FORECASTING WITH LARGE SCALE ECONOMETRIC SYSTEMS

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1. INTRODUCTION

Large scale econometric systems cause special problems for estimation, forecasting, and control. Common feature of large scale systems is a hierarchical structure which recommends to make use of decentralized procedures in system applications. The paper investigates the characteristic features of hierarchical systems and focuses on decentralized solution and forecasting methods. The concepts developed serve as a starting point for other system applications like the investigation of stochastic system properties or optimal control experiments. All results stress the importance of stochastic aspects in the analysis of large scale systems.

Hierarchical systems are characterized by a number of subsystems which are connected via interaction channels. Due to these interactions a forecast produced by the individual subsystems in general will not meet some overall system constraints. In case of a world trade model, e.g., the import forecasts of the individual countries based on their assumptions about export demand might not add up to the volume of world trade calculated either by adding up country imports or adding up country exports. Therefore the role of a coordinator has to be defined to take into account the interactions between the subsystems.

Attempts to formulate a statistical forecasting theory for hierarchical systems indicate that the advantages of a decentralized decision process in a forecasting procedure depend on the information structure of the system. In case of an improper information exchange either no forecast is found which meets all system restrictions or the coordinator "wipes out" the subsystem forecasts.

Based on Kalman filter theory suggestions are made for the information exchange between the subsystems and a coordinator in a forecasting exercise. Explicitely the relative precision of the subsystem forecasts and the coordinator is taken into account.

A number of potential applications become evident for the LINK System - a worldwide research project to study the international transmission of economic fluctuations by linking national econometric models. The basic research strategy of LINK stresses the importance of the model builders' knowledge in the process of model specification and model application. The theory of decentralized solution procedures for large scale systems provides us with a better understanding of the present information exchange pattern used within the LINK System and with proposals for improved solution methods both in numerical and statistical terms.

2. STRUCTURE OF HIERARCHICAL SYSTEMS

2.1. Description of Hierarchical Systems

Let Figure 2.1 illustrate an econometric system with mathematical representation

$$h(y,x,u) = 0 \quad . \tag{2.1}$$

Three categories of variables are involved, namely,

- y the vector of system output (or endogenous) variables,
- x the vector of system input (or exogenous) variables,
- u the vector of system error (or disturbance) variables.

For the sake of generality we will assume that besides the system error u also the system input x is a random variable. This is particularly true if the system is used for forecasting where for most system inputs only unprecise information is available. Obviously the case of a precisely known input is also contained in this assumption.

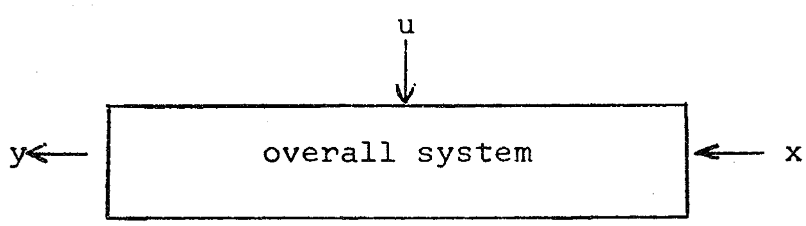

Figure 2.1

Large scale econometric systems are characterized by a more or less pronounced hierarchical structure where subsystems are connected via interaction systems at various hierarchical levels. We will, therefore, investigate the hierarchical system structure to improve our analysis of large scale econometric systems.

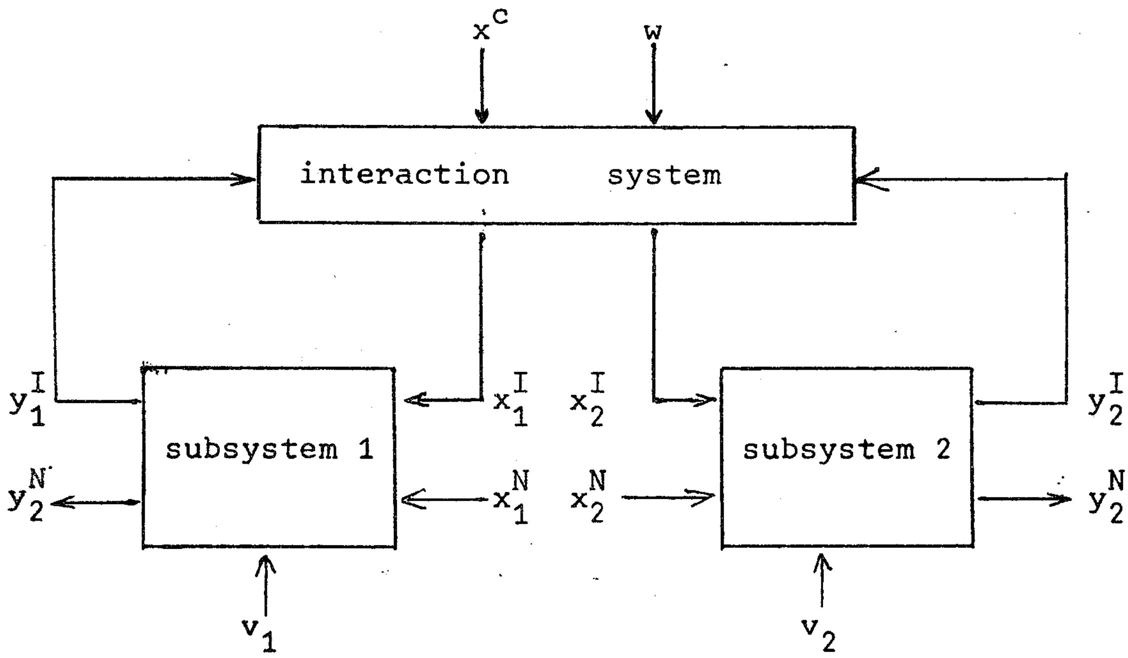

Figure 2.2

To understand the general characteristics of a complex hierarchical system it is sufficient to consider the case of a two-level structure. Figure 2.2 illustrates the simplest case with two subsystems connected via an interaction system. The hierarchical system structure suggest the following partition of the system variables:

Among the endogenous variables y we distinguish between y^I, the vector of interacting endogenous variables and y^N, the vector of non-interacting endogenous variables, depending if they feed back into the interaction system or not.

