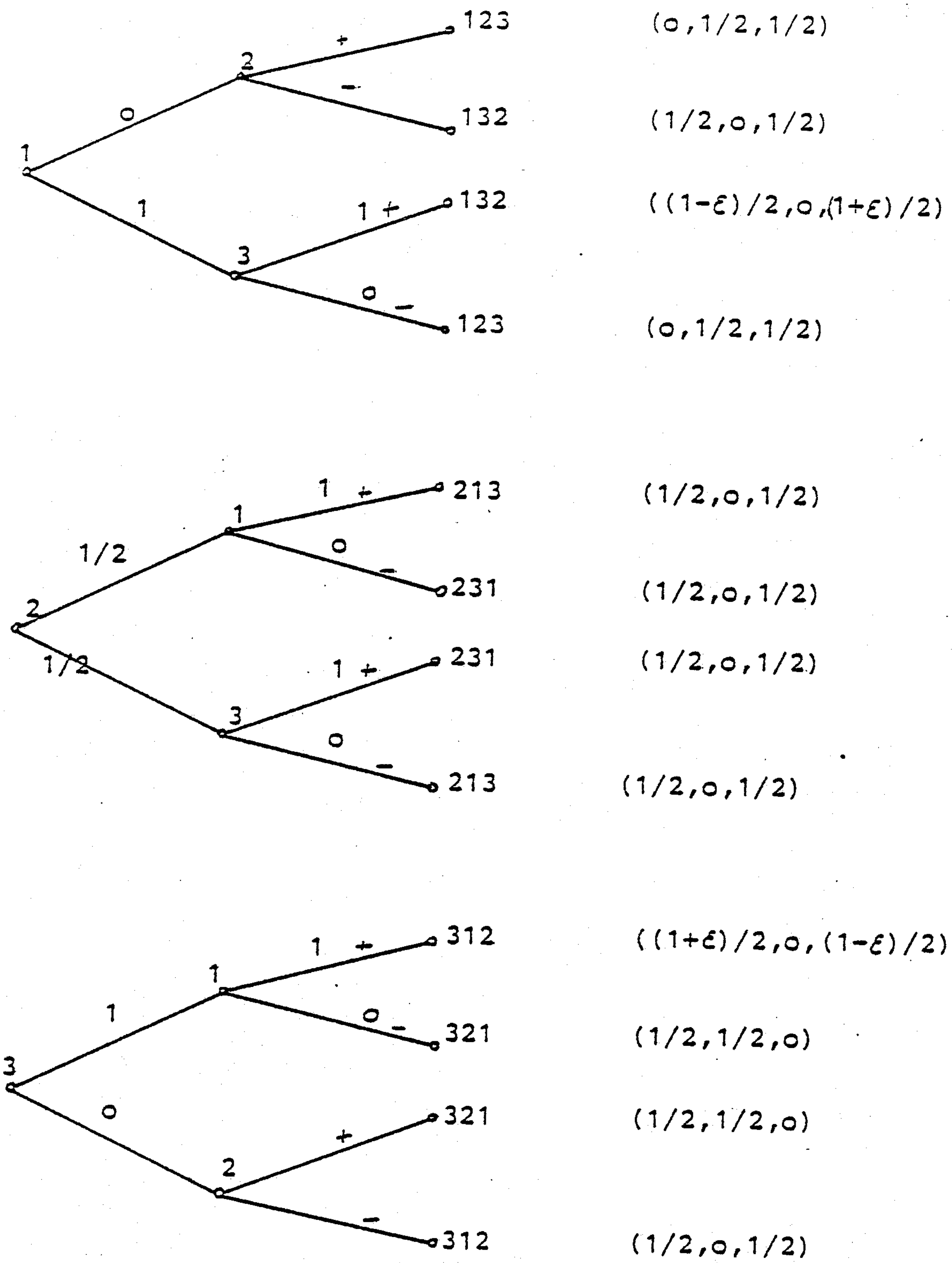


Table II.1.1: 3-person game with characteristic function v(13) = 1 and O otherwise

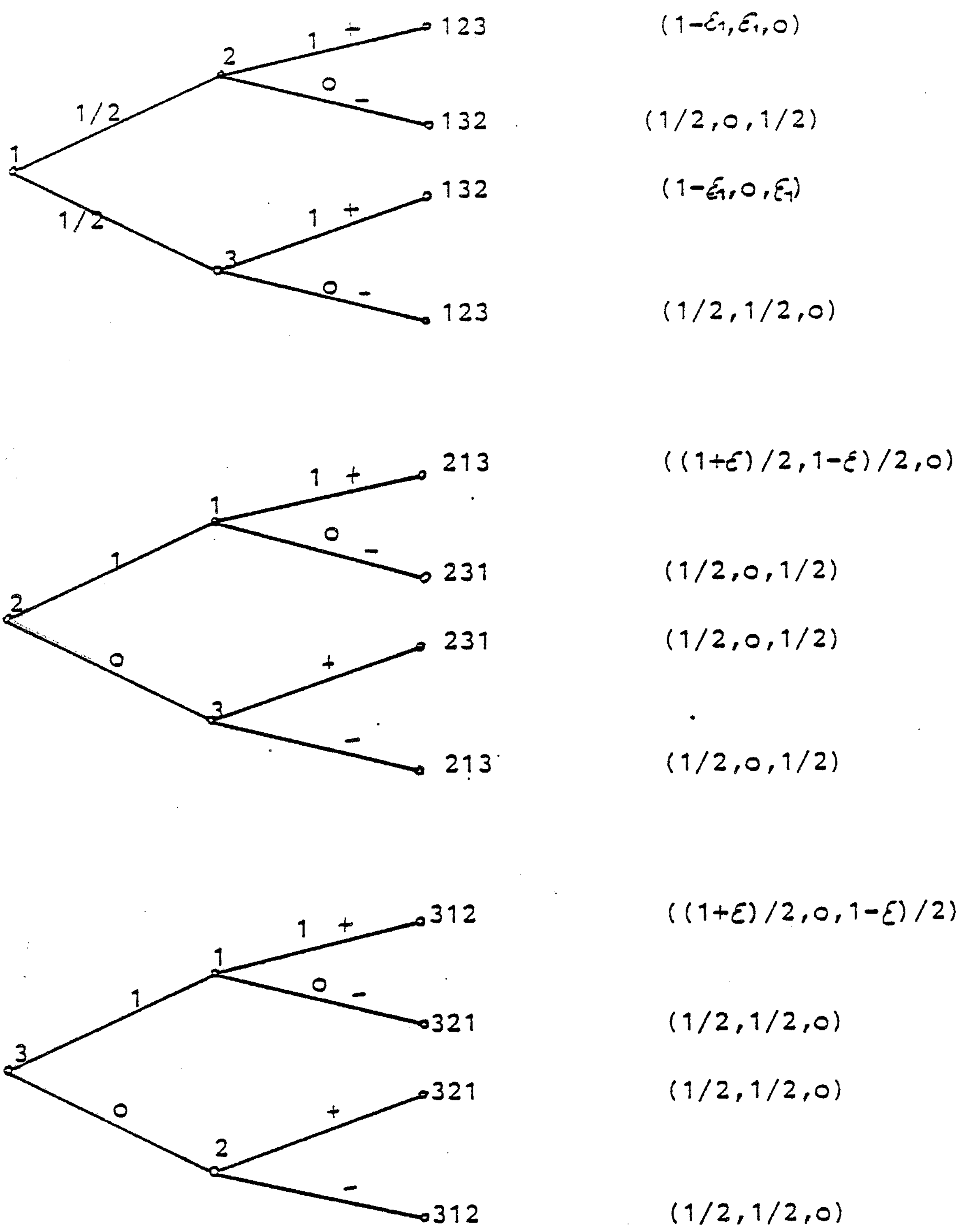


Table II.1.2: 3-person game with characteristic function
v(12) = v(13) = 1 and 0 otherwise

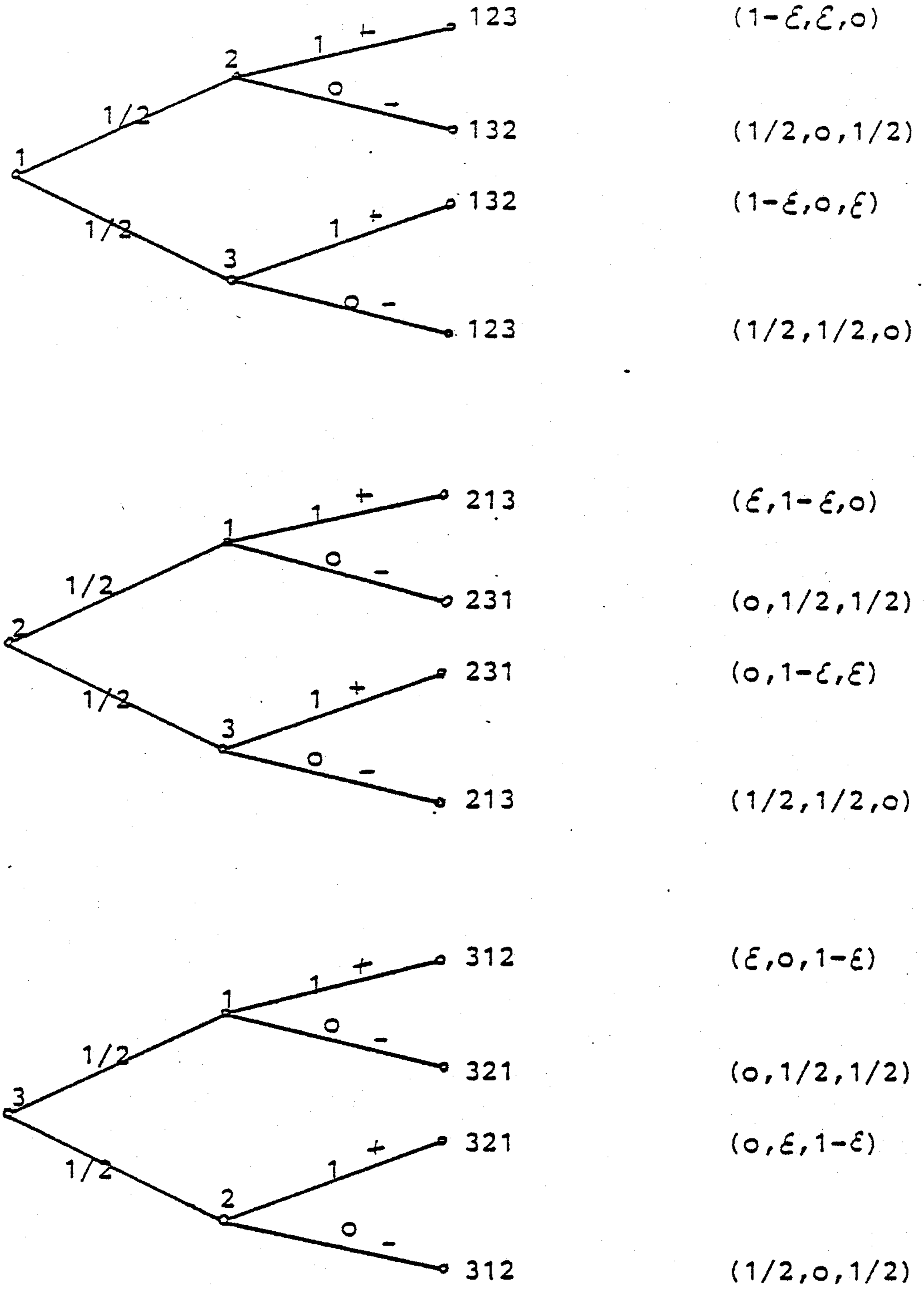


Table II.1.3: 3-person game with characteristic function
v(12) = v(13) = v(23) = 1 and 0 otherwise

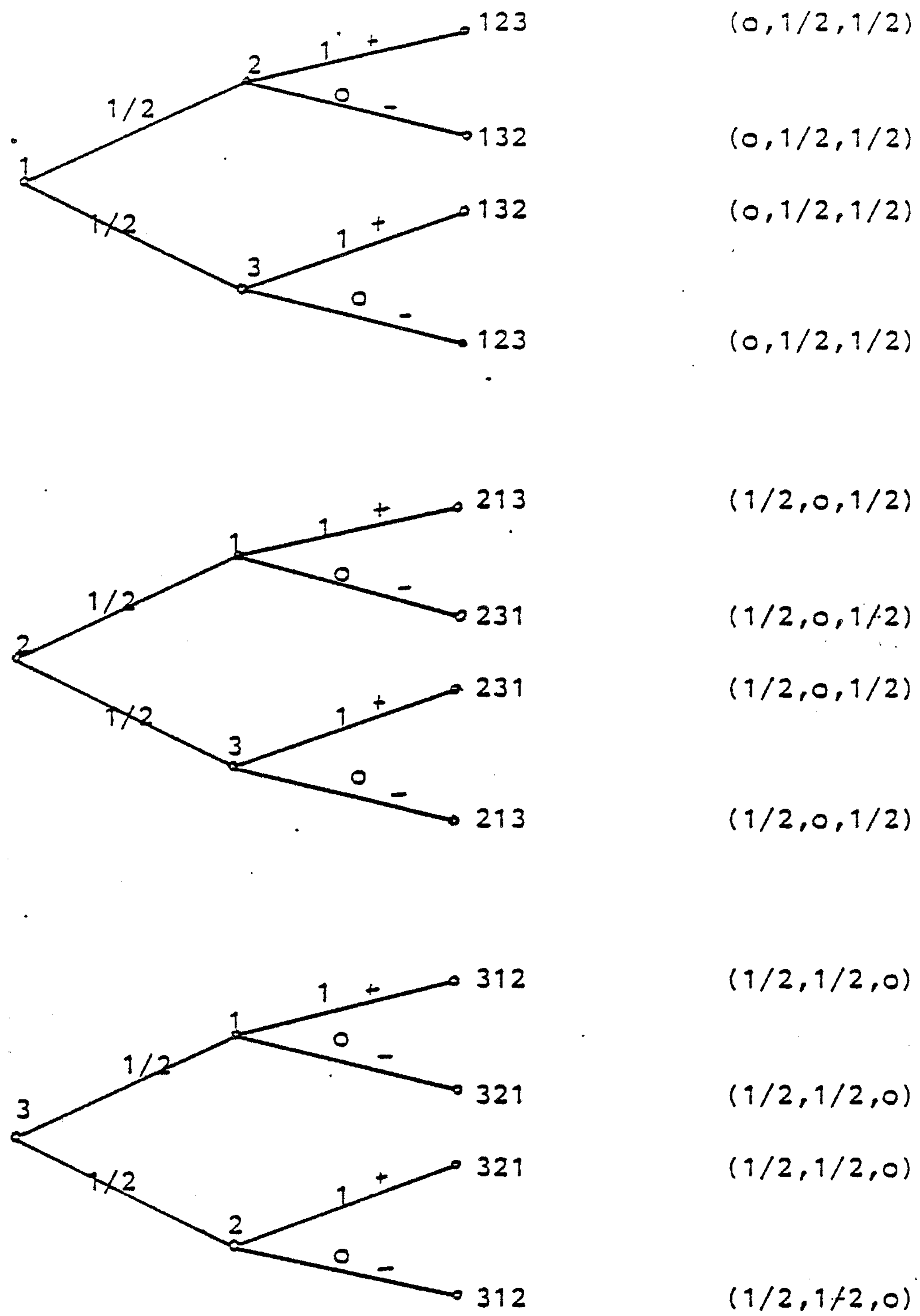


Table II.1.4: 3-person game with characteristic function v(123) = 1 and O otherwise

2. 4-person games

In the sequel we denote by E_1 the expected value of a game that is played corresponding to the newly defined rules.
The expected value E refers to a game that is played corresponding to the rules given by A.Rapoport.
S denotes the Shapley-value.

