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Finite Memory Distributed Systems

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Abstract

A distributed system model is studied, where individual agents play repeatedly against each other and change their strategies based upon previous play. It is shown how to model this environment in terms of continuous population densities of agent types. A complication arises because the population densities of different strategies depend upon each other not only through game payoffs, but also through the strategy distributions themselves. In spite of this, it is shown that when an agent imitates the strategy of his previous opponent at a sufficiently high rate, the system of equations which governs the dynamical evolution of agent populations can be reduced to one equation for the total population. In a sense, the dynamics 'collapse' to the dynamics of the entire system taken as a whole, which describes the behavior of all types of agents. We explore the implications of this model, and present both analytical and simulation results.

Keywords
Fixed strategy, prisoner's dilemma, Fokker-Plank, distributed system

JEL Classification
C61, C73

Comments

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Contents

1 Introduction

Distributed models of heterogeneity, in which many agents engage in repeated play against each other in a spatial or network environment, are pervasive in the social sciences and particularly in Economics (see e.g. Brock and Durlauf [2002, 2003], Kirchkamp and Nagel [2001], Kirchkamp [2000], and Epstein and Axtell [1996] for a small subset of such models, and Tesfatsion and Judd [forthcoming 2006] *Handbook of Computational Economics Vol. 2* for an overview). These models are of interest because they allow complicated behavior to emerge, as many different types of agents can interact with each other in a complex variety of ways over time. As this behavior is usually too complicated to admit a closed form, analytical solutions of a model's quantitative and/or qualitative properties are often obtained by appealing to the ergodicity of the system, and by resorting to numerical, Monte Carlo simulations.

Recently, Dorofeenko and Shorish [2005] have shown how to take one class of models of this type and formulate a close analytical approximation of its behavior. This approach takes as fundamental not the individual players, but rather the probability distribution of the types of players over some area of interaction. By appealing to this level of abstraction, analytically quantifiable conclusions may be generated even though the underlying discrete dynamics are very complicated. Our prototypical example demonstrated that the demographic prisoner's dilemma model of the form given in Epstein (1998) admits an analytical representation, and that the stable spatial structures found in Epstein's paper lose their stability and become metastable in the continuous space environment. The structure of the continuous approximation employed allowed for both analytical and numerical simulations on the approximation to be performed, which served as a usual benchmarking procedure when compared to the original model.

One feature of the demographic prisoner's dilemma as formulated above is that agents are endowed with a fixed strategy type. That is, they do not adjust their game strategy in the face of prior (or expected future) experience. Although agents are heterogeneous with respect to both playing location and to 'wealth' (a measure of their aggregate payoff from all previous games played), once born they inherit the game strategy which was played by their forbearers. While this assumption is certainly useful from a technical standpoint, it does render the economic agents more as ‘zero-intelligence’ traders rather than as players capable of adaptive behavior, which is more realistic. This paper addresses the issue by analyzing an extension of the demographic prisoner’s dilemma which allows individual agents to switch strategies based upon the outcome of previous play. That is, agents are allowed to adapt their behavior according to what they have previously encountered.

The analysis proceeds along similar lines as in Dorofeenko and Shorish [2005], but the introduction of adaptive behavior changes the continuous approximation in several ways. Thus, although the abstract formulation as a partial differential equation system remains the same, solving the system analytically requires a new approach. Using techniques similar to those found in e.g. Krall and Trivelpiece [1986], we approximate the dynamics of the system by noting that, provided there is ‘enough’ memory of what was previously played, the different strategies available to the agents tend to evolve in the same way. That is, we can approximate the multi-dimensional system of many strategies by a single relation, which (in a sense to be made precise) summarizes the ‘total’ movement of the entire system. Within this approximation’s domain of applicability (viz., models which contain two strategies), the resulting dynamical equation can then be numerically simulated.

In addition, we present (as yet) speculative work on obtaining qualitative results of the resulting approximated system, by considering the evolution of an entropy measure from information theory and the physical sciences. This measure, known as ‘negentropy’, could provide future analytical conclusions to be drawn about the approximated system, and hence provide a unifying framework for both our approach and for previous simulation work performed with e.g. cellular automata and other discrete systems.

The paper is organized as follows. Section 2 introduces the model, and derives the underlying system evolution (or ‘law of motion’) for a finite set of strategies. Section 3 describes the model approximation techniques used, focusing upon the single-variable approximation applied to two strategies, and also introduces speculative work on deriving a qualitative stability measure from the negentropy concept. Section 4 then presents numerical simulation results for the model, and Section 5 concludes. The validity of the single-variable approximation given in Section 3 is proven in Appendix A, while Appendix B summarizes the parameter selection and initial conditions for the numerical simulations.

2 The Model

The model is an extension of the demographic prisoner's dilemma model of Epstein [1998], modeled as a continuous stochastic process over a circle as in Dorofeenko and Shorish [2005]. The model is an abstraction of an environment where individual agents move randomly over a grid, and interact with other agents by playing a prisoner's dilemma game. Conditional upon the outcome of the game, agents either increase or decrease their wealth according to the game's payoff. If wealth falls below some threshold, the agent dies, whereas if wealth is high enough, the agent has some probability of cloning itself and generating another agent of the same 'type', i.e. generating another agent who will play the same strategy. One of the interesting questions one may ask about such a model is whether after a large number of interactions has passed, one of the prisoner's dilemma strategies (measured as a proportion of the population) is dominant, and whether or how the various parameters of the model (such as the spatial extent of the grid, the payoffs to the prisoner's dilemma game, the probability of generating a 'clone', among others) influence this proportion.

The underlying model is thus a finite state automata, which generates a complicated system of interactions. The questions listed above are, then, questions about aggregate features of the economy than about the characteristics of any given individual. Common to models of such type, the 'primitives' are the behavior of the agent, the rules of interaction, and any initial conditions of e.g. wealth or other endowments.

For brevity we refer the reader to Dorofeenko and Shorish [2005] for an in-depth discussion of the approximation method used to move from the discrete finite-state system to the continuous approximation used. We focus instead upon this approximation and its extension to a model in which the individual agents can adapt their strategy according to the outcomes of previous game encounters, i.e. to the strategies of other players previously met.