Among the exogenous variables x we distinguish between those which enter only the interaction system and are subject to the control of the coordinator x^c, and those which enter only the subsystems and are either interacting exogenous variables x^I or non-interacting exogenous variables x^N, depending if they are output of the interaction system or not.

Among the disturbance variables u we distinguish between those which enter the subsystems v and those which enter the interaction system w.

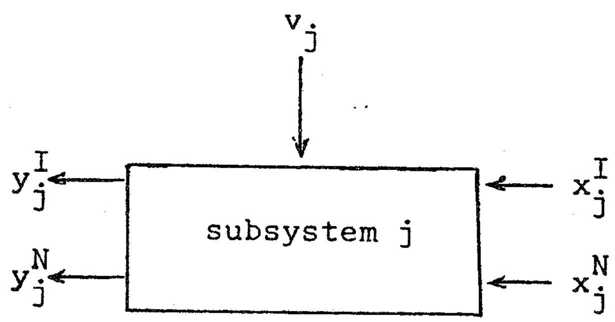

Figure 2.3

Let us further assume that on a certain hierarchical level a system is composed of n subsystems, then the <u>j-th sub-</u>

system, illustrated in Figure 2.3, is dealing with the following vectors of variables,

- y_j^I interacting endogenous variables of the j-th subsystem,
- y_j^N non-interacting endogenous variables of the j-th subsystem,
- x_j^I interacting exogenous variables of the j-th subsystem,
- x_j^N non-interacting exogenous variables of the j-th subsystem,
- v_j disturbances of the j-th subsystem,

which are connected via the following functional relationship describing the j-th subsytem,

$$f_j(y_j, x_j, v_j) = 0 \quad , \tag{2.2}$$

where

$$y_j = \begin{bmatrix} y_j^I \\ y_j^N \end{bmatrix} \quad \text{and} \quad x_j = \begin{bmatrix} x_j^I \\ x_j^N \end{bmatrix} \quad . \tag{2.3}$$

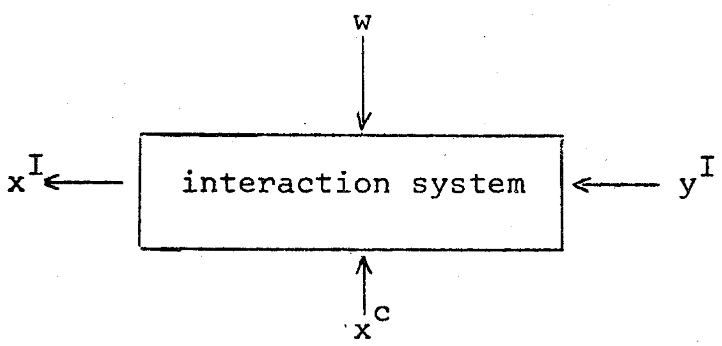

Figure 2.4

The vectors of variables in the interaction system, illustrated in Figure 2.4, are

y^I interacting endogenous variables,
x^I interacting exogenous variables,
x^c exogenous variables of the interaction system,
w disturbances of the interaction system,

which enter the following functional relationship describing the interaction system,

$$g(y^I, x^I, x^c, w) = 0 \quad , \tag{2.4}$$

where

$$y^I = \begin{bmatrix} y_1^I \\ \cdot \\ \cdot \\ \cdot \\ y_n^I \end{bmatrix} \quad \text{and} \quad x^I = \begin{bmatrix} x_1^I \\ \cdot \\ \cdot \\ \cdot \\ x_n^I \end{bmatrix} \quad . \tag{2.5}$$

2.2 Hierarchical Structure of the LINK System

The LINK system is an excellent example of a large scale exonometric system with a hierarchical structure. The individual country or regional models form subsystems which are connected via an interaction system, the trade model.

Interacting endogenous variables y^I comprise import volumes and export prices of each country or region. The trade model describes the conversion of these variables into export volumes and import prices, the interacting exogenous variables x^I. Exogenous variables which enter only the interaction system x^c are at the present state of the LINK system exchange rates, c.i.f./f.o.b. conversion ratios, and data base conversion factors (from OECD to IMF data, e.g.). Examples for non-interacting exogenous variables x^N are, e.g., public expenditures and tax rates.

The LINK system can be imagined as a system with two hierarchical levels, namely the trade model and the country or regional models. A third level would be added if we would consider the regional models consisting of country models coordinated by a regional coordinator.

3. SOLUTION PROCEDURES FOR LARGE SCALE SYSTEMS

3.1. Basic Concepts of Decentralized Solution Procedures

In all kinds of system applications, e.g. forecasting or policy simulation, we are confronted with the problem of calculating the system solution.

