A METHOD OF UPDATING INPUT-OUTPUT MATRICES

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I. Introduction

Empirical analyses of the behavior of economies are leaning more and more heavily on input-output methodology with the passage of time. The major reasons for this development are self-evident. Index-number problems are an inevitable by-product of macroeconomic models. (For example, the dependence of the "real GNP" statistic on the base-year prices chosen has always attracted considerable attention.) These problems can be ameliorated only through the process of disaggregation -- in other words, index-number problems can be eliminated only through the elimination of the indexes themselves. Input-output analysis provides a convenient way in which macroeconomic price and output aggregates may be translated into their industrial components. Besides, goals of economic policy direct attention to disaggregated economic variables. We are not only concerned with the overall level of economic activity and factor use, but also with production by particular industries (e.g. branches of agriculture or resource extraction) and employment of particular factors (e.g. skilled and unskilled labor) in those industries. Input-output analysis permits the empirical treatment of these detailed policy concerns. The individual entrepreneur, even more than the economic policy maker, needs information about disaggregated industrial interrelations. Macroeconomic variables such as aggregate output and employment are not always satisfactory indicators of economic activity in individual industries.

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These obvious advantages of input-output analysis are accompanied by several disadvantages -- many of which turn out to be more apparent than real. Let it suffice to mention only three of the most significant of these disadvantages here. First the possibility of joint production may seem to pose problems for the formulation of an input-output system: it may be impossible to assign a given input unambiguously to a single output. This difficulty may be obviated by mapping a vector of inputs into a vector of outputs. Such an approach fits naturally into an activity-analysis framework and has found its way into many empirical input-output studies. Another modelling problem lies in the spacial setting of inputs and outputs, which necessitates the incorporation of transportation and communication services in our analysis. One way in which this difficulty may be met is by distinguishing as separate goods the same physical good in different locations and interpreting transportation and communication services as inputs necessary to transform these goods into one another. It must be noted, however, that the present store of empirical evidence on capitalist economies permits only a very rough implementation, if any, of such a scheme.

The most common and possibly the most significant criticism to be leveled at input-output analysis is concerned with its assumption of fixed production coefficients[1)]. This assumption is often considered to imply that each good (or set of jointly produced goods) is produced through only a single technique of production, that each production process is characterized by constant returns to scale, and that there is no technological progress. These are indeed restrictive assumptions to place on a description of the supply-side structure of an economy, but it must be emphasized that they are <u>not</u> inherent implications of the input-output approach.

An output produced by x production techniques may be broken into x goods, one for each technique. These x goods may then be treated in the usual input-output fashion, each good associated with a single input vector. The vector of inputs i used to produce a given output j may be pictured as a point on an isoquant. In fact, the standard neoclassical isoquant -- for which continuous factor substitution is possible -- may be interpreted as the locus of efficient points taken from a continuous spectrum of techniques, each of which may be characterized by a vector of input-output coefficients.

In an economy, an output j may be produced by several techniques for a variety of reasons. For example, the input endowments of the different regions producing output j may be different, and the consequent diversity of relative input prices may give rise to a diversity of techniques chosen. However, the data necessary to describe each of these techniques as a vector of input-output coefficients is usually not available. Estimated input-output coefficients usually relate a given output (or a set of jointly produced outputs) to a single vector of inputs. Consequently, this vector of inputs must be seen as an "average" of all the actual techniques used in the production of that output. (It is not necessarily an average of points from a single isoquant. For instance, if technological information does not flow freely between regions, different regions may face different isoquants.) Naturally, the composition of techniques may change with the passage of time, and thus the values of the estimated input-output coefficients may change even though no change in the underlying technology has taken place.

Not only does the input-output approach not imply a zero elasticity of factor substitution, it also does not imply constant returns to scale. For each level of output j, there exists a set of input vectors that defines the corresponding isoquant. If the level of output changes with the passage of time, the set of input vectors may change as well. These changes may map out increasing, constant, or decreasing returns to scale. Thus, the movement from isoquant to isoquant along a production function may be pictured by a changing set of input vectors.

