

BARGAINING GAMES AND THE EXISTENCE  
OF UTILITY FUNCTION

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## Abstract

We prove, under suitable assumptions, the existence of such utility functions  $u_1, u_2$  for players 1 and 2, respectively, that player 1 prefers a simple two-person bargaining game with the status quo  $a$  to that with the same outcome set and the status quo  $b$  if and only if

$$u_1(a) - u_1(b) > u_2(a) - u_2(b) .$$

The utility functions are unique up to their zero points and a common positive multiplier. We also extend this result to an arbitrary set of players. The derivation of utility does not involve any assumption concerning behaviour of individuals under risk.



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## 1. Introduction

Probability is often visualized as a subjective concept. It is therefore of interest to develop a utility theory in which no concept of objective probability is assumed. SAVAGE (1954) first derive utility on the basis of behaviour of a person faced with games against nature. The main purpose of this paper is to derive utility on the basis of simple two-person bargaining games. To achieve such a result, we assume that a person is able to express preferences not only between every possible pair of alternatives but also between every pair of bargaining games that have the same outcome set but, maybe, differ in the status quo.

We prove, under suitable assumptions, the existence of such utility functions  $u_1, u_2$  for players 1 and 2, respectively, that player 1 prefers a game with the status quo a to that with the status quo b if and only if

$$u_1(a) - u_1(b) > u_2(a) - u_2(b) .$$

Player 2's preferences are converse.

The utility functions are unique up to their zero points and a common positive multiplier. We also extend this result to an arbitrary set of players.

Since our derivation of utility is free of the notion of probability, it should enable one to realize the ideas of RAMSEY (1931) to derive probabilities on the basis of utilities.

As far as two-person case is concerned, our result is similar to that of MYERSON (1977), although our construction and

assumptions are different.

Some words concerning the presentation of assumptions are appropriate. The main axioms are denoted by A1 - A3, B1 - B8, S1 - S2. Instead of referring to these every time, we assume that any of them, once stated, remains valid for the rest of the paper.

## 2. Preliminary definitions and assumptions

A two-person bargaining game is characterized by the set of outcomes and a distinguished outcome  $q$  - the status quo. If no trade occurs, the outcome is  $q$ . A trade takes place if and only if both players agree upon a unique outcome.

Let  $A$  denote the set of all possible outcomes in various bargaining games. Given any two outcomes  $a, b \in A$  we use symbolism  $a \succeq_i b$  ( $b \preceq_i a$ ) to denote that player  $i$  prefers  $a$  to  $b$  or he is indifferent between them. We assume that the preference-indifference relation holds between any two outcomes, and it is transitive. For any player  $i$ , we have thus the axioms:

- A1. It holds  $a \succeq_i a$  for any  $a \in A$ .
- A2. Given any two outcomes  $a, b \in A$ , either  $a \succeq_i b$  or  $a \preceq_i b$  holds.
- A3. For any  $a, b, c \in A$ , if  $a \succeq_i b$  and  $b \succeq_i c$ , then  $a \succeq_i c$ .

Let  $X_i$  denote the set of equivalence classes of  $A$  under  $\succeq_i$ , i.e.  $x_i \in X_i$  if and only if there exists  $a \in A$  such that

$$x_i = \{b \in A : b \succeq_i a \text{ and } b \preceq_i a\}.$$

We will use letters  $x_i, y_i, z_i$  to denote elements of  $X_i$ . The set  $X_i$  can be ordered in the natural way. We denote this order by  $\geq$ . It will not lead to misunderstanding as the elements of  $X_i$  have subscripts.

We will use the capital letters  $S, T, V$  to denote the outcome sets of various bargaining games.

Definition 1. A two-person bargaining game is a triple  $(S, x_1, x_2)$  where  $S$  is the outcome set and  $x_1, x_2$  are the equivalence classes generated by the status quo.

We denote by  $B$  the set of all possible two-person bargaining games. Given any two games  $(S, x_1, x_2), (S, y_1, y_2) \in B$ , we use the symbolism  $(S, x_1, x_2) \rightarrow (S, y_1, y_2)$  to denote that the first player prefers the first game to the second or he is indifferent between them. We will write  $(S, x_1, x_2) \leftrightarrow (S, y_1, y_2)$  if  $(S, x_1, x_2) \rightarrow (S, y_1, y_2)$  and  $(S, x_1, x_2) \leftarrow (S, y_1, y_2)$ . Similarly,  $(S, x_1, x_2) \twoheadrightarrow (S, y_1, y_2)$  means that  $(S, x_1, x_2) \rightarrow (S, y_1, y_2)$  but not  $(S, x_1, x_2) \leftarrow (S, y_1, y_2)$ .

Note that preference-indifference relation  $\rightarrow$  is defined only for games which differ only in the status quo and have the same outcome set.

It will be convenient to introduce the symbolism  $(x_1, x_2) \rightarrow (y_1, y_2)$  to denote the fact that  $x_1 \geq y_1, x_2 \leq y_2$  and, for at least one of them, the equality does not hold.

Some results concerning ordered groups will be necessary. An additive group  $K$  is called ordered if  $K$  is a linear ordered set under relation  $\geq$  and, for any  $x, y, z \in K$

$$x \geq y \text{ implies } x+z \geq y+z .$$

Given two ordered groups  $K$  and  $L$ , an injective mapping  $g: K \rightarrow L$  is said to be an o-embedding if

$$g(x+y) = g(x) + g(y) \text{ for all } x, y \in K$$

and

$$x > y \text{ if and only if } g(x) > g(y) .$$

A bijective o-embedding is said to be an o-isomorphism.

We denote by  $mx$  the sum  $x+x+\dots+x$  ( $m$  times).

An ordered group  $K$  with the neutral element  $e$  is said to be Archimedean if, for arbitrary  $x, y \in K$ ,

$$y > e \text{ implies } my > x \text{ for a suitable integer } m \geq 1.$$

Theorem (HÖLDER (1901)). An ordered group  $K$  is Archimedean if and only if there exists an o-embedding from  $K$  to the additive group of the real numbers with the natural ordering.

Theorem (HION (1954)). Let  $K \neq 0$  and  $L$  be subgroups of the additive group of real numbers, endowed with the natural ordering, and  $g$  an o-homomorphism (or o-isomorphism) from  $K$  into  $L$ . Then there exists a real number  $\alpha \geq 0$  such that

$$g(x) = \alpha x \text{ for all } x \in K .$$

### 3. Two-person bargaining games

In this section we shall restrict our attention to bargaining games among two fixed players. We assume that class B of all possible bargaining games and the preference-indifference relation  $\rightarrow$  of the first player satisfy the following axioms.

B1. For any  $(S, x_1, x_2), (S, y_1, y_2) \in B$ , either  $(S, x_1, x_2) \rightarrow (S, y_1, y_2)$  or  $(S, y_1, y_2) \rightarrow (S, x_1, x_2)$ .

B2 (Rationality). If  $(x_1, x_2) \rightarrow (y_1, y_2)$ , then  $(S, x_1, x_2) \leftrightarrow (S, y_1, y_2)$  for any  $(S, x_1, x_2), (S, y_1, y_2) \in B$ .

