EXISTENCE OF ECONOMIC EQUILIBRIA
IN
DISCRETE NON-TATONNEMENT MODELS

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ABSTRACT

This paper is intended to describe a monetary pure exchange economy in mathematical terms. Three main characteristics of the approach presented are to be mentioned;

(i) analysis is carried out in a discrete time framework,

(ii) a non-tâtonnement process is assumed to represent the dynamics of the economic system,

(iii) the problems of the existence and the economic properties of equilibrium states are mainly focused on.

By this special choice of methods some results are obtained which justify the application of fixed point algorithms for the purpose of approximate calculation of particular non-tâtonnement equilibria.

Investigations start with a general model of a non-tâtonnement economy for which the existence of an equiblibria can be proved. In order to get economically interpretable results a special non-tâtonnement model is constructed by transforming a set of economic assumptions into mathematical relations. An auxiliary system derived from this special model is then thoroughly analyzed using classical fixed point theorems.

From the properties established for the auxiliary model equilibrium theorems concerning the "underlying" economy are obtained .

Contents

0. Introduction.

Quite recently various attempts have been made to generalize the well-known equilibrium and stability results obtained for the classical Walrasian economy ([1] ,pp.1o7-128, 263-323) in order to include economic systems which do not meet all the "Walrasian" conditions ([6] , [7] , [8] , [9] , [12] , [13] , [24]). Among these the so-called non-tatonnement case deserves special interest ([1] ,pp.324-346; 9 ,pp.2o7-227; 25 ,pp.339-347). The economic systems which are considered in this paper are economies of the non-tâtonnement type which means that in addition to the price mechanism adjustment of quantities through goods transactions is assumed to take place. We confine ourselves to the most simple case of pure exchange economies with money being incorporated. Chapter 13 of the Arrow-Hahn book ([1]) may be taken as a starting point of the analytic formulation and treatment, though the important question of stability of the adjustment process is not touched in what follows. Investigations are exclusively concerned with the problem of the existence of an equilibrium and with its economic properties. The approach presented in the following was mainly chosen for the purpose of enabling actual computation of equilibria by use of recently developed fixed point algorithms ([4] , [5] , [17] , [18] , [23]), though computational treatment of the models considered is left to further work.

The most important economic assumptions for our model can be stated briefly as follows: As already mentioned the economy to be considered is one of the pure exchange type which means that production and consumption do not appear explicitly in the model. A certain number of goods (m) is distributed among a certain number of individuals (n). Individual demands for those goods are supposed to be derived originally from a Cobb-Douglas utility function, but several other types of utility functions could be used too. The phenomenon which is to be analyzed is a dynamic adjustment process taking place within the "economy" and being determined by individual and aggregate demands. This process consists of simultaneous and independent adjustment of prices of goods - expressed in money terms - and quantities which are goods endowments of individuals - and will be described in a discrete-time framework. During a "unit" time interval of adjustment (the length of which will not be specified)

(i) commodity prices move according to a "law of supply and demand"

(ii) quantities (individual goods assets) change through transactions which are thought to be carried out on each of the single commodity markets.

The special good money is used as a numeraire and as an exchange medium. Of course certain transaction rules have to be adhered to which will be discussed later on.

For an economic model constructed under these assumptions the existence of a distinguished set of prices and goods endowments can be shown, this state being at least a "trading equilibrium" and at most a "complete economic equilibrium" (pp. 45, 51) of the economy. Given the paramenters of the model this particular state can be determined approximately by computation, whence its economic meaning is obtained immediately.

Chapter 1 is preliminary in the sense that it provides the mathematical tools of description. Regardless of computational aspects a general form of the model is given and an almost evident existence statement is formulated.

Chapter 2 is exclusively concerned with the translation of certain economic assumptions into functional relations. The functions are given in explicit analytic form in order to render possible computability of a desired fixed point. Because of mathematical reasons an auxiliary system is considered using modified "demand" functions and a very special transaction function.

In Chapter 3 the existence of an equilibrium state of the artificial system of Chapter 2 is proved. Then the consequences of this result for the underlying "original" economy are stated.

Chapter 4 contains another version of the non-tâtonnement model including a trading correspondence instead of transaction functions. It is shown that the results of the previous chapter hold still in that case.

In Chapter 5 a general type of price adjustment is introduced and a corresponding equilibrium result is proved. Finally some hints are given how to perform the computation of the discussed euqilibria by relatively simple fixed point algorithms.

Thus the discussion starts with the most general formulation of the problem, then turns to a very special case in order to obtain results of the desired "computable" form and finally goes back towards increased generality by relieving some of the restrictive assumptions of Chapter 2.

Aknowledgement
I am grateful to Mrs.I.Hafner who has typed the manuscript with all its tedious mathematical notation.

1. GENERAL MODEL

The economic system to be investigated is a rather simple one, but it turns out that the thorough mathematical treatment of even such "primitive" economies tends to become somewhat complex.

What we are to consider in what follows is a so-called monetary exchange economy, that is a pure trading economy without production and consumption of goods but with money being existent. Using another interpretation an "exchange economy" may be described as an economic system, where - concerning aggregate amounts of goods - production and consumption remain in a stationary state over time ([1], pp.337).

This extremely simplified model of an economy will allow to focus on the analytic and numerical treatment of the relations between demands for goods, price movements and transaction activities. Of course it can only be taken as a starting point for further investigations of economic models being closer to reality.

The present chapter is concerned in its first part with the mathematical formulation of a general model of the monetary pure exchange economy including the general functional form of the various adjustment processes.
A second part contains an almost evident existence theorem for a certain kind of economic equilibrium of the non-tâtonnement type.

1.1. Definitions

Let the economy under consideration be constituted by (m-1) non-monetary commodities ($m \in \mathbb{N}$), one particular good namely money and n trading individuals ($n \in \mathbb{N}$). The good money is denoted by the integer 1, each non-monetary good by an integer i ($i=2,3,\ldots,m$), each of the individuals let be denoted by an integer j ($j=1,2,\ldots,n$).

The following definitions seem to be useful for a concise description of the model in mathematical terms:

i) Scalars:

For $i = 1,2,\ldots,m$ and $j = 1,2,\ldots,n$ let

$\bar{x}_{ij} \in \mathbb{R}_+$ be j's endowment with good i ,

$x_{ij} \in \mathbb{R}$ j's demand for i,

$z_{ij} \in \mathbb{R}$ j's planned (long-term) excess demand for i (called "target excess demand") ([1] , pp. 339),

$a_{ij} \in \mathbb{R}$ j's active excess demand for i ([1], pp.340),

$p_i \in \mathbb{R}_+$ the price of good i in money terms.

ii) Vectors:

For each j ($j=1,2,\ldots,n$) one may define the following vectors of amounts and prices of non-monetary goods (for which the name commodities will be reserved):

$$\underline{\bar{x}}_j = (\bar{x}_{.j}) \in \mathbb{R}_+^{m-1} ,$$

$$\underline{x}_j = (x_{.j}) \in \mathbb{R}^{m-1} ,$$

$$\underline{z}_j = (z_{.j}) \in \mathbb{R}^{m-1} ,$$

$$\underline{p} = (p_{.}) \in \mathbb{R}_+^{m-1} ,$$

where $(x_{.j})$ is to be understood as the (m-1) - vector

$$(x_{2j}, x_{3j}, \dots, x_{mj})' ,$$

iii) Aggregate amounts:

For $i = 1,2,\dots,m$ let $\bar{x}_i$ be given by

$$\bar{x}_i = \sum_{j=1}^{n} \bar{x}_{ij}$$

and x_i, z_i, a_i be defined analogously.

Because of the pure exchange property of the economy $\bar{x}_i$ has a fixed value over the entire time range.

Finally let the (m-1) - vector $\bar{\underline{x}}$ be defined according to

$$\bar{\underline{x}} = (\bar{x}_2 , \bar{x}_3 , \dots, \bar{x}_m)' ,$$

Using the above listed definitions one may try to describe the dynamics of a monetary pure exchange economy in terms of a discrete - time model establishing the dynamic relations in first order difference equation form. Section 1.2. will give the most general form of those difference equation models derived from a set of assumptions about price dynamics and trading behaviour of individuals. Specification based on further assumptions is left to Chapter 2.

1.2. Price - Quantity Set $\mathcal{C}$.

Dynamics of the model to be described will be understood as adjustment of both commodity prices and assets according to existing demands. An elementary step of dynamic movement consists of a transformation of one combination of commodity prices and goods assets at time τ into another one at time $(\tau+1)$. Time points τ and $(\tau+1)$ may be taken e.g. as the beginnings of two consecutive time intervals of "unit length" during which adjustement takes place.

Since good 1 (money) is to serve as a numeraire, its price is assumed to remain constant over time ($p_1(\tau)=1$ for $\tau=1,2,\ldots$), and therefore the variable p_1 is excluded from the description of dynamics.

The space to be considered is Euclidean space $\mathfrak{X} = \mathbb{R}^{m(n+1)-1}$, the vectors $\underline{x} \in \mathfrak{X}$ being subject to the following interpretation:

$$\underline{x} = (x_1, x_2, \ldots, x_{m(n+1)-1})' =$$

$$= (p_2, p_3, \ldots, p_m, \bar{x}_{11}, \bar{x}_{21}, \ldots, \bar{x}_{mn})' .$$

Thus $\mathfrak{X}$ is to be understood economically as the space of cominations of commodity prices and goods assets, briefly called the price-quantity space. Denoting time by the superscript τ the elementary step of adjustment may be described by a transformation of a vector $\underline{x}^{\tau} \in \mathfrak{X}$ into another vector $\underline{x}^{\tau+1} \in \mathfrak{X}$ according to transformation rules which have to be specified later on.

Mathematically the adjustment step will be given by a function f mapping a subset $\mathcal{C} \subset \mathfrak{X}$ into $\mathfrak{X}$

(1) $$f: \mathcal{C} \longrightarrow \mathfrak{X}, \qquad \underline{x}^{\tau} \longmapsto \underline{x}^{\tau+1}$$

$$\underline{x}^{\tau+1} = f(\underline{x}^{\tau}),$$

$$(\tau = 1,2,\ldots),$$

where $\mathcal{C}$ has to be defined below.

Because of the economic meaning of the vector $\underline{x}^{\tau} \in \mathfrak{X}$ mentioned above the following assumptions are supposed to hold for every $\tau=1,2,\ldots$

Assumption 1:

Commodity prices $p_2, p_3, \ldots, p_m$ measured in terms of money ($p_1 = 1$) are normalized subject to

$$(2) \qquad \sum_{i=2}^{m} p_i^\tau \bar{x}_i = \bar{x}_1 \qquad (\tau = 1,2,\ldots)$$

guaranteeing equality between the total money value of commodities and total money supply of the economy.

Assumption 2:

The total quantity of commodity - i - assets equals a constant value $\bar{x}_i$, namely the total amount of commodity i available in the economy (i=1,2,...,m)

$$(3) \qquad \sum_{j=1}^{n} \bar{x}_{ij}^\tau = \bar{x}_i \qquad (i = 1,2,\ldots,m \; ; \; \tau = 1,2,\ldots)$$

These two assumptions together with the self-explaining non-negativity of commodity assets lead to the following definitions of subsets of $\mathcal{X}$:

$$(4a) \qquad \mathcal{A} = \{\underline{x} \in \mathcal{X} \mid \sum_{i=2}^{m} p_i \bar{x}_i = \bar{x}_1 ; p_i \geq 0, \; i = 2,3,\ldots,m\}$$

$$(4b) \qquad \mathcal{B}_i = \{\underline{x} \in \mathcal{X} \mid \sum_{j=1}^{n} \bar{x}_{ij} = \bar{x}_i \; , \; \bar{x}_{ij} \geq 0, \; j = 1,2,\ldots,n\}$$

$i = 1,2,\ldots,m$,

where $\bar{x}_i$ is the given constant value of the total quantity of good i.

$$(4c) \qquad \mathcal{C} = \mathcal{A} \cap \bigcap_{i=1}^{m} \mathcal{B}_i$$

Assumptions 1 and 2 imply that the function f is a mapping of the set $\mathcal{C}$ into itself.

1.3. Functional Relationships

So far we have only postulated adjustment between two successive time points to take place as a transformation of a price-quantity-vector $\underline{x}^{\tau}$ into a vector $\underline{x}^{\tau+1}$ by $\underline{x}^{\tau+1} = f(\underline{x}^{\tau})$ without saying anything about the nature of the function f. For the purpose of further analysis of the model it will be necessary to specify f in an economically admissible way using a set of behavioural assumptions.

First of all it may be of interest to give a short verbal description of the economic behaviour the individuals of the system are supposed to show. It seems to be unnecessary to point out that the following statements reflect mere assumptions about the type of economic activity to be analyzed.

The starting point of dynamic movement in the system is a given set-up of commodity prices and goods assets held by the individuals represented by a vector $\underline{x} \in \mathcal{X}$. In a first step each individual j may be supposed to express demand for good i (i=1,2,...,m) x_{ij} subject to an individual budget constraint which takes into account the total money value of j's assets of commodities and his money holdings. Demand x_{ij} may be obtained by maximization of an individual utility function of j under that budget constraint. Target excess demand $z_{ij} = x_{ij} - \overline{x}_{ij}$ for i by j is consequently given as a function of prices and commodity assets.

In order to keep the model simple the rôle of money is restricted to serving as an exchange medium. The crucial assumption that every transaction has to take place as an exchange of a commodity for money - bartering of commodity for commodity being impossible - gives rise to a so-called financial constraint for each individual. This means that individual j's ($j=1,2,\ldots,n$) purchases are limited by his actual holding of money. Owing to this additional condition target excess demand z_{ij} is transformed into so-called active excess demand a_{ij} giving the extent of transactions of i in which j could engage willingly if he faced only his own individual constraints. Active excess demands are assumed to be identical with the demands expressed by the individuals in the market. Therefore active excess demands determine both goods transactions and price movements. How one could think of the nature of this determination will be demonstrated in Chapter 2. The behavioural rules outlined so far can easily be written in mathematical terms by use of some auxiliary functions.

A thorough discussion of the assumptions sketched above can be found in [1] , pp.337-346 , where also the terminology is taken from.

1.3.1. Functions Constituting the Adjustment Process.

For the construction of the desired function which describes the movement from $\underline{x}^{\tau}$ to $\underline{x}^{\tau+1}$ we need some auxiliary functions which map certain subsets of $\mathcal{X}$ into $\mathbb{R}$. The following functional relations are based on the economic arguments of the foregoing section:

(5a) $$z_{ij} : \mathcal{C} \longrightarrow R \qquad (\mathcal{C} \subseteq \mathcal{X})$$

$$\underline{x} \longmapsto z_{ij}(\underline{x})$$

z_{ij} determines target excess demand for good i by individual j as a function of $\underline{x} \in \mathcal{C}$,

$$\underline{x} = (x_1, x_2, \ldots, x_{m(n+1)-1})' =$$

$$= (p_2, p_3, \ldots, p_m; \bar{x}_{11}, \bar{x}_{21}, \ldots, \bar{x}_{mn})'$$

with $\mathcal{C}$ being the set of feasible price-quantitiy-vectors defined by (4c).

(5b) $a_{ij}: \mathcal{D} \longrightarrow \mathbb{R} \qquad (\mathcal{D} \subseteq \mathcal{X})$

$\underline{x} \longmapsto a_{ij}(\underline{x})$

$\mathcal{D} \subseteq \mathcal{X}$ is the set of all feasible price-target excess demand-vectors.

$$\mathcal{D} = \{\underline{x} \mid \underline{x} = (\tilde{p}_2, \tilde{p}_3, \ldots, \tilde{p}_m;\ z_{11}(\underline{\tilde{x}}), z_{21}(\underline{\tilde{x}}), \ldots, z_{mn}(\underline{\tilde{x}}))';\ \underline{\tilde{x}} \in \mathcal{C}\}.$$

Thus a_{ij} represents active excess demand for i by j as a function of $\underline{x} \in \mathcal{D}$.

(5c) $t_{ij}: \mathcal{E} \longrightarrow \mathbb{R} \qquad (\mathcal{E} \subseteq \mathcal{X})$

$\underline{x} \longmapsto t_{ij}(\underline{x})$

$\mathcal{E} \subseteq \mathcal{X}$ is defined as the set of all feasible price-active excess demand-vectors.

$$\mathcal{E} = \{\underline{x} \mid \underline{x} = (\tilde{\tilde{p}}_2, \tilde{\tilde{p}}_3, \ldots, \tilde{\tilde{p}}_m;\ a_{11}(\underline{\tilde{\tilde{x}}}), a_{21}(\underline{\tilde{\tilde{x}}}), \ldots, a_{mn}(\underline{\tilde{\tilde{x}}}))';\ \underline{\tilde{\tilde{x}}} \in \mathcal{D}\}.$$

Thus t_{ij} determines the quantity t_{ij} of good i transacted by individual j as a function of prices and active excess demands.