Lastly, the production function itself may shift over time, i.e. technological progress may make itself felt. Under these conditions, the set of input vectors may change even if the level of output produced remains the same. It is thus apparent that input-output analysis can describe movement along a given isoquant, movement from one isoquant to another along a given production function, and movement from one production function to another.

An input-output matrix which characterizes an economy's supply-side structure at one point in time may be different from that which characterizes the same economy's structure at another point in time. Empirical interindustry models are, to my knowledge, always formulated in discrete time terms and each period calls for a new input-output matrix. Each estimated coefficient of the matrix may be understood as an "average" in two senses:

(1) an "average" of points on an isoquant(s) (i.e. of techniques) at a given point in time, and
(2) an "average" of the transformation of a given input vector (i.e. of a given technique) through time on account of changes in the level of output or technological progress.

In short, the formulation of an input-output matrix as a description of an economy's supply-side structure does not imply the restrictions usually associated with fixed production coefficients, viz. no factor substitution, constant returns to scale, and no technological progress. To ensure that these restrictions are absent, the input-output matrix must change through time. For, with the passage of time, the level of output, the state of technical know-how, and the composition of techniques to produce a given output all change. Empirically estimated input-output coefficients are "averages" with respect to these states, and hence their values may be expected to change as well.

Thus, an annual interindustry model requires a different input-output matrix for each year. Unfortunately, input-output tables are not estimated directly on a yearly basis. For example, the last Austrian input-output table was computed for 1964; the last United States table was computed for 1963. Those input-output coefficients which cannot be estimated directly year by year must be updated through indirect computation procedures. We have given a rough indication of the importance of this task. In the next section we examine the updating method most commonly used by economists thus far. In Sections 3 and 4, we present a new updating method, evaluate its significance, and inquire how it relates to the method above.

2. The Stone-Brown Updating Method

The Stone-Brown[2] method, more commonly known as the rAs method, is the most popular way of updating input-output tables. Let $\underset{m \times m}{\underline{A_t}}$ be a directly estimated table of interindustry input-output coefficients for the base year t. Its representative element a_t^{ij} may be described as the amount of input i necessary to produce one unit of output j. Let $\underset{m \times 1}{\underline{Q_t}}$ be a vector of outputs (by industry) for year t. (The constants beneath $\underline{A_t}$ and $\underline{Q_t}$ indicate the dimensions of the matrix and vector, respectively.)

Then the following identities may be formed:

$$(1) \quad \underset{m \times m}{\underline{A_t}} \cdot \underset{m \times 1}{\underline{Q_t}} = \underset{m \times 1}{\underline{QT_t}} \quad \text{and}$$

$$(2) \quad \underset{1 \times m}{\underline{Q'_t}} \cdot \underset{m \times m}{\underline{A_t}} = \underset{1 \times m}{\underline{QB'_t}} ,$$

where $\underline{QT_t}$ is a column vector of intermediate good purchases by type of product and $\underline{QB'_t}$ is a row vector of intermediate good purchases by buyer. Equation (1) means that, for any industry i, the sum of all purchases of its product by other industries (j = 1, ..., m) is equal to total intermediate good purchases of product i:

$$\sum_{j=1}^{m} a_t^{ij} \cdot Q_t^j = QT_t^i .$$

Equation (2) means that, for any industry j, the sum of all its purchases from other industries (i = 1, ..., m) is equal to total intermediate good purchases by that industry:

$$\sum_{i=1}^{m} a_t^{ij} \cdot Q_t^i = QB_t^j .$$

The summations of equation (1) may proceed, conceptually, independent of prices if each good Q_t^i is homogeneous.

However, the summations of equation (2) are price dependent, because different types of goods are added to one another. Thus, we will assume that the matrix $\underline{A_t}$ is given in value terms, not in real terms. (The empirical estimation of $\underline{A_t}$ is always performed in value terms.)