Referring to the stochastic system (2.1) a system solution specifies the stochastic characteristics of the system output y (or endogenous) variables in terms of the system inputs. Thus we are interested in the probability distribution of y conditioned on the system input x and system error u:

$$p(y|x,u) \quad . \tag{3.1}$$

In case of a hierarchical system structure described by (2.2) and (2.4) the solution is expressed by the conditional probability distribution of the endogenous variables of all subsystems $y_1,\ldots,y_n$ and all interacting exogenous variables $x_1^I,\ldots,x_n^I$, for given non-interacting exogenous variables $x_1^N,\ldots,x_n^N$, exogenous variables of the interaction system x^c, subsystem disturbances $v_1,\ldots,v_n$, and interaction system disturbance w:

$$p(y_1,\ldots,y_n,x_1^I,\ldots,x_n^I|x_1^N,\ldots,x_n^N,x^c,v_1,\ldots,v_n,w). \tag{3.2}$$

Solving a system is equivalent to calculate the characteristics of the above specified distributions (e.g. mean and variance). In case of a hierarchical system structure the question arises if the computational complexity can be simplified by splitting up the computational burden between the subsystems and a coordinator. Such a decentralized solution procedure would require to define the information exchange between the coordinator and the subsystems and their computational tasks. As in most cases the coordinator and the subsystems represent organisational units, solution algorithms for large scale systems which share the computational work between the subsystems and a coordinator seem to be very appealing. We will devote therefore our attention to decentralized solution procedures for large scale systems.

With reference to (2.2) we realize that the j-th subsystem solution y_j depends on the disturbance v_j, the interacting exogenous variables x_j^I, and the non-interacting exogenous variables x_j^N. If the functional relationship of the j-th subsystem f_j specifies a "weak" influence of the interacting exogenous variables, then even under wrong assumptions for x_j^I the j-th subsystem solution calculated by (2.2) will still yield satisfactory results.

Obviously the j-th subsystem is independent in calculating its solution, if no interacting exogenous variables effect the j-th subsystem.

Serious problems, however, for a decentralized solution procedure based on subsystem solutions arise, if via the interacting exogenous variables not negligible interdependencies occur. In this case the role of a coordinator has to be defined who collects information from the subsystems, processes this information and feeds new information back to the subsystems in order to arrive at subsystem solutions which are compatible with all the constraints imposed by the interaction system.

As we are dealing with stochastic systems the system solution, denoted by $\hat{y}$, is always a random variable. The random character of the solution stems from stochastic inputs and random disturbances. Thus a system solution is described by the probability distribution of the endogenous variables or some characteristics of this distribution (e.g. the first two moments).

In the sequel we will introduce three types of decentralized solution procedures for hierarchical systems. The stochastic character of the solution is evident in Solution Method 3. Solution Methods 1 and 2 yield stochastic solutions as soon as random inputs (disturbances or exogenous variables) enter the system. The stochastic characteristics of the solution can either be calculated analytically (in case of linear functional relationships and distributions described by the first two moments) or obtained by stochastic simulation. Solution Methods 1 and 2 will, however, also produce deterministic solutions, if the disturbances are neglected and all exogenous variables are considered deterministic. In this case the solution is concerned only with deterministic equation systems.

The advantage of this convention is, that the same notation and description of the algorithms holds both for the stochastic and deterministic case. A stochastic solution is obtained as soon as stochastic inputs are present.

3.2. Solution Method 1 (Overall System Solution)

3.2.1. Preliminaries

A plausible starting point for a solution procedure for a large system is the suggestion, that each subsystem mails its equation system and the necessary subsystem inputs to a coordinator who takes over the task of solving the complete system. We will formulate the algorithm for this solution method and discuss its advantages and drawbacks.

The following notational conventions are introduced: If a variable, say x_j^N, has to be assigned a numerical value (either deterministic or stochastic) by a subsystem or the coordinator, then this variable will be marked like $\underline{x}_j^N$. Further a solution value for a variable, say y_j^I, will be indicated like $\hat{y}_j^I$.

3.2.2. Algorithm

Step 1. Each subsystem mails to the coordinator

|1.1a| f_j the complete equation system of the j-th subsystem,

|1.1b| $\underline{v}_j$ the stochastic characteristics of the corresponding disturbance terms of the j-th subsystem,

|1.1c| $\underline{x}_j^N$ the (stochastic) assumptions about non-interacting exogenous variables of the j-th subsystem.

Step 2. The coordinator determines

|1.2a| g the equation system of the interaction system,

|1.2b| $\underline{w}$ the (stochastic) characteristics of the corresponding disturbance terms,

|1.2c| $\underline{x}^C$ the (stochastic) assumptions about the exogenous variables of the interaction system.

Step 3. The coordinator uses the following (stochastic) equation system

$$f_1(y_1^I, y_1^N, x_1^I, \underline{x}_1^N, \underline{v}_1) = 0$$

|1.3a|

$$\vdots$$

$$f_n(y_n^I, y_n^N, x_n^I, \underline{x}_n^N, \underline{v}_n) = 0$$

|1.3b| $g(y_1^I,\ldots,y_n^I;x_1^I,\ldots,x_n^I,\underline{x}^c,\underline{w}) = 0$

to calculate the solution for

$\hat{y}_1^N,\ldots,\hat{y}_n^N$	the non-interacting endogenous variables,
$\hat{y}_1^I,\ldots,\hat{y}_n^I$	the interacting endogenous variables,
$\hat{x}_1^I,\ldots,\hat{x}_n^I$	the interacting exogenous variables,

based on

$\underline{x}_1^N,\ldots,\underline{x}_n^N$	the non-interacting exogenous variables,
$\underline{x}^c$	the exogenous variables of the interaction system
$\underline{v}_1,\ldots,\underline{v}_n$	the subsystem disturbance terms,
$\underline{w}$	the interaction system disturbance term

Step 4. Each subsystem receives from the coordinator at least the following portion of the (stochastic) characteristics of the system solution:

|1.4a|	$\hat{y}_j^N$	the non-interacting endogenous variables of the j-th subsystem,
|1.4b|	$\hat{y}_j^I$	the interacting endogenous variables of the j-th subsystem,
|1.4c|	$\hat{x}_j^I$	the interacting exogenous variables of the j-th subsystem.