B3 (Continuity). Let  $(S, x_1, x_2), (S, y_1, y_2) \in B$ . If  $(T, z_1, z_2) \leftrightarrow (T, y_1, y_2)$  for any  $(T, z_1, z_2), (T, y_1, y_2) \in B$  where  $(z_1, z_2) \rightarrow (x_1, x_2)$ , then  $(S, x_1, x_2) \rightarrow (S, y_1, y_2)$ .

B4. Let  $x_1, y_1 \in X_1$  and  $y_2 \in X_2$ . Then there exist  $(S, x_1, x_2), (S, y_1, y_2) \in B$  such that  $(S, x_1, x_2) \leftrightarrow (S, y_1, y_2)$ .

B5. Let  $x_1 \in X_1$  and  $x_2, y_2 \in X_2$ . Then there exist  $(S, x_1, x_2), (S, y_1, y_2) \in B$  such that  $(S, x_1, x_2) \leftrightarrow (S, y_1, y_2)$ .

Suppose now, it holds

$$(S, x_1, x_2) \rightarrow (S, y_1, y_2) \text{ and } (S', x'_1, x'_2) \rightarrow (S', y'_1, y'_2)$$

for some bargaining games. If  $y_1 = x'_1$ , it is natural to assume that there exists a "bigger" outcome set T such that

$$(T, x_1, z_2) \rightarrow (T, y'_1, z'_2) .$$

If the original two games satisfy the analogous condition also with respect to the second player, we postulate the existence of a set T such that the appropriate properties

hold with respect to both players even when the second player considers the original games in the replaced order. Formally:

B6. Let  $(S, x_1, x_2), (S, y_1, y_2), (T, y_1, y_2), (T, z_1, z_2) \in B$ . If  $(S, x_1, x_2) \succ (S, y_1, y_2)$  and  $(T, y_1, y_2) \succ (T, z_1, z_2)$ , then there exist  $(V, x_1, x_2), (V, z_1, z_2) \in B$  such that

$$(V, x_1, x_2) \succ (V, z_1, z_2) .$$

B7. Let  $(S, x_1, y_2), (S, y_1, z_2), (T, y_1, x_2), (T, z_1, y_2) \in B$ . If  $(S, x_1, y_2) \succ (S, y_1, z_2)$  and  $(T, y_1, x_2) \succ (T, z_1, y_2)$ , then there exist  $(V, x_1, x_2), (V, z_1, z_2) \in B$  such that

$$(V, x_1, x_2) \succ (V, z_1, z_2) .$$

Given  $(S, x_1, x_2), (S, y_1, y_2) \in B$ , we will write

$$(S, x_1, x_2) \succeq_2 (S, y_1, y_2)$$

to denote that the second player prefers the first game to the second or he is indifferent between them.

B8. Let  $(S, x_1, x_2), (S, y_1, y_2) \in B$ . Then  $(S, x_1, x_2) \succ (S, y_1, y_2)$  if and only if  $(S, y_1, y_2) \not\succeq_2 (S, x_1, x_2)$ .

LEMMA 1. Let  $(S, x_1, x_2), (S, y_1, y_2), (T, x_1, x_2), (T, z_1, z_2) \in B$ . Let  $(S, x_1, x_2) \leftrightarrow (S, y_1, y_2)$  and  $(T, x_1, x_2) \leftrightarrow (T, z_1, z_2)$ .

Then

$$y_1 = z_1 \text{ if and only if } y_2 = z_2 .$$

PROOF: Necessity. Suppose,  $y_2 > z_2$ . According to B6, there exist  $(V, y_1, y_2), (V, z_1, z_2) \in B$  such that

$$(V, y_1, y_2) \rightarrow (V, z_1, z_2) ,$$

contradicting to B2.

Now, suppose,  $y_2 > z_2$ . According to B6, there exist  $(V, z_1, z_2), (V, y_1, y_2) \in B$  such that

$$(V, y_1, y_2) \leftarrow (V, z_1, z_2) ,$$

which is again a contradiction to B2. Hence, we must have  $y_2 = z_2$ .

We have thus shown the necessity. The proof of the sufficiency is analogous, i.e., in both cases,  $y_1 > z_1$  and  $y_1 < z_1$ , we obtain a contradiction to B2. Q.E.D.

Given  $x_1, y_1 \in X_1$  and  $x_2 \in X_2$ , we denote

$$f_2(x_1, x_2 // y_1) = \{y_2 \in X_2 : (S, x_1, x_2) \leftrightarrow (S, y_1, y_2)$$

$$\text{for some } (S, x_1, x_2), (S, y_1, y_2) \in B\} .$$

According to B4,  $f_2(x_1, x_2 // y_1) \neq \emptyset$  for any  $x_1, y_1 \in X_1$  and  $x_2 \in X_2$ . By Lemma 1,  $f_2(x_1, x_2 // y_1)$  contains only one equivalence class. Hence,

$$f_2(x_1, x_2 // y_1) : X_1 \times X_2 \times X_1 \rightarrow X_2$$

is a mapping which assigns an equivalence class to any  $x_1, y_1 \in X_1$  and  $x_2 \in X_2$ .

Analogously,  $f_1(x_1, x_2 // y_2) : X_1 \times X_2 \times X_2 \rightarrow X_1$  is defined as follows:

$$f_1(x_1, x_2 // y_2) = \{y_1 \in X_1 : (S, x_1, x_2) \leftrightarrow (S, y_1, y_2)\}$$

for some  $(S, x_1, x_2), (S, y_1, y_2) \in B$  .

The following properties follow immediately from the definition and Lemma 1:

$$f_1(x_1, x_2 // y_2) = y_1 \text{ if and only if } f_2(x_1, x_2 // y_1) = y_2; \quad (1)$$

$$f_1(x_1, x_2 // y_2) = y_1 \text{ if and only if } f_2(y_1, y_2 // x_1) = x_2; \quad (2)$$

$$f_1(x_1, x_2 // y_2) = y_1 \text{ if and only if } f_1(y_1, y_2 // x_2) = x_1 . \quad (3)$$

LEMMA 2: For any  $x_1 \in X$  and  $x_2, y_2, z_2 \in X_2$ , it holds

$$f_1(x_1, x_2 // y_2) = f_1(f_1(x_1, x_2 // z_2), z_2 // y_2) .$$

PROOF: Let us denote  $y_1 = f_1(x_1, x_2 // y_2)$ . We have

$$(S, x_1, x_2) \leftrightarrow (S, y_1, y_2) \quad (4)$$

for some  $(S, x_1, x_2), (S, y_1, y_2) \in B$  .

Denote  $z_1' = f_1(x_1, x_2 // z_2)$  and  $z_1 = f_1(z_1', z_2 // y_2)$ .