The model is defined over an interval of the real line with endpoints identified, i.e. over a circle. Time is continuous, and at each time there exists a population density of agents. This population density depends upon a *strategy* $s \in \mathcal{S}$,[1] where $\mathcal{S}$ is a finite set of strategies available to each player, a *wealth level* $w \in \mathbb{R}_+$ held by a player, and finally a *location* $x \in [-L/2, L/2]$ (endpoints identified), which indicates where on the circle an agent lies:

Definition 1. *A population density function for those agents with strategy s, wealth level w and location x at time t is a function $f(s, w, x, t)$ such that given a subset of strategies $\mathcal{S}_1 \subset \mathcal{S}$, an interval $[a, b] \in [-L/2, L/2]$ and a set of wealth levels $[\underline{w}, \bar{w}]$, the number of agents which possess strategy $s \in \mathcal{S}_1$, location $x \in [a, b]$ and wealth level $w \in [\underline{w}, \bar{w}]$ at t is*

$$\mathbb{N}(t) = \int_{[-L/2,L/2]} \int_{\underline{w}}^{\bar{w}} \int_a^b f(s, w, x, t) dx dw ds. \tag{2.1}$$

Agents may live or die in this environment. There are two ways for an agent to die. First, there exists a probability mass $d > 0$ such that $df(s, w, x, t)$ represents the mass of agents at (s, w, x, t) who die at time t. Second, if an agent's wealth drops to zero, it is assumed to be unable to survive and so dies.

We also assume that there exists a rate of diffusion in the wealth space of the population distribution $f(s, w, x, t)$. We formalize this diffusion by defining a diffusion parameter M, which measures how fast wealth diffuses from agents to their closest neighbors with lower wealth. The diffusion in the wealth space in our model generates a non-uniform wealth distribution between agents.

If an agent survives, it engages in random play with a neighbor. For each strategy pair (s, s'), $s, s' \in \mathcal{S}$ we associate a **payoff** $\nu_{ss'}$ from playing the game, where the agent plays

[1] In the analysis and simulations, agents play the prisoner's dilemma game upon interaction, so the strategy set is given by $\mathcal{S} = \{c, d\}$, where c stands for 'cooperate' and d 'defect'. We keep to the more general formulation in setting up the model, however.

strategy s and the opponent plays s'. Note that the first subscript denotes the recipient of the payoff–hence $\nu_{ss'}$ is the payoff for the agent who plays strategy s when an opponent plays strategy s'. Generally, $\nu_{ss'} \neq \nu_{s's}$.

Immediately after playing a game the agent may randomly clone itself by sacrificing a level of wealth and adding to the population mass of its own strategy–this level of wealth lost is also the threshold wealth for cloning, so that agents without at least this wealth level do not clone. The wealth lost is fixed at an exogenous value w_1, and for those agents with at least w_1 in wealth, the cloning probability is set exogenously to $\nu_b \in [0, 1]$.

Once game play and cloning have been performed, the agent moves to a nearby location, and the process repeats itself. This causes diffusion of the population density in location space, in the same way as having zero wealth causes diffusion in the wealth space. For simplicity, we normalize the location diffusion parameter to one.

All newborn agents have their initial wealth levels drawn from a distribution defined over the probability density function $g(w)$. In order to keep the level of wealth in the model stationary, we must assume that:

$$w_1 := \bar{w} = \int_0^\infty w(g(w)dw. \tag{2.2}$$

If $w_1 < (>) \bar{w}$, where $\bar{w}$ is the mean wealth level of the population, then the cloning process itself will add to (subtract from) the wealth level of the population.

Following Dorofeenko and Shorish [2005] the continuous approximation outlined above is only valid when the time of interaction between agents (i.e. the time it take them to play the prisoner's dilemma game) is effectively instantaneous when compared with the time of a 'period' δt. This ensures that for any two-player interaction, the densities of the strategies before and after the interaction remain essentially the same. This features is also what allows us to model what it means for'adaptive play' in this framework.

Recall that in a discrete setting, an individual agent may adapt their strategy if they condition their current (and/or future) play upon those strategies which were observed as being played by previous competitors. Using the time assumption given above, we infer that in the continuous setting given here, adaptive play means that the population density function of one strategy will be conditioned (within the time interval δt) upon the population density functions of all other strategies. More properly, this means that the change in the population density function of one strategy is dependent upon the level of the population densities of the other strategies in the population.

For exposition a few further definitions are in order. First denote the total mass of agents with strategy s at location x to be

$$n(s,x,t) := \int_0^\infty f(s,w,x,t)dw, \tag{2.3}$$

while the mass of those (s,x) agents with wealth level at least equal to w_1, i.e. those agents which have enough wealth to clone themselves is given by

$$n_1(s,x,t) := \int_{w_1}^\infty f(s,w,x,t)dw. \tag{2.4}$$

In addition, it is also useful to define the residual of the mass of agents located at a point x, using the suggestive notation 'e' for 'empty':

$$n_e(x,t) := 1 - \sum_{s\in\mathcal{S}} n(s,x,t). \tag{2.5}$$

Finally, we define for every strategy s, the *net wealth transfer* $V(s,t)$ of the strategy at time t. This wealth transfer incorporates the wealth gain due to the payoff from interacting with other agents (playing the game), minus any loss of wealth due to cloning (which costs the threshold wealth level w_1). This net wealth transfer, again expressed as a population aggregate, is:

$$V(s,x,t) := \sum_{s'\in\mathcal{S}} \nu_{ss'} n(s',x,t) - \nu_b w_1 n_e(x,t) \mathbf{1}_{w\geq w_1}, \tag{2.6}$$

where $\mathbf{1}_{w\geq w_1}$ is an indicator function, equal to one if wealth w is greater than the threshold wealth w_1, and zero otherwise.