3.2.3 Remarks

1. Sometimes the subsystems in Step 1 may be asked to submit also a control solution $(\underset{\sim}{y}_j^I, \underset{\sim}{y}_j^N)$ based on a guess about the interacting exogenous variables $\underset{\sim}{x}_j^I$. It is important to note that such a control solution in the framework of Solution Method 1 does not have any value besides error check considerations. The coordinator may try to reproduce the control solution to check if there was any error in the transmission of f_j, $\underset{\sim}{v}_j$, and $\underset{\sim}{x}_j^N$.

Essentially this solution method assumes that the interaction among the subsystems is so strong, that an individual subsystem is unable to make reasonable assumptions about the interacting exogenous variables. Therefore a subsystem solution does not contain any useful information.

2. An advantage of this solution method seems to be that the coordinator produces a "complete" solution. The subsystems provide the coordinator with the information indicated in Step 1 and receive in Step 4 the complete subsystem solution. No further calculations are required by the subsystems.

3. The main disadvantage of this solution method is the enormous amount of information to be mailed to the coordinator in Step 1. The coordinator consequently has to administer and to solve a huge system with numerous error potentials. Almost no advantage is taken of the hierarchical

system structure, except in the model building phase, but all computational work in the solution phase is carried out by the coordinator.

4. Conceptually the precision of the individual subsystem models and the coordinator model can be taken into account if all stochastic characteristics (for system inputs and disturbances) are known to the coordinator. Then the coordinator could produce a stochastic solution of equation system |1.3|. In practice, however, this will turn out to be an almost unmanageable task because of the size of this equation system.

3.3. Solution Method 2 (Condensed System Solution)

3.3.1. Preliminaries

The main disadvantage of Solution Method 1 is the administration and solution of a huge equation system by the coordinator. Therefore, we will try to look for a solution procedure which reduces the computational burden of the coordinator and the information exchange with the subsystems. Is there a way to send in a more compact form all necessary subsystem information to the coordinator instead of mailing the complete subsystem equation system with all subsystem inputs?

3.3.2. Reaction Functions

From (2.4) and Figure 2.4 we observe that the interaction system is connected with the subsystems only via the interacting endogenous variables y^I and the interacting exogenous variables x^I. The coordinator only has to know how variations in the interacting exogenous variables of a subsystem x_j^I effect the interacting endogenous variables of the subsystem y_j^I. This information together with the interaction model is sufficient to produce a solution for the interacting endogenous and the interacting exogenous variables which meets all system constraints.

Each subsystem therefore is asked to provide the coordinator with what we call <u>reaction functions</u>,

$$f_j^I(y_j^I, x_j^I, \tilde{y}_j^I, \tilde{x}_j^I) = 0 \qquad (3.3)$$

which are determined as follows:

The subsystems produce with (2.2) a control solution based on the (stochastic) characteristics of the subsystem disturbances $\underline{v}_j$, the (stochastic) assumptions about non-interacting exogenous variables $\underline{x}_j^N$, and a control solution assumption about the interacting exogenous variables $\tilde{x}_j^I$. In addition the subsystems determine the functional relationship f_j^I between the interacting endogenous and exogenous variables y_j^I and x_j^I for deviations of x_j^I from the control solution value $\underset{\sim}{x}_j^I$. Instead of the large number of all non-interacting exogenous variables $\underset{\sim}{x}_j^I$ only the much smaller vector with control solution value of the interacting endogenous variables $\underset{\sim}{y}_j^I$ is suggested to enter the reaction function as $\underset{\sim}{y}_j^I$ implicitely carries all information about $\underset{\sim}{x}_j^N$. The reaction function f_j^I can fairly easy be obtained by a multiplier analysis of each subsystem model around the control solution assumption $\underset{\sim}{x}_j^I$ for the interacting exogenous variables.

For reasons which will be discussed later the control solution assumption about the interacting endogenous variables $\underset{\sim}{x}_j^I$ has to be deterministic. The control solution of the interacting endogenous variables $\underset{\sim}{y}_j^I$ will be stochastic or deterministic depending if stochastic or deterministic subsystem inputs are used.

3.3.3. Algorithm

Step 1. Each subsystem mails to the coordinator

|2.1a| f_j^I the reaction functions of the j-th subsystem,

|2.1b| $\underset{\sim}{x}_j^I$ the deterministic control solution assumption about the interacting exogenous variables of the j-th subsystem,

|2.1c| $\underset{\sim}{y}_j^I$ the (stochastic) control solution of the interacting endogenous variables of the j-th subsystem based on $\underline{x}_j^N$ and $\underline{v}_j$.

Step 2. The coordinator determines

|2.2a| g the equation system of the interaction system,

|2.2b| $\underline{w}$ the (stochastic) characteristics of the corresponding disturbance terms,

|2.2c| $\underline{x}^c$ the (stochastic) assumptions about the exogenous variables of the interaction system.

Step 3. The coordinator uses the following (stochastic) equation system

|2.3a|
$$\begin{aligned} &f_1^I(y_1^I, x_1^I, \underset{\sim}{y}_1^I, \underset{\sim}{x}_1^I) = 0 \\ &\vdots \\ &f_n^I(y_n^I, x_n^I, \underset{\sim}{y}_n^I, \underset{\sim}{x}_n^I) = 0 \end{aligned}$$

|2.3b|
$$g(y_1^I, \ldots, y_n^I, x_1^I, \ldots, x_n^I, \underline{x}^c, \underline{w}) = 0$$

to calculate the solution for

$\hat{y}_1^I, \ldots, \hat{y}_n^I$ the interacting endogenous variables,

$\hat{x}_1^I, \ldots, \hat{x}_n^I$ the interacting exogenous variables based on

$\underset{\sim}{y}_1^I, \ldots, \underset{\sim}{y}_n^I$	the control solution of the interacting endogenous variables,
$\underset{\sim}{x}_1^I, \ldots, \underset{\sim}{x}_n^I$	the control solution assumption about the interacting exogenous variables,
$\underline{x}^c$	the exogenous variables of the interaction system,
$\underline{w}$	the interaction system disturbance term.