4-person games allow much more insight in the nature of the game. We present in detail one 4-person game which has some interesting features. We then show the structure of all the other classes of 4-person games.
Consider the simple 4-person game with characteristic function

$v(12) = v(134) = 1$

$v(S) = 0$ otherwise.

Suppose player 2 starts the game (see table II.2.1.). He can make a bid either to player 1, player 3 or player 4. Suppose player 2 invites player 3. Suppose player 3 rejects the invitation. Then the coalition [21] or [24] will form. In any case player 3 will get a zero payoff. On the other hand, if player 3 accepts the invitation, he can next make a bid to player 1 and thus has a chance of getting a positiive payoff. This states, that player 3 will certainly not reject an invitation of player 2.
Now suppose player 3 makes a bid to player 1 and player 1 rejects the invitation. In this case the coalition [2341] will form and players 1 and 4 get 1/2 each, but player 3 gets a zero payoff.
In order to get a positive payoff player 3 has got to

offer player 1 a bit more than 1/2, say $(1+\varepsilon)/2$.
Player 1 will accept and the resulting payoff vector is $((1+\varepsilon)/2,0,(1-\varepsilon)/2,0)$.
If on the other hand player 3 makes a bid to player 4 and player 4 rejects, player 3 can guarantee himself a payoff of 1/2, and player 4 gets a zero payoff.
It is clear that player 4 will accept an invitation of player 3, as player 1 must join the coalition in the forth place and the gain is equally distributed between players 1 and 4.
Comparing the two possibilities of player 3 it is clear, that it is much more profitable for him to make a bid to player 1.

Players 3 and 4 behave equally in this game.

We have seen that if player 2 invites player 3 or 4 his payoff will be zero. The best payoff that player 1 can reach in these situations is $(1+\varepsilon)/2$. If player 2 offers exactly this amount to player 1, player 1 will accept which leaves a payoff of $(1-\varepsilon)/2$ for player 2.
So it is clear that player 2 will certainly make a bid to player 1.
The coalition [21] having formed, there is no gain left. Therefore player 1 will invite players 3 or 4 with equal probability. Players 3 and 4 being indifferent will accept the invitation.
For the game where players 1 and 3 start see tables II.2.2. and II.2.3.
The expected values E_1 for this game are:

$$E_1 = (17/24, 5/24, 1/24, 1/24) .$$

The Shapley-value S and the expected value E for the corresponding game in extensive form are:

$$E = (7/12, 1/3, 1/24, 1/24)$$
$$S = (7/12, 1/4, 1/12, 1/12) .$$

In the sequel we only consider 4-person games with no dummy players. For every 4-person game we compare the expected value E_1 to the Shapley-value S and to the expected value E of the corresponding game in extensive form.
The structures of each game are given at the end of this chapter.

(1) $v(123) = v(124) = 1$
$v(S) = 0$ otherwise
$E_1 = E = S = (5/12, 5/12, 1/12, 1/12)$

(2) $v(123) = v(124) = v(134) = 1$
$v(S) = 0$ otherwise
$E_1 = E = S = (1/2, 1/6, 1/6, 1/6)$

(3) $v(123) = v(124) = v(134) = v(234) = 1$
$v(S) = 0$ otherwise
$E_1 = E = S = (1/4, 1/4, 1/4, 1/4)$

(4) $v(12) = v(13) = v(14) = 1$
$v(S) = 0$ otherwise
$E_1 = ((\varepsilon+5)/8, (3-\varepsilon)/24, (3-\varepsilon)/24, (3-\varepsilon)/24)$
$\lim_{\varepsilon \to 0} E_1 = (5/8, 1/8, 1/8, 1/8)$
$E = S = (3/4, 1/12, 1/12, 1/12)$

(5) $v(12) = v(134) = 1$
$v(S) = 0$ otherwise
$E_1 = (17/24, 5/24, 1/24, 1/24)$
$E = (7/12, 1/3, 1/24, 1/24)$
$S = (7/12, 1/4, 1/12, 1/12)$

(6) $v(12) = v(134) = v(234) = 1$

$v(S) = 0$ otherwise

$E = E_1 = (5/12, 5/12, 1/12, 1/12)$

$S = (1/3, 1/3, 1/6, 1/6)$

(7) $v(12) = v(13) = v(234) = 1$

$v(S) = 0$ otherwise

$E_1 = (1/3 + \varepsilon/4, 1/3 - \varepsilon/8, 1/3 - \varepsilon/8, 0)$

$\lim_{\varepsilon \to 0} E_1 = (1/3, 1/3, 1/3, 0)$

$E = (7/12, 5/24, 5/24, 0)$

$S = (5/12, 1/4, 1/4, 1/12)$

(8) $v(12) = v(13) = v(14) = v(234) = 1$

$v(S) = 0$ otherwise

$E_1 = (1/4 + \varepsilon/2, 1/4 - \varepsilon/6, 1/4 - \varepsilon/6, 1/4 - \varepsilon/6)$

$\lim_{\varepsilon \to 0} E_1 = (1/4, 1/4, 1/4, 1/4)$

$E = (3/4, 1/12, 1/12, 1/12)$

$S = (1/2, 1/6, 1/6, 1/6)$

(9) $v(1234) = 1$

$v(S) = 0$ otherwise

$E_1 = E = S = (1/4, 1/4, 1/4, 1/4)$

3. Comments on the solutions

With the new rules only a few paths through the game tree are possible.
There are two reasons for this phenomenon:

1.- bids are only made to profitable partners.
2.- With only one exception it is always better for players to accept an invitation rather than to reject it.

To get some insight into the behaviour of the game, let us look how the game is played.
As there are at most two players with a positive payoff in one game, the best a player can do is to invite a player with whom he and the preceeding players form a winning coalition. Let us call those players 'profitable' players.
If there are more possible players to make the present coalition a winning coalition then the last player chooses with equal probability among the 'profitable' players.
If there are only winning coalitions with more than two players then there is no chance for the starting player to get a positive payoff. Therefore he is also indifferent and chooses with equal probability among all the other players.

There is only one game where the winning coalition consists of 4 players, that is game (9). In all other games the winning coalitions have less than 4 players. This yields the explanation for the fact that invitations are always accepted except in the game where only the all-player coalition is a winning coalition. For the moment let us restrict to 4-person games where the winning coalitions have less than 4 players.

The general strategy of the game can be described in the following way:
A player who invites a profitable partner offers

him somewhat, say ε, more than the worse this player can hope for to get if he rejects the invitation.
In the 4-person games it turns out that the worse a player can get is either 0 or 1/2. In order to accept an invitation proposed to him the player is offered ε or $(1+\varepsilon)/2$ respectively.

Consider for example game (1) with characteristic function $v(123) = v(124) = 1$, $v(S) = 0$ otherwise. Suppose player 1 starts the game. As he cannot get any positive payoff he will invite any of the remaining players. Suppose a bid is made to player 2. One might expect that player 2 is indifferent because neither player 3 nor player 4 can get any positive payoff without him. However, it is better for player 2 to accept an invitation, as in this case he can force either player 3 or player 4 to give him almost the total payoff, that is $(1-\varepsilon)$. On the other hand, if coalition [13] or [14] forms, the payoff for player two will only be $(1+\varepsilon)/2$, which for a small is less than $(1-\varepsilon)$.

Or consider game (4) with characteristic function $v(12) = v(13) = v(14) = 1$, $v(S) = 0$ otherwise. If player 1 starts the game he can force any of the other players to give him almost the total payoff $(1-\varepsilon)$. Here it is more profitable for players 2,3 and 4 to start the game, as they can then get a payoff of $(1-\varepsilon)/2$.

Notice that in games with only 2-player and 4-player coalitions the new expected value E_1, the expected value E and the Shapley-value S are all equal. In game (6) E and E_1 are equal. All other games have different values for the three solutions. Moreover E_1 of game (7) behaves as if it was a symmetric 3-person game with player 4 as a dummy player. Similarily game (8) behaves like a purely symmetric game assigning equal payoff to each player.

The solutions for 4-person games suggest that in general the expected value E_1 will be different from the expected value E and the Shapley-value S. Moreover E_1 seems to be less sensitive to the structure of the game. E_1 sometimes attributes the same power to players to whom E and S give different power-indices. It never happens that E or S give equal payoff to some players to whom E_1 attributes different values.
We did not expect E_1 to be equal S or E as both E and S use the marginal contribution of each player to the coalition whereas E_1 considers the minimal payoff a player can get given he rejects the present invitation.
We do not pretend that any of the 3 solution concepts E_1,E or S is a better one, as each reflects different features. Notice for example that it may well happen in real life situations that, by being the first one or the last one in the play of a game, players have a disadvantage or an advantage. This situation is in some way reflected by E_1.

Interpretation

The game described might be interpreted in the following way. Suppose one player has been chosen to start the game and has invited another player. The second player now considers among the possible outcomes those where no agreement has been reached and the gain has been distributed equally among the last two players. Given these payoff vectors the second player now starts bargaining with one of the remaining two players. In the proposed game the bargaining players are always better off if they reach an agreement.

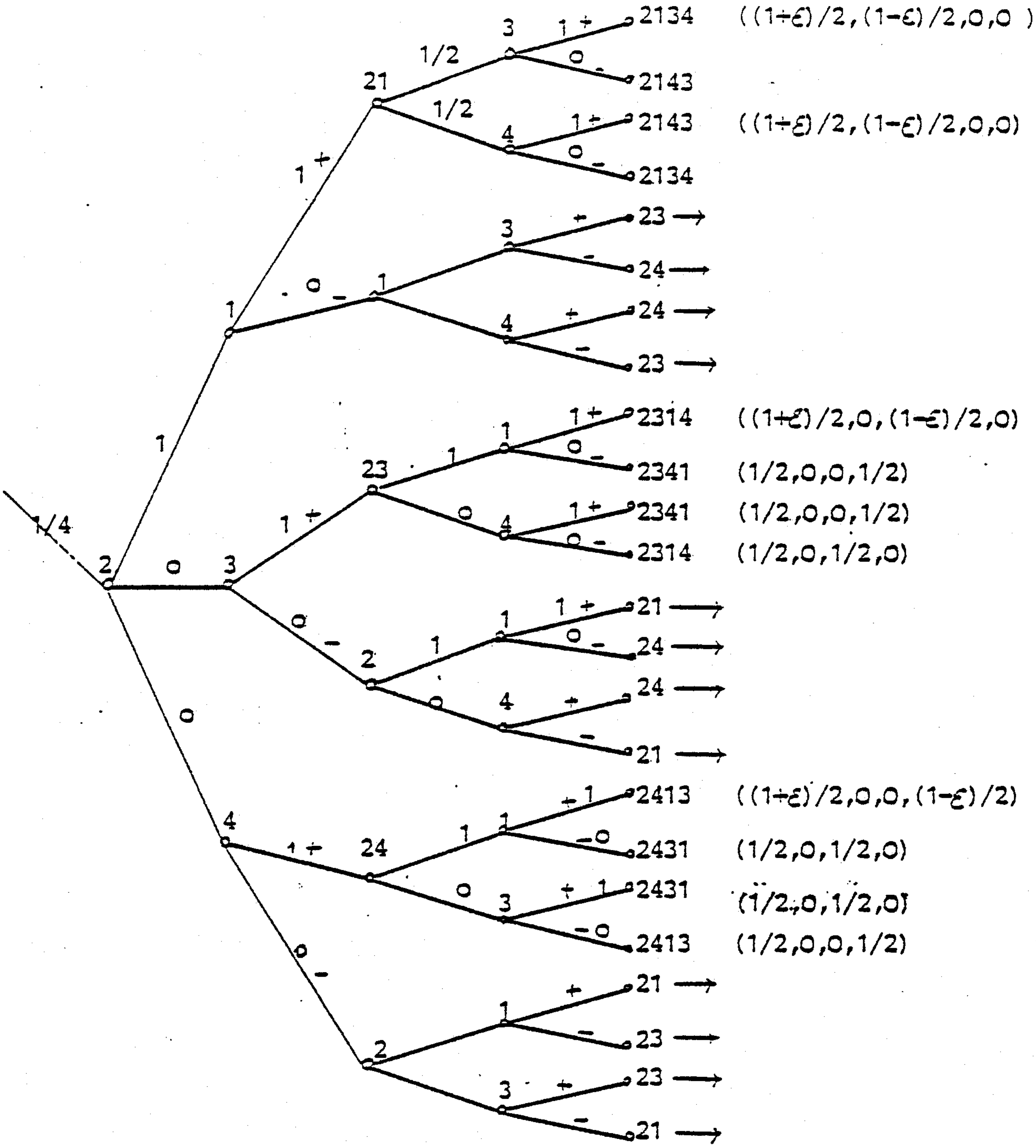


Table II.2.1: game (5)
4-person game with charcteristic function
v(12) = v(134) = 1 and 0 otherwise,
where player 2 starts the game