Armed with these definitions, we may specify the law of motion of the population densities of this economy (for exposition we shall often suppress the dependence on (x,t) and shorten $f(s,w,x,t)$ to f_s, $n(s,x,t)$ to n_s, $n_1(s,x,t)$ to n_{1s} and $V(s,x,t)$ to V_s when doing so will cause no confusion):

$$\frac{\partial f_s}{\partial t}+\frac{\partial}{\partial w}\left(V_s f_s\right)-M\frac{\partial^2 f_s}{\partial w^2}-\frac{\partial^2 f_s}{\partial x^2}=\nu_b n_e n_{1s} g(w)-df_s+n_s\sum_{s'\in\mathcal{S}}\lambda_{s's}f_{s'}-f_s\sum_{s'\in\mathcal{S}}\lambda_{ss'}n_{s'}. \quad (2.7)$$

We shall interpret this system (and define the terms $\lambda_{ss'}$) shortly. In order to solve this system, both initial and boundary conditions must be specified. The initial condition for the population density of strategy s is

$$f(s,w,x,0):=f_0(s,w,x), \quad (2.8)$$

where f_0 is the initial population density of strategy s.[2]

There are also boundary conditions for wealth and location. The wealth boundary conditions ensure that no one possesses exactly zero wealth, and unbounded wealth is also impossible:

$$f(s,0,x,t)=0\forall x,t;\ \lim_{w\to\infty}f(s,w,x,t)=0\forall x,t. \quad (2.9)$$

The location boundary conditions ensure that the population densities are periodic over the circle, so that both the values and the first derivatives agree everywhere:

$$f(s,w,-L/2,t)=f(s,w,L/2,t);\ \left.\frac{\partial f_s}{\partial x}\right|_{x=-L/2}=\left.\frac{\partial f_s}{\partial x}\right|_{x=L/2}\forall w,t. \quad (2.10)$$

[2]In the numerical simulations it was assumed that for each strategy the joint density $f_0(s,w,x)$ over wealth and space could be decomposed into the product of two marginal densities, one defined over wealth and one defined over space. See Appendix B.

2.1 Interpretation of the PDE representation

The PDE representation of the population density $f(s, w, x, t)$ is similar to that studied in Dorofeenko and Shorish (2005), but it contains a few notable differences. Most importantly, the last two terms of (2.7) together describe the 'memory' of the population from previous play, and $\lambda_{ss'}$ is a coefficient used to indicate how strong memory plays a role in deciding which strategy to choose. For example, the first of these terms, given by

$$n(s, x, t) \sum_{s' \in \mathcal{S}} \lambda_{s's} f_{s'}, \tag{2.11}$$

shows how many new agents have decided to switch to strategy s from strategy s' based upon previous play. The term $\lambda_{s's} f_{s'}$ is the proportion of the density of those agents playing $f_{s'}$ who would, if facing strategy s, decide to switch to s. The total mass of such agents who actually switch are those who actually face agents playing strategy s, which is given by $n(s, x, t)$. Thus, the total mass of agents switching from s' to s is simply the product of $n(s, x, t)$ and $\lambda_{s's} f_{s'}$. Summing over all possible strategies $s' \in \mathcal{S}$ yields (2.11).

Similarly, the second term indicates how much population mass is lost from strategy s, where $\lambda_{ss'} f(s, w, x, t)$ shows how many strategy s players have the potential to switch to s', and the product $\lambda_{ss'} f(s, w, x, t) n(s', x, t)$ indicates how many actually switch. Summing over all possible strategies s' yields the desired relation, which is subtracted from the population mass of strategy s.

The coefficients $\lambda_{ss'}$ thus measure the importance of 'memory' in determining which strategy adds to (or subtracts from) its population density. These coefficients depend upon the payoffs of the game being played, and are by no means stationary in general. However, for ease of analysis we assumed that they take on fixed values in time, so that the problem remains stationary. In addition, we presume that the matrix of λ's is symmetric, so that $\lambda_{ss'} = \lambda_{s's}$. This matrix may be thought of as a matrix of 'switching rates' or 'coupling coefficients' linking one strategy set to another–note finally that own-switching carries a coefficient of zero, so that $\lambda_{ss} = 0 \ \forall s \in \mathcal{S}$.

3 Model Approximation

The term

$$\phi_s := \sum_{s' \in \mathcal{S}} (\lambda_{ss'}(n_s f_{s'} - n_{s'} f_s)) \tag{3.1}$$

has an analogous relationship to a type of friction in fluid dynamics. In particular, if the coefficients $\lambda_{ss'}$ are large enough, they overwhelm the dynamics of (2.7) and the problem becomes one of defining the evolution of Markov switching between strategies independent of direct game payoffs, cloning and death rates, etc. It may be the case that this switching between strategies continues until a stationary solution is found (see Section 3.2 on 'Negentropy' and our speculative efforts to prove that such a solution is stable under this dynamical system). If this is the case, then each strategy mass f_s is a stable proportion of a 'total density' $f := \sum_{s\in\mathcal{S}} f_s$, such that

$$f_s = \frac{n_s}{n} f + \delta f_s,\ |\delta f_s| \ll f\ \forall s, \tag{3.2}$$

where $n = n(x,t) := \sum_{s\in\mathcal{S}} n_s$.[3]

We show in Section 3.1 below that this simplification can in fact be performed if the system has only two strategies. Anticipating what follows, let us continue and show how the system can be reduced to a single law of motion.

By aggregating (2.7) over s and substituting (3.2) into the result, and noting that

$$\sum_{s\in\mathcal{S}} \phi_s = 0,$$

we find that

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial w}(Vf) - M\frac{\partial^2 f}{\partial w^2} - \frac{\partial^2 f}{\partial x^2} = \nu_b n_e n_1 g(w) - df, \tag{3.3}$$

where $V := \frac{1}{n}\sum_{s\in\mathcal{S}} n_s V_s$ is the average net gain or loss from game-playing in the economy, since V may also be written as $V = \frac{1}{n}\sum_{s,s'\in\mathcal{S}} \nu_{ss'} n_s n_{s'}$.

[3]For future reference, we also define $n_1 = n_1(x,t) := \sum_{s\in\mathcal{S}} n_{1s}$ as the total density of all agents who have enough wealth to clone themselves.