Step 4. Each subsystem receives from the coordinator at least the following position of the (stochastic) characteristics of the system solution

|2.4| $\hat{x}_j^I$ the interacting exogenous variables of the j-th subsystem.

Step 5. Each subsystem uses the following (stochastic) equation system

$$|2.5| \qquad f_j(y_j^I, y_j^N, \hat{x}_j^I, \underline{x}_j^N, \underline{v}_j) = 0$$

to calculate a complete subsystem solution $\hat{y}_j^I$ and $\hat{y}_j^N$.

3.3.4. Remarks

1. In this solution method the subsystem control solution is not redundant. The control solution vector of the interacting endogenous variables implicitely transmits in compact form all information about the subsystem assumptions for the non-interacting exogenous variables x_j^N.

The only purpose of the control solution assumption about the interacting exogenous variables $\tilde{x}_j^I$ is to construct the subsystem reaction function. As this assumption is completely overwritten by the solution with the interaction system the algorithm implicitely contains the assumption that the selection of $\tilde{x}_j^I$ does not express any probabilistic statement with respect to the true value of the interacting exogenous variables.

2. The big advantage of this solution method is the reduced information exchange between coordinator and subsystems. The subsystem control solution for the interacting endogenous variables together with the control solution assumptions for the interacting exogenous variables and the reaction functions contain in compact form all necessary subsystem information for the coordinator in order to obtain an overall system solution. Instead of transmitting the whole subsystem equation system only reaction functions for the usually few interacting endogenous variables have to be mailed.

3. The coordinator has to administer much less information. Information exchange should be quicker and easier with extremely reduced error potentials.

4. The computational work is shared between the coordinator and the subsystems. The subsystems play an active role in the solution process. Advantage is taken of the hierarchical system structure to formulate a decentralized solution procedure.

5. The coordinator need not even explicitely know all the submodels but only implicitely via the response functions. This may be an important aspect in cases where proprietors of submodels for some reason don't want to reveal all details of their model and non-interacting exogenous assumptions.

6. This solution procedure is recommended even if the coordinator receives the complete subsystem information |1.1| as indicated in Step 1 of Solution Method 1. Because of cost consideration the coordinator is advised to solve the condensed model |2.3| and than successively the subsystem models.

7. Because of the reduced size of the condensed system |2.3| it is much easier to calculate either analytically (in case of linear functions) or by stochastic simulation (in case of nonlinear functions) a stochastic system solution if stochastic system inputs are available.

3.4. Solution Method 3 (Anticipation System Solution)

3.4.1. Preliminaries

A completely different situation arises if the subsystem model proprietors feel that they are able to make some probabilistic statement about the interacting exogenous variables x_j^I. Then, together with their assumptions about the noninteracting exogenous variables $\underset{\sim}{x}_j^N$ and the stochastic characteristics of the subsystem disturbances $\underset{\sim}{v}_j$, the subsystems are able to compute what we call an informative control solution $\underset{\sim}{y}_j$ for the vector of endogenous subsystem variables.

The main difference of this control solution in comparison with control solutions in Solution Method 1 and 2 is its anticipatory character as it contains anticipatory information about the realisation of the interacting exogenous variables. These anticipations may reflect, e.g., information from trade agreements or time series analysis of monthly trade figures.

We recall that the only purpose of a control solution in Solution Method 1 is to check transmission errors between the subsystems and the coordinator. In Solution Method 2 the subsystem assumption about the interacting exogenous variables serves to determine the reaction function. In both cases, however, the subsystem assumptions about the interacting exogenous variables are overwritten as it is implicitely assumed in the solution methods that the subsystems are unable to make any informative statement about these variables.

In most practical cases, however, this will not be true. The submodel proprietors usually are able to make a more or less precise judgement about the interacting exogenous variables and the question arises how to process this information.

A first look at the subsystem control solutions typically will reveal that some overall system constraints, like balance constraints, as formulated in the interaction system are violated. One way to overcome this problem is to "believe" either the subsystem or the interaction system and to make a choice between one of this models. The choice is obvious if we have reason to assume that one model is absolutely correct.

More realistic is the point of view that we are dealing with systems of limitted accuracy because of the presence of error terms and unprecise assumptions about exogenous variables. As a consequence for decentralized solution procedures in hierarchical systems we have to evaluate two competing sources of information between subsequent hierarchical levels. One source are the subsystems, the other is the interaction system. Both sources are usually unprecise.

Statistical theory suggests that instead of choosing between various sources of information it is favourable to combine these sources by a weighting scheme which takes into account the relative precision of the various information channels.

3.4.2. Combining Unprecise Information

Now we will outline the statistical concepts for combining unprecise information. The results follow easily from Kalman filter theory, a sequential estimation method for stochastic dynamic models in state space representation.