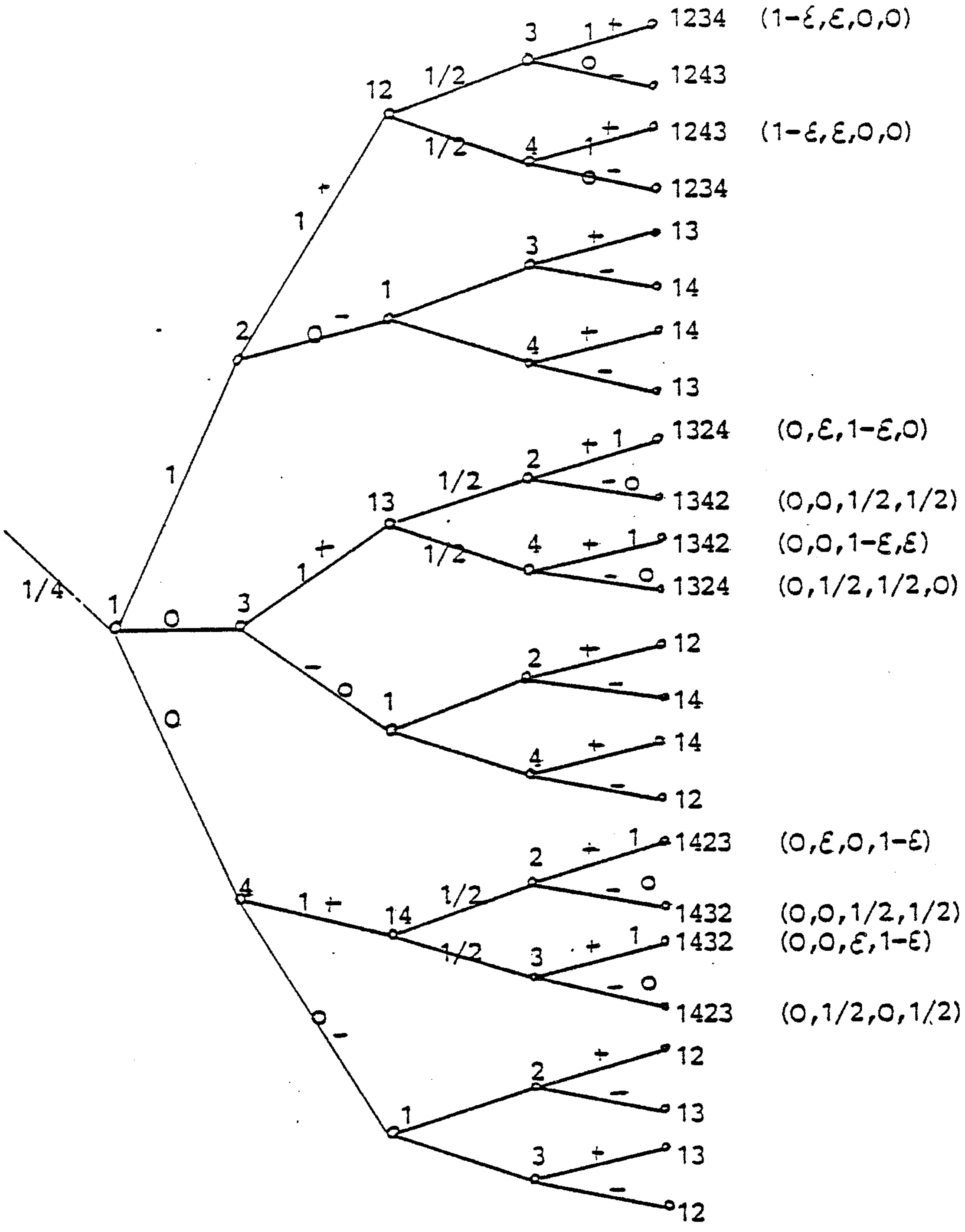


TAble II.2.2: game (5)
4-person game with characteristic function
v(12) = v(134) = 1 and 0 otherwise,
where player 1 starts the game

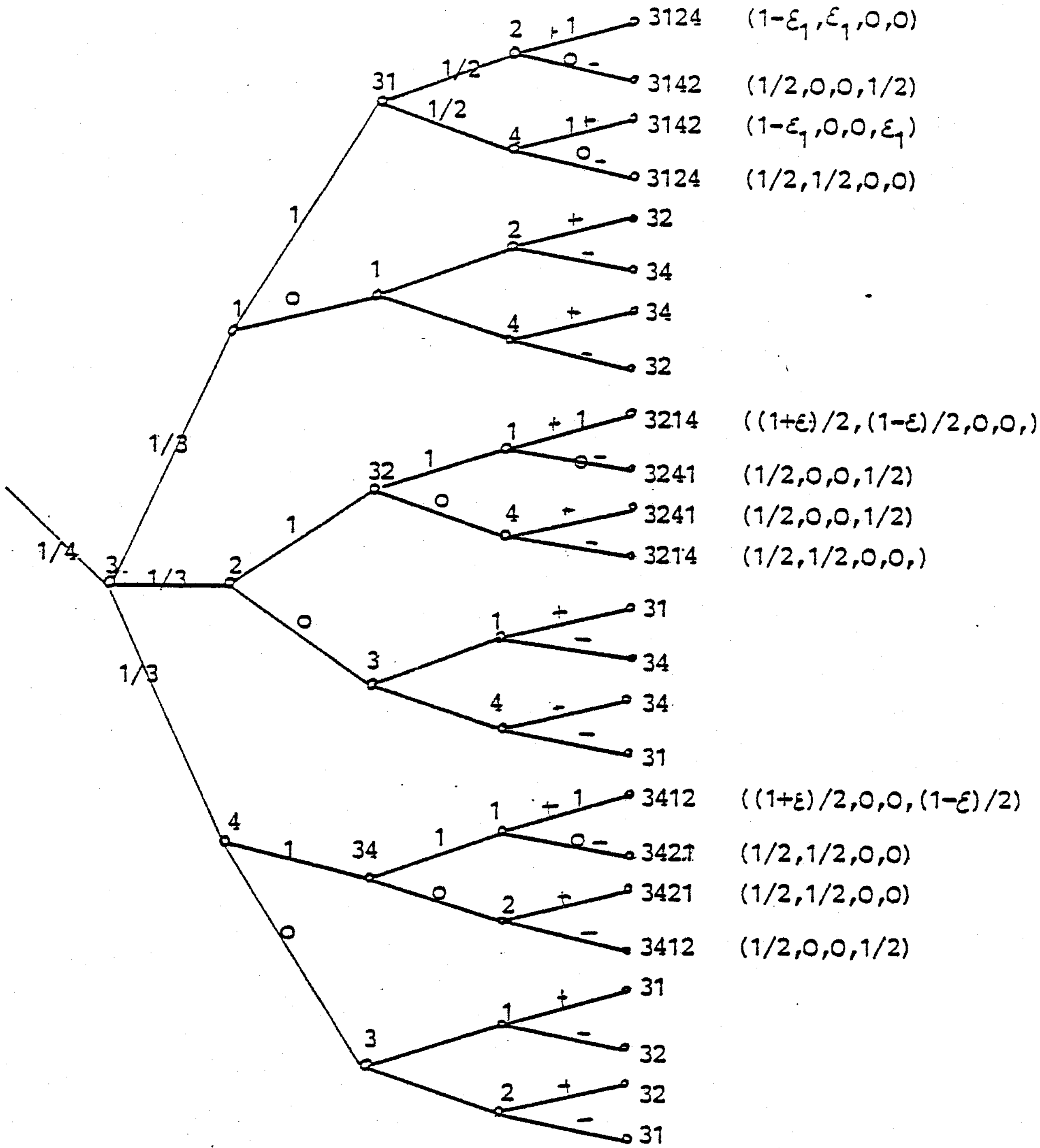


Table II.2.3: game (5)
4-person game with characteristic function
v(12) = v(134) = 1 and 0 otherwise,
where player 3 starts the game

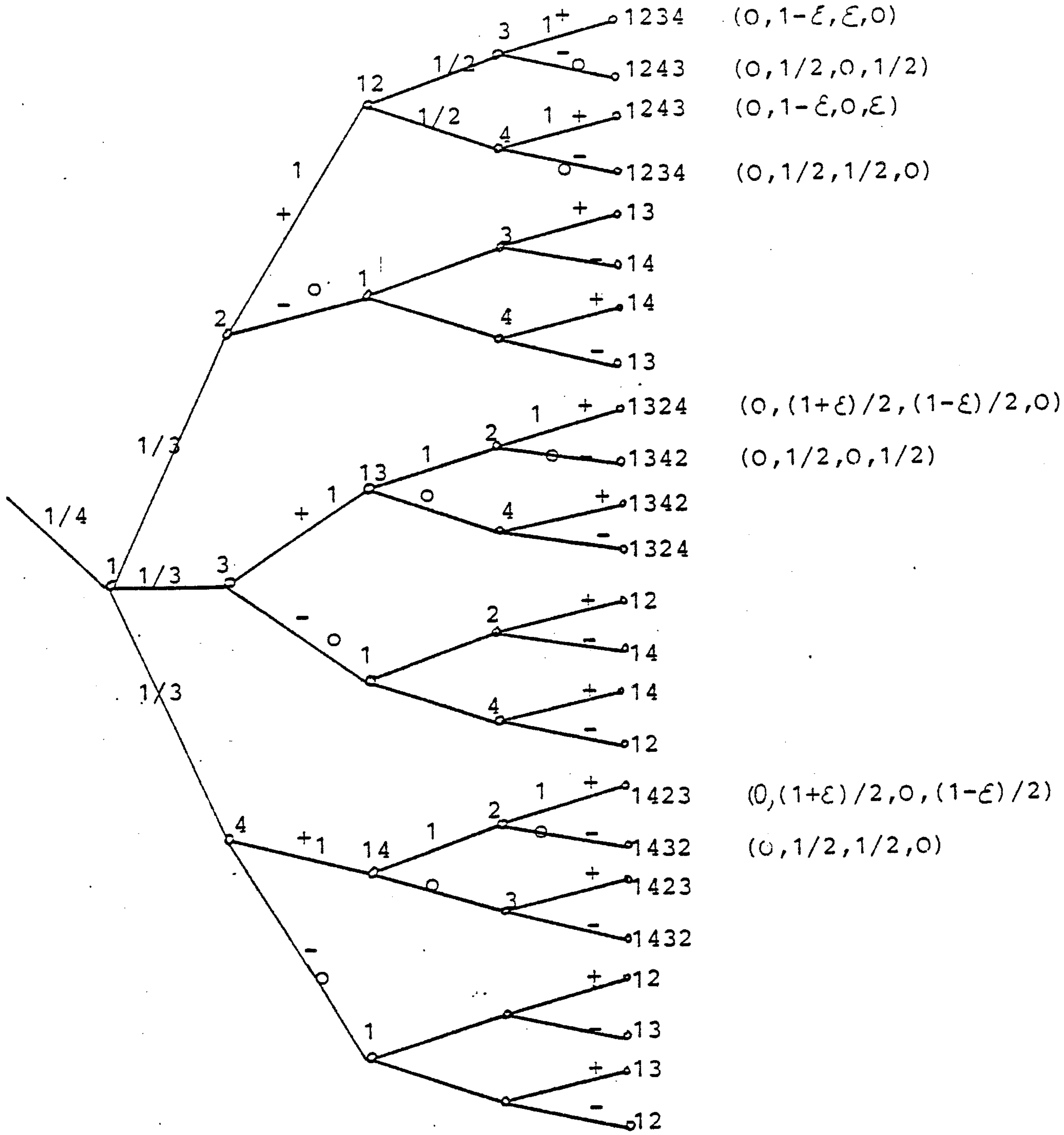


Table II.2.4: game (1)
4-person game with characteristic function
v(123) = v(124) = 1 and 0 otherwise,
where player 1 starts the game

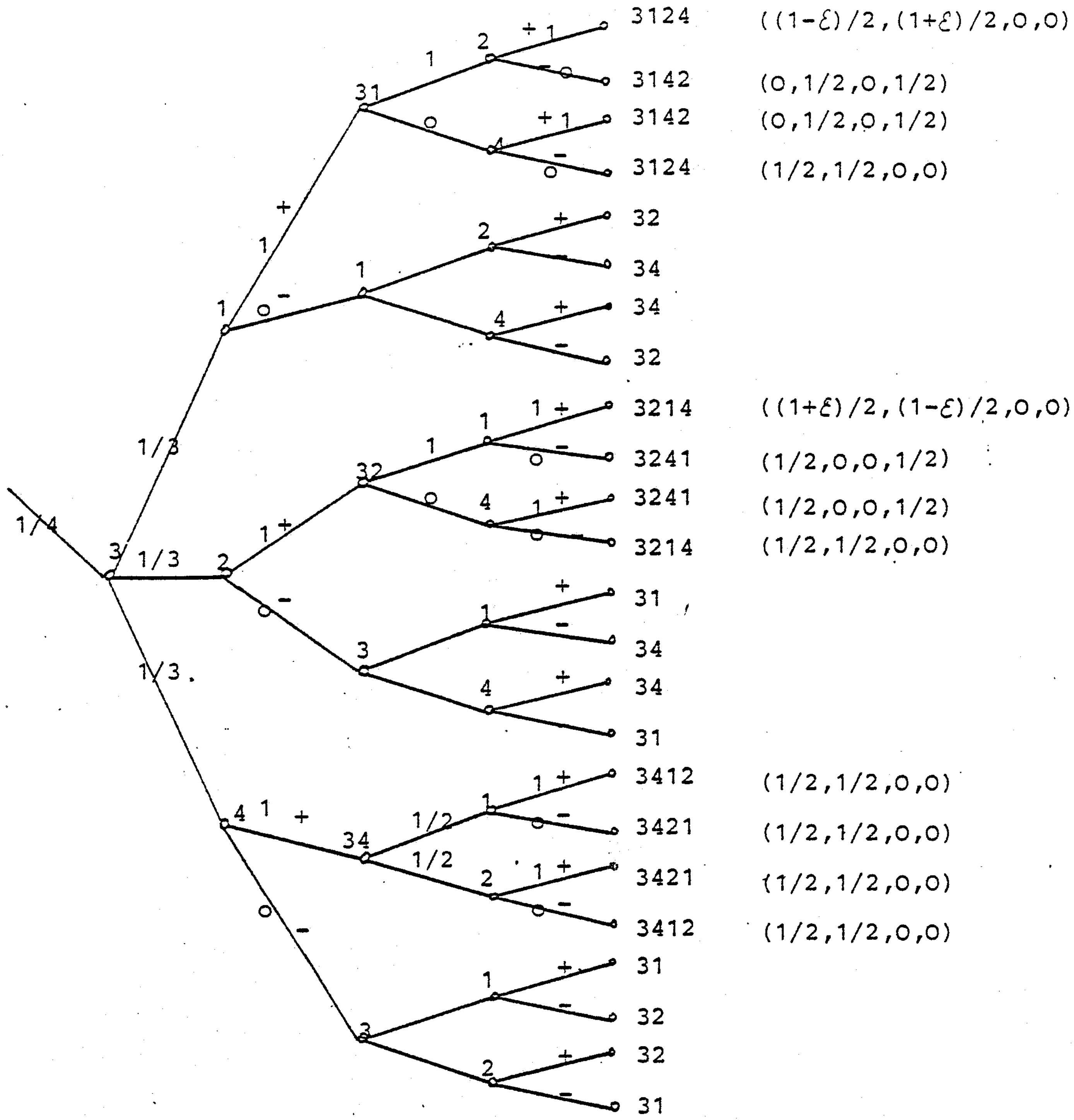


Table II.2.5: game (1)
4-person game with characteristic function
v(123) = v(124) = 1 and 0 otherwise, where
player 3 starts the game