Suppose that data are available for the vectors $\underline{Q_t}$, $\underline{QT_t}$, and $\underline{QB_t}$ on a yearly basis, but that the input-output table is available only at year t^3. Then, given that identities (1) and (2) hold for every time period, the table may be updated on the basis of these identities. Let us derive the updated input-output table for year t + h. From equation (1) it is clear that $\underline{A_{t+h}} \cdot \underline{Q_{t+h}} = \underline{QT_{t+h}}$, where $\underline{Q_{t+h}}$ and $\underline{QT_{t+h}}$ are known and $\underline{A_{t+h}}$ remains to be calculated. This system contains m equation and m^2 unknowns (the number of coefficients in the input-output matrix). We artificially eliminate our degrees of freedom through formulating the problem in the following terms: $\underset{m \times m}{\underline{r_{t+h}}} \cdot \underset{m \times m}{\underline{A_t^*}} \cdot \underset{m \times 1}{\underline{Q_{t+h}}} = \underset{m \times 1}{\underline{QT_{t+h}}}$, where $\underline{r_{t+h}}$ is a diagonal matrix and $\underline{A_t^*}$ is the matrix $\underline{A_t}$ expressed in the relative prices of year t+h (i.e. $\underline{A_t^*} = \underline{P_{t+h}} \cdot \underline{A_t} \cdot \underline{P_{t+h}^{-1}}$, where $\underline{P_{t+h}}$ is a diagonal matrix of the relative prices of years t+h and t). The m diagonal elements of the matrix $\underline{r_{t+h}}$ (the unknowns of the new equation system) are called absorption factors: for each seller of good i, all inputs "absorbed" in the production of one unit of good i change in the proportion r_{t+h}^i. In other words, the rows of A_t^* are adjusted (i.e. the technology embodied in each type of good is adjusted) so that identity (1) holds. Consequently, the first approximation of the updated input-output matrix may be written as $\underline{A_{t+h}^1} = \underline{r_{t+h}} \cdot \underline{A_t^*}$.

It is now necessary to ensure that identity (2) holds for the matrix $\underline{A_{t+h}^1}$ derived above. For this purpose we follow an analogous procedure: $\underset{1 \times m}{\underline{Q'_{t+h}}} \cdot \underset{m \times m}{\underline{A^1_{t+h}}} \cdot \underset{m \times m}{\underline{s_{t+h}}} = \underset{1 \times m}{\underline{QB'_{t+h}}}$, where $\underline{s_{t+h}}$ is a diagonal matrix to be determined. The m elements of this matrix are called fabrication factors: the input composition of each output (the "degree of fabrication") changes by the proportions given in $\underline{s_{t+h}}$. Thus, the columns of $\underline{A^1_{t+h}}$ are now adjusted (i.e. the technology of each buyer of intermediate goods is adjusted) so that identity (2) holds. The second approximation of the updated input-output matrix may be written as $\underline{A^2_{t+h}} = \underline{A^1_{t+h}} \cdot \underline{s_{t+h}}$.

This adjustment, however, breaks identity (1) and thus the elements of each row of $\underline{A^2_{t+h}}$ must be scaled to the correct row sum, QT^i_{t+h} (i = 1, ..., m). Thenthe elements of each column must be scaled to the correct column sum, QB^j_{t+h} (j = 1, ..., m), and so on. This procedure is continued until further iteration produces no significant coefficient change. The absorption and fabrication factors are given by the following equations:

$$(3) \quad r^i_{t+h} = \frac{QT^i_{t+h}}{\sum_{j=1}^{m} {}^k a^{ij}_{t+h} \cdot Q^j_{t+h}} \quad \text{for all } i;$$

$$(4) \quad s^j_{t+h} = \frac{QB^j_{t+h}}{\sum_{i=1}^{m} {}^{k+1} a^{ij}_{t+h} \cdot Q^i_{t+h}} \quad \text{for all } j.$$

where the left superscripts of the input-output coefficients stand for the iteration number.