We have

$$(T', x_1, x_2) \leftrightarrow (T', z_1', z_2) \text{ and } (T'', z_1', z_2) \leftrightarrow (T'', z_1, y_2) \quad (5)$$

for some

$$(T', x_1, x_2), (T', z_1', z_2), (T'', z_1', z_2), (T'', z_1, y_2) \in B .$$

Suppose  $z_1 > y_1$  . According to B6, there exist

$(T, x_1, x_2), (T, z_1, y_2) \in B$  such that

$$(T, x_1, x_2) \rightarrow (T, z_1, y_2) .$$

From (4) and B6, it follows that

$$(V, y_1, y_2) \rightarrow (V, z_1, y_2)$$

for some  $(V, y_1, y_2), (V, z_1, y_2) \in B$ , contradicting to B2.

On the other hand, suppose  $z_1 < y_1$ . According to B6 and (5), we obtain

$$(T, x_1, x_2) \leftarrow (T, z_1, y_2)$$

for some  $(T, x_1, x_2), (T, z_1, y_2) \in B$ . From (4) and B6, it follows that

$$(V, y_1, y_2) \leftarrow (V, z_1, y_2)$$

for some  $(V, y_1, y_2), (V, z_1, y_2) \in B$ , which is again a contradiction to B2. Hence,  $y_1 = z_1$ . Q.E.D.

LEMMA 3: For any  $x_1 \in X_1$  and  $x_2, y_2, z_2 \in X_2$ , it holds

$$f_1(x_1, x_2 // y_2) = f_1(f_1(x_1, z_2 // y_2), x_2 // z_2) .$$

PROOF: Denote  $y_1 = f_1(x_1, x_2 // y_2)$ . We have

$$(S, x_1, x_2) \leftrightarrow (S, y_1, y_2) \tag{6}$$

for some  $(S, x_1, x_2), (S, y_1, y_2) \in B$ .

Denote  $z_1' = f_1(x_1, z_2 // y_2)$  and

$$z_1 = f_1(f_1(x_1, z_2 // y_2), x_2 // z_2) = f_1(z_1', x_2 // z_2) .$$

That means

$$(T', x_1, z_2) \leftrightarrow (T', z_1', y_2) \quad \text{and} \quad (T'', z_1', x_2) \leftrightarrow (T'', z_1, z_2) \tag{7}$$

for some

$$(T', x_1, z_2), (T', z_1', y_2), (T'', z_1', x_2), (T'', z_1, z_2) \in B .$$

Suppose  $z_1 > y_1$ . According to B7, there exist  $(T, x_1, x_2), (T, z_1, y_2) \in B$  such that

$$(T, x_1, x_2) \rightarrow (T, z_1, y_2).$$

From (6) and B6, it follows that

$$(V, y_1, y_2) \rightarrow (V, z_1, y_2)$$

for some  $(V, y_1, y_2), (V, z_1, y_2) \in B$ , which is a contradiction to B2.

Suppose now,  $z_1 < y_1$ . From B7 and (7), we obtain

$$(T, x_1, x_2) \leftarrow (T, z_1, y_2)$$

for some  $(T, x_1, x_2), (T, z_1, y_2) \in B$ . From (6) and B6, it follows that

$$(V, y_1, y_2) \leftarrow (V, z_1, y_2)$$

for some  $(V, y_1, y_2), (V, z_1, y_2) \in B$ , a contradiction to B2. Consequently,  $y_1 = z_1$ . Q.E.D.

Let us choose arbitrary  $e_1 \in X_1$  and  $e_2 \in X_2$ . We keep them fixed throughout the further discussion.

Definition 2. Given any  $x_1, y_1 \in X_1$ , we define the sum and the difference as follows:

$$\begin{aligned} x_1 + y_1 &= f_1(x_1, e_2 // f_2(e_1, e_2 // y_1)), \\ x_1 - y_1 &= f_1(x_1, f_2(e_1, e_2 // y_1) // e_2). \end{aligned}$$

THEOREM 1. The ordered set  $X_1$  together with the addition + is an ordered Abelian group, and  $e_1$  is the neutral element.

PROOF: Commutativity. Given  $x_1, y_1 \in X_1$ , denote

$$x_2 = f_2(e_1, e_2 // x_1) \text{ and } y_2 = f_2(e_1, e_2 // y_1).$$

By (1), we must have

$$x_1 = f_1(e_1, e_2 // x_2) \text{ and } y_1 = f_1(e_1, e_2 // y_2).$$

According to Lemma 2, it follows

$$x_1 = f_1(f_1(e_1, e_2 // y_2), y_2 // x_2) = f_1(y_1, y_2 // x_2).$$

Hence,

$$\begin{aligned} x_1 + y_1 &= f_1(x_1, e_2 // f_2(e_1, e_2 // y_1)) = \\ &= f_1(f_1(y_1, y_2 // x_2), e_2 // y_2) . \end{aligned}$$

From Lemma 3, we obtain

$$x_1 + y_1 = f_1(y_1, e_2 // x_2) = y_1 + x_1 .$$

We have thus proved the commutativity.

Associativity. Given  $x_1, y_1, z_1 \in X_1$ , denote

$$z_2 = f_2(e_1, e_2 // z_1) \text{ and } x_2 = f_2(y_1, e_2 // x_1 + y_1) .$$

By (1), we must have

$$x_1 + y_1 = f_1(y_1, e_2 // x_2) . \tag{8}$$

By Lemma 3, we have

$$\begin{aligned}(x_1+y_1)+z_1 &= f_1(x_1+y_1, e_2 // z_2) = \\ &= f_1(f_1(x_1+y_1, x_2 // z_2), e_2 // x_2). \quad (9)\end{aligned}$$

From (8) and Lemma 2, we obtain

$$\begin{aligned}f_1(x_1+y_1, x_2 // z_2) &= f_1(f_1(y_1, e_2 // x_2), x_2 // z_2) = \\ &= f_1(y_1, e_2 // z_2) = y_1+z_1. \quad (10)\end{aligned}$$

On the other hand, by the commutativity,

$$x_2 = f_2(y_1, e_2 // x_1+y_1) = f_2(y_1, e_2 // y_1+x_1).$$

But

$$y_1+x_1 = f_1(y_1, e_2 // f_2(e_1, e_2 // x_1)).$$

Hence, by (1)

$$x_2 = f_2(e_1, e_2 // x_1).$$

According to (10), this means that (9) can be re-written in the form

$$\begin{aligned}(x_1+y_1)+z_1 &= f_1(y_1+z_1, e_2 // f_2(e_1, e_2 // x_1)) = \\ &= (y_1+z_1)+x_1.\end{aligned}$$

By the commutativity, it follows

$$(x_1+y_1)+z_1 = x_1+(y_1+z_1).$$

Thus, we have proved the associativity.

Our definition of the difference is correct, since, by Lemma 3.

$$\begin{aligned}(x_1 - y_1) + y_1 &= f_1(f_1(x_1, f_2(e_1, e_2 // y_1)) // e_2, e_2 // f_2(e_1, e_2 // y_1)) = \\ &= f_1(x_1, e_2 // e_2)\end{aligned}$$

and, by the rationality axiom,

$$f_1(x_1, e_2 // e_2) = x_1 .$$

On the other hand, if  $z_1 \in X_1$  and

$$z_1 + y_1 = f_1(z_1, e_2 // f_2(e_1, e_2 // y_1)) = x_1 = f_1(x_1 - y_1, e_2 // f_2(e_1, e_2 // y_1)) ,$$

then, from (3), it follows  $z_1 = x_1 - y_1$  .