The dynamics given by (3.3) determine the law of motion of the entire system, via the approximation given by (3.2). This problem is easier to solve than the original system (2.7). Approximation (3.2) also implies that the 'spatial densities' n_s also obey certain restrictions:

$$\frac{\partial n_s}{\partial t} - \frac{\partial^2 n_s}{\partial x^2} = \nu_b n_e n_1 - d n_s - M \left.\frac{\partial f_s}{\partial w}\right|_{w=0}. \tag{3.4}$$

Since from (3.2) we know that

$$n_{1s} = \frac{n_1}{n} n_s, \tag{3.5}$$

$$\left.\frac{\partial f_s}{\partial w}\right|_{w=0} = \frac{n_s}{n} \left.\frac{\partial f}{\partial w}\right|_{w=0}, \tag{3.6}$$

we may substitute these relations into (3.4) to yield

$$\frac{\partial n_s}{\partial t} - \frac{\partial^2 n_s}{\partial x^2} = \left[\nu_b n_e \frac{n_1}{n} - d - M \frac{1}{n} \left.\frac{\partial f}{\partial w}\right|_{w=0}\right] n_s. \tag{3.7}$$

This relation may also be expressed in terms of the total spatial density n:

$$\frac{\partial n}{\partial t} - \frac{\partial^2 n}{\partial x^2} = \nu_b n_e n_1 - \left[d + M \frac{1}{n} \left.\frac{\partial f}{\partial w}\right|_{w=0}\right] n. \tag{3.8}$$

Defining $\alpha_s = \alpha(s, x, t) := n(s, x, t)/n(x, t)$ we may finally express the above relations as:

$$\frac{\partial \alpha_s}{\partial t} - \frac{\partial^2 \alpha_s}{\partial x^2} - 2\frac{\partial \alpha_s}{\partial x}\frac{\partial}{\partial x} \ln n = 0. \tag{3.9}$$

Equations (3.2), (3.3) and (3.7) thus comprise the complete system for the total density approximation.

3.1 Two Strategy Approximation

It remains to be shown that the approximation given in (3.2) is actually valid. To do so, we restrict our attention to systems with only two strategies (such as the canonical repeated prisoner's dilemma game) and define a domain of applicability for the approximation.

First, we define the conditional probability densities at a given spatial point x, i.e. the 'normalized' probability density for each strategy $s = c, d$:

$$\varphi_s(x, w, t) := \frac{f_s(x, w, t)}{n_s(x, t)}. \tag{3.10}$$

Using this definition, the general system (2.7) may be rewritten in the form:

$$\frac{\partial \varphi_s}{\partial t} + \frac{\partial}{\partial w}(V_s \varphi_s) - M\frac{\partial^2 \varphi_s}{\partial w^2} - \frac{\partial^2 \varphi_s}{\partial x^2} - 2\frac{\partial \varphi_s}{\partial x}\frac{\partial}{\partial x}\ln n_s = \tag{3.11}$$
$$\nu_b n_e \beta_s g(w) - \left(\nu_b n_e \beta_s + M\left(\frac{\partial \varphi_s}{\partial w}\right)_{w=0}\right)\varphi_s + \sum_{s' \in S} \lambda_{s's} n_{s'} \varphi_{s'} - \varphi_s \sum_{s' \in S} \lambda_{ss'} n_r,$$

where

$$\beta_s = \frac{n_{1s}}{n_s} = \int_{w_1}^{\infty} \varphi_s \, dw$$

is the mass of type s agents who can clone themselves relative to the total number of agents of that strategy type.

As in the unmodified law of motion (2.7), it is the relative size and frequency of the 'memory terms' $\lambda_{ss'}$ which determines how differently the normalized densities $\varphi_s, \varphi_{s'}$ behave. If there are only two strategies, in fact, the difference between the two densities φ_c and φ_d can be expressed as a difference term which fluctuates on a scale faster than the mean transition time $\tau \sim \lambda^{-1}$ between strategies. In other words, the behavior of the two densities can be captured (to this degree of precision) by the dynamics of a 'total density' function, denoted φ, and a 'difference density' denoted $\tilde{\varphi}$:

$$\varphi := \frac{n_c}{n}\varphi_c + \frac{n_d}{n}\varphi_d,$$
$$\tilde{\varphi} := \varphi_c - \varphi_d, \tag{3.12}$$

with $n(x, t) := n_c + n_d$ representing the total mass of all individuals at (x, t), regardless of strategy.

Notice that if $\tilde{\varphi}$ varies symmetrically (so that there is no bias) and quickly (so that the law of large numbers applies) during the mean transition time τ, then for all intents and purposes it vanishes during the time scale which drives the dynamics. This means that over this time scale, the dynamics of φ_c and φ_d, and hence of φ itself, are essentially the same, i.e. $\varphi_c \simeq \varphi_d \simeq \varphi + O(\lambda^{-1})$, and that all three distributions follow the law of motion governed by (2.7). The proof that $\tilde{\varphi}$ vanishes to order τ is given in the Appendix.

The resulting dynamical system after imposing this approximation is

$$\frac{\partial}{\partial t}(n\varphi) + \frac{\partial}{\partial w}(Vn\varphi) - \left(M\frac{\partial^2}{\partial w^2} + \frac{\partial^2}{\partial x^2}\right)(n\varphi) = \nu_b n_e \beta n g(w) - dn. \tag{3.13}$$

This system is the two strategy approximation of the system (2.7), and is equivalent to (3.3) with restrictions (3.7). It is this approximation that is used to derive the simulations of the system in Section 4.

3.2 Negentropy

It must be shown that the approximation (3.2) of the dynamics of the system as represented by the general formulation (2.7) is stable, that is, for an open and dense set of initial conditions, the system (2.7) will converge to a stationary distribution given by 3.2). Similar to the stability properties of a Lyapunov function, we introduce a 'negentropy' function $H(t)$ for this system. Such a function $H(t)$ can be obtained under the assumption of the symmetry of the coefficients $\lambda_{ss'} = \lambda_{s's}$.

The negentropy function is defined as:

$$H(t) = \int_0^\infty \int_{-L/2}^{L/2} \sum_{s\in\mathcal{S}} G(f_s)\, dwdx \equiv \int \sum_{s\in\mathcal{S}} G(f_s)\, dwdx, \tag{3.14}$$

where the function $G(f)$ satisfies the conditions:

$$G(0) = G'(0) = 0, (G(f), G'(f), G''(f)) \geq 0, \forall f \geq 0 \tag{3.15}$$

The simple example of such a function is $G(f) = f^\alpha$, $1 \leq \alpha$. The more conventional example: $G(f) = (1+f)[ln(1+f) - 1]$.