Let us assume that we want to obtain an estimate $\hat{s}$ of a (multivariate) random variable s . One source of information (the state model) tells us that the distribution of s is described by mean $\bar{s}$ and variance V :

$$s \sim (\bar{s}, V) \tag{3.4}$$

With now other information available statistical estimation theory suggests that for a wide class of loss functions the optimal estimate, without additional information, denoted by $\hat{s}_-$, is given by the expectation

$$\hat{s}_- = \bar{s} \tag{3.5}$$

with estimation error

$$\Sigma_- = V \quad . \tag{3.6}$$

Let us assume that an observable (multivariate) random variable m represents an additional source of information for the directly unobservable random variable s . It is known that m is related to s via the following, for convenience linear, equation (the measurement model)

$$m = Hs+w$$

where w is a disturbance term with known variance indicating the precision of the measurement m with respect to s :

$$w \sim (0,W) \quad . \tag{3.7}$$

Then the optimal estimate for the same class of loss functions which makes use of both information channels, denoted by $\hat{s}_+$, is

$$\hat{s}_+ = \bar{s}+c \quad , \tag{3.8}$$

where c represents an additive correction term which is proportional to the descrepancy of both informations,

$$c = K(m-H\bar{s}) \tag{3.9}$$

and the proportionality factor K is determined by the relative precision of the two information sources:

$$K = VH'(HVH'+W)^{-1} \quad . \tag{3.10}$$

The error reduction by adding the additional information can be judged by comparing (3.6) with the estimation error of the combined estimate

$$\Sigma_+ = (I-KH)V \quad . \tag{3.11}$$

For the detailed proof see Schleicher (1977). These results will be used to formulate a decentralized solution method for hierarchical systems where the subsystems submit informative control solutions.

3.4.3. Algorithm

Step 1. Each subsystem mails to the coordinator

|3.1a| f_j^I the reaction functions of the j-th subsystem,

|3.1b| $\underline{x}_j^I$ the stochastic control solution assumption about the interacting exogenous variables of the j-th subsystem: $\underline{x}_j^I \sim (\bar{\underline{x}}_j^I, \underline{X}_j^I)$

|3.1c| $\tilde{y}_j^I$ the stochastic control solution of the interacting endogenous variables of the j-th subsystem based on $\underline{x}_j^N$ and $\underline{v}_j$: $\tilde{y}_j^I \sim (\bar{\tilde{y}}_j^I, \tilde{Y}_j^I)$

Step 2. The coordinator determines

|3.2a| g the equation system of the interaction system,

|3.2b| $\underline{w}$ the stochastic characteristics of the corresponding disturbance terms,

|3.2c| $\underline{x}^c$ the stochastic assumptions about the exogenous variables of the interaction system.

Step 3. The coordinator combines

(1) the subsystem control solution (state model)

|3.3a| $s \sim (\bar{s}, V)$

where

$$s = \begin{bmatrix} y^I \\ x^I \end{bmatrix} \quad , \quad s = \begin{bmatrix} \bar{\underset{\sim}{y}}^I \\ \bar{\underset{\sim}{x}}^I \end{bmatrix} \quad , \quad V = \begin{bmatrix} Y^I & 0 \\ 0 & X^I \end{bmatrix}$$

with

(2) the coordinator model (measurement model)

|3.3b| $\quad m = Hs+w \quad , \quad w \sim (0,W)$

which is composed of the linearized interaction system and the linearized reaction functions (the linearized version of the condensed system |2.3a| and |2.3b| , and where m, H and W follow from the information collected in Step 1 and Step 2 and therefore are known,

to calculate the solution for

$\hat{y}_1^I,\ldots,\hat{y}_n^I$ the interacting endogenous variables,

$\hat{x}_1^I,\ldots,\hat{x}_n^I$ the interacting exogenous variables,

using the following formula for the mean of the solution

$$\hat{s} = \bar{s}+K(m-H\bar{s}) \qquad (3.12)$$

$$\text{where } K = VH'(HVH'+W)^{-1} \qquad (3.13)$$

and the following formula for the corresponding error covariance matrix

$$S = (I-KH)V \qquad (3.14)$$

Step 4. Each subsystem receives from the coordinator at least the following portion of the stochastic characteristics of the system solution

|3.4a| $\hat{x}^I_j$ the interacting exogenous variables of the j-th subsystem.

Step 5. Each subsystem uses the following stochastic equation system

|3.5| $f_j(y^I_j, y^N_j, \hat{x}^I_j, \underline{x}^N_j, \underline{v}_j) = 0$

to calculate a complete subsystem solution $\hat{y}^I_j$ and $\hat{y}^N_j$.

3.4.4. Remarks

1. This solution method stresses the stochastic character of a stochastic system solution. Each information entering the system is evaluated for accuracy by specifying its mean and variance.

2. Whenever the subsystems feel able to make an informative probabilistic statement about their interacting exogenous variables and thus can submit an informative control solution to the coordinator the relative precision of the submitted subsystem control solution and the interaction system should be taken into account. As it is difficult to imagine that subsystems are unable to make such an informative judgement

we draw the conclusion that it is highly recommended to include stochastic elements into a solution method for hierarchical systems. The gain in precision in comparison with Solution Method 1 and 2 results from the fact that in addition anticipatory information about the interacting exogenous variables is processed.

3. This solution method not only evaluates the relative precision between subsystem models and interaction model but also the relative precision between various subsystem models. Thus a comparatively unprecise subsystem control solution will be given less weight in the calculation of the overall system solution. If, on the other hand, a subsystem submits a control solution with comparatively low variance only slight modifications will be made by the coordinator.

4. Intuitively the basic principle of this solution method can be explained as follows: Discrepancies between the subsystem control solutions and the overall system constraints expressed by the interaction system stem from two error sources, namely either from an error in the subsystem model or from an error in the interaction model. Instead of allocating the discrepancy only to the subsystems (that means overwriting the subsystem control solution) or only to the interaction system (that means neglecting the validity of the overall system constraints) the discrepancies are split among subsystems and interaction system according to the relative precision of the two information sources.