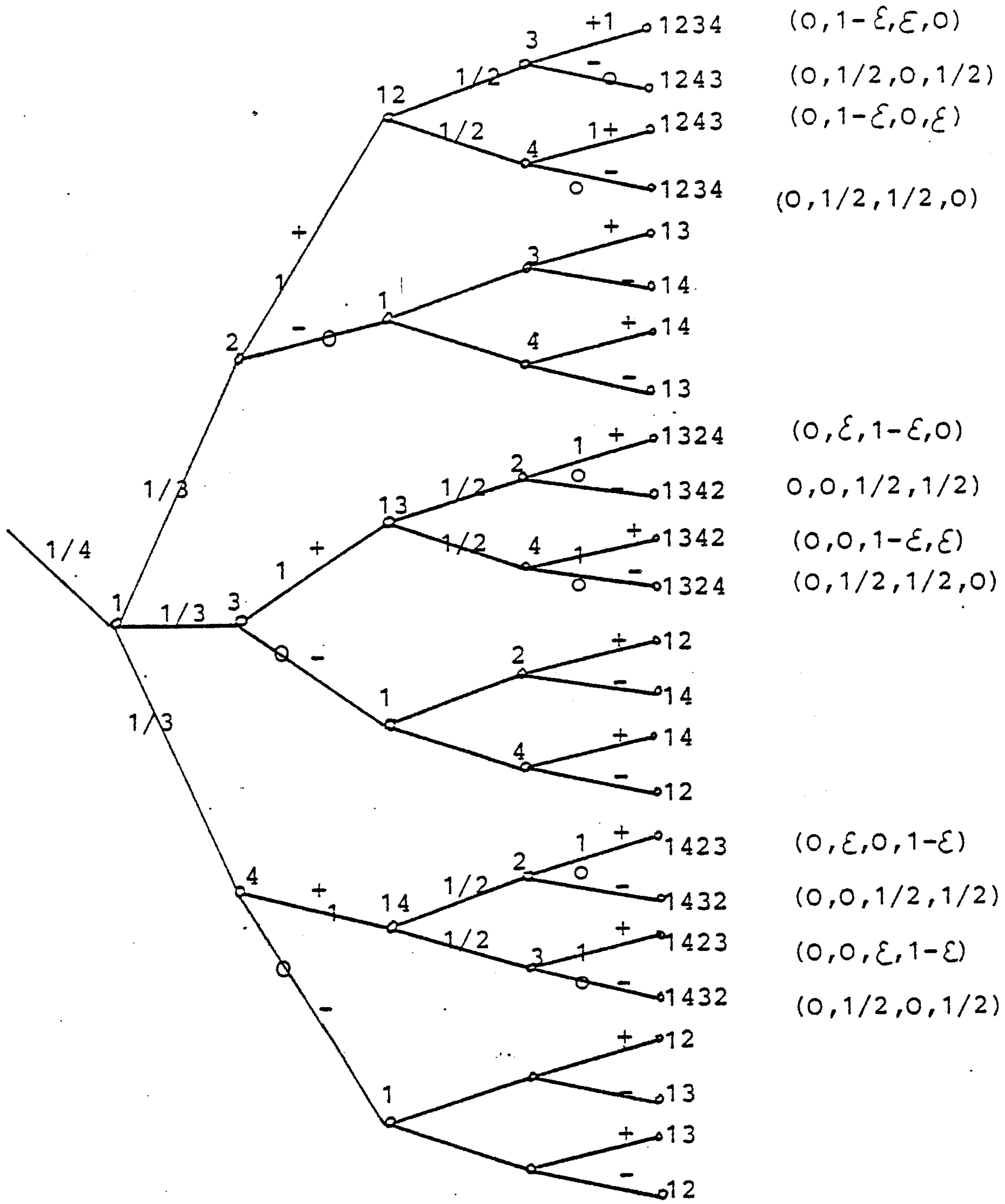


Table II.2.6: game (2)
4-person game with characteristic function
v(123) = v(124) = v(134) = 1 and 0 otherwise,
where player 1 starts the game

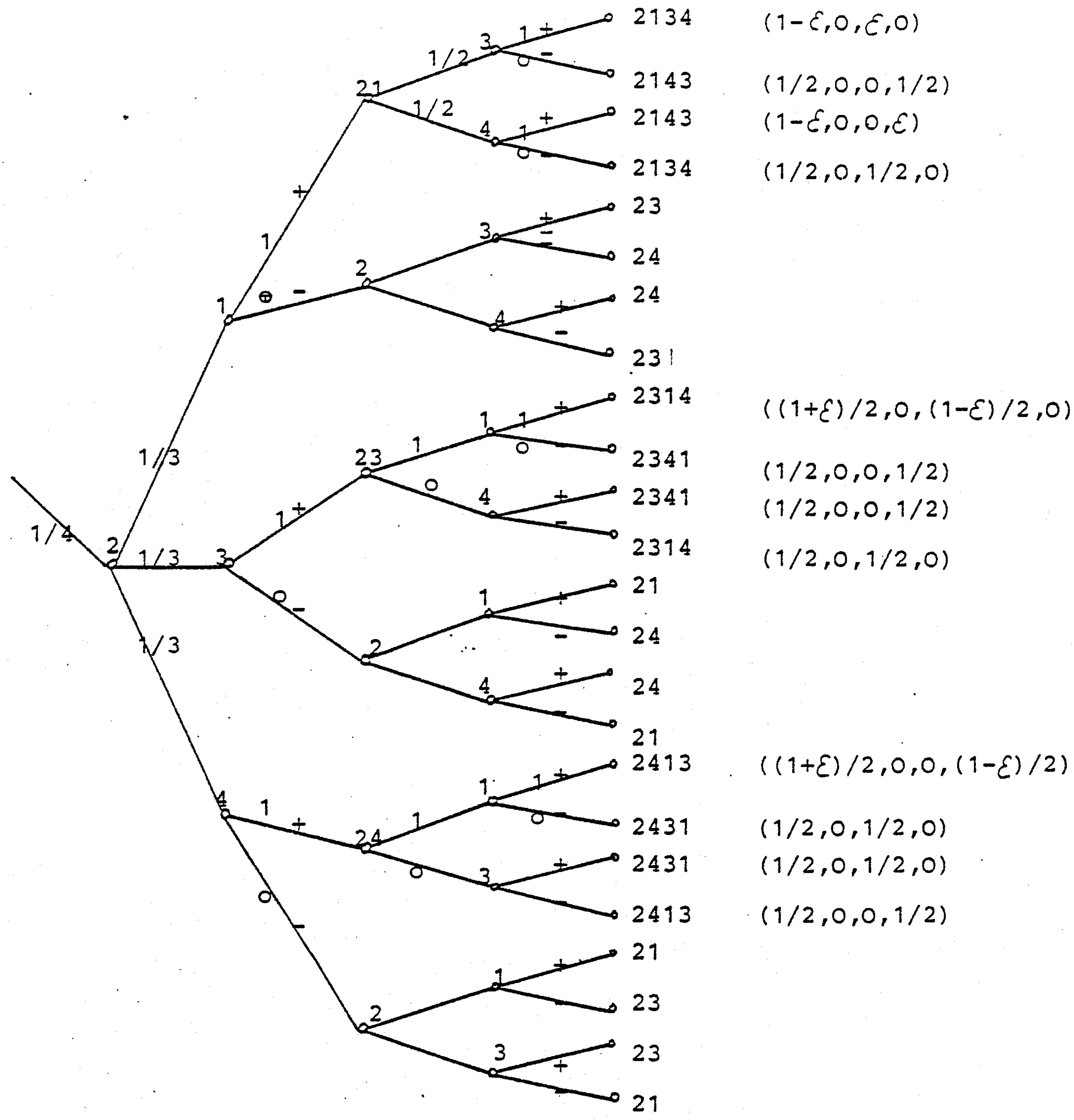


Table II.2.7: game (2)
4-person game with characteristic function
v(123) = v(124) = v(134) = 1 and 0 otherwise,
where player 2 starts the game

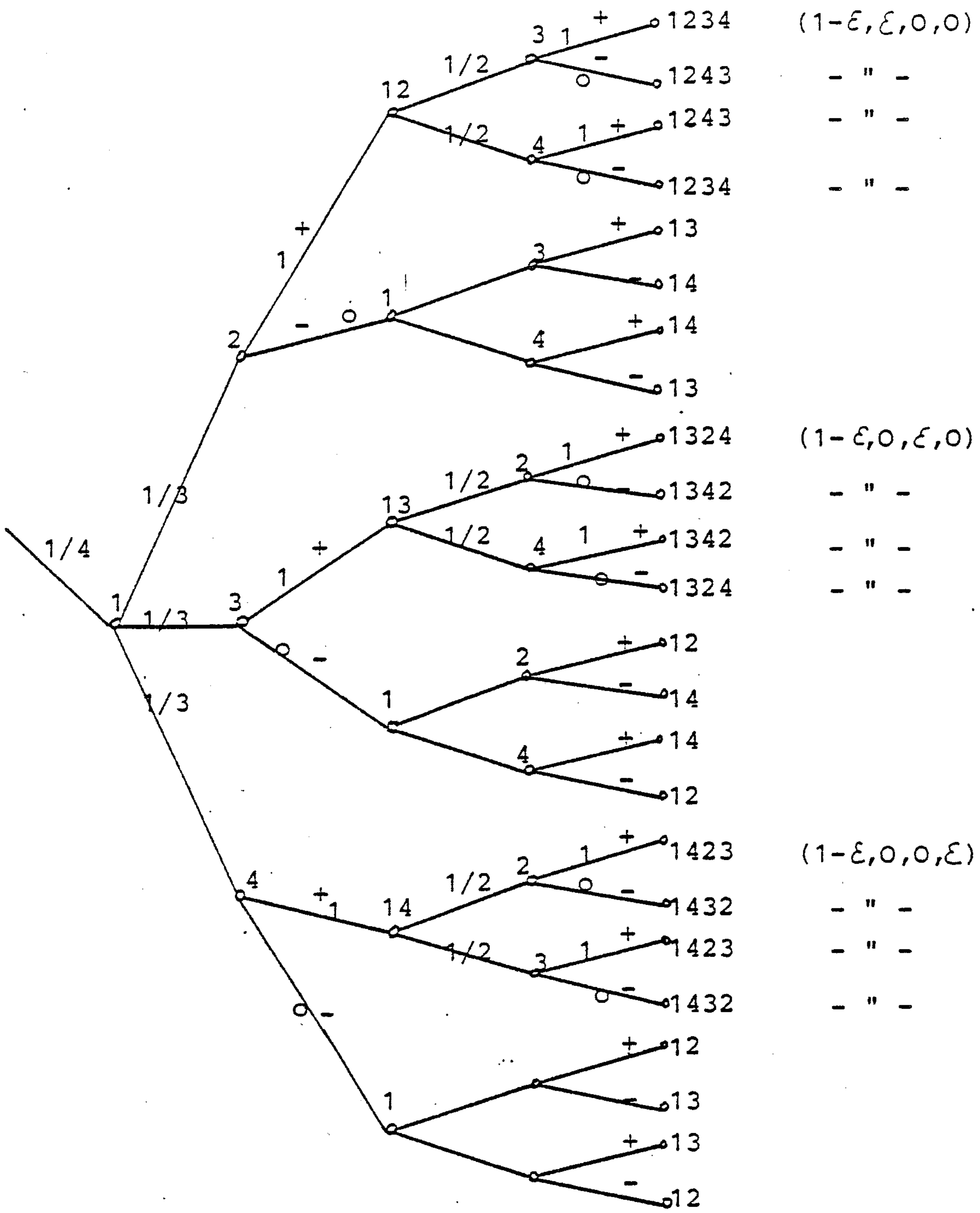


Table II.2.8: game (4)
4-person game with characteristic function
v(12) = v(13) = v(14) = 1 and 0 otherwise,
where player 1 starts the game