Multiplying each of equations (2.7) by $G'(f_s)$, integrating by w and x over the solution domain $0 < w < \infty$, $-L/2 < x < L/2$ and taking a sum over the index s, we obtain:

$$\dot{H} = \nu_b \left(w_1 \int_{-L/2}^{L/2} n_e \sum_{s\in\mathcal{S}} \left[f_s(w_1) G'(f_s(w_1)) - G(f_s(w_1))\right] dx + \int n_e g(w) \sum_{s\in\mathcal{S}} n_{1s} G'(f_s)\, dwdx \right)$$
$$- \int \sum_{s\in\mathcal{S}} \left[M \left(\frac{\partial f_s}{\partial w}\right)^2 + \left(\frac{\partial f_s}{\partial x}\right)^2 \right] G''(f_s)\, dwdx - d \int \sum_{s\in\mathcal{S}} f_s G'(f_s)\, dwdx \qquad (3.16)$$
$$- \frac{1}{2} \int \sum_{s,s'\in\mathcal{S}} \lambda_{ss'} n_s n_{s'} \left[G'(f_s) - G'(f_{s'})\right] \left(\frac{f_s}{n_s} - \frac{f_{s'}}{n_{s'}}\right) dwdx$$

The first term in parentheses is non-negative, because $fG'(f) - G(f) = \int_0^f fG''(f)df \geq 0$. However, the sign of the frictional term is not defined.

Thus, the birth process yields the non-negative negentropy production, while all the other processes excluding the last one yield the non-positive negentropy production. This function can be used in the total density approximation, where the frictional term is absent.

We may introduce the other negentropy function using $G(f/n)$ in (3.14) instead of $G(f)$. Then the frictional term produces the non-positive contribution:

$$-\frac{1}{2} \int \sum_{s,s'\in\mathcal{S}} \lambda_{ss'} n_s n_{s'} \left[G'\left(\frac{f_s}{n_s}\right) - G'\left(\frac{f_{s'}}{n_{s'}}\right) \right] \left(\frac{f_s}{n_s} - \frac{f_{s'}}{n_{s'}}\right) dwdx \qquad (3.17)$$

but the sign of contribution of some other terms (namely, spatial diffusion and birth rate terms) is not defined in that case. Future analysis may allow us to use this negentropy function to derive analytical stability results for the total density approximation of Section 3.1.

4 Simulations

The dynamical system given by (2.7) can be simulated once the parameters of the model have been specified. These simulations used a discretized grid to approximate the PDE system in the wealth and location space. For illustrative purposes, the simulation results presented here are based upon one specification of the parameter set, the values of which are defined in the Appendix. These results, while not generalizable, nonetheless present the reader with some mode of comparison between an environment where memory is relevant for distributed systems, and an environment where memory is absent.

The simulations of the full system with memory were compared to a 'benchmark' system without memory. As shown in Dorofeenko and Shorish [2005] the environment without memory admits a metastable cooperative structure, which later decays to zero. Defectors, while initially profiting from the cooperators in a similar fashion to 'predatory-prey' interaction, eventually also die out as cooperators become ever more scarce.

In order to remain within the limits of the single-component approximation, a scale value for the memory switching term $\lambda = \lambda_{cd} = \lambda_{dc}$ was chosen which demonstrates how the cooperator and defector probability densities converge to the same 'total' density after a characteristic time period at the order of $1/\lambda$. For the simulations, this memory term was set to $\lambda = 20$. For the benchmark model without memory, $\lambda = 0$.

Figures 1 and 2 compare the long-run evolution of the system with and without memory, for the spatial densities of cooperators (diamonds, in red) and defectors (stars, in blue).[4] As can be seen, the dynamical systems with and without memory behave in a similar fashion–given a small initial concentration of cooperators in a uniform 'sea' of defectors, there is first the development of a tight central cluster of cooperators surrounded by defectors on the sides. As time progresses, the cooperator cluster waxes and then wanes, and both densities eventually converge to the stationary state where all agents die out.

Judging from the spatial densities alone one would conclude that the two systems

[4]Note that in order to display these animations and the animation of Figure 6 correctly, this PDF document must be opened on a system which can handle the AVI movie format.

(Loading density evolution...)

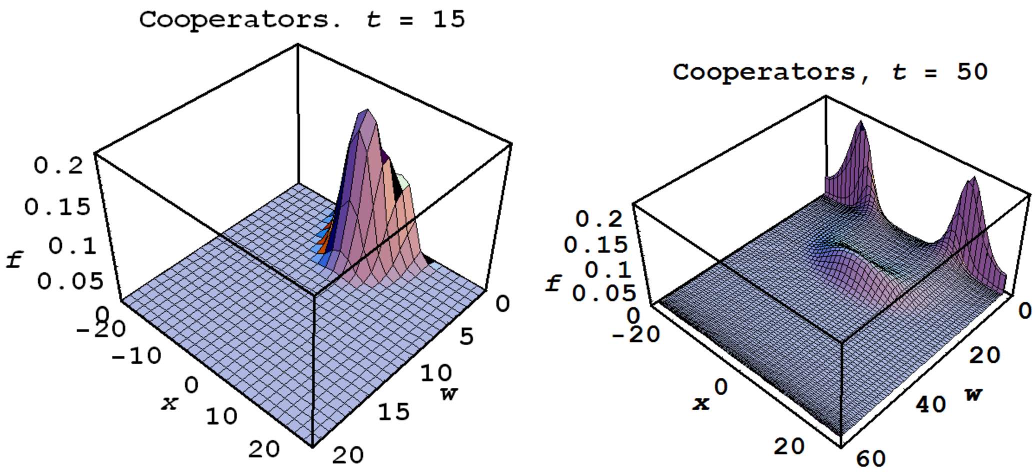
Figure 1: Density Evolution Without Memory ($\lambda = 0$)
Play NormalPlay Slow Pause/Resume

(Loading density evolution...)