5. This solution method may very well describe the administrative procedures in a forecasting exercise of a large organisation. Forecast proposals submitted by a division are evaluated at the next hierarchical level and combined with any other available information. Then the relevant portions of corrected forecasts are mailed back to the divisions.

6. All advantages of Solution Method 2 as reduced information exchange between coordinator and subsystems, active participation of the subsystems in the computational work, and reduced computational burden for the coordinator are retained also in Solution Method 3.

7. The need to exchange information about error variances may be considered as a disadvantage of this method. As, however, the statistical analysis reveals this is the price to be paid for a reduced solution error.

4. FORECASTING WITH THE LINK SYSTEM

4.1. Description of the LINK System

Our discussion about decentralized solution methods suggest a number of applications for the LINK System. We will present in this section some proposals for forecasting procedures which take into account the hierarchical system structure.

The LINK system is composed of a number of country or regional models

$$f_j(y_j^N, m_j, x_j^N, e_j, v_j) = 0 \ , \quad j=1,\ldots,n \tag{4.1}$$

which are connected via a trade model

$$e = Am + w_e \ . \tag{4.2}$$

For simplicity we are only considering imports $m' = (m_1', \ldots, m_n')$ as interacting endogenous variables and exports $e' = (e_1', \ldots, e_n')$ as interacting exogenous variables. We denote with A the trade share matrix, with y_j^N the non-interacting endogenous variables and with x_j^N the non-interacting exogenous variables.
The stochastic characteristics of the disturbance terms are

$$v_j \sim (0, V_j) \ , \qquad j=1,\ldots,n \tag{4.3}$$

and

$$w_e \sim (0, W_e) \ . \tag{4.4}$$

4.2. Use of Anticipatory Information in Country Models

Having made a stochastic specification of the assumptions about exogenous variables a solution of all endogenous variables can be calculated for the j-th country or region by (4.1) which can be used as a forecast.

In most cases, however, the model proprietor will not be content with this result. The forecaster's intuition and experience leads him to make adjustments, usually by changing the values of some exogenous variables or by varying the constant terms in the stochastic equations. The ad hoc character of such adjustments is drawing forth a lot of criticism about too much subjectivity in forecasting with econometric models (Christ (1975)).

Any kind of such adjustments obviously reflects some anticipatory information of the forecaster about some endogenous variables because otherwise he could not evaluate a particular model forecast. This information may originate from different sources. Time series analysis on monthly reported data like industrial production, trade balance, and consumer prices may provide useful information about the corresponding model variables at least for the subsequent quarters. Consumer and investment survey data may shed, after a correction for reporting biases, some light on the actual level of private consumption and investment expenditures. It is intuitively plausible and confirmed by statistical methodology that the introduction of such anticipatory information will increase the precision of the forecast. Our proposal of combining unprecise information in section 3.4.2 provides us with a tool to make systematic and efficient use of anticipatory information in a forecasting exercise.

Let us assume that for a country model a forecast was calculated by (4.1) based on assumptions about all model inputs. For convenience we ommit the country subscript j and denote this forecast by

$$\underset{\sim}{y} \sim (\bar{\underset{\sim}{y}}, V) \tag{4.5}$$

where V indicates the estimated forecasting error.

Let us further assume that y^a describes anticipatory information about some endogenous model variables which is related to the endogenous model variables y via the following linear stochastic relationship

$$y^a = Gy + w_a \tag{4.6}$$

where the disturbance term w

$$w \sim (0, W_a) \tag{4.7}$$

indicates the precision of this anticipatory information. Then our analysis of 3.4.2 recommends to combine the model forecast with the anticipatory information in the following way:

The expectation of the combined forecast is

$$\hat{y} = \bar{y}+c \tag{4.8}$$

with $$c = K(y^a-G\bar{y}) \tag{4.9}$$

and $$K = VG'(GVG'+W_a)^{-1} \quad . \tag{4.10}$$

The error covariance matrix of the combined forecast is

$$S = (I-KG)V \quad . \tag{4.11}$$

The interpretation of this result seems to be very appealing. According to (4.8) the optimal forecast which makes both use of the econometric model, its input assumptions, and all available anticipatory information is based on the model projection $\bar{y}$ plus an additive correction term. This result may be considered as a justification of the widespread constant-term adjustment practice among model proprietors. Instead of ad hoc adjustments (4.9) and (4.10) yield corrections which meet statistical optimality properties and make the forecasting process more transparent as the model proprietor is induced to specify explicitely all his information used to produce a forecast.

Two conclusions are suggested: Firstly, whenever a model proprietor uses constant adjustment in a forecasting exercise this indicates the presense of explicitely or implicitely used anticipatory information. Secondly, there are methodological sound procedures to include anticipatory information into an econometric forecast.

4.3. Present Forecasting Procedure of the LINK System

The present forecasting procedure of the LINK System is identical with Solution Method 1. The regional or country model proprietors are asked to submit their complete equation system with all inputs to the project coordinator. The coordinator adds the trade model and produces a forecast by solving the overall system.

The regional or country model proprietors are also asked to submit a control solution but the role of this information is not well defined. The algorithm just overwrites and thus ignores the control solutions. On the other hand the control solutions seem to have an informative value for the project coordinator beyond error check considerations as the control values are used for making adjustments in the trade model.

4.4. Forecasting with a Condensed System

Considerable computational savings can be gained by switching to Solution Method 2. In case of the LINK System the condensed system to be solved by the coordinator becomes extremely simple. Neglecting stochastic inputs the following information has to be exchanged:

Each regional or country model proprietor mails to the coordinator

- B_j the impact multiplier matrix of country exports on country imports,
- $\underset{\sim}{e}_j$ the control solution assumption about country exports,
- $\underset{\sim}{m}_j$ the control solution of country imports.