The iterative process is convergent for nonegative input-output coefficients.[4)]

In summary, the Stone-Brown updating method uses absorption and fabrication factors to make the base-year matrix consistent with the two identities above. Even if there are no errors in measurement, however, there is no way to guarantee that the input-output matrix updated by the rAs method is an accurate representation of $\underline{A_{t+h}}$. The reason is that the input-output coefficients are underdetermined (for input-output matrices of order greater than 2): there are m^2 coefficients to be determined by means of two vectors, each containing m elements. In other words, there are $(m^2 - 2m)$ degrees of freedom.

To gain a more reliable picture of input-output matrices which have not been estimated directly, one may proceed in two ways. First, another method of updating input-output matrices may be devised. We then would have two estimates of each updated matrix, and a choice or a compromise between the two could be made on the basis of known interindustry flows. (Usually some coefficients of intermediate good input-output matrices can be estimated directly on a yearly basis and these coefficients may provide dues to evaluate the alternative updated matrices). Second, the updating method above may be amended in such a way that the degrees of freedom are reduced. Both of these lines of attack are provided in the following section.

3. A New Updating Method

We begin ourexposition by outlining a new method of updating input-output matrices -- an alternative to the rAs method. Then we will examine the relation between these methods and explore the possibility of reducing their respective degrees of freedom.

Just as the rAs method rests on the existence of row and column controls derived from two input-output identities, our new method -- which we shall call the TAU method -- relies on row and column controls derived from the equation systems of the static open Leontief models for outputs and prices. The Leontief output framework may be expressed as a system of identities: the gross output of each good is equal to the output of that good going to intermediate uses plus the output going to final demand.

$$(5) \quad \underset{m \times 1}{\underline{Q_t}} = \underset{m \times m}{\underline{A_t}} \cdot \underset{m \times 1}{\underline{Q_t}} + \underset{m \times 1}{\underline{{}^{D}Q_t}} ,$$

where $\underline{Q_t}$ is a vector of gross outputs (by industry), $\underline{A_t}$ (as above) is the intermediate-good input-output matrix, and $\underline{{}^{D}Q_t}$ is a vector of final demands.

Suppose that the input-output matrix has been directly estimated for year t but not for any other year. Furthermore, suppose that the vector of final demands and the vector of gross outputs are available on a yearly basis. Through the empirical estimates of these two vectors, matrix $\underline{A_t}$ may be updated.

The elements of the final demand vector may be measured directly or predicted from demand functions. In either case, it is plausible to assume that they contain disturbance terms.

We take it that the measurement or estimation of the final demands ensures that the disturbance terms, ε_t^i (for industries i = 1, ..., m) are all "well-behaved":

$$(6) \quad E\,(\varepsilon_t^i) = 0$$

$$E\,(\varepsilon_t^i - E\,(\varepsilon_t^i))^2 = 0 \qquad\qquad i = 1, \ldots, m;$$

$$\mathrm{cov}\,(\varepsilon_{t=r}^i, \varepsilon_{t=s}^i) = 0 \text{ for } r \neq s$$

i.e. the expected value of each disturbance term is zero, the variance is a constant, and the values of each disturbance term at different time periods are independent of one another. If each disturbance term satisfies these three conditions, we shall say that there are no "systematic errors" produced in the system.