The functions defined by (5a-5c) shall be given for $i=1,2,\ldots,m$; $j=1,2,\ldots,n$. Excess supply of a commodity is written as negative excess demand, and of course purchases (sales) of goods are represented by positive (negative) transaction quantities. Definitions (5a-5c) imply the assumption that target excess demand z_{ij}, active excess demand a_{ij}, and transaction quantity t_{ij} are functions - not correspondences - of the respective variables. For the case of transaction quantities t_{ij} a generalization using a trading correspondence will be carried out in one of the following chapters.

For i=2,3,...,m we define

(5d) $$dp_i : \mathcal{E} \longrightarrow \mathbb{R} \qquad (\mathcal{E} \subseteq \mathcal{X})$$

$$\underline{x} \longmapsto dp_i(\underline{x}) \ ,$$

where the change of the price of commodity i $dp_i(\underline{x})$ shall be determined as a function of current prices and actual active excess demands. The well-known "law of supply and demand" can be taken as an example for such a relationship.

Resonable conditions to be fulfilled by the given functions and an example of specifying their exact form will be discussed later on.

1.3.2. Composition of the Adjustment Function f .

Using Definitions (5a-5d) new functions can be constructed, each of them mapping a subset of $\mathcal{X}$ into a subset of $\mathcal{X}$. A vector $\underline{x} \in \mathcal{X} = \mathbb{R}^{m(n+1)-1}$ is again written $\underline{x} = (x_1, x_2, \dots, x_{m(n+1)-1})'$ Then the following functions are well-defined:

(6a)

$$Z : \mathcal{C} \longrightarrow \mathcal{X}$$

$$\underline{x} \longmapsto Z(\underline{x}) \qquad (\underline{x} \in \mathcal{C})$$

$$Z_k(\underline{x}) = \begin{cases} x_k = p_{k+1} & \text{if } 1 \leq k < m \\ z_{ij}(\underline{x}) \quad j = \left[\!\left[\frac{k-m}{m}\right]\!\right] + 1 & \text{if } m \leq k < m(n+1)-1 \\ \qquad\quad i = k - jm + 1 & \end{cases}$$

From (5a) and (5b) it is readily seen that $Z(\mathcal{C}) = \mathcal{D}$.

(6b) $\quad A: \quad \mathcal{D} \longrightarrow \mathcal{X}$

$$\underline{x} \longmapsto A(\underline{x}) \qquad (\underline{x} \in \mathcal{D})$$

$$A_k(\underline{x}) = \begin{cases} x_k & \text{if } 1 \leqslant k < m \\ a_{ij}(\underline{x}) & \text{if } m \leqslant k \leqslant m(n+1)-1 \end{cases}$$

j,i as described above

From (5b) and (5c) we get $A(\mathcal{D}) = \mathcal{E}$.

(6c) $\quad D: \quad \mathcal{E} \longrightarrow \mathcal{X}$

$$\underline{x} \longmapsto D(\underline{x}) \qquad (\underline{x} \in \mathcal{E})$$

$$D_{k(\underline{x})} = \begin{cases} dp_{k+1}(\underline{x}) & \text{if } 1 \leqslant k < m \\ t_{ij}(\underline{x}) & \text{if } m \leqslant k \leqslant m(n+1)-1 \end{cases}$$

j,i as described above

The economic meaning of the functions (6a) - (6c) is evident: Z e.g. associates with every feasible price-assets-vector $\underline{x} \in \mathcal{C}$ a vector of (unaltered) prices and target excess demands $Z(\underline{x})$ in a unique way. The mappings A and D may be interpreted analogously.

From the foregoing remarks it is clear that the functions Z,A and D permit composition in the following way yielding a new function

(7) $\quad \triangle : \mathcal{C} \longrightarrow \mathcal{X}$

$$\triangle = D \circ A \circ Z$$

Recalling the economic meaning of the various functions and assuming that all changes of economic variables considered so far are changes from a value at time τ to a new value at time $(\tau+1)$ the function $\triangle$ is seen to represent the difference between the price-quantity vector $\underline{x}^{\tau+1}$ and its predecessor $\underline{x}^{\tau}$. Starting with a vector $\underline{x} \in \mathcal{E}$ adjustment of prices and quantities can be achieved by the mapping

(7a) $$F : \mathcal{E} \longrightarrow \mathcal{X}$$

$$\underline{x} \longmapsto F(\underline{x}) = \underline{x} + \triangle(\underline{x}) \ ,$$

where time superscripts are omitted.
(The difference equation form of the model is now given by

$$\underline{x}^{\tau} - \underline{x}^{\tau-1} = \triangle(\underline{x}^{\tau-1}) \ ) \ .$$

It can be shown that under certain additional assumptions about the functions defined above F becomes an adjustment function for the special case of a monetary pure exchange economy. Then it will be called f according to (1) .

1.4. Existence of a Fixed Point of f.

Because of the pure exchange property of the economy considered transaction functions t_{ij} of Definition (5c) are assumed to be subject to the linear relations

$$\sum_{j=1}^{n} t_{ij} \equiv 0 \qquad\qquad (i = 1,2,\ldots,m) \ .$$

Further let us assume the price change functions dp_i of Definition (5d) to fulfil

$$\sum_{i=2}^{m} dp_i \ \bar{x}_i \equiv 0$$

which ensures that the new prices are subject to the normalizing conditions of Assumption 1.

The economic assumptions can be written in terms of the function $\triangle$ of Definition (7) as

Assumption 3:

For every $\underline{x} \in \mathcal{C}$ and for every $i=1,2,\ldots,m$

$$\sum_{k=0}^{n-1} \triangle_{(m-1)+i+km}(\underline{x}) = 0$$

Assumption 4:

For every $\underline{x} \in \mathcal{C}$

$$\sum_{k=1}^{m-1} \bar{x}_{k+1} \triangle_{k}(\underline{x}) = 0$$

Assumption 5:

For every $\underline{x} \in \mathcal{C}$ and for every $k=1,2,\ldots,m(n+1)-1$

$$x_k + \triangle_k(\underline{x}) \geqslant 0 .$$

From (4a) - (4c) and (7), (7a) it follows that with Assumptions 3,4 and 5 being valid the function F maps the set $\mathcal{C}$ into itself. Hence with the three assumptions F is seen to be a dynamic adjustment function of a monetary pure exchange model as described in Section (1.2.) and therefore it will be written f .

Recalling the fact that f describes the elementary step of a dynamic process one might be interested in finding the stationary states or equilibria of the system investigated. A vector $\hat{\underline{x}}$ will be called a dynamic equlibrium of the adjustment process f if

$$(8) \quad \underline{x}^{\tau} = \hat{\underline{x}} \Longrightarrow \underline{x}^{\tau+1} = f(\underline{x}^{\tau}) = \hat{\underline{x}} \qquad (\tau \in \mathbb{N})$$

or equivalently

(9) $f(\hat{\underline{x}}) = \hat{\underline{x}}$

If a time path of $\underline{x}^\tau$s reaches $\hat{\underline{x}}$, it will stick to that value for the whole future of development. From (9) it is clear that for $\underline{x} \in \mathcal{E}$ the property of being an equlibrium point of adjustment is equivalent to being a fixed point of the single-valued function f. Thus the question of the existence of a stationary state of the adjustment process reduces to the problem of the existence of a fixed point of the function f.

One gets the main result:

Theorem 1:

Assume that for the functions defined above

a) Assumptions 3,4 and 5 hold ,

b) the functions of

target excess demand	z_{ij}	for $1 \leq i \leq m$, $1 \leq j \leq n$
active excess demand	a_{ij}	
transaction	t_{ij}	
price change	dp_i	

defined according to (5a-d) are continuous on their respective domains.

Then there exists a fixed point $\hat{\underline{x}}$ of the adjustment function f yielding

$$f(\hat{\underline{x}}) = \hat{\underline{x}} \quad .$$

Proof:

To prove this assertion Brouwer's Fixed Point Theorem ([21] , pp.3o3-3o8) is to be used. Therefore one has to show the following three facts:

α) f maps the set $\mathcal{C}$ into itself ,

β) $\mathcal{C} \subseteq \mathcal{X}$ is a bounded, closed, convex set ,

γ) the function f is continuous on $\mathcal{C}$.

α) From the Assumptions 3 and 4 we have for $\underline{x} \in \mathcal{C}$

$$\sum_{k=1}^{m-1} \bar{x}_{k+1} \cdot f_k(\underline{x}) = \bar{x}_1 \quad ,$$

and

$$\sum_{k=0}^{n-1} f_{(m-1)+i+km}(\underline{x}) = \bar{x}_i \qquad (1 \leq i \leq m) \quad ,$$

and because of Assumption 5

$$f(\underline{x}) \geq \underline{0} \; .$$

Hence

$$f(\underline{x}) \in \mathcal{A}, \; f(\underline{x}) \in \mathcal{B}_i \quad \text{for} \quad i=1,2,\ldots,m$$

and consequently (see (4a-c))

$$f(\underline{x}) \in \mathcal{C} \; , \text{ if } \underline{x} \in \mathcal{C} \; .$$

β) Let $\underline{x}$ be e point of $\mathcal{C}$. Then the coordinates of $\underline{x}$ are subject to the (m+1) linear constraints

$$\sum_{k=1}^{m-1} \bar{x}_{k+1} \cdot x_k = \bar{x}_1 \qquad (\bar{x}_i > 0 \text{ for } 1 \leq i \leq m)$$

$$\sum_{k=0}^{n-1} x_{(m-1)+i+km} = \bar{x}_i \qquad (1 \leq i \leq m)$$

with $\underline{x} \geq \underline{0}$.

Therefore each coordinate x_k ($1 \leq k \leq m(n+1)-1$) of the points of $\mathcal{C}$ has to be bounded and $\mathcal{C}$ is a bounded subset of $\mathcal{X} = \mathbb{R}^{m(n+1)-1}$.

Definitions (4a-c) show immediately that $\mathcal{A}$ and $\mathcal{B}_i$ $(1 \leq i \leq m)$ are closed in $\mathcal{X}$ and such is $\mathcal{C}$. Finally the intersection $\mathcal{C}$ of the convex sets $\mathcal{A}$, $\mathcal{B}_i$ $(1 \leq i \leq m)$ is convex.

γ) From the Definitions (6a-c) and from Assumption b) the functions Z, A and D are seen to be continuous on their respective domains. Hence their composition Δ = DoAoZ is a continuous function on $\mathcal{C}$ and likewise $f=I+\Delta$ (where I denotes the identity function on $\mathcal{C}$) is continuous on $\mathcal{C}$.

Brouwer's Theorem then ensures the existence of a fixed point $\hat{\underline{x}}$ of f with

$$f(\hat{\underline{x}}) = \hat{\underline{x}} \qquad \square$$

1.5.) Summary

A short review of the foregoing chapter might be useful to determine the stage of development of the present investigation.

First of all it was intended to provide for the mathematical framework in which a simple pure exchange economy could be described. Then this decription was performed in two steps: first giving the discrete time model of economic dynamics in its general difference equation form, second focusing on the elementary adjustment step which was written as a function mapping Euclidian space of appropriate dimension into itself. Within the given model this adjustment function was constructed in its most general form reflecting the effect of demands on changes in prices and goods assets. Imposing the pure exchange conditions explicitly upon the system and assuming continuity of the constituting economic functions theexistence of a dynamic equilibrium of the economic adjustment process was shown.

2. EXCESS DEMAND, TRANSACTIONS AND PRICE ADJUSTMENT.

In Chapter 1 the model of a monetary pure exchange economy was given in fairly general terms without specifying the precise functional form of the various relationships.

Now a concise definition of the analytic form of the functions used will be given based on a detailed explanation of economic behaviour in this context. Thus an example of a dynamic economic system shall be constructed which both reflects plausible behavioural conditions and allows the application of Theorem 1 in order to prove the existence of an equilibrium state of the economy being analyzed.

Bearing in mind the convenience of notation the general economic system treated so far may be denoted as

(1) $$\mathcal{E} = (\mathcal{C}, f)$$

the ordered pair of the price-quantity-set $\mathcal{C}$ and the adjustment function f. $\mathcal{E}$ will briefly be called an economy.

2.1. Demand Functions.

With respect to the application of Theorem 1 to a specified model it will be necessary to determine the definite form of the functions z_{ij}, a_{ij}, t_{ij} and dp_i under the condition of economic significance.

Let us start with z_{ij} the function of individual target excess demand.

Denoting demand for good i by individual j as x_{ij} target excess demand takes on the form

(2) $$z_{ij} = x_{ij} - \bar{x}_{ij}$$

Therefore one has to obtain x_{ij} as a function of $\underline{x}$, that is of prices and current goods assets. The present section will be concerned with this problem. First of all two crucial assumptions about the rôle of money - already mentioned in the previous chapter - are to be restated. Though reflecting rather primitive monetary theory they seem to be acceptable in our case.

Assumption 1:

Money (good 1) is referred to as a "numeraire" the price of which is held constant over time at the value $p_1=1$ and with respect to which relative prices of commodities are calculated. Total money supply $\overline{x}_1$ is also a constant.

Assumption 2:

Money serves as a medium of exchange such that every transaction is performed as an exchange of a certain quantity of a commodity for an "equivalent" amount of money equivalence being determined with respect to current prices.

Except these two properties of the good money there shall be no other properties from which the owner of money could derive utility directly.

2.1.1. Demand for Money.

As a basic rule of behaviour it is assumed that each individual intends to reach a "better" commodity endowment - in terms of individual utility - in the future by means of transactions. Hence for the purpose of being able to improve their endowments individuals are induced to hold a certain minimum amount of money. Following a simple microeconomic concept ([15],p.174) desired money stock is assumed to be a fixed proportion of individual "commodity wealth" evaluated at the current price system:

(3) $$x_{1j} = \alpha_{1j} \langle \underline{p} , \overline{\underline{x}}_j \rangle \qquad 0 < \alpha_{1j} < 1$$

$$(j = 1,2,\ldots,n) ,$$

where α_{1j} is a measure of j's inclination to hold money.

2.1.2. Demand for Commodities.

In order to obtain $x_{ij}(\underline{x})$ - j's demand for good i ($2 \leq i \leq m$) as a function of current prices and commodity assets - the following assumptions seem to make sense:
Let us think of individual commodity demand as being derived from a Cobb-Douglas-utility function

(4) $$u_j = \prod_{i=2}^{m} x_{ij}^{\alpha_{ij}} \quad ; \quad \alpha_{ij} \geq 0, \ \sum_{i=2}^{m} \alpha_{ij} = 1$$

(u_j being j's individual utility)

by maximization under the well-known budget constraint

(5) $$\langle \underline{p} , \underline{x}_j \rangle + x_{1j} = \langle \underline{p} , \overline{\underline{x}}_j \rangle + \overline{x}_{1j} .$$

This procedure leads to the demand function

(6) $$x_{ij}(\underline{x}) = \left[(1-\alpha_{1j}) \langle \underline{p} , \overline{\underline{x}}_j \rangle + \overline{x}_{1j} \right] \frac{\alpha_{ij}}{p_i} ,$$

where α_{ij} characterizes the intensity of demand for i by j. ([23],pp.19).

The fact that $x_{ij}(\underline{x})$ tends to infinity if $\alpha_{ij} > 0$, $\overline{x}_{1j} > 0$ and $p_i \rightarrow 0$ causes severe difficulties for the analytic treatment of equilibrium properties of the model. Therefore $x_{ij}(\underline{x})$ as defined in (6) will be replaced by an auxiliary "demand" function $x^*_{ij}(\underline{x})$ which is well-behaved but does not satisfy all postulates theoretically claimed for a demand function. Justification and consequences of this apparently unavoidable modification of demand will be discussed later on.

Finally one assumption has to be made which cannot be discard in the present context:

Assumption 3:

None of the trading individuals will spend all his money in purchasing commodities; j will carry out transactions only to such an extent that his money stock $\bar{x}_{1j}$ will not fall below an individual lower bound $\varepsilon_j > 0$ being close to zero.