To prove the theorem, it remains to show that, for any  $x_1, y_1, z_1 \in X_1$  ,

$$x_1 \leq y_1 \text{ implies } x_1 + z_1 \leq y_1 + z_1 .$$

It is sufficient to show that

$$x_1 \leq x_1 + y_1 \text{ if and only if } y_1 \geq e_1 .$$

Denote  $y_2 = f_2(e_1, e_2 // y_1)$  .

(11)

Let  $x_1 \leq x_1 + y_1$  . That means,

$$x_1 \leq f_1(x_1, e_2 // y_2) .$$

According to B2, we must have  $y_2 \geq e_2$ , and, from (11) and B2, it follows,  $y_1 \geq e_1$  .

Suppose now that  $y_1 \geq e_1$ . By (11) and B2, it follows  $y_2 \geq e_2$  and

$$x_1 \leq f_1(x_1, e_2 // y_2) = x_1 + y_1 .$$

Note that, by (1), it immediately follows that  $e_1$  is the neutral element of the group  $X_1$ . Q.E.D.

We define the sum for  $X_2$  analogously.

Definition 3. For any  $x_2, y_2 \in X_2$ , we define the sum and the difference as follows:

$$\begin{aligned} x_2 + y_2 &= f_2(e_1, x_2 // f_1(e_1, e_2 // y_2)), \\ x_2 - y_2 &= f_2(f_1(e_1, e_2 // y_2), x_2 // e_1) . \end{aligned}$$

Since our definitions and the axioms which we have used are completely symmetric with respect to the players, the following corollary is obvious.

COROLLARY: The ordered set  $X_2$  together with the above defined addition is an ordered Abelian group, and  $e_2$  is the neutral element.

Let us define a mapping  $h: X_2 \rightarrow X_1$  in the following way:

$$h(x_2) = f_1(e_1, e_2 // x_2). \tag{12}$$

LEMMA 4: The mapping  $h: X_2 \rightarrow X_1$  is an o-isomorphism.

PROOF: According to B4 and (1), the mapping  $h$  is bijective.

We shall prove that

$$h(x_2 + y_2) = h(x_2) + h(y_2) .$$

By Lemma 3, we must have

$$h(x_2 + y_2) = f_1(e_1, e_2 // x_2 + y_2) = f_1(f_1(e_1, x_2 // x_2 + y_2), e_2 // x_2) .$$

$$\text{Denote } y_1 = f_1(e_1, x_2 // x_2 + y_2) .$$

Since

$$x_2 + y_2 = f_2(e_1, x_2 // f_1(e_1, e_2 // y_2)) ,$$

it follows from (1) that

$$y_1 = f_1(e_1, e_2 // y_2) = h(y_2) .$$

Hence,

$$h(x_2 + y_2) = f_1(h(y_2), e_2 // x_2) .$$

From (1) and (12), we must have

$$x_2 = f_2(e_1, e_2 // h(x_2)) .$$

Hence,

$$h(x_2 + y_2) = f_1(h(y_2), e_2 // f_2(e_1, e_2 // h(x_2))) = h(y_2) + h(x_2)$$

and, by the commutativity,

$$h(x_2 + y_2) = h(x_2) + h(y_2) .$$

On the other hand, by (1), we have

$$h(e_2) = e_1 .$$

It remains to prove that, for arbitrary  $x_2, y_2 \in X_2$ ,

$$x_2 \geq Y_2 \text{ if and only if } h(x_2) \geq h(y_2) \quad (13)$$

By the definition of  $h$  and  $f_1$ , there must exist

$$(S, e_1, e_2), (S, h(x_2), x_2), (T, e_1, e_2), (T, h(y_2), y_2) \in B$$

such that

$$(S, e_1, e_2) \leftrightarrow (S, h(x_2), x_2) \text{ and } (T, e_1, e_2) \leftrightarrow (T, h(y_2), y_2).$$

According to B6, there must exist  $(V', h(x_2), x_2),$

$$(V', h(y_2), y_2), (V'', h(x_2), x_2), (V'', h(y_2), y_2) \in B$$

such that

$$(V', h(x_2), x_2) \leftrightarrow (V', h(y_2), y_2) \text{ and } (V'', h(x_2), x_2) \leftrightarrow (V'', h(y_2), y_2).$$

Thus, (13) follows from B2.

Q.E.D.

LEMMA 5: For any  $x_1, y_1 \in X_1$  and  $x_2, y_2 \in X_2$ ,

$$x_1 = f_1(y_1, y_2 // x_2) \text{ if and only if } x_1 - y_1 = h(x_2) - h(y_2) .$$

PROOF: We have

$$h(x_2) - h(y_2) = f_1(h(x_2), f_2(e_1, e_2 // h(y_2)) // e_2) .$$

According to (1) and (12), we must have

$$f_2(e_1, e_2 // h(y_2)) = y_2 .$$

Hence, by Lemma 2,

$$\begin{aligned} h(x_2) - h(y_2) &= f_1(h(x_2), y_2 // e_2) = \\ &= f_1(f_1(h(x_2), y_2 // f_2(e_1, e_2 // y_1)), f_2(e_1, e_2 // y_1) // e_2) \end{aligned} \quad (14)$$

According to Lemma 3,

$$\begin{aligned} f_1(h(x_2), y_2 // f_2(e_1, e_2 // y_1)) &= \\ &= f_1(f_1(h(x_2), x_2 // f_2(e_1, e_2 // y_1)), y_2 // x_2) \end{aligned} \quad (15)$$

By Lemma 2,

$$\begin{aligned} f_1(h(x_2), x_2 // f_2(e_1, e_2 // y_1)) &= \\ &= f_1(f_1(h(x_2), x_2 // e_2), e_2 // f_2(e_1, e_2 // y_1)) . \end{aligned} \quad (16)$$

Since  $h(x_2) = f_1(e_1, e_2 // x_2)$ , we must have from (3)

$$f_1(h(x_2), x_2 // e_2) = e_2 .$$

Combining this with (1), we can rewrite (16) in the form

$$\begin{aligned} f_1(h(x_2), x_2 // f_2(e_1, e_2 // y_1)) &= \\ &= f_1(e_1, e_2 // f_2(e_1, e_2 // y_1)) = y_1 \end{aligned} \quad (17)$$

Combining (14), (15) and (17), we obtain

$$\begin{aligned} h(x_2) - h(y_2) &= f_1(f_1(y_1, y_2 // x_2), f_2(e_1, e_2 // y_1) // e_2) = \\ &= f_1(y_1, y_2 // x_2) - y_1 . \end{aligned}$$

The statement of the lemma follows immediately. Q.E.D.

**THEOREM 2:** There exists an o-isomorphism  $h: X_2 \rightarrow X_1$  such that, for any  $(S, x_1, x_2), (S, y_1, y_2) \in B$ ,  $(S, x_1, x_2) \rightarrow (S, y_1, y_2)$  if and only if  $x_1 - y_1 \geq h(x_2) - h(y_2)$ .