This information is sufficient to put together the reaction functions of the system

$$m = \underset{\sim}{m}+B(e-\underset{\sim}{e}) \quad (4.12)$$

where

$$B = \begin{bmatrix} B_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & B_n \end{bmatrix} . \quad (4.13)$$

This reaction functions form together with the trade model

$$e = Am \quad (4.14)$$

the deterministic version of the condensed LINK system. Substituting (4.12) into (4.14) and defining

$$d = AB\underset{\sim}{e} - A\underset{\sim}{m} \quad , \tag{4.15}$$

$$D = AB \quad , \tag{4.16}$$

$$F = D - I \; , \tag{4.17}$$

the coordinator only has to solve

$$d = Fe \tag{4.18}$$

in terms of exports e to obtain the export vector which is compatible with all system constraints.

The export solution then is mailed to the country model proprietors to enable them to calculate a complete country solution which takes care of all system constraints.

This solution procedure for a forecasting exercise is even recommended if the coordinator has access to the complete country models and all their inputs as it avoids iterating back and forth between country models and trade model. The only additional computations required are the impact multiplier calculations. Then (4.17) is solved for e and subsequently all country models.

In (4.12) we proposed a linear reaction function which of course can only be regarded as a first order Taylor series approximation in case of a nonlinear model. As the approximation is made around the control solution the approximation error probably can be considered negligible.

4.5. Forecasting with Stochastic Information

Whenever the country or regional model proprietors feel able to make a probabilistic statement about their interacting exogenous variables - in our simplified LINK model the country exports - then the submitted control solution carries valuable information which explicitely should be processed by the forecasting procedure. Then Solution Method 3 should be applied which in case of the LINK system requires only a few simple matrix calculations.

The regional or country model proprietors mail to the coordinator

B_j the impact multiplier matrix of country exports on country imports,

$e_j \sim (\bar{e}_j, E_j)$ the stochastic control solution assumptions about country exports,

$m_j \sim (\bar{m}_j, M_j)$ the stochastic control solution of the country imports.

The coordinator uses this information to put together the stochastic reaction functions

$$m = \underset{\sim}{\bar{m}}+B(e-\bar{\underline{e}})+w_m \quad , \quad w_m \sim (0,M) \qquad (4.19)$$

where

$$M = \begin{bmatrix} M_1 & \cdot & \cdot & 0 \\ \vdots & & \cdot & \vdots \\ 0 & \cdot & \cdot & M_n \end{bmatrix} \qquad (4.20)$$

which together with the stochastic trade model

$$e = Am+w_e \quad , \quad w_e \sim (0,We) \qquad (4.21)$$

form the stochastic version of the condensed system. This system can be further reduced by substituting (4.18) into (4.20) yielding

$$f = Fe+w \quad , \qquad (4.22a)$$

$$w \sim (0,W) \quad \text{with } W = W_e+AMA' \quad , \qquad (4.22b)$$

where f is defined as follows

$$f = AB\underline{\bar{e}}-A\underset{\sim}{\bar{m}} \quad . \qquad (4.22c)$$

Then the coordinator combines the country control solution assumption about country exports

$$e \sim (\underline{\bar{e}},E) \qquad (4.23)$$

where

$$E = \begin{bmatrix} E_1 & \cdot & \cdot & 0 \\ \vdots & & \cdot & \vdots \\ 0 & \cdot & \cdot & E_n \end{bmatrix} \qquad (4.24)$$

with the information contained in the trade model (4.22) by means of the statistical technique outlined in section 3.4.2. Thus we get as mean for the combined forecast

$$\hat{e} = \underline{\bar{e}}+K(f-F\underline{\bar{e}}) \quad (4.25a)$$

where

$$K = EF'(FEF'+W)^{-1} \quad (4.25b)$$

with error covariance matrix

$$S = (I-KF)E. \quad (4.25c)$$

This forecasting method which is based on an informative country control solution can be given on intuitiv plausible interpretation. (4.25a) states that the expectation for the overall system export forecast $\hat{e}$ is based on the expectation of the export assumptions made by the country model proprietors $\underline{e}$ and an additive correction term. This correction term is basically proportional to the discrepancies of country model and trade model projections. To what extend this discrepancy is allocated to the country model estimate is determined by the weighting matrix K which reflects the relative accuracy of the two information sources expressed by the error covariance matrix of the trade model W and the error covariance matrix of the country export estimates E.

The calculations in (4.25) are very simple. The maximal dimension of all matrices and vectors is equal to the number of countries times the number of commodity categories.

5. CONCLUSIONS

The analysis of large scale econometric systems by investigating the hierarchical system structure bears useful applications also for estimation and control problems. Using a condensed system version it is also possible to study even for very large systems the stochastic properties along the concepts of Howrey (1967) and Howrey and Klein (1967). Mention should be made that the proposed forecasting method for the LINK system which makes explicit use of stochastic country information can easily be extended to include any other kind of anticipatory information and is a means to overcome forecasting problems when only incomplete country information is available.

The basic conclusion of our analysis of large scale econometric system is that in general neglecting of the stochastic system properties yields very unsatisfactory results in statistical terms. This becomes particularly evident if a decentralized forecasting procedure is used. Then a purely deterministic system solution damages most of the advantages of a decentralized forecasting method as important subsystem information is not processed.

Two suggestions were made for forecasting with the LINK system. Firstly, if the present pattern of information exchange between country model proprietors and coordinator is maintained the computational burden can be considerably reduced by using a condensed system which preserves all system constraints. Secondly, if the country control solutions are considered to be informative (this is not evident in the present forecasting practice) then a solution algorithm with explicit stochastic elements should be considered. Although this algorithm is computationally very simple a prerequisite for its applicability is a knowledge of the error properties both of the country models and the trade model.

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