Hence money holdings $\bar{x}_{1j}$ are supposed to meet the condition

(7) $$\bar{x}_{1j} \geqslant \mathcal{E}_j > 0 \quad (j=1,2,\ldots,n) .$$

Obviously (7) brings about a slight variation of the domain of the demand functions. Regarding Assumption 3 as the domain of demand functions we have to consider now

(8) $$\mathcal{C}^* = \mathcal{C} \cap \mathcal{B}^* \quad , \quad \mathcal{C}^* \subseteq \mathcal{X} ,$$

where $\mathcal{B}^* = \{\underline{x} \in \mathcal{X} \mid x_{km} \geqslant \mathcal{E}_k \ , \ 1 \leqslant k \leqslant n\}$

with $\mathcal{E}_k > 0$ being given constants close to 0.

Since confusion is unlikely the desired auxiliary "demand" function $x^*_{ij}(\underline{x})$ will be called demand function although it does not represent demand in the strict sense of demand theory ([22], pp.32-56; [2]).

A possible form of $x^*_{ij}(\underline{x})$ may be the following one:

(9) For $2 \leq i \leq m$ and $1 \leq j \leq n$

$$x^*_{ij}(\underline{x}) = \begin{cases} \text{a)}\ 0 & , \text{ if } \alpha_{ij} = 0 ; \\ \text{b)}\ [(1-\alpha_{1j}) \langle \underline{p}, \underline{\overline{x}}_j \rangle + \overline{x}_{1j}] \frac{\alpha_{ij}}{p_i} & , \text{ if } p_i \geq h_{ij}(\underline{p}_i,\underline{\overline{x}}_j) \text{ and } \alpha_{ij} > 0 ; \\ \text{c)}\ K.\overline{x}_i & , \text{ if } p_i < h_{ij}(\underline{p}_i,\underline{\overline{x}}_j) \text{ and } \alpha_{ij} > 0 ; \end{cases}$$

where $\alpha_{ij} \geq 0, \quad \sum_{i=2}^{m} \alpha_{ij} = 1 \ (j=1,2,\ldots,n)$,

$K \gg 1$,

$\underline{p}_i = (p_2, p_3, \ldots, p_{i-1}, p_{i+1}, \ldots, p_m)'$,

$$h_{ij}(\underline{p}_i,\underline{\overline{x}}_j) = \frac{[(1-\alpha_{1j}) \sum_{k \neq i} p_k \overline{x}_{kj} + \overline{x}_{1j}] . \alpha_{ij}}{K.\overline{x}_i - (1-\alpha_{1j}) \alpha_{ij} \overline{x}_{ij}}$$

and $\underline{x} \in \mathcal{C}^*$.

The function $x^*_{ij}(\underline{x})$ is seen to be identical with the original demand function of (6) in case b). Only for a very low price p_i of good i demand x^*_{ij} for i is artificially bounded from above by the constant $K.\overline{x}_i$ according to (9c). Obviously the higher the constant K is chosen the "larger" is the set of $\underline{x}$vectors for which $x^*_{ij}(\underline{x}) = x_{ij}(\underline{x})$. Straightforward calculations show that the demand function $x^*_{ij}(\underline{x})$ is uniformly bounded on the set $\mathcal{C}^*$ with the least upper bound being given by $K.\overline{x}_i$ if $\alpha_{ij} > 0$.

2.1.3. Target Excess Demand.

Taking into account his present wealth both of commodities and of money and disregarding real transaction conditions individual j may express his so-called target excess demand ([1], pp.339) for good i in the form

(1o)

$$z^*_{ij}(\underline{x}) = \begin{cases} \max \left[x^*_{ij}(\underline{x}), \varepsilon_j\right] - \bar{x}_{ij}, & \text{if } i=1 ; \\ x^*_{ij}(\underline{x}) - \bar{x}_{ij}, & \text{if } 2 \leqslant i \leqslant m \end{cases}$$

$$\text{for } \underline{x} \in \mathcal{C}^*, \quad 1 \leqslant j \leqslant n .$$

Money target excess demand $z^*_{1j}(\underline{x})$ $(1 \leqslant j \leqslant n)$ is constructed in this special manner with respect to the "non-bankruptcy-condition" (7).

Recalling Definition (1.6a) by (1o) an example of specifying the function $Z(\underline{x})$ is provided. The special function obtained by combining the individual target excess demands z^*_{ij} in the obvious way will be denoted $Z^*(\underline{x})$. The image of $\mathcal{C}^*$ under Z^* is written $Z^*(\mathcal{C}^*) = \mathcal{D}^*$.

2.1.4. Active Excess Demand .

The notion of target excess demand used in the previous section can be interpreted as individual j's long term plan for buying and selling goods which is based on his present total wealth evaluated at current prices. It reflects merely the private interests of the individual without regarding real transaction possibilities.

Since in our economy only one type of transaction is possible, namely that of exchanging commodity for money, a trader has to determine his intended purchases in accordance with his present money stock. Therefore positive excess demand quantities have to be imposed a "financing constraint" upon. Thus we are led to define so-called active excess demands ([1] ,pp.34o) a^*_{ij} as those demands for goods which are actually working in the market. These demands will be subject to the financing constraints

$$(11) \qquad \sum_{a^*_{ij}>0} p_i \, a^*_{ij} \leq \max(-z^*_{1j}, 0) \qquad (1 \leq j \leq n)$$

with the right hand side of (11) representing the amount of money being available for individual j for the purpose of purchases.

One possibilitiy of designing active excess demands is given by

(12)

a) $a^*_{1j}(\underline{x}) = z^*_{1j} \qquad (1 \leq j \leq n)$

b) for $2 \leq i \leq m$

$$a^*_{ij}(\underline{x}) = \begin{cases} z^*_{ij}, & \text{if } z^*_{ij} \leq 0 \\ g_j(\underline{x}) \cdot k_j(\underline{x}) \cdot z^*_{ij}, \text{ if } \underline{p}^+_j \geq \underline{0}, & \text{if } z^*_{ij} > 0 \\ k_j(\underline{x}) \cdot z^*_{ij}, \text{ if } \underline{p}^+_j = \underline{0}, & \text{if } z^*_{ij} > 0 \end{cases}$$

$$(1 \leq j \leq n) ,$$

where $\underline{x} \in \mathcal{D}^*$, $\underline{x} = (x_1, x_2, \dots, x_{m(n+1)-1})'$ is written alternatively $\underline{x} = (p_2, p_3, \dots, p_m, z^*_{11}, z^*_{21}, \dots, z^*_{mn})'$,

$\underline{p}^+_j = (p_{k_1}, p_{k_2}, \dots, p_{k_r})'$ with

$\{k_1, k_2, \dots, k_r\} = \{k \in \mathbb{N} \mid 2 \leq k \leq m , \quad z^*_{kj} > 0\}$

depending on a given $\underline{x} \in \mathcal{D}^*$

$$g_j(\underline{x}) = \min \left\{ 1, \frac{\max(-z^*_{1j}, \mathcal{E}_j/2)}{\sum\limits_{z^*_{kj} > 0} p_k z^*_{kj}} \right\}$$

$$k_j(\underline{x}) = \min \left\{ 1, \frac{\max(-z^*_{1j}, 0)}{\mathcal{E}_j/2} \right\}$$

The elaborate terms in case b) are obtained through reducing j's target excess demand for each of the commodities i=2, 3,...,m in the same proportion. The use of $k_j(\underline{x})$ is to yield continuity of $a^*_{ij}(\underline{x})$ as will be demonstrated soon.

By means of (12) a specified form $A^*(\underline{x})$ of the function $A(\underline{x})$ defined in (1.6b) is determined with $\underline{x} \in \mathcal{D}^*$. The image of $\mathcal{D}^*$ under A^* is written $A^*(\mathcal{D}^*) = \mathcal{E}^*$ in analogy to (1.6b).

2.2. Transactionsof Goods.

Concerning actual transactionsof goods the main condition lies in the pure exchange property of the economy. Since the economic system considered is closed in the sense that neither production nor consumption do exist, for each good i transactions must sum up to zero:

$$(13) \qquad \sum_{j=1}^{n} t^*_{ij} = 0 \qquad\qquad (1 \leq i \leq m)$$

To derive transaction functions one may argue as follows: For a fixed commodity $i (2 \leq i \leq m)$ the set of individuals j={1,2,...,n} can be partitioned into two subsets corresponding to the position each individual occupies relative to aggregate ecxess demand a^*_i for i. Thus the short side of excess demand for i is defined by

(14a) $$a_i^{s*} = \sum_{l \in \mathcal{S}_i} a_{il}^*$$

where $$\mathcal{S}_i = \{l \in j \mid a_{il}^* a_i \leq 0\}$$

$$a_i^* = \sum_{j=1}^{n} a_{ij}^* \quad ;$$

the long side of excess demand for i by

(14b) $$a_i^{l*} = a_i^* - a_i^{s*} = \sum_{l \in \mathcal{L}_i} a_{il}^*$$

$$\mathcal{L}_i = j \setminus \mathcal{S}_i$$

As a further principle of transactions it will be assumed that individuals at the short side of demand or supply (which is covered by the special case of negative demand) are fully satisfied, whereas individuals at the long side can only transact a fraction of the intended quantities (expressed by active excess demands). One simple way of performing this long side reduction is a proportional reduction of long side active excess demands.

Money - because of its particular rôle as an exchange medium-is not transacted in a separate money market, but its transaction quantities result from the various commodity transactions involving money.

One may summarize the whole situation as follows: For each non-monetary good i ($2 \leq i \leq m$) there exists a market where each individual announces his active excess demand for that good representing his planned transaction quantity. The amounts of commodity i to be actually transacted are then determined by the relationship between the short side and the long side of demand.

A simple mathematical expression of these assumptions about transaction behaviour can be found in the following way:

(15a)

$$t^*_{ij}(\underline{x}) = \begin{cases} a^*_{ij} & \text{if } j \in \mathcal{S}_i \\ \dfrac{|a^{s*}_i|}{|a^{l*}_i|} \cdot a^*_{ij} \,, & \text{if } j \in \mathcal{L}_i \end{cases}$$

$$\text{for } i=2,3,\ldots,m; \quad j=1,2,\ldots,n \;;$$

$$x \in \mathcal{E}^* \,,\; \underline{x} = (p_2,p_3,\ldots,p_m,a^*_{11},a^*_{21},\ldots,a^*_{mn})' \;;$$

(15b)

$$t^*_{1j}(\underline{x}) = -\sum_{k=2}^{m} p_k \, t^*_{kj} \qquad (1 \leqslant j \leqslant n)$$

It can be seen readily that the transactions (15a,b) meet Condition (13) and that they do not offend the principle of "voluntary exchange".

2.3. Price Adjustment.

In the previous section active excess demands were assumed to be the main determinants of quantity movements, since those demands are expressed officially and therefore they are effective in the market. By the same reasoning active excess demands may be considered as the working forces of price dynamics.

Price changes taking place within one time period are supposed to obey the following two laws:

(i) Money price p_1 remains constant over time at the level $p_1=1$.

(ii) The changes of commodity prices p_i are exclusively determined by aggregate active excess demands a_i^* according to the following price mechanism: In general aggregate active excess demand $a_i^* \geq 0$ (aggregate active excess supply $a_i^* < 0$) for (of) commodity i increases (decreases) the price p_i of that good. The price change dp_i is influenced additionally by spill over effects from other markets.

A reasonable mathematical form of this well-known mechanism of price adjustment could be provided by ([17] ,pp.11)

(16a) $$dp_1^*(\underline{x}) = 0$$

(16b) $$dp_i^*(\underline{x}) = \frac{p_i + \delta[\max(0,a_i^*)]}{1+ \delta \sum_{k=2}^{m} \gamma_k(\max(0,a_k^*))} - p_i$$

$$(2 \leq i \leq m)$$

where $0 < \delta < 1$

$$\gamma_k = \frac{\bar{x}_k}{\bar{x}_1} \qquad (2 \leq k \leq m),$$

$$\underline{x} \in \mathcal{E}^* \, , \; \underline{x} = (p_2,p_3,\ldots,p_m,a_{11}^*,a_{21}^*,\ldots,a_{mn}^*)'$$

The positive factor δ is an appppropriately chosen damping constant of the price movement. Through different choices of δ different degrees of "price rigidity" can be introduced.

Recalling Definition (1.6c) the function $D^* : \mathcal{E}^* \to \mathcal{X}$ can be constructed yielding the changes in prices and goods assets between two successive time points.

In a last step the various special functions discussed in this chapter can be composed in the obvious way to form the function

(17) $\Delta^{*} : \mathcal{E}^{*} \rightarrow \mathcal{X}$, $\Delta^{*}(\underline{x}) = (D^{*} \circ A^{*} \circ Z^{*})(\underline{x})$

according to Definition (1.7).

2.4. Summary.

From a set of assumptions about pure exchange activities - based on classical utility theory - an explicit analytic form of the functions constituting economic dynamics in our model was derived as an example of specifying the general model of Chapter 1.

The system just obtained will be denoted

(18) $\epsilon^{*} = (\mathcal{C}^{*}, f^{*})$ with f^{*} given by

$f^{*} = I^{*} + \Delta^{*}$ (I^{*} being the identity on $\mathcal{C}^{*}$).

ϵ^{*} will be called an economy too keeping in mind that its performance does not fully agree with the basic axioms of demand theory because of the use of the auxiliary demand functions x^{*}_{ij} . In spite of this the discussion of the auxiliary model turns out to be useful, for the results obtained for ϵ^{*} will throw some light on the properties of the underlying economic model ϵ .

3. EQUILIBRIUM ANALYSIS.

The aim of the present chapter will be a detailed description of equilibrium properties of the special pure exchange model $\mathcal{E}^*$ designed in Chapter 2. The main results will be afforded by the application of Theorem 1 to the adjustment function f^* specified in (2.18).

For technical reasons let us restate the definitions contained in (1.8) and (1.9) formally:

Definition 1:

A vector $\hat{\underline{x}} \in \mathcal{C}(\mathcal{C}^*)$ is called a dynamic equilibrium of the economy $\mathcal{E}(\mathcal{E}^*)$ if $\hat{\underline{x}}$ is a fixed point of the adjustment function f (f^*).
$\mathcal{C}$, $\mathcal{C}^*$, f, f^*, $\mathcal{E}$ and $\mathcal{E}^*$ are defined according to (1.4c), (2.8), (1.7a), (2.18), (2.1) and (2.18).

Later on it will become clear that the notion of a dynamic equilibrium of an economy in the above sense does not coincide with the conventional concept of a competitive economic equlibrium of models of the tâtonnement type ([3], pp.74-89). In one of the following sections the relations between these two equilibrium concepts will be discussed thoroughly. The extensive treatment of the dynamic equilibrium problem is expected to be justified by the economic results which are to be obtained subsequently.

The mathematical question to be focused on is the problem of the existence of a dynamic equilibrium of the economy $\mathcal{E}^*$. The affirmative answer may be stated as

Theorem 2:

Let $\mathcal{E}^* = (\mathcal{C}^*, f^*)$ be the economy defined by (2.18).
Then there exists a dynamic equlibrium $\hat{\underline{x}}^*$ of $\mathcal{E}^*$.

The proof of this theorem will be a straightforward application of Theorem 1 to the function f^* and it will be carried out in full detail in the following sections of the present chapter.

3.1. Positive Money Stocks.

The transition from the original price-quantity set $\mathcal{C}$ to the "artificial" domain $\mathcal{C}^*$ of the function f^* because of introducing the non-bankruptcy-condition (2.7) must be taken into account in further considerations.

From (2.8) $\mathcal{C}^*$ is seen to be bounded, closed in $\mathfrak{X}$ and convex, since $\mathcal{C}$ has these properties. With respect to Theorem 1 we have to prove

Lemma 1:

a) The function $\triangle^*$ of (2.17) meets the Assumptions 1.3 and 1.4 (with $\mathcal{C}^*$ instead of $\mathcal{C}$ and $\triangle^*$ instead of $\triangle$).

b) For the function f^* of (2.18) the condition

$$(1) \qquad f^*_k(\underline{x}) \geqslant \begin{cases} \varepsilon_j & \text{if } k=m+jm \\ & \quad (0 \leqslant j \leqslant n-1), \\ 0 & \text{else,} \end{cases}$$

$$(\underline{x} \in \mathcal{C}^*)$$

is valid.