**PROOF:** Let  $h: X_2 \rightarrow X_1$  be defined by (12). Necessity. According to B5, there exist  $(T, z_1, x_2), (T, y_1, y_2) \in B$  such that

$$(T, z_1, x_2) \leftrightarrow (T, y_1, y_2) .$$

By B6 and B2, it follows  $z_1 \leq x_1$ .

On the other hand, by Lemma 5,

$$z_1 - y_1 = h(x_2) - h(y_2) .$$

Hence,

$$x_1 - y_1 \geq h(x_2) - h(y_2)$$

If we take into account Lemma 4, then the necessity has been proved.

Sufficiency. If  $(S, y_1, y_2) \neq (S, x_1, x_2)$ , then, by B1, it follows immediately

$$(S, x_1, x_2) \rightarrow (S, y_1, y_2) .$$

Suppose now that

$$(S, y_1, y_2) \rightarrow (S, x_1, x_2) .$$

Let  $(T, z_1, x_2)$ ,  $(T, y_1, y_2)$  be as in the proof of the necessity. From Lemma 5, we have

$$x_1 \geq z_1$$

According to B6, there exist  $(V, x_1, x_2), (V, z_1, x_2) \in B$  such that

$$(V, x_1, x_2) \leftarrow (V, z_1, x_2) .$$

From B2 and (18), it follows  $x_1 = z_1$ , i.e.,

$$(T, x_1, x_2) \leftrightarrow (T, y_1, y_2) .$$

Suppose that, for some  $(T', y_1, y_2), (T', z'_1, z'_2) \in B$ , it holds

$$(T', y_1, y_2) \rightarrow (T', z'_1, z'_2)$$

and  $(z'_1, z'_2) \rightarrow (x_1, x_2)$ . By B6, there exist  $(V', x_1, x_2), (V', z'_1, z'_2) \in B$  such that

$$(V', x_1, x_2) \rightarrow (V', z'_1, z'_2),$$

contradicting to B2. Hence,  $(T', z'_1, z'_2) \rightarrow (T', y_1, y_2)$  for any  $(T', z'_1, z'_2), (T', y_1, y_2) \in B$  if  $(z'_1, z'_2) \rightarrow (x_1, x_2)$ . It follows from B3 that

$$(S, x_1, x_2) \rightarrow (S, y_1, y_2) \quad \text{Q.E.D.}$$

By B8, we immediately obtain the following corollary:

COROLLARY. Let  $(S, x_1, x_2), (S, y_1, y_2) \in B$  and  $h$  be as in Theorem 2. Then

$$(S, x_1, x_2) \rightarrow (S, y_1, y_2) \text{ if and only if } x_1 - y_1 \leq h(x_2) - h(y_2) .$$

REMARK. We have used axioms B1 and B3 only in the proof of the sufficiency of Theorem 2. If they are dropped out, then the weaker statement holds:

$$\begin{aligned} \text{if } (S, x_1, x_2) \rightarrow (S, y_1, y_2), \text{ then } x_1 - y_1 &\geq h(x_2) - h(y_2), \\ \text{if } (S, x_1, x_2) \leftrightarrow (S, y_1, y_2), \text{ then } x_1 - y_1 &= h(x_2) - h(y_2) . \end{aligned}$$

Definition 4. A finite sequence of bargaining games  $(S^i, x_1^i, x_2), (S^i, x_1^{i+1}, y_2) \in B, i=1, \dots, m$  is called a Player 1's saving sequence if  $y_2 > x_2$  and, for any  $i$ ,

$$(S^i, x_1^i, x_2) \rightarrow (S^i, x_1^{i+1}, y_2) .$$

A saving sequence can be interpreted in the following way. Suppose players are involved into the game  $(S^i, x_1^i, x_2)$ . As the game  $(S^i, x_1^{i+1}, y_2)$  is preferred by player 2, player 1 can expect to gain in the game  $(S^i, x_1^i, x_2)$  at least  $x_1^{i+1}$ . If they start from the game

$(S^1, x_1^1, x_2)$ , then, after  $m$  stages, player 1 can expect to gain at least  $x_1^{m+1}$  since the status quo in any stage is at least as good as the gain in the previous stage. On the other hand, player 2 loses the whole gain before the next stage starts.

A saving sequence for player 2 is defined analogously.

Definition 5. A sequence of bargaining games  $(S^i, x_1, x_2^i)$ ,  $(S^i, y_1, x_2^{i+1}) \in B$ ,  $i=1, \dots, m$ , is called a Player 2's saving sequence if  $x_1 < y_1$  and, for any  $i$ ,

$$(S^i, x_1, x_2^i) \leftarrow (S^i, y_1, x_2^{i+1}).$$

Definition 6. Player 1 is called thrifty if, for any  $x_1, y_1 \in X_1$  and  $x_2, y_2 \in X_2$  where  $x_2 < y_2$ , there exists his saving sequence  $(S^i, x_1^i, x_2)$ ,  $(S^i, x_1^{i+1}, y_2)$ ,  $i=1, \dots, m$ , such that

$$x_1^1 < x_1 \quad \text{and} \quad x_1^{m+1} > y_1 .$$

Analogously, player 2 is called thrifty if, for any  $x_1, y_1 \in X_1$  such that  $x_1 < y_1$  and for any  $x_2, y_2 \in X_2$ , there exists his saving sequence  $(S^i, x_1, x_2^i)$ ,  $(S^i, y_1, x_2^{i+1})$ ,  $i=1, \dots, m$ , such that

$$x_2^1 < x_2 \quad \text{and} \quad x_2^{m+1} > y_2 .$$

In other words, if a player is thrifty, then it is possible that, after time long enough, he achieves a gain as good as he wants, even if the other player gains "very small amount" at each stage.

LEMMA 6. Let  $(S, x_1, x_2), (S, y_1, y_2), (T, x'_1, x_2), (T, y'_1, y_2) \in B$  and  $x'_1 \succ x_1$ . If

$(S, x_1, x_2) \rightarrow (S, y_1, y_2)$  and  $(T, x'_1, x_2) \leftarrow (T, y'_1, y_2)$ , then  $y'_1 \succ y_1$ .

PROOF: According to B5, there exist  $(S', x_1, x_2), (S', z_1, y_2), (T', x'_1, x_2), (T', z'_1, y_2) \in B$  such that

$$(S', x_1, x_2) \leftrightarrow (S', z_1, y_2) \text{ and } (T', x'_1, x_2) \leftrightarrow (T', z'_1, y_2). \quad (19)$$

Combining B6 and B2, we obtain

$$y_1 \leq z_1 \text{ and } y'_1 \geq z'_1. \quad (20)$$

On the other hand, according to Theorem 2, (19) means that

$$x_1 - z_1 = h(x_2) - h(y_2) \text{ and } x'_1 - z'_1 = h(x_2) - h(y_2).$$

By the hypothesis of the lemma, it follows  $z'_1 \geq z_1$ .

Combining this and (20), we obtain  $y'_1 \succ y_1$ .