Figure 2: Density Evolution With Memory ($\lambda = 20$)
Play NormalPlay Slow Pause/Resume

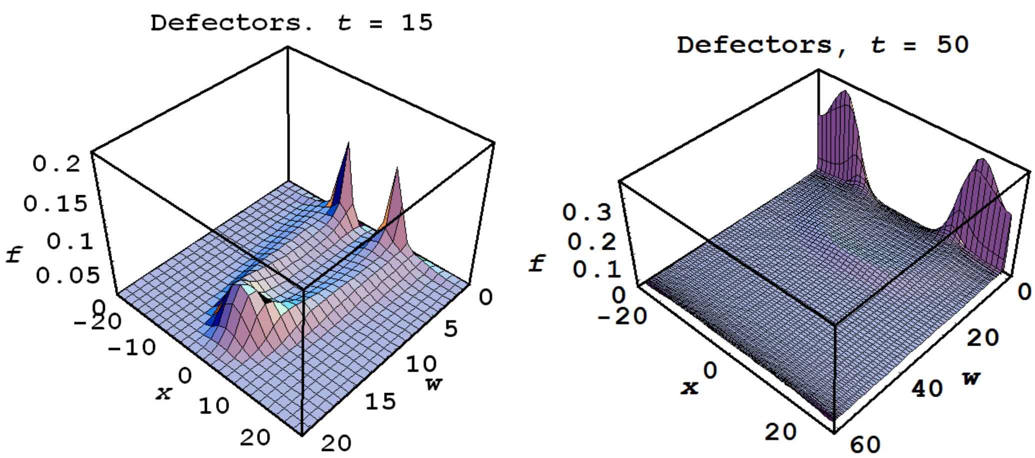
are roughly equivalent, and the existence of memory does not substantially alter the conclusions that one would draw in the absence of the switching terms given by λ–the main difference is in speed, as the system with memory takes longer to dissipate. But an examination of the probability density functions over both the location and wealth space tell a different story. Although it is true that the long run time evolution of the system is identical with and without memory, the densities evolve in a different way.

Figure 3 for cooperators and Figure 4 for defectors each compare the probability densities for the system without memory to the system with memory. Although the time evolution for systems with and without memory appear similar, they still exhibit differences along their respective time paths. For example, Figure 3(b) indicates that the cooperator density with memory has ‘tails’ at relatively low wealth levels, which differs from the density without memory given in Figure 3(a). In addition, comparing Figure 3(b) with Figure 4(b) shows that at these low wealth levels, the probability density of cooperators looks very similar to the probability density for defectors in the environment with memory. That is, there exists a tendency for the probability mass of cooperators to cluster at points in common with the probability mass of defectors when memory is present.



(a) Cooperator Density, $\lambda = 0$ (b) Cooperator Density, $\lambda = 20$

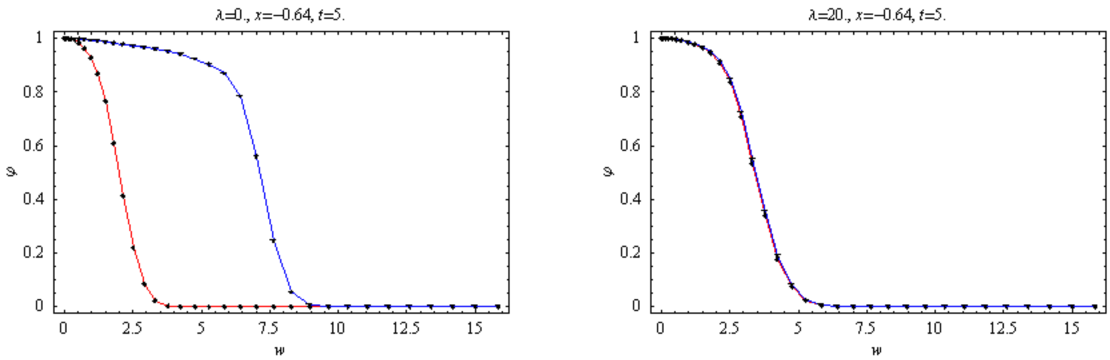
Figure 3: Cooperator Densities Without and With Memory



(a) Defector Density, $\lambda = 0$ (b) Defector Density, $\lambda = 20$

Figure 4: Defector Densities Without and With Memory

This tendency is due to the fact that after a certain period of time, the 'relaxation time' of the system, both normalized probability densities φ_c and φ_d converge toward each other. Figure 5 compares the behavior of these normalized densities at a given location in space, as they evolve over a range of wealth values. Figure 5(a) shows that even after a sufficiently long period of time, the normalized densities without memory remain different from each other–indeed, this difference remains throughout the evolution of the system, until both populations die out. This is due to the fact that when the switching rate $\lambda = 0$, i.e. when memory effects are absent, the characteristic relaxation time is $1/0 = \infty$. It takes an infinitely long period of time for the normalized densities to converge to each other.



(a) Normalized Density, $\lambda = 0$

(b) Normalized Density, $\lambda = 20$

Figure 5: Differences Between Normalized Densities, Without and With Memory

By contrast, Figure 5(b) demonstrates that in the environment with memory, after sufficient time has passed the normalized densities converge toward each other, and a one-dimensional system emerges. The amount of time which must pass, the relaxation time, is here on the order of $1/20 = 0.05$, and the observed time to convergence is around $\tau \simeq 0.2$. This convergence is also demonstrated in Figure 6, which shows an animated time evolution of the normalized densities over wealth–around the characteristic time of $\tau \simeq 0.2$, the two densities converge to the same 'total density' representation.

(Loading density evolution...)

Figure 6: Evolution of Normalized Density, $\lambda = 20$
Play NormalPlay Slow Pause/Resume

5 Conclusion

In using the PDE representation to explore multiple agent interaction with memory, we have relied upon the observation that when examining aggregate behavior, it is not vital to associate memory with a specific individual, or collection of individuals. Rather, the evolution of the likelihood of adopting one or another strategy can be influenced by a 'collective memory' that affects the population as a whole.