Hence the image of $\mathcal{C}^*$ under f^* is contained in $\mathcal{C}^*$.

$$(2) \qquad f^*(\mathcal{C}^*) \subseteq \mathcal{C}^*$$

Proof:

It may be useful to return to the obvius "economic" notation of the various vectors occurring in the different steps of reasoning.

a)

(i) Assumption 1.3 (with the appropriate changes of notation) is met:

For $\underline{x} \in \mathcal{C}^*$ we have for $1 \leq i \leq m$

$$\sum_{k=0}^{n-1} \Delta^*_{(m-1)+i+km}(\underline{x}) = \sum_{k=0}^{n-1} (D^* \circ A^* \circ Z^*)_{(m-1)+i+km}(\underline{x}) =$$

$$= \sum_{j=1}^{n} (t^*_{ij} \circ A^* \circ Z^*)(\underline{x}) = 0,$$

since the pure exchange condition (2.13) holds for i=1, 2,...,m .

(ii) With respect to Assumption 4 we have for $\underline{x} \in \mathcal{C}^*$

$$\sum_{k=1}^{m-1} \bar{x}_{k+1} \Delta^*_k(\underline{x}) = \sum_{k=2}^{m} \bar{x}_k \, dp^*_k(\underline{x}) =$$

$$= \sum_{k=2}^{m} \bar{x}_k \frac{p_k + \delta[\max(0,a^*_k)]}{1 + \delta \sum_{l=2}^{m} \gamma_l [\max(0,a^*_l)]} - \sum_{k=2}^{m} \bar{x}_k p_k =$$

$$= \bar{x}_1 - \bar{x}_1 = 0$$

using the definition of γ_l ($2 \leq l \leq m$) and the fact that $\underline{x} \in \mathcal{A}$ (see (1.4a)).

b)

The economic meaning of Condition (1) - which replaces Assumption 1.5 - is simply that through transaction j's money stock $\overline{x}_{1j}$ cannot fall below the individual "security bound" $\mathcal{E}_i$ and that commodity assets have to be nonnegative.

If $k = m + j.m \quad (0 \leq j \leq n-1)$,

$$f^*_k(\underline{x}) = x_k + \Delta^*_k(\underline{x}) = \overline{x}_{1j} + (t^*_{1j} \circ A^* \circ Z^*)\,(\underline{x})$$

Writing $T^*_{ij}(\underline{x}) = (t^*_{ij} \circ A^* \circ Z^*)\,(\underline{x})$ we have

$$f^*_k(\underline{x}) = \overline{x}_{1j} + T^*_{1j}(\underline{x}) = \overline{x}_{1j} - \sum_{i=2}^{m} p_i\, T^*_{ij}(\underline{x})$$

using (2.15b).

Consequently one gets

$$\overline{x}_{1j} - \sum_{i=2}^{m} p_i\; T^*_{ij}(\underline{x}) \geq \overline{x}_{1j} - \sum_{T^*_{ij}>0} p_i\, T^*_{ij}(\underline{x}) \geq$$

$$\geq \overline{x}_{1j} - \sum_{a^*_{ij}>0} p_i\,(a^*_{ij} \circ Z^*)\,(\underline{x}) \geq \overline{x}_{1j} - (\overline{x}_{1j} - \mathcal{E}_i) = \mathcal{E}_j$$

from (2.15a), (2.11) and (2.1o).

For arbitrary i,j and $\underline{x} \in \mathcal{E}^*$ (2.4) and (2.6) imply $x_{ij} \geq 0$ and hence $x^*_{ij} \geq 0$, therefore we have $z^*_{ij} \geq -\overline{x}_{ij}$, $a^*_{ij} \geq -\overline{x}_{ij}$ and $t^*_{ij} \geq -\overline{x}_{ij}$ yielding $\overline{x}_{ij} + (t^*_{ij} \circ A^* \circ Z^*)\,(\underline{x}) \geq 0$ which completes the proof of (1).

The assertion that f^* is self-mapping on $\mathcal{E}^*$ is then immediately clear from (2.8) and the foregoing arguments.

□

Thus Lemma 1 shows that Assumption a) of Theorem 1 is met by the adjustment function f^* of $\mathcal{E}^*$.

3.2. Continuity of the Constituting Functions.

The main part of the proof of Theorem 2 will be concerned with Assumption b) of Theorem 1. First of all the economic functions of the system $\mathcal{E}^*$ seem to be constructed in order to reflect certain premises about economic behaviour and about the market mechanism. Now their mathematical properties have to be examined upon which some important features of the economy depend. Regarding Theorem 1 and Theorem 2 continuity considerations are to be emphasized.

3.2.1. Continuity of z^*_{ij} $(1 \leq i \leq m;\ 1 \leq j \leq n)$.

Recalling (2.1o) it is easily seen that for the proof of the continuity of z^*_{ij} it suffices to show that $x^*_{ij}: \mathcal{C}^* \longrightarrow \mathbb{R}$ is continuous on $\mathcal{C}^*$.

<u>Lemma 2:</u>

For $1 \leq i \leq m$, $1 \leq j \leq n$ the demand functions

$$x^*_{ij} : \mathcal{C}^* \longrightarrow \mathbb{R}$$

defined by (2.3) and (2.9) respectively are continuous on their domain $\mathcal{C}^*$.

Proof:

(i) $i=1$:

Continuity of $x^*_{1j}(\underline{x}) = x_{1j}(\underline{x})$ $(1 \leq j \leq n)$ on the set $\mathcal{C}^*$ is trivially clear from (2.3).

(ii) $2 \leq i \leq m$:

Using the notational conventions of (2.9) for a fixed ordered pair (i,j) with $\alpha_{ij} > 0$ the set $\mathcal{C}^*$ may be partitioned into

$$\mathcal{C}^* = \mathcal{C}^*_1(i,j) \cup \mathcal{C}^*_2(i,j)$$

where $\mathcal{C}^*_1(i,j) = \{\underline{x} \in \mathcal{C}^* \mid p_i \geq h_{ij}(\underline{p}_i, \underline{\bar{x}}_j)\}$

$$\mathcal{C}^*_2(i,j) = \mathcal{C}^* \setminus \mathcal{C}^*_1(i,j)$$

a) Let $\underline{x} \in \mathcal{C}^*_1(i,j)$ be such that $p_i > h_{ij}(\underline{p}_i, \underline{\bar{x}}_j)$.

Then

$$p_i > \frac{\left[(1-\alpha_{1j})\sum_{k\neq i} p_k \bar{x}_{kj} + \bar{x}_{1j}\right] \alpha_{ij}}{K.\bar{x}_i - (1-\alpha_{1j})\, \alpha_{ij}.\bar{x}_{ij}} \geq \frac{\varepsilon_j \cdot \alpha_{ij}}{K \cdot \bar{x}_i} > 0$$

Thus $x^*_{ij}(\underline{x})$ is defined for each such $\underline{x}$ and it is readily seen to be continuous at $\underline{x}$ from (2.9).

b) Take an arbitrary $\underline{x} \in \mathcal{C}^*_2(i,j)$:

Being a constant on $\mathcal{C}^*_2(i,j)$ $x^*_{ij}(\underline{x})$ is continuous at every point of $\mathcal{C}^*_2(i,j)$ as is easily seen from the definition of $\mathcal{C}^*_2(i,j)$.

c) Let $\underline{x} \in \mathcal{C}^*_1(i,j)$ be such that $p_i = h_{ij}(\underline{p}_i, \underline{\bar{x}}_j)$.

For such a point on the boundary of $\mathcal{C}^*_1(i,j)$ we have from (2.9)

$$x^*_{ij}(\underline{x}) = \left[(1-\alpha_{1j}) \langle \underline{p}, \underline{\bar{x}}_j \rangle + \bar{x}_{1j}\right] \frac{\alpha_{ij}}{h_{ij}} =$$

$$= \frac{\left[(1-\alpha_{1j})\sum_{k\neq i} p_k \bar{x}_{kj} + \bar{x}_{1j}\right] \alpha_{ij}}{h_{ij}} + (1-\alpha_{1j})\, \alpha_{ij} \cdot \bar{x}_{ij} =$$

$$= K.\bar{x}_i - (1-\alpha_{1j})\, \alpha_{ij}\, \bar{x}_{ij} + (1-\alpha_{1j})\, \alpha_{ij}\, \bar{x}_{ij} = K.\bar{x}_i$$

Taking an arbitrarily small $\varepsilon > 0$ one can find a neighbourhood $\mathcal{U}_1(\underline{x})$ of $\underline{x}$ in the induced topology of $\mathcal{C}^*$ such that

$$|x^*_{ij}(\underline{y}) - K\bar{x}_i| < \varepsilon \qquad \text{for every } \underline{y} \in (\mathcal{U}_1(\underline{x}) \cap \mathcal{C}^*_1(i,j))$$

because of (2.9).

Analogously there exists a neighbourhood $\mathcal{U}_2(\underline{x})$ such that

$|x^*_{ij}(\underline{z}) - K.\bar{x}_i| < \varepsilon$ holds trivially for every

$\underline{z} \in (\mathcal{U}_2(\underline{x}) \cap \mathcal{C}^*_2(i,j))$ since x^*_{ij} is a constant on

$\mathcal{C}^*_2(i,j)$ with the value $x^*_{ij}(\underline{x}) = K.\bar{x}_i$.

Hence with $\mathcal{U}(\underline{x}) = \mathcal{U}_1(\underline{x}) \cap \mathcal{U}_2(\underline{x})$ we have

$$x^*_{ij}(\mathcal{U}(\underline{x})) \subseteq (K\bar{x}_i - \varepsilon \ , K\bar{x}_i + \varepsilon)$$

and therefore x^*_{ij} is continuous at $\underline{x}$.

Thus for every ordered pair (i,j) with $\alpha_{ij} > 0$ $x^*_{ij}(\underline{x})$ is a continuous function on $\mathcal{C}^*$.

The case $\alpha_{ij}=0$ is trivial because of $x^*_{ij}(\underline{x}) \equiv 0$.

□

At this point it seems to be necessary to emphasize the fact that continuity of the excess demand functions $z^*_{ij}(\underline{x})$ was achieved by the introduction of modified demand functions $x^*_{ij}(\underline{x})$.

3.2.2. Continuity of a^*_{ij} $(1 \leq i \leq m \ ; 1 \leq j \leq n)$.

Next the functions of active excess demand $a^*_{ij}(\underline{x})$ given according to (2.12) shall be analyzed carefully. Before discussing details it may be of use to review the ideas which led to the special form of $a^*_{ij}(\underline{x})$.

The underlying concept is determined by the financial constraint (2.11). Of course it has to be taken into account only for purchases of goods (that is for positive excess demands), whereas supplies of goods (negative excess demands) remain unaffected. If individual j plans to buy a positive amount of good i ($z^*_{ij} > 0$), he has to compare his total planned expenditures on goods with the amount of money available for purchases (which equals $\max(-z^*_{1j}, 0)$). If financial constraints are binding, target excess demand $z^*_{ij} > 0$ has to be reduced appropriately.

One can show the following

Lemma 3:

For $1 \leq i \leq m$, $1 \leq j \leq n$ the active excess demand functions

$$a^*_{ij} : \mathcal{D}^* \longrightarrow \mathbb{R} \qquad \text{given by (2.12)}$$

are continuous on $\mathcal{D}^*$.

Proof:
A vector $\underline{x} \in \mathcal{D}^*$ will be written $\underline{x} = (p_2, p_3, \ldots, p_m, z^*_{11}, z^*_{21}, \ldots, z^*_{mn})'$.

(i) $i=1$:
This case is evident from (2.12a).

(ii) $2 \leq i \leq m$:
Let the ordered pair (i,j) be fixed.
Then by the hyperplane $z^*_{ij} = 0$ the set $\mathcal{D}^* \subseteq \mathcal{X}$ is partitioned into two subsets $\mathcal{D}^*_1$ and $\mathcal{D}^*_2$, where

$$\mathcal{D}^*_1 = \{\underline{x} \in \mathcal{D}^* \mid z^*_{ij} \leq 0\}$$

$$\mathcal{D}^*_2 = \mathcal{D}^* \setminus \mathcal{D}^*_1$$

Let us first consider $\mathcal{D}^*_2$ characterized by $z^*_{ij} > 0$.

1) $\underline{x} \in \mathcal{D}^*_2 \quad (z^*_{ij} > 0)$

$\mathcal{D}^*_2$ may again be partitioned into

$$\mathcal{D}^*_2 = \mathcal{D}^*_{21} \cup \mathcal{D}^*_{22} \quad \text{where}$$

$$\mathcal{D}^*_{21} = \{\underline{x} \in \mathcal{D}^*_2 \mid \underline{p}^+_j \geq \underline{0}\}$$

$$\mathcal{D}^*_{22} = \mathcal{D}^*_2 \setminus \mathcal{D}^*_{21}$$

a) $\underline{x}^0 \in \mathcal{D}^*_{21}$

Then according to (2.12b) active excess demand is given by

$$a^*_{ij}(\underline{x}^0) = g_j(\underline{x}^0) \cdot k_j(\underline{x}^0) \cdot z^*_{ij}(\underline{x}^0)$$

Because of $\underline{x}^0 \in \mathcal{D}^*_{21}$ there exists at least one l $(2 \leq l \leq m)$ with $p_l^0 \cdot z^{*\,0}_{lj} > 0$ yielding $\sum_{z^{*\,0}_{kj} > 0} p^0_k \cdot z^{*\,0}_{kj} > 0$.

This result together with the continuity of the functions

$$\sum_{z^*_{kj}>0} p_k \cdot z^*_{kj} \quad \text{and} \quad \max(-z^*_{1j}, \varepsilon_j/2) \text{ at the point}$$

$\underline{x} \in \mathcal{D}^*_{21}$ shows the function

$$g_j(\underline{x}) = \min \left\{ 1, \frac{\max(-z^*_{1j}, \varepsilon_j/2)}{\sum_{z^*_{kj}>0} p_k \cdot z^*_{kj}} \right\}$$

to be continuous at $\underline{x}^0$.

Since for

$$k_j(\underline{x}) = \min \left\{ 1, \frac{\max(-z^*_{1j},0)}{\mathcal{E}_j/2} \right\}$$

continuity at $\underline{x}^0$ is evident, the function $a^*_{ij}(\underline{x})$ is continuous at $\underline{x}^0 \varepsilon \mathcal{D}^*_{21}$.

b) $\underline{x}^1 \varepsilon \mathcal{D}^*_{22}$

For $\underline{x}^1$ we have

$$a^*_{ij}(\underline{x}^1) = k_j(\underline{x}^1). \; z^*_{ij} \; (\underline{x}^1)$$

Obviously the function $k_j(\underline{x}) \cdot z^*_{ij}(\underline{x})$ is continuous at $\underline{x}^1$. Hence it is sufficient to find a neighbourhood $\mathcal{U}_1(\underline{x}^1)$ of the point $\underline{x}^1$ such that for every $\underline{y} \; \varepsilon \; (\mathcal{U}_1(\underline{x}^1) \cap \mathcal{D}^*)$

$$a^*_{ij}(\underline{y}) = k_j(\underline{y}) \cdot z^*_{ij}(\underline{y})$$

This can be achieved in the following way:

Given $\underline{x}^1 \; \varepsilon \; \mathcal{D}^*_{22}$ let the set $\mathcal{K}$ be defined by

$$\mathcal{K} = \{k \; \varepsilon \; \mathbb{N} | 2 \leq k \leq m, \quad z^{*1}_{kj} \geq 0\}$$

Since $\underline{p}^+_j = \underline{0}$ at $\underline{x}^1$ (see (2.12b)) we have

$$z^{*1}_{kj} \cdot p^1_k = 0 \quad \text{for} \quad k \; \varepsilon \; \mathcal{K}.$$

Because of the continuity of the function $\sum_{k \varepsilon \mathcal{K}} | z^*_{kj} \cdot p_k |$ at the point $\underline{x}^1$ we can choose a neighbourhood $\mathcal{U}_1(\underline{x}^1)$ of $\underline{x}^1$ such that for every $\underline{y} \; \varepsilon \; (\mathcal{U}_1(\underline{x}^1) \cap \mathcal{D}^*)$, $\underline{y} = (p_2, p_3, \ldots, p_m, z^*_{11}, z^*_{21}, \ldots, z^*_{mn})'$ the conditions

$$\sum_{k \varepsilon \mathcal{K}} |z^*_{kj} p_k| < \mathcal{E}_j/2 \quad ; \qquad z^*_{kj} > 0, \text{ if } z^{*1}_{kj} > 0$$

$$\text{and } z^*_{kj} < 0, \qquad \text{if } k \notin \mathcal{K}$$

are met.