LEMMA 7. The ordered group  $X_1$  as well as  $X_2$  is Archimedean if and only if at least one player is thrifty.

PROOF: Since both  $X_1$  and  $X_2$  are isomorphic, we need to consider only one of them, say  $X_1$ .

Sufficiency. Without loss of generality, we can assume that the first player is thrifty. That means, given  $y_1 \in X_1$  and  $x_2, y_2 \in X_2$ , if  $y_2 > x_2$ , then we can find a player 1's saving sequence  $(S^i, x_1^i, x_2), (S^i, x_1^{i+1}, y_2), i=1, \dots, m$ , such that

$$x_1^1 \leq e_1 \text{ and } x_1^{m+1} > y_1.$$

On the other hand, given such a sequence, according to B5, we can construct another saving sequence  $(T^i, z_1^i, x_2), (T^i, z_1^{i+1}, y_2), i=1, \dots, m$ , such that

$$(T^i, z_1^i, x_2) \leftrightarrow (T^i, z_1^{i+1}, y_2) \text{ and } z_1^1 = e_1 .$$

By Lemma 6, we have  $z_1^i \geq x_1^i$  for all  $i$ . Hence,

$$z_1^{m+1} > y_1 \tag{21}$$

From Lemma 5, we have

$$z_1^{i+1} = i(h(y_2) - h(x_2)).$$

Denote  $z_1 = h(y_2) - h(x_2) > e_1$ . By (21), it follows

$$m z_1 > y_1 . \tag{22}$$

Since  $h$  is an isomorphism, we can find appropriate  $x_2, y_2$  for any  $z_1 \in X_1$  if  $z_1 > e_1$ . In other words, given arbitrary  $y_1, z_1 \in X_1, z_1 > e_1$ , the existence of a saving sequence insures us that

$$m z_1 > y_1$$

where  $m$  is the amount of stages in the sequence. Necessity. Let  $X_1$  be Archimedean. That means, given arbitrary  $y_1, z_1 \in X_1, z_1 > e_1$ , it holds

$$m z_1 > y_1$$

for a positive integer  $m$  large enough. Hence, given  $x_1, y_1 \in X_1$  and  $x_2, y_2 \in X_2$ , if  $y_2 > x_2$ , we must have

$$m(h(y_2) - h(x_2)) > y_1 - x_1$$

for some  $m$ . According to B5, we can construct a player 1's saving sequence  $(S^i, x_1^i, x_2), (S^i, x_1^{i+1}, y_2) i=1, \dots, m$  such that

$$(S^i, x_1^i, x_2) \leftrightarrow (S^i, x_1^{i+1}, y_2)$$

and  $x_1^i = x_1$ . By Lemma 5,  $x_1^{i+1} - x_1^i = (h(y_2) - h(x_2))$ .

Hence,  $x_1^{m+1} = x_1 + m(h(y_2) - h(x_2)) > y_1$ .

i.e., the first player is thrifty.

Q.E.D.

THEOREM 3. If at least one of players is thrifty, then there exist such a utility function  $u_1: A \rightarrow R$  as well as a utility function  $u_2: A \rightarrow R$  that, for any  $(S, x_1, x_2), (S, y_1, y_2) \in B$ ,

$$(S, x_1, x_2) \rightarrow (S, y_1, y_2) \text{ if and only if } u_1(a) - u_1(b) \geq u_2(a) - u_2(b)$$

where  $a$  is the status quo of the first game and  $b$  is that of the second.

The proof follows immediately from Lemma 7, Hölder's theorem and Theorem 2.

Suppose that in Theorem 3 we have

$$u_1(a) - u_1(b) = u_2(a) - u_2(b)$$

and  $b$  is a Pareto optimal outcome of the set  $S$ . That means that players are indifferent between both games. Hence,  $b$  is a natural solution for the game with the outcome set  $S$  and the status quo  $b$ .

4. Linear with bargaining utility functions

Let  $i, j$  be two players. We denote by  $B_{ij}$  the set of all two-person bargaining games where  $i$  is the first player and  $j$  is the second.

Given any two games  $(S, x_i, x_j), (S, y_i, y_j) \in B_{ij}$ , we write

$$(S, x_i, x_j) \stackrel{i}{\sim} (S, y_i, y_j)$$

to denote that player  $i$  prefers the first game to the second or he is indifferent between them.

Definition 7. A set  $I$  of players is called a society if it includes at least two different players and, for any two of them, the set of all two-person bargaining games between them satisfies the axioms B1 - B8.

S1(Transitivity of players). Let  $i, j, k \in I$ ,  $i \neq j, j \neq k$  and  $i \neq k$ . Let  $(S, x_i, x_j), (S, y_i, y_j) \in B_{ij}$  and  $(T, x_j, x_k), (T, y_j, y_k) \in B_{jk}$ . If  $(S, x_i, x_j) \stackrel{i}{\sim} (S, y_i, y_j)$  and  $(T, x_j, x_k) \stackrel{j}{\sim} (T, y_j, y_k)$ , then there exist  $(V, x_i, x_k), (V, y_i, y_k) \in B_{ik}$  such that

$$(V, x_i, x_k) \stackrel{i}{\sim} (V, y_i, y_k) .$$

Let us fix  $e_i \in X_i$  for any player  $i \in I$ . Given any two players  $i, j \in I$ , we can define the addition according to Definition 2. Denote this addition by  $+_j$  and the resulting group by  $(X_i, +_j)$ .

LEMMA 8. It holds  $(X_i, +_j) = (X_i, +_k)$  for any  $i, j, k \in I$ ,  $i \neq j, i \neq k$ .

PROOF: The statement of the lemma is obvious for  $k=j$ .

Let  $k \neq j$ . According to Lemma 4, there exists an order preserving isomorphism

$$h_{ij} : (X_j, +_i) \rightarrow (X_i, +_j) .$$

Moreover, according to Lemma 5,

$$(S, x_i, x_j) \xleftrightarrow{i} (S, y_i, y_j) \text{ if and only if } x_i - y_i = h_{ij}(x_j) - h_{ij}(y_j) .$$

In particular

$$(S, x_i, x_j) \xleftrightarrow{i} (S, e_i, e_j) \text{ if and only if } x_i = h_{ij}(x_j) .$$

Analogously,

$$(T, x_j, x_k) \xleftrightarrow{j} (T, e_j, e_k) \text{ if and only if } x_j = h_{jk}(x_k) .$$

If both  $x_j = h_{jk}(x_k)$  and  $x_i = h_{ij}(x_j)$  hold, according to S1, we can find  $(V', x_i, x_k), (V', e_i, e_k), (V'', x_i, x_k), (V'', e_i, e_k) \in B_{ik}$  such that

$$(V', x_i, x_k) \xrightarrow{i} (V', e_i, e_k) \text{ and } (V'', x_i, x_k) \xrightarrow{i} (V'', e_i, e_k) .$$

That means

$$x_i = h_{ik}(x_k)$$

where  $h_{ik} : (X_k, +_i) \rightarrow (X_i, +_k)$  is the isomorphism mentioned in Lemma 4 and 5.