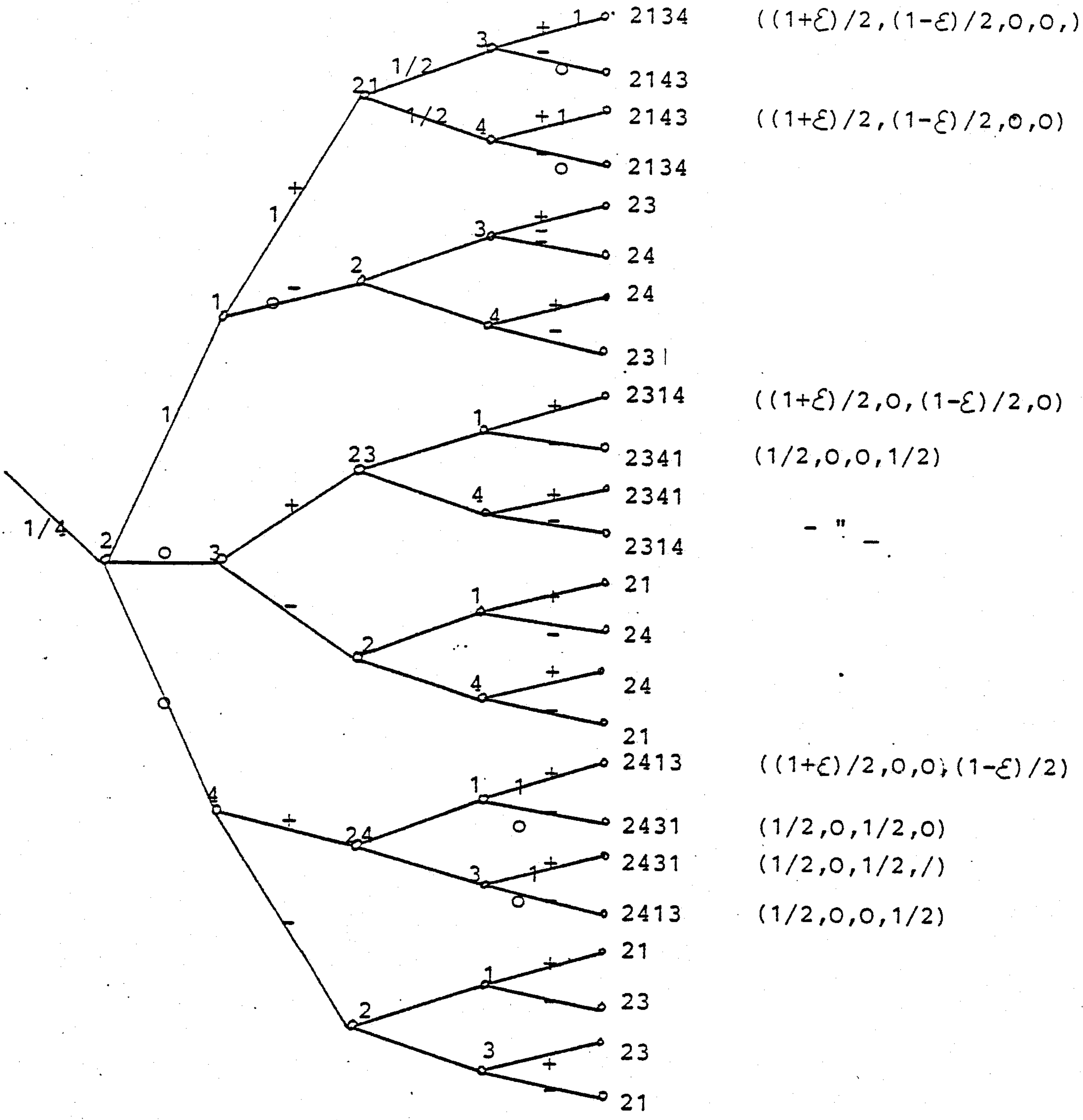


Table II.2.9: game (4)
4-person game with characteristic function
$v(12) = v(13) = v(14) = 1$ and 0 otherwise,
where player 2 starts the game

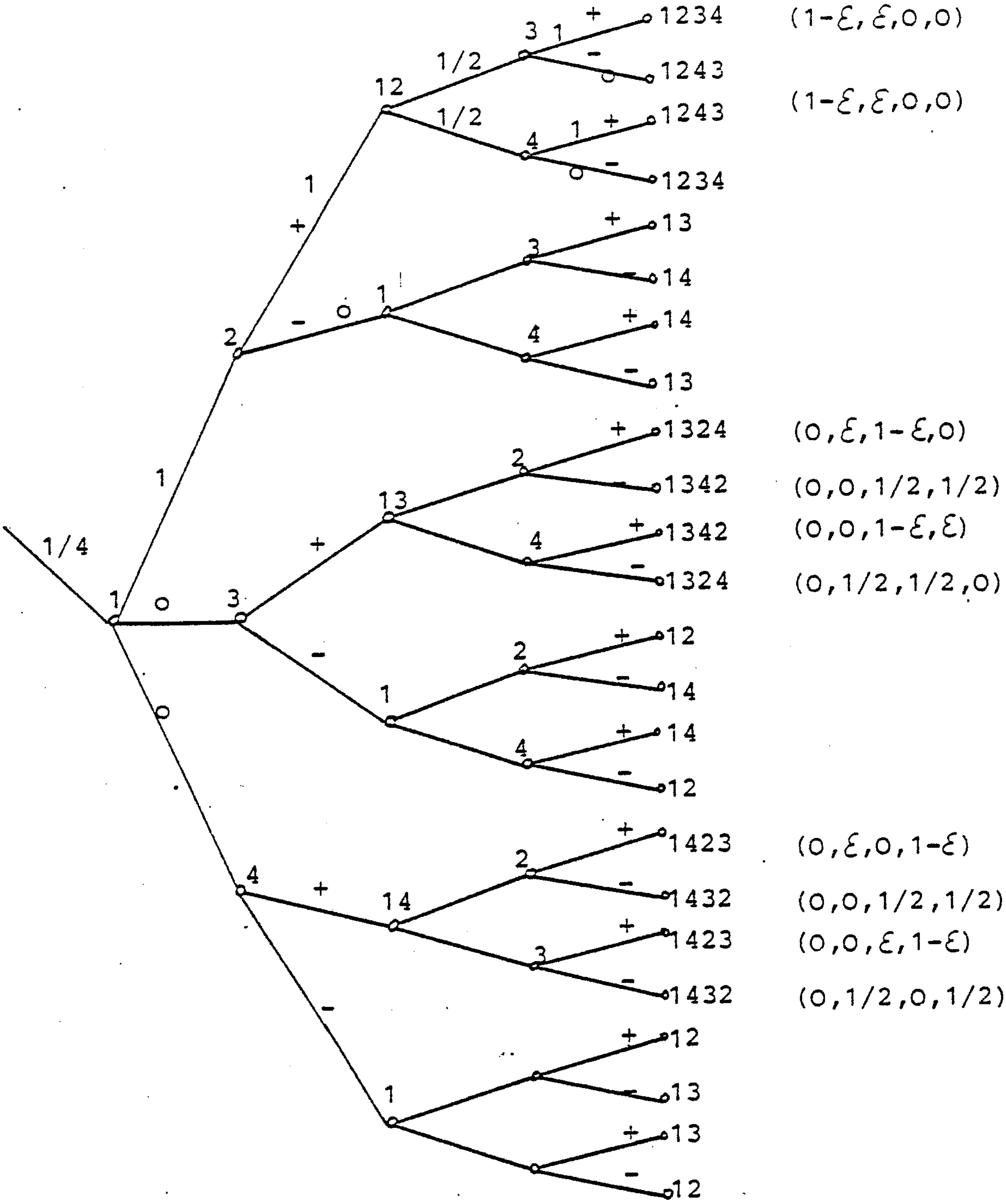


Table II.2.10: game (6)
4-person game with characteristic function
v(12) = v(134) = v(234) = 1 and 0 otherwise,
where player 1 starts the game

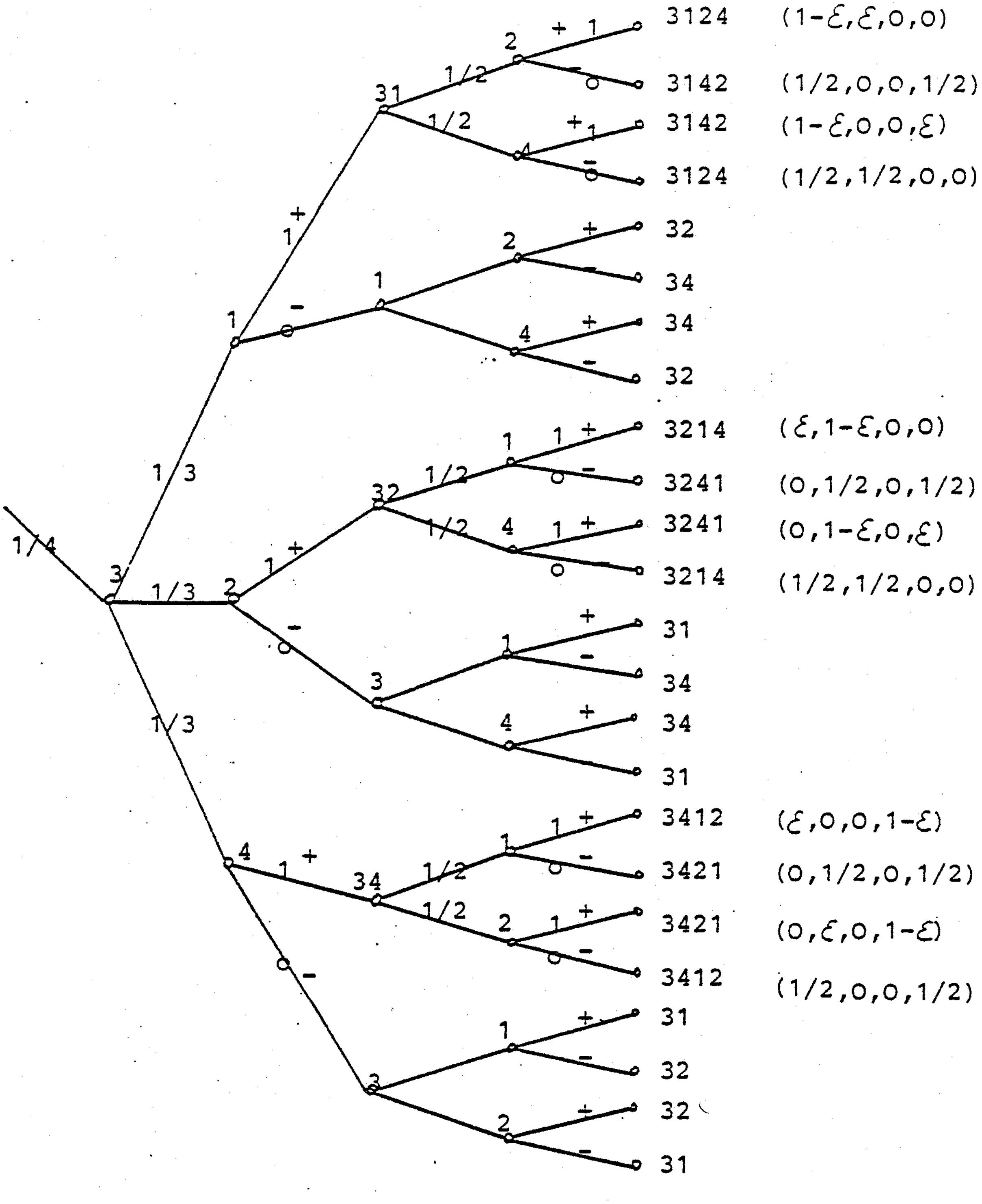


Table II.2.11: game (6)
4-person game with characteristic function
v(12) = v(134) = v(234) = 1 and 0 otherwise
where player 3 starts the game

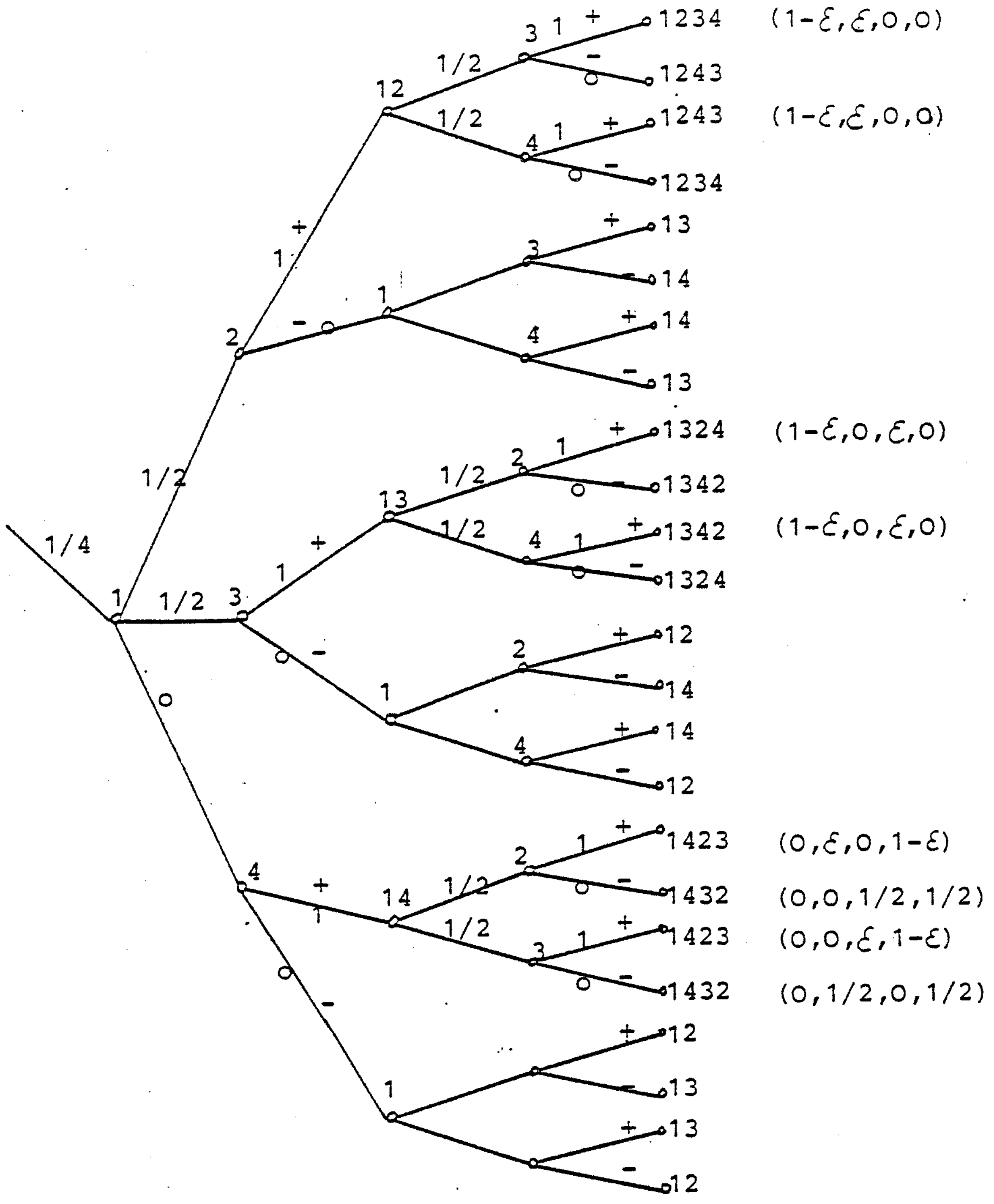


Table II.2.12: game (7)
4-person game with characteristic function
v(12) = v(13) = v(234) = 1 and 0 otherwise,
where player 1 starts the game