Then the following inequalities are easily seen to hold for every $\underline{y} \in (\mathcal{U}_1(\underline{x}^1) \cap \mathcal{D}^*)$:

$$\frac{\max(-z^*_{1j}, \varepsilon_j/2)}{\sum\limits_{z^*_{kj}>0} p_k z^*_{kj}} \geq \frac{\varepsilon_j/2}{\sum\limits_{z^*_{kj}>0} |p_k z^*_{kj}|} \geq$$

$$\geq \frac{\varepsilon_j/2}{\sum\limits_{k \in \mathcal{K}} |p_k z^*_{kj}|} > \frac{\varepsilon_j/2}{\varepsilon_j/2} = 1$$

Thus for $\underline{y} \in (\mathcal{U}_1(\underline{x}^1) \cap \mathcal{D}^*)$ we get

$$g_j(\underline{y}) = \min \left\{ 1, \frac{\max(-z^*_{1j}, \varepsilon_j/2)}{\sum\limits_{z^*_{kj}>0} p_k z^*_{kj}} \right\} = 1$$

and consequently

$$a^*_{ij}(\underline{y}) = k_j(\underline{y}) . z^*_{ij}(\underline{y}) ,$$

as desired.

2) $\underline{x} \in \mathcal{D}^*_1 \qquad (z^*_{ij} \leq 0)$:

a) If $\underline{x}$ is such that $z^*_{ij} < 0$ nothing is left to prove.

b) Let $\underline{x}^2$ lie in the hyperplane $z^*_{ij} = 0$ implying $a^*_{ij}(\underline{x}^2) =$

$= z^{*2}_{ij} = 0.$

Because of the continuity of $z^*_{ij}(\underline{x})$ for a given $\eta > 0$ one can choose a neighbourhood $\mathcal{V}(\underline{x}^2)$ of $\underline{x}^2$ in $\mathfrak{X}$ such that $|z^*_{ij}| < \eta$ for every $\underline{x} \in (\mathcal{V}(\underline{x}^2) \cap \mathcal{D}^*)$. As an immediate consequence one has

$$a^*_{ij}(\underline{x}) = \begin{cases} z^*_{ij} & \text{if } \underline{x} \in (\mathcal{V}(\underline{x}^2) \cap \mathcal{D}^*_1) \\ g_j(\underline{x}).k_j(\underline{x}).z^*_{ij} & \text{if } \underline{x} \in (\mathcal{V}(\underline{x}^2) \cap \mathcal{D}^*_{21}) \\ k_j(\underline{x}).z_{ij}{}^* & \text{if } \underline{x} \in (\mathcal{V}(\underline{x}^2) \cap \mathcal{D}^*_{22}) \end{cases}$$

Since $|g_j(\underline{x})| \leqslant 1$ and $|k_j(\underline{x})| \leqslant 1$ for $\underline{x} \in \mathcal{D}^*$

$$|a^*_{ij}(\underline{x})| \leqslant |z^*_{ij}| < \eta \qquad \text{ofr every } \underline{x} \in (\mathcal{V}(\underline{x}^2) \cap \mathcal{D}^*) ,$$

where $\eta > 0$ was given arbitrarily. □

3.3.3. Continuity of t^*_{ij} $(1 \leqslant i \leqslant m;\ 1 \leqslant j \leqslant n)$ and $dp^*_i(2 \leqslant i \leqslant m)$.

The last step towards the application of Theorem 1 will be to prove that the components of the function $D^* : \mathcal{E}^* \rightarrow \mathcal{X}$ are continuous.

To start with the transaction functions $t^*_{ij} : \mathcal{E}^* \rightarrow \mathbb{R}$ it may be recalled that each of them maps vectors $\underline{x} = (p_2, p_3, \dots, p_m, a^*_{11}, a^*_{21}, \dots, a^*_{mn})' \in \mathcal{E}^*$ into $\mathbb{R}$.

We have to prove the following

<u>Lemma 4</u>:

For $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$ the transaction functions $t^*_{ij} : \mathcal{E}^* \rightarrow \mathbb{R}$ given by (2.15a,b) are continuous on $\mathcal{E}^*$.

Proof:

(i) $2 \leqslant i \leqslant m$

Let again the pair (i,j) be fixed arbitrarily.
For every $\underline{x} = (p_2, p_3, \dots, p_m, a^*_{11}, a^*_{21}, \dots, a^*_{mn})' \in \mathcal{E}^*$ the function

$$a^*_i(\underline{x}) = \sum_{k=1}^{n} a^*_{ik}$$

is well-defined and a partition of the set $\mathcal{E}^*$ is given by

$$\mathcal{E}^* = \mathcal{E}^*_s(i,j) \cup \mathcal{E}^*_l(i,j) \quad \text{where}$$

$$\mathcal{E}^*_s(i,j) = \{ \underline{x} \in \mathcal{E}^* \mid a^*_i(\underline{x}).a^*_{ij} \leqslant 0\}$$

$$\mathcal{E}^*_l(i,j) = \{ \underline{x} \in \mathcal{E}^* \mid a^*_i(\underline{x}).a^*_{ij} > 0\}$$

1) $\underline{x}^0 \in \mathcal{E}^*_l(i,j)$

Then, without loss of generality, $a^*_i(\underline{x}^0)$ may be supposed to have a strictly positive value implying $a^{*\,0}_{ij} > 0$. (The proof of the continuity of $t^*_{ij}(\underline{x})$ at $\underline{x}^0$ is essentially the same, if $a^*_i(\underline{x}^0) < 0$ is assumed).

Since in our case individual j is at the long side of demand for good i, the considered transaction function $t^*_{ij}(\underline{x})$ for the argument $\underline{x}^0$ is written

$$t^*_{ij}(\underline{x}^0) = \Psi_i(\underline{x}^0) \;.\; a^{*\,0}_{ij}$$

with

$$\Psi_i(\underline{x}) = \frac{|a_i{}^{s*}|}{|a_i{}^{l*}|}$$

Let $\vartheta \in \mathbb{R}$ be such that

$$0 < \vartheta < \min\{|a^{*\,0}_{il}| > 0\}$$

and let $\mathcal{W}(\underline{x}^0)$ be a neighbourhood of $\underline{x}^0$ such that for every $\underline{x} = (p_2,p_3,\ldots,p_m,a^*_{11},a^*_{21},\ldots,a^*_{mn})' \in (\mathcal{W}(\underline{x}^0) \cap \mathcal{E}^*)$ $a^*_i(\underline{x}) > 0$ and $|a^*_{il}-a^{*\,0}_{il}| < \vartheta$ $(1 \leqslant l \leqslant n)$ implying $t^*_{ij}(\underline{x}) = \Psi_i(\underline{x}).a^*_{ij}$.

For an arbitrary $\underline{x} \in (\mathcal{W}(\underline{x}^0) \cap \mathcal{E}^*)$ we have the inequalities

$$\Psi_i(\underline{x}) \leq \frac{\sum_{l \in \mathcal{S}_i(\underline{x}^0)} (-a^{*0}_{il} + \vartheta)}{\sum_{l \in \mathcal{L}_i(\underline{x}^0)} (a^{*0}_{il} - \vartheta)}$$

and

$$\Psi_i(\underline{x}) \geq \frac{\sum_{l \in (\mathcal{S}_i(x^0) \setminus \mathcal{S}^0_i(\underline{x}^0))} (-a^{*0}_{il} - \vartheta)}{\sum_{l \in \mathcal{L}_i(\underline{x}^0)} (a^{*0}_{il} + \vartheta) + \sum_{l \in \mathcal{S}_i(\underline{x}^0)} \vartheta} \quad ,$$

where

$$\mathcal{S}_i(\underline{x}^0) = \{l \mid a^{*0}_{il} \leq 0\} \quad ,$$

$$\mathcal{S}^0_i(\underline{x}^0) = \{l \mid a^{*0}_{il} = 0\} \quad ,$$

$$\mathcal{L}_i(\underline{x}^0) = \{l \mid a^{*0}_{il} > 0\} \quad .$$

If ϑ tends to zero both the upper and the lower bound of $\Psi_i(\underline{x})$ $(\underline{x} \in (\mathcal{W}(x^0) \cap \mathcal{E}^*))$ approach the limit

$$\frac{\sum_{l \in \mathcal{S}_i(\underline{x}^0)} -a^{*0}_{il}}{\sum_{l \in \mathcal{L}_i(\underline{x}^0)} a^{*0}_{il}} = \Psi_i(\underline{x}^0)$$

assuring continuity of $t^*_{ij}(\underline{x})$ at $\underline{x}^0 \in \mathcal{E}^*_1(i,j)$.

2) $\underline{x}^1 \in \mathcal{E}^*_s(i,j)$

a) $a^*_i(\underline{x}^1).a^{*1}_{ij} < 0$

If e.g. $a^*_i(\underline{x}^1) < 0$ and $a^{*1}_{ij} > 0$ one can use the continuity of the function

$$\phi(\underline{x}) = a^*_i(\underline{x}).a^*_{ij}$$

to determine a neighbourhood $\mathcal{U}(\underline{x}^1)$, of $\underline{x}^1$ such that

$$\phi(\underline{x}) < 0 \qquad \text{for } \underline{x} \in (\mathcal{U}(\underline{x}^1) \cap \mathcal{E}^*)$$

Hence for $\underline{x} \in (\mathcal{U}(\underline{x}^1) \cap \mathcal{E}^*)$ $t^*_{ij}(\underline{x})$ is given by

$$t^*_{ij}(\underline{x}) = a^*_{ij}$$

and therefore it is continuous at the point $\underline{x}^1$.

b) $a^*_i(\underline{x}^1) \cdot a^{*\ 1}_{ij} = 0$

implying $t^*_{ij}(\underline{x}^1) = a^{*\ 1}_{ij}$.

b1) $a^{*\ 1}_{ij} = 0$

The obvious inequality $\left| \dfrac{a_i^{s*}}{a_i^{1*}} \right| \leqslant 1$ leads to the relation

$$\left| t^*_{ij}(\underline{x}) \right| \leqslant \left| a^*_{ij} \right| \qquad \text{for every } \underline{x} \in \mathcal{E}^* \text{ yielding continuity}$$

of $t^*_{ij}(\underline{x})$ at $\underline{x}^1$ with $a^{*\ 1}_{ij} = 0$.

b2) $a^{*\ 1}_{ij} \neq 0$

This condition implies $a^*_i(\underline{x}^1) = 0$. Then for a sufficiently small neighbourhood $\mathcal{V}(\underline{x}^1)$ of $\underline{x}^1$ we have the following results which are to be stated verbally, since the mathematical reasoning is the same as that of part 1) of the present proof:

If $\underline{x} \in (\mathcal{V}(\underline{x}^1) \cap \mathcal{E}^*_s(i,j))$, $t^*_{ij}(\underline{x}) = a^*_{ij}$ will be "close" to $a^{*\ 1}_{ij} = t^*_{ij}(\underline{x}^1)$.

If $\underline{x} \in (\mathcal{V}(\underline{x}^1) \cap \mathcal{E}^*_1(i,j))$, $\Psi_i(\underline{x})$ will be close to 1.

Therefore $t^*_{ij}(\underline{x}) = \Psi_i(\underline{x}) \cdot a^*_{ij}$ will have a value near a^{*1}_{ij} .

Thus $t^*_{ij}(\underline{x})$ is seen to be continuous at $\underline{x}^1$.

(ii) $i = 1$

Recalling (2.15b) and the foregoing discussion continuity of $t^*_{1j}(\underline{x})$ is evident. □

The last result needed for the proof of Theorem 2 is to be stated as

Lemma 5:

For $2 \leq i \leq m$ the price adjustment functions $dp^*_i : \mathcal{E}^* \longrightarrow \mathbb{R}$ given by (2.16b) are continuous on $\mathcal{E}^*$.

Proof:
Writing $\underline{x} \in \mathcal{E}^*$ in the "economic" form $\underline{x} = (p_2, p_3, \ldots, p_m, a^*_{11}, a^*_{21}, \ldots, a^*_{mn})'$ and using the continuity of

$$a^*_k(\underline{x}) = \sum_{j=1}^{n} a^*_{kj} \qquad (2 \leq k \leq m)$$

the above assertion is seen to be valid. □

By Lemmas 1-5 Assumption b) of Theorem 1 was shown to hold for the special functions of Chapter 2. Therefore Theorem 1 can be applied to the adjustment function f^* in order to complete the proof of Theorem 2.

It may be important to stress once again that the functions constructed in Chapter 2 represent merely one particular translation of a set of economic assumptions into exact mathematical relations. To prevent our first attempt of a description of the pure exchange economy $\mathcal{E}$ from becoming too complicated some rather restrictive conditions had to be imposed upon the model. Several restrictions seem to be amenable to weakening without losing the main equilibrium results.

3.3. Some Properties of the Dynamic Equlibrium $\hat{\underline{x}}^*$ of the Economy $\mathcal{E}^*$.

After the existence of a dynamic equilibrium (see Definition 1) of the economy $\mathcal{E}^*$ has been proved the economic relevance of this result is to be discussed. In this context it will be of particular interest to compare the notion of a dynamic equilibrium of the monetary pure exchange economy with the concept of a competitive economic equilibrium of a Walrasian economy ([3] , pp.76).

We shall use the following definitions:

<u>Definition 2</u>:

A vector $\underline{x} \in \mathcal{C}\,(\mathcal{C}^*)$ is called an economic equilibrium of the economy $\mathcal{E}\,(\mathcal{E}^*)$, if

$$\sum_{j=1}^{n} (a_{ij} \circ Z)(\underline{x}) \leq 0 \qquad \left(\sum_{j=1}^{n} (a^*_{ij} \circ Z^*)(\underline{x}) \leq 0 \right)$$

$$\text{for } i=2,3,\ldots,m \,.$$

The set of economic equilibrium points of the economy $\mathcal{E}\,(\mathcal{E}^*)$ will be denoted $\mathcal{M}_e\,(\mathcal{M}^*_e)$.

<u>Definition 3</u>:

A vector $\underline{x} \in \mathcal{C}\,(\mathcal{C}^*)$ is called a complete economic equilibrium of the economy $\mathcal{E}\,(\mathcal{E}^*)$, if for

$2 \leq i \leq m \,, \quad 1 \leq j \leq n$

$$(a_{ij} \circ Z)(\underline{x}) \leq 0 \qquad\qquad ((a^*_{ij} \circ Z^*)(\underline{x}) \leqq 0).$$

The set of complete economic equilibrium points of the economy $\mathcal{E}\,(\mathcal{E}^*)$ will be denoted $\mathcal{M}_c\,(\mathcal{M}^*_c)$.

For obvious reasons a price-quantity-vector $\underline{x} \in \mathcal{C}\,(\mathcal{C}^*)$ will be called a state of the economy $\mathcal{E}\,(\mathcal{E}^*)$.

Definition 4:

A state $\underline{x} \in \mathcal{C}\,(\mathcal{C}^*)$ is called a shortage equilibrium of $\mathcal{E}\,(\mathcal{E}^*)$, if for $2 \leq i \leq m$, $1 \leq j \leq n$

$$(a_{ij} \circ Z)\,(\underline{x}) \geq 0 \qquad ((a^*_{ij} \circ Z^*)(\underline{x}) \geq 0),$$

$$(dp_i \circ A \circ Z)\,(\underline{x}) = 0 \qquad (dp^*_i \circ A^* \circ Z^*)\,(\underline{x}) = 0),$$

and if there exists at least one pair (i',j') with

$$(a_{i'j'} \circ Z)\,(\underline{x}) > 0 \qquad (a^*_{i'j'} \circ Z^*)\,(\underline{x}) > 0).$$

The set of shortage equlibria of $\mathcal{E}\,(\mathcal{E}^*)$ will be denoted $\mathcal{M}_s$ $(\mathcal{M}^*_s)$.

Then the existence of a dynamic equilibrium of the economy $\mathcal{E}^*$ permits the following economic interpretation:

Corollary 1:

If the economy $\mathcal{E}^*$ is given by (2.18) and if $\mathcal{M}^*_d$ denotes the set of dynamic equlibria of $\mathcal{E}^*$, then

$$\mathcal{M}^*_c \cup \mathcal{M}^*_s = \mathcal{M}^*_d \ .$$

This means that for $\mathcal{E}^*$ at least one distinguished state $\hat{\underline{x}}^*$ must exist, $\hat{\underline{x}}^*$ being a complete economic euqilibrium or a shortage equilibrium.