Hence, the composition  $h_{ik} \circ h_{jk}^{-1} \circ h_{ij}^{-1}$  is o-isomorphism from  $(X_i, +_j)$  to  $(X_i, +_k)$  which maps  $x_i$  to itself, i.e., the groups  $(X_i, +_j)$  and  $(X_i, +_k)$  are in fact the same group. Q.E.D.

In accordance with Lemma 8, we can drop out the subscripts in the notation of the sum in  $X_i$ .

Let us fix an arbitrary  $r \in I$  and denote  $X_r$  by  $U$ . Let  $h_i: X_i \rightarrow U$ ,  $i \neq r$ , be the  $\alpha$ -isomorphisms mentioned in Theorem 2 and  $h_r: X_r \rightarrow U$  the identical mapping. Let  $K$  be an arbitrary ordered group.

Definition 8. A family of functions  $g_i: X_i \rightarrow K$ ,  $i \in I$  is called to be linear with bargaining if it holds

$(S, x_i, x_j) \rightarrow_i (S, y_i, y_j)$  if and only if  $g_i(x_i) - g_i(y_i) \geq g_j(x_j) - g_j(y_j)$  for any  $(S, x_i, x_j), (S, y_i, y_j) \in B_{ij}$  and for any  $i, j \in I, i \neq j$ .

LEMMA 9: The family of isomorphisms  $h_i: X_i \rightarrow U, i \in I$  is linear with bargaining.

PROOF: The statement of the lemma follows immediately from Theorem 2 if  $i=r$  or  $j=r$ .

Let  $i \neq r$  and  $j \neq r$ . According to B5, there exist  $(T, x_i, x_r), (T, y_i, e_r) \in B_{ir}$  such that

$$(T, x_i, x_r) \leftrightarrow_i (T, y_i, e_r) \quad . \quad (23)$$

We must have

$$h_i(x_i) - h_i(y_i) = x_r \quad .$$

Analogously, there exist  $(V, x_r, z_j), (V, e_r, y_j) \in B_{rj}$  such that

$$(V, x_r, z_j) \leftrightarrow_r (V, e_r, y_j) \quad . \quad (24)$$

Hence,

$$h_j(z_j) - h_j(y_j) = x_r = h_i(x_i) - h_i(y_i) . \quad (25)$$

From S1, we obtain

$$(S', x_i, z_j) \stackrel{\leftarrow}{i} (S', y_i, y_j)$$

for some  $(S', x_i, z_j), (S', y_i, y_j) \in B_{ij}$  .

Suppose  $(S, x_i, x_j) \stackrel{\neq}{i} (S, y_i, y_j)$ . By B6 and B2, it follows  $z_j \geq x_j$ . From (25), we obtain

$$h_i(x_i) - h_i(y_i) \geq h_j(x_j) - h_j(y_j) .$$

On the other hand, suppose that

$$h_i(x_i) - h_i(y_i) \geq h_j(x_j) - h_j(y_j) . \quad (26)$$

If  $(S, y_i, y_j) \neq (S, x_i, x_j)$ , then, by B1, it follows

$$(S, x_1, x_2) \rightarrow (S, y_1, y_2) .$$

Let  $(S, y_i, y_j) \rightarrow (S, x_i, x_j)$  . By S1, (23) and (24), there exist  $(S'', x_i, z_j), (S'', y_i, y_j) \in B_{ij}$  such that

$$(S'', x_i, z_j) \stackrel{\neq}{i} (S'', y_i, y_j) .$$

By B6 and B2, it must hold  $z_j \leq x_j$ . From (25) and (26), we have  $z_j = x_j$ , i.e.,

$$(S'', x_i, x_j) \stackrel{\neq}{i} (S'', y_i, y_j) .$$

Let

$$(T', y_i, y_j) \stackrel{\neq}{i} (T', z_i^!, z_j^!)$$

for some  $(T', y_i, y_j), (T', z_i^!, z_j^!) \in B_{ij}$  and  $(z_i^!, z_j^!) \rightarrow (x_i, x_j)$ .

According to B6, there exist  $(V', x_i, x_j), (V', z_i', z_j') \in B_{ij}$  such that

$$(V', x_i, x_j) \not\rightarrow_i (V', z_i', z_j'),$$

contradicting to B2. Hence,  $(T', z_i', z_j') \leftrightarrow (T', y_i, y_j)$  for any  $(T', z_i', z_j'), (T', y_i, y_j) \in B_{ij}$  if  $(z_i', z_j') \rightarrow (x_i, x_j)$ .

It follows from B3 that

$$(S, x_i, x_j) \rightarrow (S, y_i, y_j) \quad \text{Q.E.D.}$$

Definition 9. We say that  $g_i: X_i \rightarrow K$  is a quasi utility function if, for any  $x_i, y_i \in X_i$ ,

$$g_i(x_i) \geq g_i(y_i) \text{ if and only if } x_i \geq y_i .$$

LEMMA 10. Let  $i \in I, i \neq r$ . Let  $g_i: X_i \rightarrow K$  and  $g_r: U \rightarrow K$  be such quasi utility functions that, for any  $(S, x_i, x_r), (S, y_i, y_r) \in B_{ir}$ ,

$$(S, x_i, x_r) \rightarrow_r (S, y_i, y_r) \text{ if and only if } g_i(x_i) - g_i(y_i) \geq g_r(x_r) - g_r(y_r).$$

Let  $\bar{g}_i: X_i \rightarrow K$  be defined as follows

$$\bar{g}_i(x_i) = g_i(x_i) - g_i(e_i).$$

Then  $\bar{g}_i$  is an o-embedding.

PROOF: From the definitions, it is clear that  $\bar{g}_i$  is injective and  $\bar{g}_i(e_i) = e_r$ .

Let  $x_i, y_i \in X_i$ . According to the definition of the sum, there exist  $(S, e_i, e_r), (S, y_i, y_r), (T, x_i, e_r), (T, x_i + y_i, y_r) \in B_{ir}$  such that

$$(S, e_i, e_r) \leftrightarrow_i (S, y_i, y_r) \text{ and } (T, x_i, e_r) \leftrightarrow_i (T, x_i + y_i, y_r) .$$