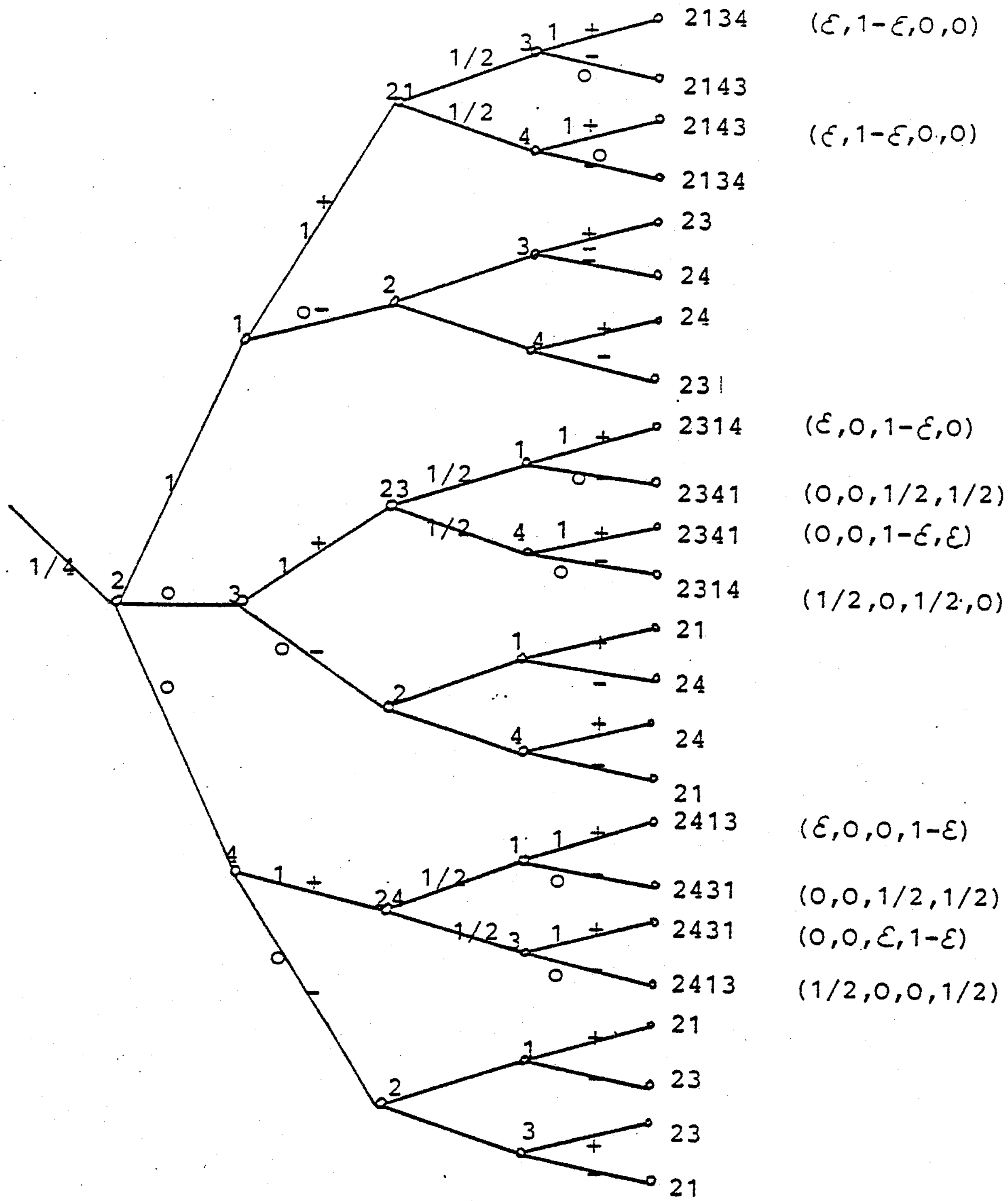


Table II.2.13: game (7)
4-person game with characteristic function
v(12) = v(13) = v(234) = 1 and 0 otherwise,
where player 2 starts the game

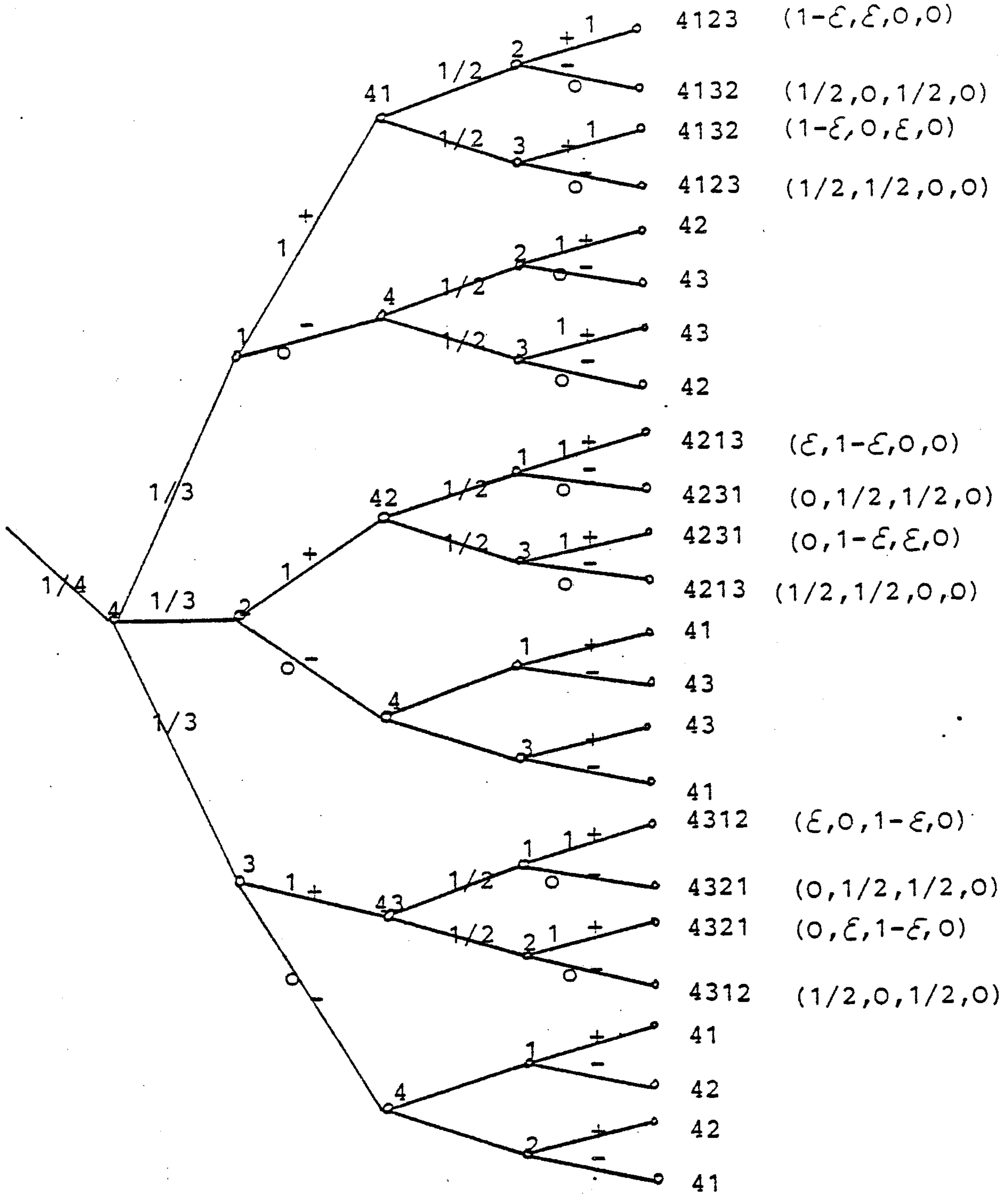


Table II.2.14: game (7)
4-person game with characteristic function
v(12) = v(13) = v(234) = 1 and 0 otherwise
where player 4 starts the game

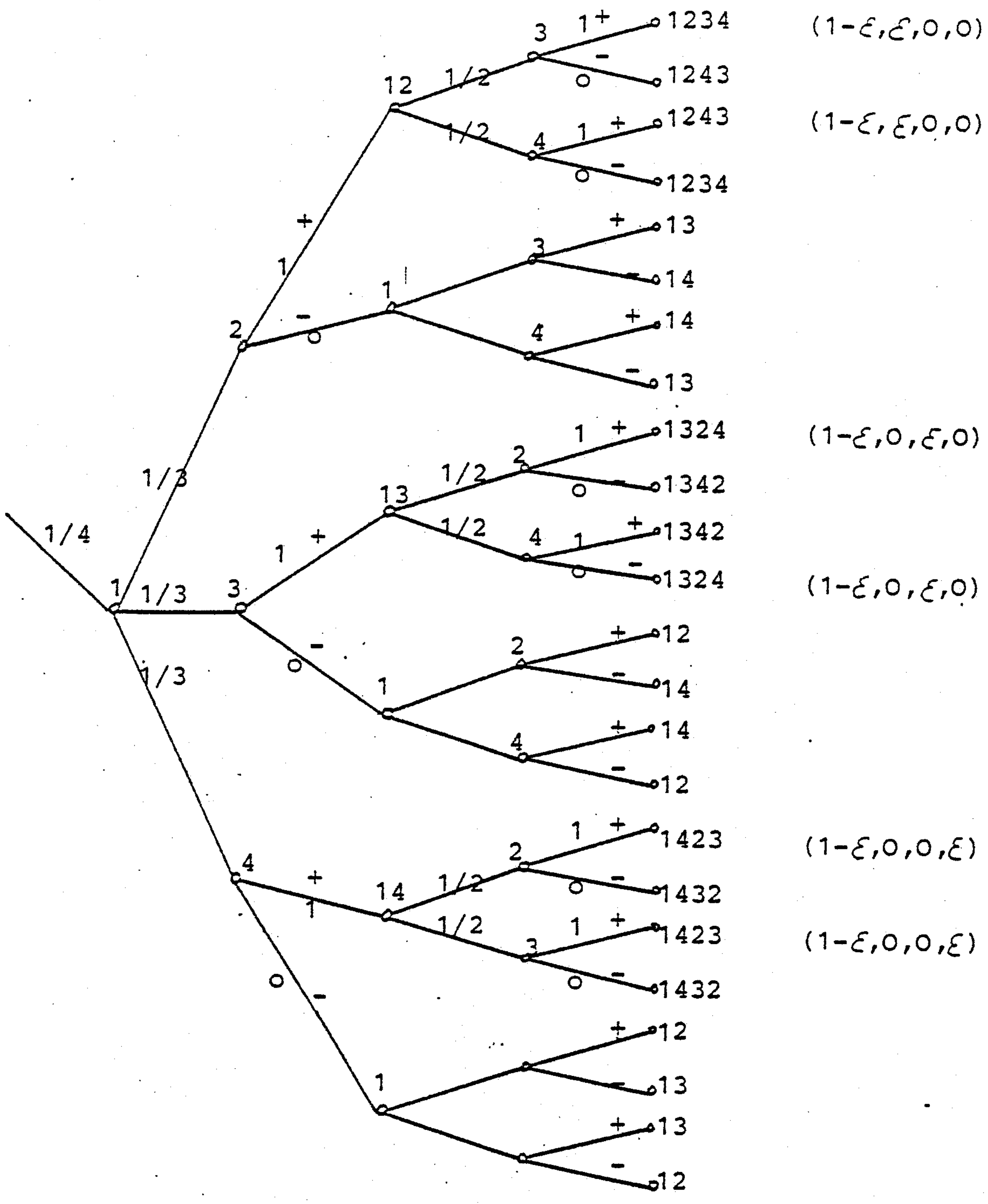


Table II.2.15: game (8)

4-person game withe charcteristic function
v(12) = v(13) = v(14) = v(234) = 1 and 0
otherwise, where player 1 starts the game

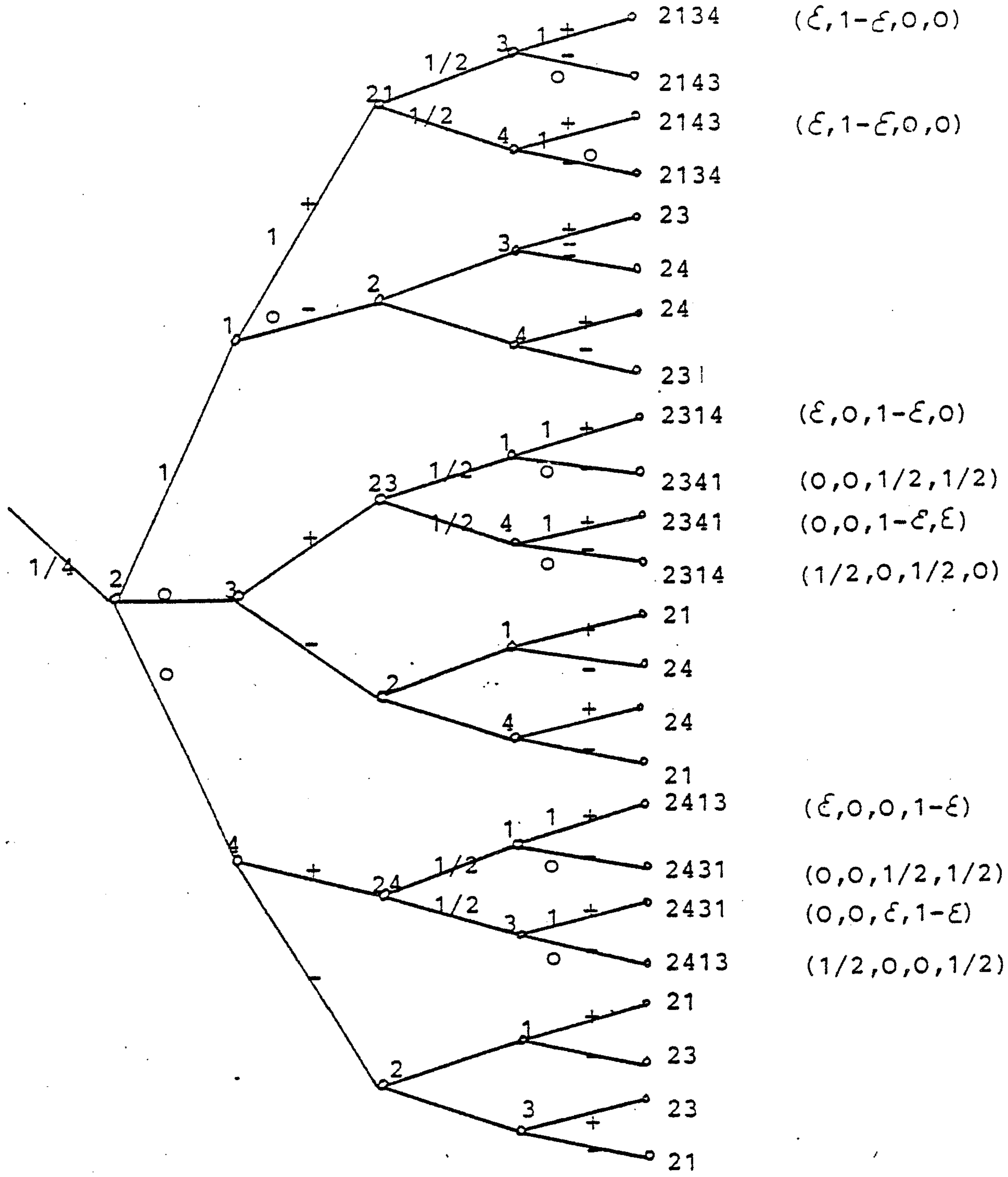


Table II.2.15: game (8)
4-person game with characteristic function
v(12) = v(13 = v(14) = v(234) = 1 and 0
otherwise, where player 2 starts the game