Proof:

Writing a dynamic euqilibrium $\underline{\hat{x}}^*$ ε $\mathcal{M}_d^*$ in the form $\underline{\hat{x}}^* = (\hat{p}_2^*, \hat{p}_3^*, \ldots, \hat{p}_m^*, \hat{x}_4^*, \hat{x}_{21}^*, \ldots, \hat{x}_{mn}^*)'$ and using the brief notation

$$\hat{a}_i^* = \sum_{j=1}^{n} (a_{ij}^* \circ Z^*)\ (\underline{\hat{x}}^*) \qquad (2 \leqslant i \leqslant m)$$

the conditions to be fulfilled by a dynamic equlibrium of ϵ^* may be stated as

$$(3) \qquad \frac{\hat{p}_i^* + \delta[\max(0,\hat{a}_i^*)]}{1 + \delta \sum_{k=2}^{m} \gamma_k [\max(0,a_k^*)]} = \hat{p}_i^* \qquad (2 \leqslant i \leqslant m)$$

$$(4) \qquad (t_{ij}^* \circ A^* \circ Z^*)\ (\underline{\hat{x}}^*) = 0 \qquad (1 \leqslant i \leqslant m;\ 1 \leqslant j \leqslant n)$$

1) Let $\underline{\hat{x}}^*$ ε $\mathcal{M}_d^*$ be a dynamic equilibrium of ϵ^*. Then the following two assertions are seen to be valid:

(5a) For $i=2,3,\ldots,m$

$$\hat{a}_{ik}^* \cdot \hat{a}_{il}^* \geqslant 0 \qquad (1 \leqslant k \leqslant n;\ 1 \leqslant l \leqslant n),$$

where $\quad a_{ik}^* = (a_{ik}^* \circ Z^*)\ (\underline{\hat{x}}^*)$.

If we assume the existence of a pair of integers (k',l') with

$$\hat{a}_{ik'}^* \cdot \hat{a}_{il'}^* < 0,$$

then the following two cases are possible.

α) If $\hat{a}_i^* = 0$, then $(t_{ik'}^* \circ A^* \circ Z^*)\ (\underline{\hat{x}}^*) = \hat{a}_{ik'}^* \neq 0$

and $(t_{il'}^* \circ A^* \circ Z^*)\ (\underline{\hat{x}}^*) = a_{il'}^* \neq 0$

because of (2.15a).

β) If $\hat{a}_i^* \neq 0$, then $\hat{a}_i^{s*} \neq 0$ and $\hat{a}_i^{1*} \neq 0$ because of our initial assumption about (k',l').

Therefore we have

$$(t^*_{ik'} \circ A^* \circ Z^*)\,(\hat{\underline{x}}^*) = \frac{|\hat{a}_i^{s*}|}{|\hat{a}_i^{1*}|} \cdot \hat{a}^*_{ik'} \neq 0$$

and

$$(t^*_{il'} \circ A^* \circ Z^*)\,(\hat{\underline{x}}^*) = \frac{|\hat{a}_i^{s*}|}{|\hat{a}_i^{1*}|} \cdot \hat{a}^*_{il'} \neq 0.$$

Both α) and β) contradict to Condition (4), and thus the proof of Assertion (5a) is completed.

(5b) For every pair (i_1, i_2) with $2 \leq i_1 \leq m, 2 \leq i_2 \leq m$

$$\hat{a}^*_{i_1} \cdot \hat{a}^*_{i_2} \geq 0 \quad \text{must hold.}$$

This is easily derived from Condition (3) for a dynamic equilibrium $\hat{\underline{x}}^*$.

Combining Assertions (5a) and (5b) we get the following result:

If $\hat{\underline{x}}^* \in \mathcal{M}^*_d$, then either

$$\hat{a}^*_{ij} \leq 0 \quad \text{for} \quad 2 \leq i \leq m,\ 1 \leq j \leq n \text{ implying } \hat{\underline{x}}^* \in \mathcal{M}^*_c$$

or $\hat{\underline{x}}^* \notin \mathcal{M}^*_c$ with

$$\hat{a}^*_{ij} \geq 0 \quad \text{for} \quad 2 \leq i \leq m,\ 1 \leq j \leq n \quad \text{and}$$

$$d\hat{p}^*_i = 0 \text{ for } 2 \leq i \leq m$$

$$\text{implying } \hat{\underline{x}}^* \in \mathcal{M}^*_s .$$

Thus $\mathcal{M}^*_d \subseteq \mathcal{M}^*_c \cup \mathcal{M}^*_s$ has been proved.

2) The second part $(\mathcal{M}^*_c \cup \mathcal{M}^*_s) \subseteq \mathcal{M}^*_d$ is an immediate consequence of the corresponding definitions. □

Another property of a dynamic equilibrium state $\hat{\underline{x}}^*$ of the economy ϵ^* may be described in terms of the so-called Hahn-condition for transactions ([25], pp. 345). This condition claims that in a trading economy transactions are carried out in such a way that at every moment active excess demands are subject to

$$(6) \quad a^{*}_{ij} \neq 0 \Longrightarrow a^{*}_{ij} \cdot a^{*}_{i} > 0 \qquad (2 \leq i \leq m;\ 1 \leq j \leq n).$$

Hahn's transaction rule supposes trades to "work" instantaneously preventing the formation of individual active excess demands opposite to the aggregate active excess demand.

It must be stressed that (6) is not an assumption about transactions in the system ϵ^* , but it becomes apparent as a necessary condition for a state $\hat{\underline{x}}^*$ to be a dynamic equilibrium of ϵ^*.

<u>Corollary 2:</u>

If $\hat{\underline{x}}^*$ is a dynamic equilibrium of the economy ϵ^*, then $\hat{\underline{x}}^*$ must display the so-called Hahn-property ([11], 465)

$$\hat{a}^{*}_{ij} \neq 0 \Longrightarrow \hat{a}^{*}_{ij} \cdot \hat{a}^{*}_{i} > 0 ,$$

where

$$\hat{a}^{*}_{ij} = (a^{*}_{ij} \circ Z^*)(\hat{\underline{x}}^*)$$

$$\hat{a}^{*}_{i} = \sum_{j=1}^{n} \hat{a}^{*}_{ij}$$

Proof:

The assertion of Corollary 2 is easily seen to be valid from the proof of Corollary 1.

Thus Theorem 2 has the following economic implications: There exists at least one state $\hat{\underline{x}}^*$ of the economy ϵ^* such that with the corresponding price system and distribution of goods all individuals are either fully satisfied or they are satisfied to the extent which overall shortage or surplus of goods permits. Of course, satisfaction is expressed in terms of active excess demands which are the demands being decisive for the market mechanism.

3.4. Equilibria of the Original Economy $\bar{\mathcal{E}}$.

So far we have only analyzed the auxiliary economic system $\mathcal{E}^*$ using modified demands x^*_{ij} instead of the original demand functions x_{ij}. Now it will be shown that our approach is useful in the sense that from the results just obtained for $\mathcal{E}^*$ one may derive interesting equilibrium statements about an "underlying" economy $\bar{\mathcal{E}}$.

In order to remove the artificial restrictions on demands expressed by (2.9) the extended real line $\bar{\mathbb{R}}$ shall be used as the range of demand functions.

<u>Definition 5</u>:

The so-called extended real line $\bar{\mathbb{R}}$ ([14] , pp.1,2) is given by

$$\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$$

where $+\infty = \infty$,

$-\infty < r < \infty$ for every $r \in \mathbb{R}$ and operations involving ∞ and $-\infty$ respectively are defined as follows:

$r+\infty=\infty, r+(-\infty)=-\infty$ for every $r \in \mathbb{R}$,

$r-\infty = -\infty, r-(-\infty)=\infty$,

$r.\infty = \infty, r(-\infty)=-\infty$, if $r > 0$,

$r.\infty =-\infty, r(-\infty)=\infty$, if $r < 0$,

$0.\infty = 0, 0.(-\infty)=0$

Then for the "original" economic system $\bar{\mathcal{E}}$ individual demand functions $\bar{x}_{ij}(\underline{x})$ are defined by

(7)

$$\bar{x}_{ij}(\underline{x}) = \begin{cases} [(1-\alpha_{1j})\langle \underline{p},\bar{\underline{x}}_j\rangle + \bar{x}_{1j}] \dfrac{\alpha_{ij}}{p_i}, & \text{if } p_i > 0 \\ \infty, & \text{if } p_i = 0 . \end{cases} \quad (\underline{x} \in \mathcal{C}^*)$$

Replacing $\check{x}^*_{ij}(\underline{x})$ by $\bar{x}_{ij}(\underline{x})$, $z^*_{ij}(\underline{x})$ by $\bar{z}_{ij}(\underline{x})$ and $a^*_{ij}(\underline{x})$ by $\bar{a}_{ij}(\underline{x})$ in (2.1o) and (2.12) and following the operation rules of Definition 5 we obtain target excess demand functions $\bar{z}_{ij} : \mathcal{C}^* \longrightarrow \bar{\mathbb{R}}$ and active excess demand functions $\bar{a}_{ij} : \bar{\mathcal{D}}^* \longrightarrow \mathbb{R}$ with $\bar{\mathcal{D}}^* = \bar{Z}(\mathcal{C}^*)$ (where $\bar{Z}$ is given in the obvious sense by (1.6a)). Concerning the notation of "original" demand $\bar{x}_{ij}$ confusion with the individual goods assets of Chapter 1 seems to be unlikely.

If $p_i=0$ and $\alpha_{ij} > 0$ active excess demand $\bar{a}_{ij}$ is given by

(9)

$$\bar{a}_{ij}(\underline{x}) = \begin{cases} \infty, & \text{if } \max(-\bar{z}_{1j},0) > 0, \\ 0, & \text{if } \max(-\bar{z}_{1j},0) = 0, \text{ where } \bar{z}_{1j}=z^*_{1j}. \end{cases}$$

Defining $\operatorname{sgn}(\infty)= 1$ one gets the following relations as immediate consequences of the respective definitions

(8)

a) $z^*_{ij}(\underline{x}) \leqslant \bar{z}_{ij}(\underline{x})$, $(\underline{x} \in \mathcal{C}^*)$

b) $\operatorname{sgn} z^*_{ij} = \operatorname{sgn} \bar{z}_{ij}$,

c) $\operatorname{sgn} a^*_{ij} = \operatorname{sgn} \bar{a}_{ij}$,

Since we may reasonably assume the set

$$j_i = \{j \in j \mid \alpha_{ij} > 0\}$$

to be non-empty for every $i=1,2,\dots,m$ (2.15a) can be modified in order to obtain the transaction functions $\bar{t}_{ij}(\underline{x})$ at those points of $\bar{\mathcal{E}}^* = \bar{A}(\mathcal{D}^*)$ where $p_i = 0$ for at least one i ($2 \leq i \leq m$):

(1oa) If $\underline{x} \in \bar{\mathcal{E}}^*$ with $p_i = 0$,

$$\bar{t}_{ij}(\underline{x}) = \begin{cases} \bar{a}_{ij} , & \text{if } j \in \mathcal{S}_i \\ -\dfrac{\bar{a}_i^s}{c(\mathcal{L}_i)} , & \text{if } j \in \mathcal{L}_i \end{cases}$$

where $c(\mathcal{L}_i)$ denotes the number of elements of $\mathcal{L}_i \subsetneq j_i$

If $p_i > 0$ $\bar{t}_{ij}(\underline{x})$ is given according to (2.15a,b) with the mentioned changes of notation. Using the brief notation $\mathcal{J}_\infty(\underline{x}) = \{i \mid \bar{a}_i(\underline{x}) = \infty\}$ for $\underline{x} \in \mathcal{C}^*$ with $\mathcal{J}_\infty(\underline{x}) \neq \emptyset$ the price adjustment function (2.16) has to be modified according to

$$\text{(1ob)} \quad \overline{dp}_i = \begin{cases} \dfrac{1}{c(\mathcal{J}_\infty(\underline{x}))} - p_i & \text{if } i \in \mathcal{J}_\infty(\underline{x}) \\ 0 - p_i & \text{if } i \notin \mathcal{J}_\infty(\underline{x}) \end{cases}$$

With the functions of the present section the so-called original economy

$$\text{(11)} \qquad \bar{\mathcal{E}} = (\mathcal{C}^*, \bar{f})$$

can be constructed as a counterpart to (2.18).

Although the system $\mathcal{E}^*$ was not at all satisfactory from the point of view of demand theory the results obtained for $\mathcal{E}^*$ can be interpreted in terms of the underlying competitive economic model $\bar{\mathcal{E}}$ in the following way:

Definition 6:

A state $\underline{x} \in \mathcal{C}^*$ is called a trading equilibrium of the economy $\epsilon^*(\bar{\epsilon})$ if for $1 \leq i \leq m$; $1 \leq j \leq n$

$$(t^*_{ij} \circ A^* \circ Z^*)(\underline{x}) = 0 \qquad ((\bar{t}_{ij} \circ \bar{A} \circ \bar{Z})(\underline{x}) = 0).$$

The set of trading equilibria of the economy $\epsilon^*(\bar{\epsilon})$ will be denoted $\mathcal{M}^*_t(\bar{\mathcal{M}}_t)$.

Writing $\bar{\mathcal{M}}_c$ for the set of complete economic equilibria and $\bar{\mathcal{M}}_s$ for the set of shortage equilibria of $\bar{\epsilon}$ we have

Theorem 3:

If the economy $\bar{\epsilon}$ is given by (11)

$$\bar{\mathcal{M}}_c \cup \bar{\mathcal{M}}_s \cup \bar{\mathcal{M}}_t = \bar{\mathcal{M}}_c \cup \bar{\mathcal{M}}_t \neq \emptyset$$

For the economy $\bar{\epsilon}$ there exists at least one distinguished state $\underline{x} \in \mathcal{C}^*$ which is a complete economic equilibrium or a shortage equilibrium or a trading equlibrium of $\bar{\epsilon}$.

Proof:

Theorem 2 ensures the existence of a dynamic equilibrium $\hat{\underline{x}}$ of the system ϵ^*. Using Corollary 1 we have to distinguish two different cases.

(i) $\hat{\underline{x}} \in \mathcal{M}^*_c$

This implies $\hat{a}^*_{ij} \leqq 0$ for every pair (i,j) $(2 \leq i \leq m;\ 1 \leq j \leq n)$ which is only possible if $\hat{z}^*_{ij} \leqq 0$ or if $\hat{z}^*_{1j} \geqq 0$. Both cases imply $\hat{a}_{ij} \leqq 0$ as is readily seen from the definitions and therefore $\hat{\underline{x}}$ is a complete economic equilibrium of $\bar{\epsilon}$ $(\bar{\mathcal{M}}_c \neq \emptyset)$.

(ii) $\hat{\underline{x}} \in \mathcal{M}^*_s$

a) If for every pair (i,j) $\hat{x}^*_{ij} \leq K.\bar{x}_i$, we have $\bar{f}(\hat{\underline{x}}) = f^*(\hat{\underline{x}})$, (see p.23) and $\hat{\underline{x}}$ is seen to be a shortage state of the economy $\in$ ($\bar{\mathcal{M}}_s \neq \emptyset$).

b) If there exists at least one pair (i,j) with $\hat{x}^*_{ij} > K\bar{x}_i$ $\hat{\underline{x}}$ can be shown to represent at least a trading equilibrium of $\bar{\in}$. Since $\hat{\underline{x}}$ is a dynamic equilibrium of $\in^*$ we have

$$\hat{a}^*_{ij} \geqslant 0 \text{ and therefore } \hat{z}^*_{ij} \geqslant 0 \quad (2 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n);$$

(8a) leads then to $\hat{\bar{z}}_{ij} \geqslant \hat{z}^*_{ij} \geqslant 0$ and hence

$$\hat{\bar{a}}_{ij} \geqslant 0 \quad \text{for} \quad 2 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n.$$

Because of the existence of at least one pair (i',j') with $\hat{x}^*_{i'j'} > K\bar{x}_{i'}$ and therefore $x^*_{i'j'}(\hat{\underline{x}}) < \bar{x}_{i'j'}(\hat{\underline{x}})$ the $\bar{a}_i$ resulting from $\hat{x}$ need not guarantee price equilibrium in whereas a trading equilibrium is assured by the totally "one-sided" demands $\hat{\bar{a}}_{ij} \geqslant 0$ ($2 \leqslant i \leqslant m$; $1 \leqslant j \leqslant n$). ($\bar{\mathcal{M}}_t \neq \emptyset$).

□

By the results of Theorem 3 the elaborate discussion of the auxiliary economic system $\in^*$ seems to be justified.

4. ADJUSTMENT CORRESPONDENCE.

Undoubtedly the weakest point of the pure exchange model treated so far lies in the utmost restrictive formulation of the transaction part of the system. Assuming proportionate satisfaction of long side traders a very special kind of market mechanism is supposed to work in transaction activities. The present chapter will be concerned with relieving those unrealistic restrictions where analysis will again be done in terms of the auxiliary model $\mathcal{E}^*$.