In short, $\underline{{}^{D}Q_t} = \underline{{}^{D}Q_t^*} + \underline{{}^{D}\varepsilon_t}$, where $\underline{{}^{D}Q_t}$ is a vector of actual final demands, $\underline{{}^{D}Q_t^*}$ is a vector of predicted or measured final demands, and $\overline{{}^{D}\varepsilon_t}$ is the corresponding vector of disturbance terms, which satisfy conditions (6). From equation (5), the gross outputs may be derived from the final demands:

$$(7) \quad \underline{\hat{Q}_t} = (\underline{I}-\underline{A_t})^{-1} \cdot \underline{{}^{D}Q_t^*}.$$

If there are no systematic errors in the vector ${}^{D}Q_t^*$, then there can be no systematic errors in the derived vetor of gross outputs, $\underline{\hat{Q}_t}$, either. In other words:

$$(8) \quad \underline{Q_t^*} = \underline{\hat{Q}_t} + \underline{\hat{\varepsilon}_t},$$

where $\underline{Q_t^*}$ is the vector of measured gross outputs and $\underline{\hat{\varepsilon}_t}$ is the corresponding vector of disturbance terms, which also satisfy conditions (6). (Furthermore, we assume that $\underline{Q_t} = \underline{Q_t^*} + \underline{\varepsilon_t}$, where Q_t is the vector of actual gross outputs and each of the corresponding elements of $\underline{\varepsilon_t}$ satisfies conditions (6).)

The reasons underlying this conclusion are apparent. As the model is specified, $^{D}\underline{Q}_{t}^{*}$ is exogenously given and determines $\underline{\hat{Q}}_{t}$ via the matrix $(\underline{I}-\underline{A}_{t})^{-1}$. There are no omitted variables in this model (which could give rise to systematic variation in $\underline{Q}_{t}^{*}$ not explained by the variation in $^{D}\underline{Q}_{t}^{*}$); $^{D}\underline{Q}_{t}^{*}$ and $(\underline{I}-\underline{A}_{t})^{-1}$ are meant to explain $\underline{\hat{Q}}_{t}$ **entirely**. We assume that the coefficients of the input-output matrix contain no disturbance terms. (Clearly, these coefficients cannot be given with perfect certainty in practice. Yet there is little justification for considering disturbance terms in this context, because there are invariably insufficient data available to permit an evaluation of the properties of these terms.) The final demand disturbance terms -- arising, say, out of the unpredictability and variablity of human behavior and the lack of precision in measurement -- satisfy conditions (6). Thus, the gross output disturbance terms must satisfy these conditions as well.

Given this relation between $\underline{\hat{Q}}_{t}$ and $\underline{Q}_{t}^{*}$, let us examine how $^{D}\underline{Q}^{*}$ and $\underline{Q}^{*}$ (empirical estimates of which are available year by year) may be employed to update $\underline{A}$ (which is estimated only for year t). In particular, let us derive an updated input-output table for year t+h. Given $^{D}\underline{Q}_{t+h}^{*}$ and $\underline{A}_{t}$, an approximation of the gross output vector may be found through system (7).

$$(9) \quad \underline{\bar{Q}}_{t+h} = (\underline{I} - \underline{A}_{t})^{-1} \cdot {}^{D}\underline{Q}_{t+h}^{*} ,$$

where $\underline{\bar{Q}}_{t+h}$ is the approximated vector of gross outputs. Let us calculate $\underline{\bar{Q}}$ for all years in which empirical estimates of $^{D}\underline{Q}^{*}$ and $\underline{Q}^{*}$ are available (from year t through year $t+\tau$). We now compare the vector of directly estimated gross outputs, $\underline{Q}_{t+h}^{*}$, with the vector of derived gross outputs, $\underline{\bar{Q}}_{t+h}$, for all years t+1 through $t+\tau$.

If the input-output matrix $\underline{A}$ remains unchanged from year to year (for all years t through $t+\tau$), then -- as a glance at systems (7), (8) and (9) immediately reveals -- the two vectors are related to one another as follows:

$$\underline{\bar{Q}_{t+h}} = \underline{\hat{Q}_{t+h}} = \underline{Q^{*}_{t+h}} - \underline{\hat{\varepsilon}_{t+h}}, \qquad h = 1, \ldots, \;;$$

where $\underline{\hat{\varepsilon}_{t+h}}$ satisfies the conditions (6).