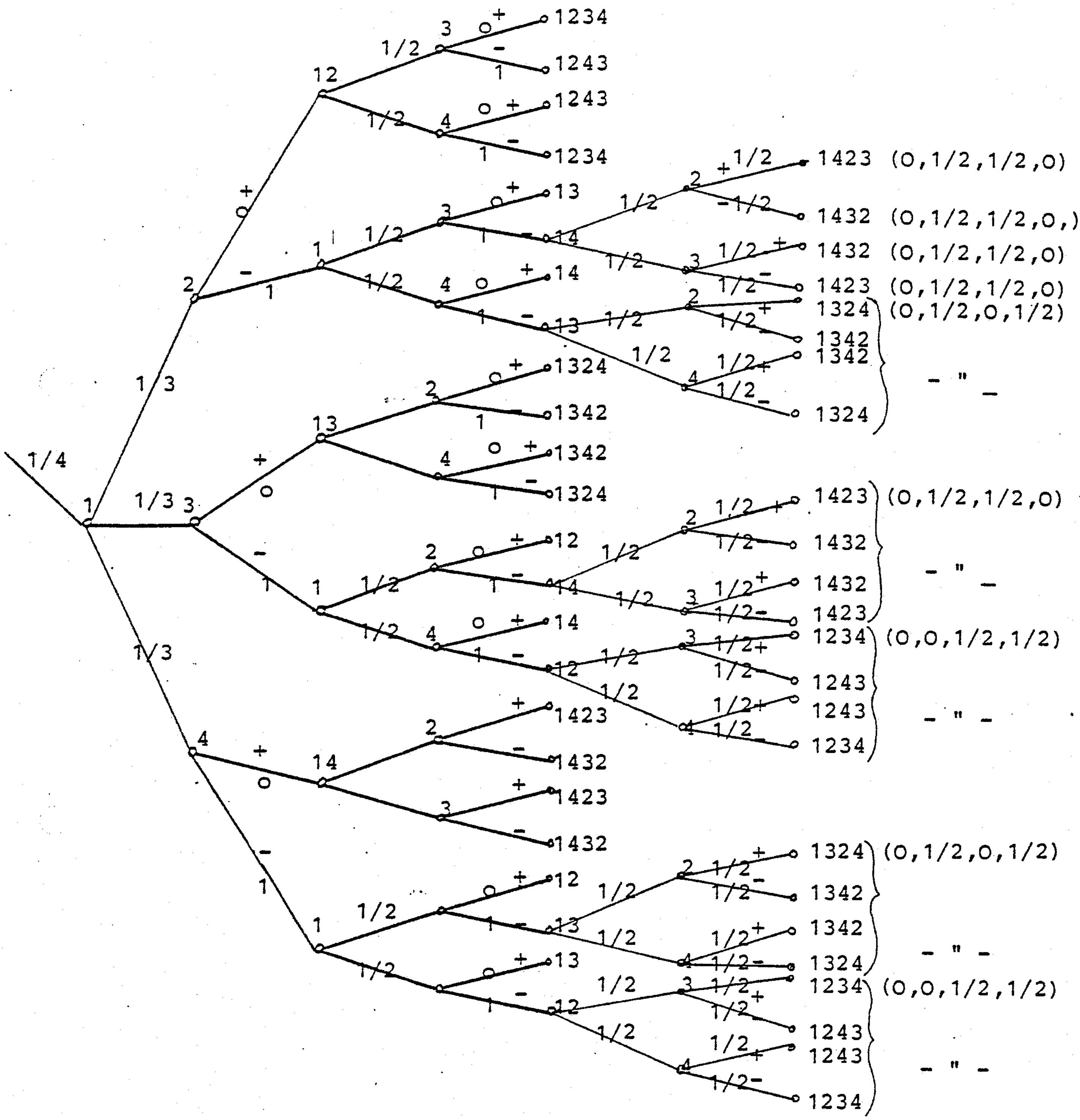


Table II.2,17: game (9)
4-person game with characteristic function
v(1234) = 1 and 0 otherwise
where player 1 starts the game

III. Aspirations and Shapley-value

In this chapter we ask about relations between Shapley-value and the set of aspirations.
For $x \in R^n$ denote by $x(S) = \sum_{i \in S} x_i$

Definition: The payoff vector $x \in R^n$ for the game <N,v> is an aspiration if

1. $x(S) >= v(S)$ for every S in N and
2. for each i in N there exists a coalition S containing i such that $x(S) = v(S)$.

The first condition states that players behave rational. The second condition asserts that there exists at least one coalition S in N which can afford the amount x(S).

1. Shrinked aspirations

The Shapley-value is defined on the set of imputations which is different from the set of aspirations. As the set of aspirations and the set of imputations are two different domains, we introduce a vector x' to make a comparison of Shapley-value and aspirations possible.
If x is an aspiration let $x(N) = \sum_{i \in N} x_i$

be the sum of aspirations of all players.
The imputation $x'=(x_1',x_2',\ldots,x_n')$ is defined by

$$x_i' = x_i/x(N) .$$

Obviously $\sum_i x_i' = v(N)$, the amount the all-player coalition can in fact afford.

The vector x' might be interpreted in the following way:
Think of a game where the available payoff should be distributed only among all players. In other words, suppose the process of coalition formation goes on until the all-player coalition has formed. As people demand more than the all-player coalition can afford, the players compare the fictitious payoff vector x' to a fair payoff vector as the Shapley-value is, which gives them the answer how to share the expected payoff.

The question we want to answer is the following:
Is it always possible to transform an aspiration such that the 'shrinked' aspiration, that is the vector of aspirations devided by v(N), equals the Shapley-value?

Consider as an example the game with characteristic function $v(12) = v(13) = v(14) = 1$ and
$v(S) = 0$ otherwise.
The set of aspirations x for this game is

$$x = (\alpha, 1-\alpha, 1-\alpha, 1-\alpha) \quad \text{where } 0 \leq \alpha \leq 1 .$$

For the value x(N) we get: $x(N) = 3-2\alpha$,
which results in the transformed aspiration x':

$x' = 1/(3-2\alpha).(\alpha, 1-\alpha, 1-\alpha, 1-\alpha)$ with $0 \leq \alpha \leq 1$.

The Shapley-value S for this game equals

$S = (3/4, 1/12, 1/12, 1/12)$

Obviously there exists a solution for the equations

$x_i / x(N) = S_i$, namely $\alpha = 9/10$,

resulting in the aspiraton x_1:

$x_1 = (9/10, 1/10, 1/10, 1/10)$.

For many simple 4-person games there exists a solution to the equaton $x' = S$.

But consider the simple game:

$v(12) = v(134) = 1$.

The set of aspirations for this game is

$x = (\alpha, 1-\alpha, \beta, 1-\alpha-\beta)$ where $0 \leq \alpha \leq 1$
and $0 \leq \beta \leq 1-\alpha$

and $x(N) = 2-\alpha$. Therefore

$x' = 1/(2-\alpha).(\alpha, 1-\alpha, \beta, 1-\alpha-\beta)$.

The Shapley-value S for this game has the form

$S = (7/12, 1/4, 1/12, 1/12)$.

As $x_2' - x_3' = x_4'$,

1/4 - 1/12 should be equal 1/12, which obviously does not hold.

One might argue that the aspirations of players 3 and 4 should be equal as their Shapley-values are equal too. The set of aspirations would then have the form:

$$x = (\alpha, 1-\alpha, (1-\alpha)/2, (1-\alpha)/2) \quad \text{with } 0 \leq \alpha \leq 1.$$

But as $1/12 \neq (1/2)(1/4)$ there is no solution to the equation $x_i/x(N) = S$.

For the general case we state the following result:

In general it is not possible to transform an aspiration x to an imputation x' where $x_i' = x_i/x(N)$, such that x' = S.

The set of aspirations not being convex and not even starshaped, it does not seem surprising that the Shapley-value in general is not included in the set of normalized aspirations.

Presumably the equation $x_i' = x_i/x(N)$ is solvable for games where the expected value equals the Shapley-value.

If for every game the transformation from x to x' was possible with x' being equal the Shapley-value, we could have combined the two concepts of demanding 'much' as the aspirations do and 'fair play' as the Shapley-value proposes.

2. Shapley-value and aspiration core in restricted games

The vector $x \in R^n$ is in the aspiration core if and only if x is a solution to the linear program:

$$\text{Minimize} \sum_{i \in N} x_i$$

subject to a constraint for each coalition S:

$$x(S) \geq v(S)$$

for x unrestricted in sign [1].

Let $v/_S$ be the game <N,v> restricted to a subcoalition S. Let $x/_S$ be in the aspiration core of the restricted game $v/_S$.

Let $\tilde{v}(S) = \sum_{i \in S} x_{i/S}$,

that is, v assigns to coalition S the sum over the aspirations of members in S in the aspiration core.

Moreover $v/_S(S') = v(S')$ iff $S' \subseteq S$
$= 0$ otherwise .

As $\tilde{v}$ assigns to coalition S a value that is rational and also feasable for the whole coalition S (not only for some subcoalition of S) we expected the following assertion to hold:

For every game <N,v> there exists a restricted game $\langle S, v/_S \rangle$ such that

$$Sh(\tilde{v}) \in \text{Asp. of } v/_S$$

where $Sh(\tilde{v})$ denotes the Shapley-value of $\tilde{v}$.

To verify our expectation we investigate a game whose characteristic function is of a very general form, that is the Dictator-game [4], which was constructed for experimental purposes. The set of players is $N = \{1,2,3,4\}$.
The characteristic function is given by:

$v(i) = 0$ for all i in N
$v(12) = 50$
$v(13) = 60$
$v(14) = 40$
$v(23) = 20$
$v(24) = 30$
$v(34) = 10$

$v(123) = 110$
$v(124) = 80$
$v(134) = 70$
$v(234) = 90$
$v(N) = 120$.

Consider the subgame $\langle v|_S, S\rangle$ where $S=\{1,2,3\}$.
The set of aspirations for this game is the set of all vectors $x = (\lambda, 50-\lambda, 60-\lambda)$ where $110-\lambda \geq 110$ and therefore $\lambda = 0$, which states that the set of Aspirations coincides with the aspiration core:
$AC = (0,50,60)$.
We define $\tilde{v}(123) = 110$ and

$$\tilde{v}(S') = \begin{cases} v(S') & \text{for every } S' \subseteq \{1,2,3\}, \\ 0 & \text{otherwise.} \end{cases}$$

For the Shapley-value $Sh_1(\tilde{v})$ of player 1 we get
$Sh_1(\tilde{v}) = (1/6)110 + (1/3)90 = 145/3 \neq 0$.
As we see readily the Shapley-value of $\tilde{v}$,
$Sh(\tilde{v}) \notin$ Aspirations.

Let us consider one more subgame, say
$\langle S, v|_S \rangle$ where $S = \{1,2,4\}$.

The set of aspirations is given by
$x = (\lambda, 50-\lambda, 40-\lambda)$ with $90-\lambda \geq 80$ and therefore
$10 \geq \lambda \geq 0$.

The aspiration core has the form
$AC = (10,40,30)$,
and we define
$\tilde{v}(124) = 80$ and
$\tilde{v}(S') = v(S')$ for $S' \subsetneq \{1,2,4\}$
$= 0$ otherwise
The Shapley-value of player 1 equals
$Sh_1(v) = (1/6)90 + (1/3)50 = 95/3 > 10$.
This payoff already shows that
the Shapley-value is not included in the
set of aspirations.

Considering also the other subgames $v|_S$ where
$S=\{1,3,4\}$ or $S=\{2,3,4\}$, it turns out that the
Shapley-value never is an aspiration of any subgame.

<u>So we state the unexpected result:</u>
For a game $\tilde{v}$ defined as above
$Sh(\tilde{v}) \notin$ Aspirations.

Consider once more the subgame $v|_S$ where $S=\{1,2,4\}$.
Is it possible to redefine the characteristic function
in a way such that $Sh(\tilde{v}) \in$ Aspirations?

Let $v(12) = 50$
$v(14) = 40$
$v(24) = 30$ as before and redefine
$v(124) = 50$.

The set of aspirations of this subgame is all vectors $x = (\lambda, 50-\lambda, 40-\lambda)$ where

$$90 - 2\lambda \geq 30 \quad \text{or}$$
$$30 \geq \lambda \geq 0 \ .$$

The aspiration core is therefore given by

$AC = (30,20,10)$ and

we define

$$\tilde{v}(124) = 60$$
$$\tilde{v}(S') = v(S') \quad \text{for every } S' \subseteq \{1,2,4\} \ .$$
$$v(S') = 0 \qquad \text{otherwise} \ .$$

Notice that this game $\tilde{v}$ assigns a new value to coalition $\{1,2,4\}$.