4.1. Transaction Possibility Sets.

The starting point for the following investigations will be Section 2.2. where so-called transaction functions t^*_{ij} were defined. Those functions map a point $\underline{x} \in \mathcal{E}^*$ into a single "transaction point" of $\mathbb{R}^{mn}$ determining unique transaction quantities from a set of individual active excess demands. In order to obtain a better approximation of economic reality the uniquely given transaction vector with components $t^*_{ij}(\underline{x})$ will be replaced by a set of vectors of $\mathbb{R}^{mn}$ representing the so-called transaction possibility set which describes the transaction activities being possible under a given set of active excess demands of the individuals.

Denoting $\underline{x} \in \mathcal{E}^*$

$$\underline{x} = (p_2, p_3, \ldots, p_m; a^*_{11}, a^*_{21}, \ldots, a^*_{mn})'$$

and

$$\underline{t}^* \in \mathbb{R}^{mn}$$

$$\underline{t}^* = (t^*_{11}, t^*_{21}, \ldots, t^*_{mn})'$$

the following definition will play a fundamental role:

Definition 7:

For $\underline{x} \in \mathcal{E}^*$ let the transaction possibility set $T^*(\underline{x})$ be defined by

$$T^*(\underline{x}) = \{\underline{t}^* \in \mathbb{R}^{mn} \mid t^*_{ij} = a^*_{ij} \text{ if } j \in \zeta_i \quad (2 \leq i \leq m);$$

$$\sum_{j \in \alpha_i} t^*_{ij} = -a^{*s}_i \ , \ t^*_{ij} \cdot a^*_{ij} \geq 0, |t^*_{ij}| \leq |a^*_{ij}|$$

$$\text{if } j \in \alpha_i \quad (2 \leq i \leq m);$$

$$t^*_{1j} = -\sum_{k=2}^{m} p_k t^*_{kj} \quad \text{for } 1 \leq j \leq n\}$$

Remark 1:

The above definition establishes transactions to take place in such a way that short-side-individuals are fully satisfied, whereas among long-side-traders aggregate short-side-quantities are distributed somehow in accordance with the voluntary exchange-postulate. Money transactions are only the "second step" of commodity transactions.

By Definition 7 a correspondence

$$T^* : \mathcal{E}^* \longrightarrow 2^{\mathbb{R}^{mn}} \quad , \quad \underline{x} \longmapsto T^*(\underline{x})$$

is given the properties of which are to be examined.

Lemma 6:

For every $\underline{x} \in \mathcal{E}^*$ the corresponding transaction possibility set $T^*(\underline{x}) \subset \mathbb{R}^{mn}$ is a non-empty, closed convex set.

Proof:

(i) First of all it is easily seen that for a given price-excess demand-vector $\underline{x} \in \mathcal{E}^*$ $T^*(\underline{x})$ is well-defined by Definition 7, since for given $p_2, p_3, \ldots, p_m, a^*_{11}, a^*_{21}, \ldots, a^*_{mn}$ the sets ζ_i and α_i and the quantities $a^{*s}_i (2 \leq i \leq m)$ are well-defined.

Now it has to be shown that there exists at least one vector $\underline{t}^* \in T^*(\underline{x})$. This is trivially true, if $\xi_i \neq \emptyset$ for $i = 2,3,\ldots,m$, because in this case t^*_{ij} is uniquely determined for $j \in \zeta_i$, and if $a_i^{*s} \neq 0$ for each $j \in \alpha_i$ (which must then be a non-empty set) there exist transactions t^*_{ij} subject to

$$\sum_{j\in\alpha_i} t^*_{ij} = - a_i^{*s}, \quad t^*_{ij} \cdot a^*_{ij} \geq 0, \; | t^*_{ij} | \leq |a^*_{ij}| \quad (2 \leq i \leq m).$$

If $a_i^{*s} = 0$ obviously $t^*_{ij} = 0$ $(1 \leq j \leq n)$. The same unique "non-transactions" $t^*_{ij} = 0$ $(1 \leq j \leq n)$ appear if $\zeta_i = \emptyset$ implying $a_i^{*s} = 0$. Money transactions t^*_{1j} are given as a consequence of commodity transactions at the current price system. Thus for every $\underline{x} \in \mathcal{E}^*$ there exists at least one transaction vector $\underline{t}^* \in T^*(\underline{x})$ possibly including non-transaction components $t^*_{ij} = 0$.

(ii) Convexity of $T^*(\underline{x})$ ($\underline{x} \in \mathcal{E}^*$) is evident from Definition 7.

(iii) Being the set of solution vectors of a system of linear equations under " $\leq$"-restrictions for certain componente $T^*(\underline{x})$ is a closed subset of $\mathbb{R}^{mn}$.

□

The existence of an equilibrium of an economy incorporating this new transaction concept will be proved in two different ways. First Kakutani's Fixed Point Theorem ([16]) will be applied to obtain the desired result independently from the equilibrium properties of the economy $\mathcal{E}^*$. Second the existence of an equilibrium for the more general model will be derived from Theorem 2.

4.2. Upper - Semicontinuity of the Trading Correspondence T^*.

Proceeding from Definition 7 one may establish

Definition 8:

The correspondence

$$T^* : \mathcal{E}^* \longrightarrow 2^{\mathbb{R}^{mn}}, \quad \underline{x} \longmapsto T^*(\underline{x})$$

is called a trading correspondence.
(The same symbol $T^*(\underline{x})$ is used for the transaction possibility set and for the trading correspondence generated by it, since confusion seems to be unlikely).

The point-to-set-mapping T^* has the following important property:

Lemma 7:

The correspondence $T^*(\underline{x})$ is upper-semicontinuous on the set $\mathcal{E}^*$.

Proof:
(i) Let $\underline{x}^0 \in \mathcal{E}^*$ be such that

(1) $\quad a_i^*(\underline{x}^0) \neq 0 \quad$ for $i=2,3,\ldots,m$.

We shall prove T^* to be upper-semicontinuous at $\underline{x}^0$ (usc. at $\underline{x}^0$). Since in our case upper-semicontinuity of T^* is equivalent to closedness of T^*, we have to show the following fact: If $\{\underline{x}^\upsilon\}_{\upsilon\in\mathbb{N}}$ is a sequence of vectors of $\mathcal{E}^*$ tending to $\underline{x}^0$ for $\upsilon \to \infty$ and if $\{\underline{t}^\upsilon\}_{\upsilon\in\mathbb{N}}$ is a sequence of arbitrarily chosen vectors $\underline{t}^\upsilon \in T^*(\underline{x}^\upsilon)$ converging to a vector $\underline{t}^0 \in \mathbb{R}^{mn}$ then $\underline{t}^0$ is an element of the set $T^*(\underline{x}^0)$ ([2o], pp.65-69).

Because of $a_i^{*0} = a_i^*(\underline{x}^0) \neq 0$ it makes sense to consider

$$(2) \qquad a_i^* \cdot a_i^{*s} = a_i^* \sum_{j \epsilon \xi_i} a_{ij}^* = \sum_{j \epsilon \xi_i} a_i^* \cdot a_{ij}^* =$$

$$= \sum_{j=1}^{n} \min \left[0, a_i^* \cdot a_{ij}^*\right] \leqslant 0$$

$$(2 \leqslant i \leqslant m)$$

in a sufficiently small neighbourhood $\mathfrak{U}(\underline{x}^0)$ of $\underline{x}^0$ such that $a_i^*(\underline{x}) \neq 0$ for $\underline{x} \epsilon \mathfrak{U}(\underline{x}^0)$ because of the continuity of $a_i^*(\underline{x})$. Since the a_{ij}^* are trivially continuous at $\underline{x}^0$ the last term of (2) is continuous at $\underline{x}^0$. Thus $a_i^{*s}(\underline{x})$ is continuous at the point $\underline{x}^0 \epsilon \mathcal{E}^*$ $(2 \leqslant i \leqslant m)$ implying the convergence $a_i^{*s}(\underline{x}^\upsilon) \longrightarrow a_i^{*s}(\underline{x}^0)$ for $\upsilon \longrightarrow \infty$.

Let the sets of long-side-individuals corresponding to the point $\underline{x}^\upsilon$ be denoted $\mathcal{L}_i^\upsilon$ $(2 \leqslant i \leqslant m;\ 0 \leqslant \upsilon < \infty)$. Then because of the continuity of a_{ij}^* and a_i^* $(2 \leqslant i \leqslant m\ ;\ 1 \leqslant j \leqslant n)$ at $\underline{x}^0$ we have

$$\mathcal{L}_i^\upsilon = \mathcal{L}_i^0$$

for υ being sufficiently large and if in addition $a_{ij}^{*o} \neq 0$ for every pair (i,j). For such υ of course

$$\xi_i^\upsilon = \xi_i^0 .$$

Therefore

$$\lim_{\upsilon \to \infty} \sum_{j \epsilon \mathcal{L}_i^o} t_{ij}^{*\upsilon} = \sum_{j \epsilon \mathcal{L}_i^o} \lim_{\upsilon \to \infty} t_{ij}^{*\upsilon} = \sum_{j \epsilon \mathcal{L}_i^o} t_{ij}^{*0} =$$

$$= \lim_{\upsilon \to \infty} \left[-a_i^{*s}(\underline{x}^\upsilon)\right] = -a_i^{*s}(\underline{x}^0)$$

and

$$t_{ij}^{*\,0} = \lim_{\upsilon\to\infty} t_{ij}^{*\,\upsilon} = \lim_{\upsilon\to\infty} a_{ij}^{*\,\upsilon} = a_{ij}^{*\,0} \quad , \text{ if } \quad j \in \mathcal{S}_i^0 .$$

Of course for $j \in \mathcal{L}_i^0$ we have

$$|t_{ij}^{*\,0}| = |\lim_{\upsilon\to\infty} t_{ij}^{*\,\upsilon}| = \lim_{\upsilon\to\infty} |t_{ij}^{*\upsilon}| \leqslant \lim_{\upsilon\to\infty} |a_{ij}^{*\,\upsilon}| = |a_{ij}^{*\,0}|$$

and

$$t_{ij}^{*\,0} \cdot a_{ij}^{*\,0} = \lim_{\upsilon\to\infty} t_{ij}^{*\,\upsilon} \cdot \lim_{\upsilon\to\infty} a_{ij}^{*\,\upsilon} = \lim_{\upsilon\to\infty} (t_{ij}^{*\,\upsilon} \cdot a_{ij}^{*\,\upsilon} \geqslant 0.$$

To relieve the strong restriction $a_{ij}^{*\,0} \neq 0$ for every pair (i,j) let for a fixed i ($2 \leqslant i \leqslant m$) the subscript set $\mathcal{N}_i^0$ be defined by

(3) $\qquad \mathcal{N}_i^0 = \{j \in j \mid a_{ij}^{*\,0} = 0\}$

a) For the short side of i the following relations hold:

a 1) If $j \in (\mathcal{S}_i^0 \setminus \mathcal{N}_i^0)$

$$t_{ij}^{*\,0} = \lim_{\upsilon\to\infty} t_{ij}^{*\,\upsilon} = \lim_{\upsilon\to\infty} a_{ij}^{*\,\upsilon} = a_{ij}^{*\,0}$$

a 2) If $j \in (\mathcal{S}_i^0 \cap \mathcal{N}_i^0)$

$$|t_{ij}^{*\,0}| = |\lim_{\upsilon\to\infty} t_{ij}^{*\,\upsilon}| = \lim_{\upsilon\to\infty} |t_{ij}^{*\,\upsilon}| \leqslant \lim_{\upsilon\to\infty} |a_{ij}^{*\,\upsilon}| =$$

$$= |a_{ij}^{*\,0}| = 0$$

implying $t_{ij}^{*\,0} = 0 = a_{ij}^{*\,0}$.

b) On the long side of i we have for υ being large enough

$$\mathcal{L}_i^0 \subseteq \mathcal{L}_i^\upsilon$$

and therefore

$$\sum_{j\in\mathcal{A}_i^0} t_{ij}^{*\,0} = \sum_{j\in\mathcal{A}_i^0} \lim_{\upsilon\to\infty} t_{ij}^{*\,\upsilon} = \lim_{\upsilon\to\infty} \Big(\sum_{j\in\mathcal{A}_i^0} t_{ij}^{*\,\upsilon} \Big) =$$

$$= \lim_{\upsilon\to\infty} \Big(\sum_{j\in\mathcal{A}_i^\upsilon} t_{ij}^{*\,\upsilon} - \sum_{j\in\mathcal{N}_i^0} t_{ij}^{*\,\upsilon} \,.\, \max\left[0,\mathrm{sgn}(a_i^{*\upsilon}.a_{ij}^{*\upsilon})\right] \Big) =$$

$$= \lim_{\upsilon\to\infty} (-a_i^{*s}(\underline{x}^\upsilon)) - \sum_{j\in\mathcal{N}_i^0} \lim_{\upsilon\to\infty}(t_{ij}^{*\upsilon}.\max\left[0,\mathrm{sgn}(a_i^{*\upsilon}a_{ij}^{*\,\upsilon})\right]) =$$

$$= -a_i^{*s}(\underline{x}^0)$$

Since the remaining conditions of Definition 7 hold trivially, we have for $\underline{x}^0 \in \mathcal{E}^*$ with $a_i^*(\underline{x}^0 \neq 0$ $(2 \leq i \leq m)$

$$\lim_{\upsilon\to\infty} \underline{t}^{*\upsilon} = \underline{t}^{*0} \in T^*(\underline{x}^0).$$

(ii) Let $\underline{x}^0 \in \mathcal{E}^*$ be arbitrarily chosen such that $a_i^*(\underline{x}^0) = 0$ may occur for some $i \in \{2,3,\ldots,m\}$.
For a fixed i with $a_i^*(\underline{x}^0) = 0$ we use the following notations:

$$a_i^{*su}(\underline{x}) = \sum_{j=1} \min\left[0,a_{ij}^*(\underline{x})\right] \qquad \text{(aggregate supply)}$$

$$a_i^{*de}(\underline{x}) = \sum_{j=1}^{n} \max\left[0,a_{ij}^*(\underline{x})\right] \qquad \text{(aggregate demand)}$$

$$\mathcal{D}(\underline{x}) = \{j\varepsilon j \mid a_{ij}^*(\underline{x}) \geq 0\}$$

$$\mathcal{S}(\underline{x}) = \{j\varepsilon j \mid a_{ij}^*(\underline{x}) < 0\}$$

$$\mathcal{N}(\underline{x}) = \{j\varepsilon j \mid a_{ij}^*(\underline{x}) = 0\}$$

Because of $a_i^*(\underline{x}^0) = 0$ we have for $\underline{t}^{*0} \varepsilon T^*(\underline{x}^0)$

$$t_{ij}^{*\,0} = a_{ij}^*(\underline{x}^0) \qquad (1 \leq j \leq n) \ .$$

The following fact has to be shown: Given $\varepsilon > 0$ there exists a $\delta > 0$ such that for an arbitrary $\underline{x} \ \varepsilon \ \mathcal{U}_\delta(\underline{x}^0)$ and an arbitrary $\underline{t}^* \ \varepsilon \ T^*(\underline{x})$

(4) $t_{ij}^* \ \varepsilon \ (a_{ij}^{*0} - \varepsilon \ , \ a_{ij}^{*0} + \varepsilon)$ for $j = 1,2,\ldots,n$.

a) $j \ \varepsilon \ \mathcal{N}(\underline{x}^0)$

Because of the continuity of $a_{ij}^*(\underline{x})$ and the inequality

$$|t_{ij}^*| \leq |a_{ij}^*|$$

together with

$$a_{ij}^*(\underline{x}^0) = 0$$

nothing is left to be proved.

b) $j \ \varepsilon \ (\mathcal{D}(\underline{x}^0) \setminus \mathcal{N}(\underline{x}^0))$

By means of the continuity of $a_{ij}^*(\underline{x})$ and $a_i^{*su}(\underline{x})$ for a given $\varepsilon > 0$ one can find a $\delta > 0$ such that for every $\underline{x} \ \varepsilon \ \mathcal{U}_\delta(\underline{x}^0)$

(5) $\mathcal{D}(\underline{x}) \subseteq \mathcal{D}(\underline{x}^0) \ , \ (\mathcal{D}(\underline{x}^0) \setminus \mathcal{N}(\underline{x}^0)) \subseteq \mathcal{D}(\underline{x})$