This interpretation of memory allows one to specify a continuous approximation to the underlying discrete system, using standard techniques in a novel way. In this fashion, we have shown how memory can be incorporated into the framework introduced in Dorofeenko and Shorish [2005]. In addition, the introduction of memory also transfers characteristics of one strategy to the population of the other–and if this rate of transfer is high enough, the distinction between the evolution of the two strategies over time becomes blurred. Even though each strategy still carries with it payoffs which are different from the other strategy, the system as a whole behaves, in some sense, as though it were governed by a single composite strategy with characteristics of both.

It is unclear whether the interpretation of memory as we have defined it is the most efficient method of applying continuous approximation methods to discrete systems. It is also not known how the global, asymptotic properties of the system behave (although work on negentropy is providing one possible direction for future research). We can argue, at least, that simulation results with some form of analytical representation of the underlying system, as we have presented here, are more useful to the researcher than such simulations results presented as a 'black box' outcome. Being able to formulate a memory model in terms of a system of probability densities sheds new light on the effects we might expect to see in real-world interactions of this sort. After all, it is very unlikely that real-world interactions can be measured at the individual level–rather, it is the *statistics* of a population of agents which proves to be most relevant to the researcher. Explaining the evolution and dynamics of these statistics, then, is of prime importance when attempting to understand the behavior of a complicated collection of heterogeneous agents.

Future research is now focusing upon developing this paradigm for use in a wide class of interaction models over a discrete space–in particular, applying our technique to endogenous network formation seems like a natural next step, as until now the local interaction properties have been specified *ex ante*. In addition, we shall free ourselves from one particular game theoretic specification (the repeated prisoner's dilemma problem) and attempt to examine more complicated interaction and payoff environments. In this we shall most likely need to draw upon methods of analysis which belong to the 'mesoscopic' scale as defined in the natural sciences, whereby individual interactions are too numerous to completely specify, but the system taken as whole is still too small for population aggregation to provide the dynamics of interest.

At the present moment, our tool is still of limited use–it may be likened to a rather dim and wavering flashlight beam that we use to tread, step by step, upon a darkened path strewn with those small problems we can illuminate. The goal is to see not just this path, but the entire landscape of complex, dynamic, decentralized heterogenous agent interaction models, in a way which allows us to make meaningful analytical statements about the global and aggregate properties of the system. It is true that to achieve this goal, we may very well need a lighthouse–but our small flashlight may have scaling properties which have hitherto been unexploited.

Appendices

A Perturbation Expansion for the Two Strategy Approximation

The dynamical system (2.7) is first rewritten in terms of φ and $\tilde{\varphi}$), and expressed as expansion terms in those perturbations which are of smaller order than the relaxation time:

$$\frac{\partial}{\partial t}(n\varphi) + \frac{\partial}{\partial w}\left(Vn\varphi + \frac{n_c n_d}{n}\left(V_c - V_d\right)\tilde{\varphi}\right) - \left(M\frac{\partial^2}{\partial w^2} + \frac{\partial^2}{\partial x^2}\right)(n\varphi) = \nu_b n_e \beta n g(w) - dn\varphi, \tag{A.1}$$

$$\frac{\partial \tilde{\varphi}}{\partial t} + \frac{\partial}{\partial x}\left(2\tilde{\varphi}\frac{\partial}{\partial x}\ln\left(\frac{n_c n_d}{n}\right) - \frac{\partial \tilde{\varphi}}{\partial x}\right) = -\lambda n\tilde{\varphi} + h, \tag{A.2}$$

$$\begin{aligned} h = {} & 2\tilde{\varphi}\frac{\partial^2}{\partial x^2}\ln\left(\frac{n_c n_d}{n}\right) - \frac{\partial}{\partial w}\left(\frac{n_c V_d + n_d V_c}{n}\tilde{\varphi} + \left(V_c - V_d\right)\varphi\right) + M\frac{\partial^2 \tilde{\varphi}}{\partial w^2} + \\ & 2\frac{\partial \varphi}{\partial x}\frac{\partial}{\partial x}\ln\frac{n_c}{n_d} + \nu_b n_e\left(\tilde{\beta}g(w) - \tilde{\beta}\varphi - \beta\tilde{\varphi} + \frac{n_c - n_d}{n}\tilde{\beta}\tilde{\varphi}\right) + \\ & M\left(\frac{n_c - n_d}{n}\tilde{\varphi}\left(\frac{\partial \tilde{\varphi}}{\partial w}\right)_{w=0} - \tilde{\varphi}\left(\frac{\partial \varphi}{\partial w}\right)_{w=0} - \varphi\left(\frac{\partial \tilde{\varphi}}{\partial w}\right)_{w=0}\right), \end{aligned} \tag{A.3}$$

where (similar to the previous approximation (3.11))

$$\beta = \frac{n_1}{n} = \int_{w_1}^{\infty} \varphi \, dw,$$

and the associated difference is

$$\tilde{\beta} = \beta_c - \beta_d = \int_{w_1}^{\infty} \tilde{\varphi} \, dw.$$

The probability transition for $\tilde{\varphi}$ given in (A.2) depends upon the large parameter λ, which drives the overall dynamics. The parameter h, on the other hand, which is defined by the expression (A.3), is a small perturbation–its effects are negligible at this level of the approximation of the general system.

Equation (A.2) can be simplified by the transition to new variables $(x,t) \to (x_0,t)$:

$$\frac{\partial}{\partial t}\left(\tilde{\varphi}\frac{\partial x}{\partial x_0}\right) = -\lambda n \tilde{\varphi}\frac{\partial x}{\partial x_0} + h, \tag{A.4}$$

$$\frac{\partial x}{\partial t} = 2\frac{\partial}{\partial x}\ln\left(\frac{n_c n_d}{n}\right) - \left(\tilde{\varphi}\frac{\partial x}{\partial x_0}\right)^{-1}\frac{\partial \tilde{\varphi}}{\partial x_0}. \tag{A.5}$$

Equation (A.5) represents the definition of the variable transformation $x = x(t, x_0)$, while and equation (A.4) is derived from (A.2) by the substitution of that definition into it.

The initial conditions at $t = 0$ have the form: $x(0, x_0) = x_0$, $\tilde{\varphi}(0, x_0) = \tilde{\varphi}_0(x_0)$, where $\tilde{\varphi}_0(x)$ is the initial condition for $\tilde{\varphi}(t, x)$ in (A.2).