The Shapley-values for players 1,2 and 4 in the game $\tilde{v}$ are

$$Sh_1(\tilde{v}) = 25 < 30$$
$$Sh_2(\tilde{v}) = 20 \neq 50-25$$
$$Sh_4(\tilde{v}) = 15 = 40-25$$

and

$$Sh(\tilde{v}) = (25,20,15) \ .$$

Notice this result! The payoff for player 1, $Sh_1(\tilde{v})$, satisfies the condition $\lambda \leq 30$. The payoff for player 3, $Sh_3(\tilde{v}) = 40-\lambda$, for $\lambda = 25$. Therefore the condition of rationality is satisfied for coalition $\{1,4\}$, $x(14) \geq 40$, and for coalition $\{2,4\}$, $x(24) \geq 30$. The condition of rationality is also satisfied for coalition $\{1,2,4\}$, as $x(124) = 50 \geq 50$. The only coalition for which the condition of rationality is not satisfied, is coalition $\{1,2\}$, as $x(12) = 45 < 50$.

Notice that for the Shapley-value to be an aspiration

$$90 - \lambda^{*} = 60$$

should hold, as the vector of the Shapley-values always adds up to $\tilde{v}(N)$, here to $\tilde{v}(124)$. But this indicates that the Shapley-value of player 1 should be equal 30, resulting in the payoff vector (30,20,10).
Comparing the payoff vector (30,20,10) and the Shapley-value (25,20,15), it can be seen that there is no big difference between the two vectors, moreover they assign an equal payoff to player 2. Although the Shapley-value is 'closer' to the set of aspirations in this example, we still have the result:

$$Sh(\tilde{v}) \notin \text{Aspirations}.$$

2.1. An example of aspirations and the Shapley-value in a restricted 3-person game

In this section we consider aspirations of a restricted 3-person game and compare them to the corresponding Shapley-value.
Let $\langle S, v_{/S}\rangle$ be the restricted game with $S=\{1,2,3\}$.
Denote by $v(S) = x$

$v(12) = y$

$v(13) = z$

$v(23) = w$

Then the set of aspirations for this game is given by:

$$x = (\lambda, y-\lambda, z-\lambda)$$

where $(y+z-w)/2 \geq \lambda \geq 0$
or $(y+z-\lambda) \geq x$ and therefore

$y+z-x \geq \lambda \geq 0$.

Suppose $y+z-x \leq (y+z-w)/2$.

We make this assumption without loss of generality. Otherwise we would have to redefine

$\tilde{v}(S) > v(S)$.

The aspiration core results for

$\lambda = y+z-x$ and is given by

$AC = (y+z-x, x-z, x-y)$.

The game $\tilde{v}$ is therefore defined as

$\tilde{v}(123) = x$.

For the Shapley-value $Sh(\tilde{v})$ to be in the set of aspirations, the following equations must be satisfied:

(i) $y + z - \lambda = x$

(ii) $\lambda = 1/6(y+z) + 1/3(x-w)$

(iii) $y-\lambda = 1/6(y+w) + 1/3(x-z)$

(iv) $z-\lambda = 1/6(z+w) + 1/3(x-y)$

Subtracting equations (iii) and (iv) we get

$y = z$, that is $v(12) = v(13)$.

Using equation (ii) for λ in equation (iii), we get

$$y - 2y/6 - (1/3)(x-w) = (1/6)(y+w) + (1/3)(x-y) ,$$

what can be transformed to

$$5y + w = 4x .$$

Now let us make our assumption strictly, that is

$y + z - x = (y + z - w)/2$ what corresponds to

$y + z + w = 2x$.

As $y = z$, it follows

$$4y + 2w = 4x \ .$$

Together with the equation $5y + w = 4x$,we get the result

$$z = y = w \ .$$

This result indicates, that for the Shapley-value of a 3-person game $\tilde{v}$ to be in the set of aspiration, symmetry is a sufficient condition.

With the help of our considerations we now are able to construct a game $\langle S, v_{|S} \rangle$, such that $Sh(\tilde{v}) \in$ Aspirations of $v_{|S}$.
Let $S = \{1,2,3\}$.
Let the characteristic function be defined by

$$v(S) = \begin{cases} 0 & |S| = 1 \\ 10 & |S| = 2 \\ 15 & |S| = 3 \ . \end{cases}$$

The game is symmetric, and the condition $5v(12) + v(23) = 4v(123)$ is satisfied, as $50 + 10 = 4*15$.
The set of aspirations for this game is given by

$$x = (\lambda, 10-\lambda, 10-\lambda) \quad \text{with} \quad 0 \leq \lambda \leq 5,$$

and the aspiration core

$$AC = (5,5,5)$$

which defines

$$\tilde{v}(123) = v(123) = 15$$

$$\tilde{v}(S') = v(S') \qquad \text{for } S' \subseteq \{1,2,3\}$$

$$= 0 \qquad \text{otherwise} \ .$$

The Shapley-value of $\tilde{v}$ is given by

$$Sh(\tilde{v}) = (5,5,5) .$$

Notice the Shapley-value is both feasable and rational. As $Sh(\tilde{v}) = AC$, it is clear, that the Shapley-value of $\tilde{v}$ is included in the set of aspirations of $\langle S, v/_S \rangle$.

But consider the game

$$v(12) = v(13) = 10$$
$$v(23) = 14$$
$$v(123) = 16 \quad ,$$

that still satisfies the condition $5v(12) + v(23) = 4v(123)$.

The set of aspirations has the form

$$(10-\lambda, \lambda, 14-\lambda) \quad \text{where} \quad 0 \leq \lambda \leq 7 ,$$

and the aspiration core

$$AC = (3,7,7) ,$$

what redefines

$$\tilde{v}(123) = 17 .$$

Now $Sh_1(\tilde{v}) = 13/3$

$Sh_2(\tilde{v}) = Sh_3(\tilde{v}) = 19/3$

and

$$Sh(\tilde{v}) = (13/3, 19/3, 19/3) \quad .$$

Obviously the vector $Sh(\tilde{v})$ is not included in the set of aspirations.

This simple example of reduced 3-person games already shows that it occurs only in very special games, that the Shapley value of $\tilde{v}$ is included in the set of aspirations of the game $(S, v/_S)$.

Conclusions

The main purpose of this paper was to get some more insight in the difference of Shapley-value and expected value as a solution concept of the extensive-form-game proposed by A. Rapoport.
The main features responsible for the difference between expected value and Shapley-value are captured in our first chapter.

The difference is reflected very well by some examples of 5-person and 6-person games, that would not have been possible without the aid of a computer program.

Although not expected we succeeded in finding some interesting relations between Shapley-value and the set of aspirations. The results are surprising, considering what each of the two solution concepts is modelling. The Shapley-value is per definitionem the payoff a player may expect if the game is played many times, whereas The aspiration approach defines a vector of payoff demands in one single play of the game.

To allow sidepayments in the game in extensive form is an interesting modification of Rapoport's model and seems to be a step toward formalization of what bargaining situations look like in real-life situations.

APPENDIX: source listing and program output

```
PROGRAM wil(input,output,ula);

(* main program for the computation of the
   expected values

variables:
ula      -  textfile
            the results of each game are written
            on file ula
a        -  number of minimal winning coalitions
ms       -  number of players
ar.k     -  coalition that has formed until now
k[i]:=j  -  player j is in the i-th place of coalition k
ar.p     -  probability with which coalition k has formed
sh       -  set of all players
gew[j]   -  j-th minimal winning coalition
gain [i]-   expected value of player i
sum      -  control variable
            the sum of all expected values always equals 1. *)

CONST a=4;
      ms=5;

TYPE  spieler = 1..ms;
      st = SET OF spieler;

VAR ar: RECORD
          k:ARRAY [spieler] OF spieler;
          p:real
        END;

    sh:st;
    ula: text;
    gew:ARRAY[1..a] OF st;
    gain:ARRAY[1..ms] OF real;
    s,j,i,player,index:spieler;
    i1,i2,gr:integer;
    sum:real;


FUNCTION prob(mge:st;I2:spieler):real;

(* The FUNCTION prob determines the probability with
   which player i2 accepts to join coalition mge

variables:
enth     -  determines whether player i2 is included
            in some minimal winning coalition gew[j]
            for j=1,...,a
sieg     -  determines whether player i2 is the last
```

```
             player to join a minimal winning coalition,
             his acceptance probability has then the
             value 1, otherwise 0.
             Indifferent players enter the coalition with
             probability 1.

s,j,help-    auxiliary variables *)

VAR enth,sieg:boolean;
    s:spieler;
    j:0..ms;
    help:real;

BEGIN
   enth:=false;
   j:=0;
   REPEAT
      j:=j+1;
      enth:=(i2 IN gew[j])
   UNTIL enth OR (j=a);
   IF enth THEN
   BEGIN
      sieg:=false;
      j:=0;
      REPEAT
         j:=j+1;
         if (gew[j]-mge=[i2]) THEN
         BEGIN
            prob:=1;
            help:=1;
            sieg:=true
         END
         ELSE BEGIN
            prob:=0;
            help:=0
         END
      UNTIL (j=a) OR sieg
   END
   ELSE BEGIN
      prob:=1;
      help:=1
   END
END;


FUNCTION pivot:spieler;

(* The FUNCTION pivot determines for each formed
   coalition the corresponding winner, that is
   the pivot player.

variables:
gef      -   becomes true as soon as the pivot player
             has been found
```

```
koal,j,k-  auxiliary variables *)

VAR koal:st;
    j:integer;
    k:spieler;
    gef:boolean;

BEGIN
   k:=1;
   koal:=[ar.k[k]];
   gef:=false;
   WHILE NOT gef AND (k<ms) DO
   BEGIN
      k:=k+1;
      koal:=koal+[ar.k[k]];
      FOR j:=1 TO a DO
         IF (gew[j] <= koal) AND NOT gef THEN
         BEGIN
            pivot:=ar.k[k];
            gef:=true
         END
   END
END;


PROCEDURE play (l:spieler;nk:spieler;sa,sp:st;np:spieler);

(* In PROCEDURE play the game is played for every order
   of coalition formation; the PROCEDURE simulates that
   -  players accept to join coalition k in place nk+1
   -  players decline to join coalition k in place nk+1
   -  the last player must join coalition k .

variables:
l        -  last player in the coalition
nk       -  player l is in place nk
sa       -  set of players that are not yet members of
            the formed coalition
sp       -  set of possible players, that is, players
            that may be invited to join the coalition
            in place nk+1
np       -  number of possible players

i,new,ph,pin,j,h -  auxiliary variables *)

VAR i:spieler;
    new : boolean;
    ph,pin :real;
    j: 1..ms;
    h:integer;
```

```
BEGIN
   new:=true;
   IF np>1 THEN
   BEGIN

(* there is more than one possible player to
   enter the coalition in the next place *)

      pin:=ar.p;
      FOR i:=1 TO ms DO
      BEGIN

(* each time the loop index is increased the
   variable new is needed to determine the pivot
   player of the last order of coalition formation *)

         IF (i IN sp) THEN
         BEGIN
            IF  new THEN
               new:=false
            ELSE BEGIN
               IF (ar.p>0) THEN
               BEGIN
                  player:=pivot;
                  gain[player]:=gain[player]+ar.p
               END
            END;
            WITH ar DO
            BEGIN

(* player i accepts to join coalition k in place nk+1 *)

               p:=pin;
               k[nk+1] := i;
               ph := p*(1/np);
               p := ph*prob(sh-sa,i)
            END;
            play (I,nk+1,SA-[I],SA-[I],ms-nk-1);
            IF (ar.p>0) THEN
            BEGIN
               player:=pivot;
               gain[player]:=gain[player] +ar.p
            END;

(* player i declines to join coalition k in place nk+1 *)

            ar.p := ph*(1-prob(sh-sa,i));
            play (1,nk,sa,sp-[i],np-1)
         END
      END
   END
   ELSE BEGIN
      i:=1;
      WHILE NOT (i IN sp) DO
         i:=i+1;
```

```
        ar.k[nk+1] :=i;
        IF (nk<ms-1) THEN
           play (i,nk+1,sa-[i],sa-[i],ms-nk-1)
     END
  END;


  (* main program *)

  (* The main program reads and writes (on file ula)
     the minimal winning coalitions.
     It initializes the SET sh, the ARRAY gain and the
     variable sum.
     For each order of coalition formation the play of
     the game starts in the main program. For every
     player i his expected value, gain[i], is written
     on file ula, as well as the sum of all expected
     values. *)

  BEGIN
     rewrite(ula);
     writeln('  Hallo, auf gehts ');
     FOR j:=1 TO a DO
     BEGIN
        read (s);
        gew[j]:=[s];
        WHILE NOT eoln DO
        BEGIN
           read (s);
           gew[j]:=gew[j]+[s]
        END
     END;
     FOR j:=1 TO a DO
     BEGIN
        write ( ula,'winning coalition=');
        FOR index:=1 TO ms DO
           IF index IN gew[j} THEN write (ula,index:1);
        writeln(ula)
     END;
     SH:=[];
     FOR j:=1 TO ms DO
        sh:=sh+[j];
     for j:=1 to ms do
        gain[j]:=0;
     sum:=0;
     FOR j:=1 TO ms DO
     BEGIN
        WITH ar DO
        BEGIN
           k[1]:=j;
           p:=1/ms
        END;
        play (j,1,sh-[j],sh-[j],ms-1);
```

```
(* Each time the first player of the order of coalition
   formation is changed the pivot player of the last
   order of coalition formation must be determined. *)

       IF (ar.p>0) THEN
       BEGIN
          player:=pivot;
          gain[player]:=gain[player]+ar.p
       END
   END;
   FOR i:=1 TO ms DO
   BEGIN
       writeln(ula,' gain(',i:1,')=',gain[i]:12:10)
       sum := sum + gain[i]
   END;
   writeln (ula,' sum=',sum:5:3)
END.
```

```
WINNING COALITION=1234
WINNING COALITION=245
WINNING COALITION=15
 GAIN(1)=0.3666666667
 GAIN(2)=0.0583333333
 GAIN(3)=0.0000000000
 GAIN(4)=0.0583333333
 GAIN(5)=0.5166666667
 SUM=1.000
```

```
WINNING COALITION=12
WINNING COALITION=13
WINNING COALITION=145
WINNING COALITION=235
 GAIN(1)=0.6500000000
 GAIN(2)=0.1583333333
 GAIN(3)=0.1583333333
 GAIN(4)=0.0166666667
 GAIN(5)=0.0166666667
 SUM=1.000
```

```
WINNING COALITION=123
WINNING COALITION=124
WINNING COALITION=134
WINNING COALITION=145
 GAIN(1)=0.5333333333
 GAIN(2)=0.1000000000
 GAIN(3)=0.1000000000
 GAIN(4)=0.2333333333
 GAIN(5)=0.0333333333
 SUM=1.000
```

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