(6) $a_{ij}^*(\underline{x}) \ \varepsilon \ (a_{ij}^*(\underline{x}^0) - \frac{\varepsilon}{2n} \ , \ a_{ij}^*(\underline{x}^0) + \frac{\varepsilon}{2n}) \ (1 \leq j \leq n)$

(7) $a_i^{*su}(\underline{x}) \ \varepsilon \ (a_i^{*su}(\underline{x}^0) - \frac{\varepsilon}{2} \ , \ a_i^{*su}(\underline{x}^0) + \frac{\varepsilon}{2})$

This implies for $\underline{x} \ \varepsilon \ \mathcal{U}_\delta(\underline{x}^0)$

(8) $0 \leq t_{ij}^* \leq a_{ij}^*(\underline{x}) < a_{ij}^*(\underline{x}^0) + \frac{\varepsilon}{2n}$,

$$(9) \quad t^*_{ij} = -a^{*su}_i(\underline{x}) - \sum_{\substack{k \varepsilon \mathcal{D}(x) \\ k \neq j}} t^*_{ik}(\underline{x}) >$$

$$> (-a^{*su}_i(\underline{x}^0) - \frac{\varepsilon}{2}) - (\sum_{\substack{k \varepsilon \mathcal{D}(\underline{x}) \\ k \neq j}} a^*_{ik}(\underline{x}^0) + n\frac{\varepsilon}{2n}) \geq$$

$$\geq -a^{*su}_i(\underline{x}^0) - \sum_{\substack{k \varepsilon \mathcal{D}(\underline{x}^o) \\ k \neq j}} a^*_{ik}(\underline{x}^0) - \mathcal{E} =$$

$$= +a^*_{ij}(\underline{x}^0) - \mathcal{E}$$

Hence (4) is seen to hold in that case.

c) $j \varepsilon \mathcal{S}(\underline{x}^0)$

Choosing a $\delta > 0$ such that for $\underline{x} \varepsilon \mathcal{U}_\delta(\underline{x}^0)$

$$(1o) \quad S(\underline{x}) \subseteq (S(\underline{x}^0) \cup \mathcal{N}(\underline{x}^0)),$$

$$(11) \quad a^*_{ij}(\underline{x}) \varepsilon (a^*_{ij}(\underline{x}^0) - \frac{\varepsilon}{2n}, a^*_{ij}(\underline{x}^0) + \frac{\varepsilon}{2n}),$$

$$(12) \quad a^{*de}_i(\underline{x}) \varepsilon (a^{*de}_i(\underline{x}^0) - \frac{\varepsilon}{2}, a^{*de}_i(\underline{x}^0) + \frac{\varepsilon}{2}),$$

we get for $\underline{x} \varepsilon \mathcal{U}_\delta(\underline{x}^0)$

$$(13) \quad 0 \geq t^*_{ij} \geq a^*_{ij}(\underline{x}) > a^*_{ij}(\underline{x}^0) - \frac{\varepsilon}{2n},$$

$$(14) \quad t^*_{ij} = -a^{*de}_i(\underline{x}) - \sum_{\substack{k \varepsilon S(\underline{x}) \\ k \neq j}} t^*_{ik}(\underline{x}) <$$

$$< (-a^{*de}_i(\underline{x}^0) + \frac{\varepsilon}{2}) - (\sum_{\substack{k \varepsilon S(\underline{x}^o) \\ k \neq j}} a^*_{ik}(\underline{x}^0) - n\frac{\varepsilon}{2n}) =$$

$$= a^*_{ij}(\underline{x}^0) + \mathcal{E}$$

From (13) and (14) (4) follows for the case $j \varepsilon S(\underline{x})$. Combining parts (i) and (ii) of the proof closedness of the correspondence $T^*(\underline{x})$ at every point $\underline{x} \varepsilon \mathcal{E}^*$ can be established.

Thus $T^*(\underline{x})$ has been shown to be upper-semicontinuous on the set $\mathcal{E}^*$. □

4.3. Equilibrium of the Generalized Economy $\tilde{\mathcal{E}}^*$.

Using the concepts of the previous sections the system $\mathcal{E}^*$ can now be generalized so as to incorporate a trading correspondence instead of a transaction function. The equilibrium properties of the new model will then be examined.

From $\mathcal{E}^*$ a new economic system $\tilde{\mathcal{E}}^* = (\mathcal{C}^*, \tilde{f}^*)$ may be derived replacing the transaction function with components t^*_{ij} by a trading correspondence T^* in the obvious way:

For $\underline{x} \in \mathcal{E}^*$ the adjustment correspondence $\tilde{f}^*: \mathcal{C}^* \longrightarrow 2^{\mathcal{X}}$ is constructed by means of the correspondences

(15) $\tilde{D}^* : \mathcal{E}^* \longrightarrow 2^{\mathcal{X}}$

$$\tilde{D}^*(\underline{x}) = \{\underline{y} \varepsilon X \mid y_k = dp^*_{k+1}(\underline{x}) \text{ if } 1 \leq k \leq m-1 ,$$

$$(y_m, y_{m+1}, \ldots, y_{m(n+1)-1})' \in T^*(\underline{x})\}$$

and

(16) $\tilde{\Delta}^* : \mathcal{C}^* \longrightarrow 2^{\mathcal{X}}$

$$\tilde{\Delta}^*(\underline{x}) = (\tilde{D}^* \circ A^* \circ Z^*)(\underline{x}) \qquad \text{for } \underline{x} \in \mathcal{C}^* .$$

Finally one may define

(17) $\tilde{f}^* : \mathcal{C}^* \longrightarrow 2^{\mathcal{X}}$

$$\tilde{f}^*(\underline{x}) = \underline{x} + \tilde{\Delta}^*(\underline{x}) \qquad \text{for } \underline{x} \in \mathcal{C}^*.$$

Thus the economic system

(18) $\tilde{\mathcal{E}}^* = (\mathcal{C}^*, \tilde{f}^*)$

is well-defined.

Definition 9:

A state $\tilde{\underline{x}} \in \mathcal{C}^*$ is called a dynamic equilibrium of the economy $\tilde{\mathcal{E}}^*$, if $\tilde{\underline{x}}$ is a fixed point of the adjustment correspondence $\tilde{f}^*$, that is

$$\tilde{\underline{x}} \in \tilde{f}^*(\tilde{\underline{x}}) \quad .$$

As a counterpart to Theorem 2 we get

Theorem 4:

Let the economy $\tilde{\mathcal{E}}^* = (\mathcal{C}^*, \tilde{f}^*)$ be defined by (18). Then there exists a dynamic equilibrium $\tilde{\underline{x}}$ of $\tilde{\mathcal{E}}^*$.

Proof:
A summary of the results obtained so far for the adjustment correspondence $\tilde{f}^*$ will show that Kakutani's Fixed-Point Theorem applies to $\tilde{f}^*$:

(i) From (1.4a-c) and (2.8) $\mathcal{C}^*$ is seen to be a nonempty compact convex subset of X.

(ii) Since Z^* and A^* are continuous functions and the correspondence T^* is upper-semicontinuous (Lemma 7) the point-to-set-mapping $\tilde{f}^*$ is upper-semicontinuous on $\mathcal{C}^*$.

(iii) Lemma 6 assures that for every $\underline{x} \in \mathcal{C}^*$ the image set $\tilde{f}^*(\underline{x})$ is a non-empty, closed convex subset of $\mathfrak{X}$ and Definition 7 together with the results of the previous chapters yields $\tilde{f}^*(\underline{x}) \subseteq \mathcal{C}^*$ for every $\underline{x} \in \mathcal{C}^*$.
Thus Kakutani's Theorem ([16]) can be applied to prove the above assertion.

□

Corollary 3:

If $\tilde{\underline{x}}$ is a dynamic equilibrium of $\tilde{\mathcal{E}}^*$, then

$$\tilde{f}^*(\tilde{\underline{x}}) = \{\tilde{\underline{x}}\}$$

Proof:

$\underset{\sim}{\tilde{x}} \in \tilde{f}^*(\underset{\sim}{\tilde{x}})$ implies $\underline{0} \in \tilde{\Delta}^*(\underset{\sim}{\tilde{x}})$ and therefore for every $\underline{x} \in \tilde{\Delta}^*(\underset{\sim}{\tilde{x}})$

$$x_k = 0 \ , \ \text{if } 1 \leq k \leq m-1 \ ,$$

must hold.

Since x_k's corresponding to short-side-transactions are uniquely determined being equal to zero, all individual transaction quantities have to be zero implying

$$\underline{x} \in \tilde{\Delta}^*(\underset{\sim}{\tilde{x}}) \quad \Longrightarrow \quad \underline{x} = \underline{0}$$

and hence

$$\tilde{f}^* \ (\underset{\sim}{\tilde{x}}) = \{\underset{\sim}{\tilde{x}}\}$$ □

An immediate consequence of Corollary 3 is

Corollary 4:

$\underset{\sim}{\tilde{x}} \in \mathcal{E}^*$ is a dynamic equilibrium of $\tilde{\mathcal{E}}^*$, if and only if $\underset{\sim}{\tilde{x}}$ is a dynamic equilibrium of $\mathcal{E}^*$.

Proof:

This asssertion follows straightforward from the definitions and from Corollary 3. □

By the last result and Theorem 2 a very short proof of Theorem 4 is provided.

Of course the results of Section 3.4 hold also for an "underlying" economy incorporating classical demand functions and a transaction correspondence as is easily seen from Corollary 4.

5. CONCLUSIONS.

Reviewing the foregoing chapters one may observe that the investigations started with a general formulation of the equilibrium problem (Chapter 1), then proceeded to a certain specification and its thorough analysis (Chapters 2 and 3) and finally went back again towards the discussion of a more general model (Chapter 4). Although the starting point of the economy $\mathcal{E}$ will not be reached again on this way back, the present chapter will provide some results about a fairly general economic model which can be derived from former results.

At last the computational aspects of our equilibrium problem will be overviewed possibly providing a basis for further work on the computation of non-tâtonnement equilibria.

5.1. Generalization of Price Adjustment.

Starting from the system $\tilde{\mathcal{E}}^*$ of Chapter 4 one further step towards a more general economic model can be done through replacement of the special function of (2.16) by a general price adjustment function according to

<u>Definition 1o:</u> (see [1] , pp.266)

A continuous function

$$\widetilde{dp}^*: \quad \mathcal{E}^* \longrightarrow \mathbb{R}^m$$

is called a price adjustment function, if

(i) $\widetilde{dp}_1^* = 0$ for every $\underline{x} \in \mathcal{E}^*$,

(ii) $(\underline{p}(\underline{x}) + (\widetilde{dp}_2^*(\underline{x}), \widetilde{dp}_3^*(\underline{x}), \dots, \widetilde{dp}_m^*(\underline{x}))') \in S^{m-2}$

for every $\underline{x} \in \mathcal{E}^*$,

(iii) $\widetilde{dp}_i^*(\underline{x}) \leqslant 0$ if $a_i^*(\underline{x}) = 0 \quad (2 \leqslant i \leqslant m)$

If in (4.15), (4.16) and (4.17) a general price adjustment function $\widetilde{dp}^*$ is used instead of dp^* we obtain a new economic system - including a general price mechanism and a trading correspondence - which is denoted

(1) $\quad \widetilde{\widetilde{\mathcal{E}}}^* = (\mathcal{C}^* , \widetilde{\widetilde{f}}^*)$

where $\widetilde{\widetilde{f}}^*$ results from the obvious composition of functions.

Before studying the consequences which the results obtained so far have for the new model $\widetilde{\widetilde{\mathcal{E}}}^*$ it seems to be useful to restate the economic assumptions being crucial for $\widetilde{\widetilde{\mathcal{E}}}^*$. First of all each of the trading individuals is supposed to be guided by a Cobb-Douglas utility function ([23], pp.19) for the commodities i= 2,3,...,m from which "demands" (see (2.9)) for the various goods are derived subject to individual budget constraints. Individual demand for money is assumed to be proportional to current individual wealth (see (2.3)). Further it is supposed that as a behavioural guideline traders take so-called active excess demands which are formulated under individual financing constraints according to (2.12). On an aggregate level these active demands are also taken as determinants of the price mechanism. Finally a non-bankrupt condition is imposed on transaction activities (see (2.7)). Within this conceptual framework a process of simultaneous adjustment of prices and individual goods endowments is assumed to take place the rules of which are given in a rather general form by Definition 1o and Definition 8. Thus $\widetilde{\widetilde{\mathcal{E}}}^*$ represents a model pf price-quantity dynamics of a fairly general type under particular assumptions about individual demands.

Theorem 5:

Let the economy $\widetilde{\widetilde{\mathcal{E}}}^* = (\mathcal{C}^*, \widetilde{\widetilde{f}}^*)$ be defined as described above. Then there exists a dynamic equilibrium $\underline{\widetilde{\widetilde{x}}}$ of $\widetilde{\widetilde{\mathcal{E}}}^*$, that is a fixed point of $\widetilde{\widetilde{f}}^*$ characterized by

$$\underline{\widetilde{\widetilde{x}}} \in \widetilde{\widetilde{f}}^* (\underline{\widetilde{\widetilde{x}}}).$$

This assertion can be proved in exactly the same way as Theorem 4.

□

The transformation of Theorem 3 to an economy including classical demand functions, a general price mechanism and a transaction correspondence can be carried out straightforward, since the demand structure of the system $\bar{\mathcal{E}}$ of Theorem 3 remains unchanged.

5.2. Equilibrium Computation.

The main results obtained so far were derived from the properties of the "auxiliary" economic model $\mathcal{E}^*$. This model is seen to be useful for equilibrium computation too, since it turns out to be accessible by several types of fixed-point algorithms.

The principal features of the economy $\mathcal{E}^*$ are given by a continuous function f^* mapping a compact convex set $\mathcal{C}^* \subset \mathfrak{X}$ into itself. By a suitably chosen homeomorphism $\mathcal{C}^*$ can be carried over to a unit simplex of appropriate dimension. Thereby the economic equilibrium problem of $\mathcal{E}^*$ is transformed into the problem of finding a fixed point of a continuous self-mapping of a unit simplex. For the computation of approximate solutions of this problem several algorithms are available ([4] , [5] , [10] , [17] , [18] , [23]).

Since a dynamic equilibrium of $\mathcal{E}^*$ provided the starting point for most of the arguments of the previous chapters, actual computation of a fixed point of f^* would yield valuable information about equilibrium properties of the economic systems considered.

A concise treatment of the computational aspects of the present problem will be given in a forthcoming paper.

Notational Conventions:

$a, b, c, \ldots$	real numbers (scalars)
$[\![a]\!] = \max \{n \mid n \text{ integer}, n \leqslant a\}$	
$\operatorname{sgn} a = \begin{cases} 1 & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -1 & \text{if } a < 0 \end{cases}$	
$\lvert a \rvert = a \cdot \operatorname{sgn} a$	
$(a,b) = \{x \mid a<x<b\}$	open real interval
$[a,b] = \{x \mid a \leqslant x \leqslant b\}$	closed real interval
$\underline{a}, \underline{b}, \underline{c}, \ldots$	vectors
$\langle \underline{a}, \underline{b} \rangle$	inner product of $\underline{a}$ and $\underline{b}$
$\lVert a \rVert = (\langle a, a \rangle)^{1/2}$	norm of $\underline{a}$
$\underline{a} \geqslant \underline{b}$ if $a_i \geqslant b_i$	for every i
$\underline{a} > \underline{b}$ if $\underline{a} \geqslant \underline{b}$	and if there exists a k with $a_k > b_k$
$\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$	sets
$\emptyset$	empty set
$\mathbb{N}$	set of positive integers
$\mathbb{R}$	real line
$\mathbb{R}^n$	n-dimensional Euclidean space
$\mathbb{R}^n_+ = \{\underline{x} \in \mathbb{R}^n \mid \underline{x} \geqslant \underline{0}\}$	non-negative orthant of $\mathbb{R}^n$
$S^{n-1} = \{\underline{x} \in \mathbb{R}^n \mid \underline{x} > \underline{0}, \sum_{i=1}^{n} x_i = 1\}$	(n-1)-dimensional unit simplex
$\mathcal{U}_\delta(\underline{x}) = \{\underline{y} \in \mathbb{R}^n \mid \lVert y-x \rVert < \delta\}$	δ-neighbourhood of $\underline{x} \in \mathbb{R}^n$
$c(\mathcal{M})$	cardinal number of $\mathcal{M}$
$2^{\mathcal{M}} = \{\mathcal{S} \mid \mathcal{S} \subset \mathcal{M}\}$	collection of all subsets of $\mathcal{M}$

□ marks the end of a proof.

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