The estimate for $\tilde{\varphi}(x,t)$ can be derived from equation of (A.4) when considering the term h as an external function:

$$\tilde{\varphi}\frac{\partial x}{\partial x_0} = \tilde{\varphi}_0(x_0)\exp\left(-\lambda\int_0^t n\left(x\left(x_0,t'\right),t'\right)dt'\right) + \int_0^t h\left(t_1\right)\exp\left(-\lambda\int_{t_1}^t n\left(x\left(x_0,t'\right),t'\right)dt'\right)dt_1. \tag{A.6}$$

Taking into account that λ is a large parameter and replacing the second integral by its asymptotic estimate, we obtain:

$$\tilde{\varphi}\frac{\partial x}{\partial x_0} \simeq \tilde{\varphi}_0(x_0)\exp\left(-\lambda\int_0^t n\left(x\left(x_0,t'\right),t'\right)dt'\right) + \lambda^{-1}\frac{h(t)}{n(x,t)} + O\left(\lambda^{-2}\right). \tag{A.7}$$

Thus for the time period of order $\tau \sim \lambda^{-1}$ the value $\tilde{\varphi}\frac{\partial x}{\partial x_0}$ is decreasing up to order $\sim \lambda^{-1}$.

To make the same statement about the function $\tilde{\varphi}$ itself, we also need to make sure that the partial derivative $\frac{\partial x}{\partial x_0}$ is not vanishing for that time period and remains of the same order as it initially was. To do that, substitute the first term of estimate (A.7) into equation (A.5):

$$\frac{\partial x}{\partial t} = \lambda \left(\frac{\partial x}{\partial x_0}\right)^{-1} \int_0^t \frac{\partial n}{\partial x}\frac{\partial x}{\partial x_0} dt' - \left(\frac{\partial x}{\partial x_0}\right)^{-1} \frac{\partial}{\partial x_0} \ln \tilde{\varphi}_0 + \frac{\partial^2 x}{\partial x_0^2}\left(\frac{\partial x}{\partial x_0}\right)^{-2} + 2\frac{\partial}{\partial x} \ln\left(\frac{n_c n_d}{n}\right). \tag{A.8}$$

Keeping in (A.8) only the main asymptotic term $\sim \lambda$, we obtain after differentiation by t:

$$\frac{\partial}{\partial t}\left(\frac{\partial x}{\partial x_0}\frac{\partial x}{\partial t}\right) \simeq \lambda \frac{\partial n}{\partial x}\frac{\partial x}{\partial x_0}. \tag{A.9}$$

The substitution $\tau = \lambda^{1/2} t$ into (A.9) removes the explicit dependence of λ, so that

$$x(t, x_0; \lambda) = x(\lambda^{1/2} t, x_0). \tag{A.10}$$

This implies that the function x as well as its derivative $\frac{\partial x}{\partial x_0}$ changes substantially at the time order $\sim \lambda^{-1/2}$, which is much greater than the relaxation time $\tau \sim \lambda^{-1}$. Thus, the estimate (A.7) can finally be rewritten as:

$$\tilde{\varphi} \simeq \tilde{\varphi}_0(x) \exp\left(-\lambda n_0(x) t\right) + O\left(\lambda^{-1}\right) \tag{A.11}$$

When writing (A.11), we take into account the obvious relation $\left(\frac{\partial x}{\partial x_0}\right)_{t=0} \equiv 1$ and that the spatial density $n(x,t)$ obeys the "slow" equation (3.8)–in this case it does not change substantially for the relaxation time τ, and can be replaced by its initial value $n_0(x)$.

B Simulation Initial Conditions and Parameters

The simulations were run using Mathematica 5, by numerically solving the partial differential equation system 2.7. In order to do so, several parameters were fixed. First, Table 1 shows the game payoffs for the repeated Prisoner's Dilemma:

Game Payoffs		
	C	**D**
C	(1.0,1.0)	(-1.8,2.0)
D	(2.0,-1.8)	(-1.3,-1.3)

Table 1: Payoff matrix for the Prisoner's Dilemma game.

(c) = cooperate, (d) = defect

The probability distribution $g(w)$ for the wealth of a cloned agent was set to

$$g\left(w\right) := \frac{15}{16\Delta w_0}\left[1-\left(\frac{w-w_0}{\Delta w_0}\right)^2\right]^2 \vartheta\left(\Delta w_0 - \left|w-w_0\right|\right), \tag{B.1}$$

where the mean parameter $w_0 = 0.5$, the spread $\Delta w_0 = 0.5/3$, and $\vartheta\left(x\right)$ is an indicator function taking the value 1 when its argument is positive, and 0 otherwise.

The initial joint probability density $f_0(s, w, x,)$ for a strategy s was decomposed into two marginal densities–this substantially simplified the numerical analysis. The marginal densities were the initial wealth density $f_s(w)$, and the initial spatial density $f_s(x)$, so that $f_0(s, w, x) := f_s(w)f_s(x)$.

The initial wealth probability densities of cooperators and defectors were identical, i.e. $f_c(w) = f_d(w)$, and had the same functional form as the wealth density for a cloned agent $g(w)$. The mean parameter was $w_0 = 1$, while the spread was again $\Delta w_0 = 0.5/3$.

The initial spatial densities of the cooperators and defectors, $f_c(x)$ and $f_d(x)$, were different–the population of cooperators was given a larger (probabilistic) concentration than defectors in a centralized region of space, to compare and replicate earlier work in Dorofeenko and Shorish [2005] and to examine the metastability of any cooperator structure which might emerge. The initial defector density was uniform over the interval. For the cooperator density,

$$f_c(x) := 0.5 + 0.25 \cos\left(\frac{2\pi}{L}x\right), \tag{B.2}$$

while the initial defector density was set to

$$f_d(x) := 0.2. \tag{B.3}$$

Other parameters: the spatial interval was defined as $[-25, 25]$ with endpoints identified, so that $L = 50$. Wealth was defined over the interval $[0, 60]$. The threshold level of wealth for cloning was $w_1 = .67$. The wealth diffusion parameter $M = 0.04$. The cloning rate of new agents $\nu_b = 1$, while the death rate for all agents was $d = 0.01$.

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