According to the hypothesis of the lemma, we must have

$$g_i(x_i+y_i) - g_i(x_i) = g_r(y_r) - g_r(e_r) = g_i(y_i) - g_i(e_i) .$$

Hence,

$$\bar{g}_i(x_i+y_i) = \bar{g}_i(x_i) + \bar{g}_i(y_i) . \quad \text{Q.E.D.}$$

LEMMA 11. A family of quasi utility functions  $g_i: X_i \rightarrow K, i \in I$ , is linear with bargaining if and only if there exist constants  $k_i \in K, i \in I$ , and an o-embedding  $h: U \rightarrow K$  such that, for any  $i \in I$ ,

$$g_i(x_i) = h(h_i(x_i)) + k_i \quad \text{for all } x_i \in X_i .$$

PROOF: The sufficiency is obvious. To prove the necessity, denote  $k_j = g_j(e_j)$ . Let  $i \in I, i \neq r$ . By Lemma 10, the composition  $h = g_i \circ h_i^{-1}$  is an o-embedding and it holds

$$g_i(x_i) = h(h_i(x_i)) + k_i .$$

Let  $j \in I, j \neq i$ , and  $x_j \in X_j$ . According to B5, we can find  $x_i \in X_i$  such that

$$(S, e_i, e_j) \stackrel{i}{\leftrightarrow} (S, x_i, x_j)$$

for some  $(s, e_i, e_j), (S, x_i, x_j) \in B_{ij}$ . Hence, by Lemma 9,

$$h_i(x_i) = h_j(x_j) .$$

By the hypothesis of the lemma,

$$\bar{g}_i(x_i) = g_i(x_i) - g_i(e_i) = g_j(x_j) - g_j(e_j) .$$

That means  $x_i = h_i^{-1}(h_j(x_j))$

and

$$h(h_j(x_j)) = \bar{g}_i(h_i^{-1}(h_j(x_j))) = \bar{g}_i(x_i) = g_j(x_j) - g_j(e_j) .$$

In other words,

$$g_j(x_j) = h(h_j(x_j)) + k_j . \quad \text{Q.E.D.}$$

We introduce now a more simple definition of a bargaining game.

Definition 10. A two-person bargaining game is a pair  $(S, a)$  where  $S$  is the outcome set and  $a$  is the status quo.

Definition 11. A family of functions  $v_i: A \rightarrow K, i \in I$  is called to be linear with bargaining if, for any  $(S, a), (S, b) \in B_{ij}$  where  $i, j \in I, i \neq j,$

$$(S, a) \succ_i (S, b) \text{ if and only if } v_i(a) - v_i(b) \geq v_j(a) - v_j(b) .$$

Let  $u_i: A \rightarrow U, i \in I,$  be the following family of functions:

$$u_i(a) = h_i(x_i)$$

where  $x_i \in X_i$  is the equivalence class under  $\succ_i$  generated by  $a$ .

The following two theorems follow immediately from Lemmas 9 and 11.

Theorem 4. The family of functions  $u_i: A \rightarrow U, i \in I$  is linear with bargaining.

Theorem 5. A family of quasiutility functions  $v_i: A \rightarrow K, i \in I$ , is linear with bargaining if and only if there exist constants  $k_i \in K, i \in I$ , and an  $o$ -embedding  $h: U \rightarrow K$  such that, for any player  $i \in I$ ,

$$v_i(a) = h(u_i(a)) + k_i \quad \text{for all } a \in A .$$

We introduce now our last assumption.

S2. Given any society  $I$ , we can find two players  $i, j \in I$  such that player  $i$  is thrifty with respect to  $j$ .

THEOREM 6. Let  $I$  be a society. There exists a linear with bargaining family of utility functions  $u_i: A \rightarrow R, i \in I$ . Moreover, a family of utility functions  $v_i: A \rightarrow R, i \in I$  is linear with bargaining if and only if, we can find numbers  $\alpha_0, \alpha_i, i \in I$ , such that, for any  $i \in I, v_i(a) = \alpha_0 u_i(a) + \alpha_i$  for all  $a \in A$ .

The existence follows from Theorems 3 and 4. Part two follows from Theorems 3, 5 and from Hions' theorem. It should be noted, that axiom S2 can be strengthened to the following form:

S2'. Given any society  $I$ , we can find an individual  $i \in I$ , whose preferences can be represented by a surjective utility function  $g_i: A \rightarrow R$ .

The assumption that preferences of individuals can be represented by a surjective utility function is made in MYERSON (1977).

Note that, from S2', it follows immediately the existence of a bijective utility function from  $X_i$  to  $R$ .

If  $S_2$  is replaced by  $S_2'$ , then we can use the following statement instead of Theorem 3 in the proof of Theorem 6:

LEMMA 12. Let  $X$  be an ordered group. If there exists an order preserving bijective mapping  $g: X \rightarrow R$  from  $X$  to the additive group  $R$  of real numbers, then  $X$  is o-isomorphic to  $R$ .

PROOF: First,  $X$  is Archimedean. If not, then, for some  $x, y \in X, x > 0$ , it holds

$$mx \leq y$$

for any inter  $m$ . Hence,  $g(mx) \leq g(y)$  for any  $m$ .

Let

$$\alpha = \sup\{g(mx)\} .$$

Since  $g(mx) \leq g(y)$  and  $g$  is surjective, there exists  $m_0$  such that

$$g(m_0 x) = \alpha$$

But

$$(m_0 + 1)x > m_0 x .$$

Hence,

$$g((m_0 + 1)x) > g(m_0 x) = \alpha,$$

contradicting the definition of  $\alpha$ .

So,  $X$  is Archimedean and, according to Hölder's theorem, there exists an o-isomorphism  $h$  from  $X$  to a subgroup  $G \subset R$ , endowed with the natural ordering. It follows that the composition  $\bar{g} = g \cdot h^{-1}$  is an order-preserving bijective mapping from  $G$  to  $R$ .

Let  $\beta \in R$ . Define

$$A = \{\alpha \in G : \alpha < \beta\} \text{ and } B = \{\alpha \in G : \alpha \leq \beta\} .$$

Since  $\bar{g}(A) \cup \bar{g}(B) = R$ , both of them are nonvoid and there exists  $\gamma \in G$  such that

$$\gamma < \alpha \text{ for any } \alpha \in A \text{ and } \gamma \geq \alpha \text{ for any } \alpha \in B.$$

There are two possibilities: either  $\gamma < \beta$  or  $\beta \leq \gamma$ . We will consider only the first case. The other can be considered analogously.

We have

$$\gamma < \beta \leq \alpha \text{ for any } \alpha \in A \tag{27}$$

If  $\gamma \in A$ , then  $\gamma = \beta$ , i.e.,  $\beta \in G$ . If  $\gamma \notin A$ , then we can construct a sequence  $\epsilon_k \in G, \epsilon_k > 0, k=1,2,\dots$  in the following way.

Choose an arbitrary  $\alpha_1 \in A$  and set  $\epsilon_1 = \alpha_1 - \gamma > 0$ .

Define

$$\alpha_{k+1} = \bar{g}^{-1} \left( \frac{\bar{g}(\gamma + \epsilon_k) + \bar{g}(\gamma)}{2} \right)$$

and

$$\epsilon_{k+1} = \min(\alpha_k - \alpha_{k+1}, \alpha_{k+1} - \gamma) .$$

It is obvious that  $\gamma < \alpha_{k+1} < \alpha_k$ . Hence,  $\alpha_k \in A$  for any  $k=1,2,\dots$  and  $0 < \epsilon_k < 2^{-k+1}(\alpha_1 - \gamma)$ . It follows from (27) that

$$\gamma < \beta < \gamma + \epsilon \text{ for any } \epsilon > 0 .$$

Consequently,  $\beta = \gamma$ , i.e.,  $\beta \in G$ . Since  $\beta \in R$  has been chosen arbitrarily, we have proved that  $G = R$ . Hence,  $X$  is  $\sigma$ -isomorphic to  $R$ . Q.E.D.

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