In general, however, it may be expected that the coefficients of the input-output matrix $\underline{A}$ change from year to year. Insofar as these variations are systematic, they are likely to induce systematic variations in $\underline{Q^{*}_{t+h}}$ which are not present in $\underline{\bar{Q}_{t+h}}$ (as computed by system (9)). In other words, the systematic variations in the final demand vector may not be able to generate all the systematic variations in the directly estimated gross output vector, because of systematic variations in the $\underline{A}$ matrix through time. Given such variations in the coefficients of the input-output matrix, the difference between the directly estimated gross output vector, $\underline{Q^{*}_{t+h}}$, and the derived gross output vector, $\underline{\bar{Q}_{t+h}}$, may be a vector of disturbance terms which do not satisfy conditions (6). In short, $\underline{\bar{\varepsilon}_{t+h}} = \underline{Q^{*}_{t+h}} - \underline{\bar{Q}_{t+h}}$ may display systematic variations through time, which may be attributed to systematic variations in the $\underline{A}$ matrix.

Naturally, it is possible that the input-output matrix varies systematically through time and that $\underline{\bar{\varepsilon}_{t+h}}$ nevertheless satisfies conditions (6). In other words, $\underline{A_{t+h}} = \underline{A_t} + \left[\varepsilon^{ij}_{t+h}\right]$, where the elements ε^{ij}_{t+h} do not satisfy conditions (6) and yet

$$\underline{\bar{Q}_{t+h}} = (\underline{I} - \underline{A_t})^{-1} \cdot \underline{{}^{D}Q^{*}_{t+h}}$$

$$\underline{Q^{*}_{t+h}} = \underline{\bar{Q}_{t+h}} + \underline{\bar{\varepsilon}_{t+h}},$$

where $\underline{\bar{\varepsilon}_{t+h}}$ <u>may</u> have no systematic component. The reason why it mayhave none is that the systematic variations in the rows of the matrix $\underline{(I - A_{t+h})^{-1}} = \left[b_{t+h}^{ij}\right]$ may "cancel out" in the summation $\sum_j b_{t+h}^{ij} \cdot {}^DQ_{t+h}^{+j}$ (for i = 1, ..., m). Consequently, on the basis of the assumptions given above, systematic variation of the vector $\underline{\bar{\varepsilon}_{t+h}}$ through time is a sufficient but not a necessary condition for systematic variation of the input-output matrix. It may be noted, however, that the absence of systematic variation of the vector $\underline{\bar{\varepsilon}_{t+h}}$ in the face of such variation of the input-output matrix seems to be a mathematical curiosity unlikely to be encountered in practice.

The next step in our analysis is to evaluate the systematic component of the vector $\underline{\bar{\varepsilon}_{t+h}}$. This may be done by running the following regressions:

$$(10) \quad \bar{\varepsilon}_{t+h}^{i} = f^{i}(t+h) + g^{i}(\bar{\varepsilon}_{t+h-1}^{i}, \bar{\varepsilon}_{t+h-2}^{i}, \ldots, \bar{\varepsilon}_{t+h-l}^{i}) + \bar{u}_{t+h}^{i}, \qquad \text{for all } i = 1, \ldots, m;$$

where t is the base year, h = 1, ..., Υ, $f^{i}(t)$ identifies the trend of the disturbance term and corrects for heteroscedasticity, and $g^{i}(\varepsilon_{t-h-1}^{i}, \ldots, \varepsilon_{t+h-l}^{i})$ corrects for autocorrelation. The functions f^{i} and g^{i} permit the identification of the disturbance term $\bar{u}_{t+h}^{i}$ which satisfies the conditions (6). The systematic component of ε_{t+h}^{i} may thus be identified as

$$(11) \quad \bar{\varphi}_{t+h}^{i} = f^{i}(t+h) + g^{i}(\bar{\varepsilon}_{t+h-1}^{i}, \ldots, \bar{\varepsilon}_{t+h-l}^{i}).$$

Thus far, we have derived the vector of gross outputs through the approximation $\underline{\bar{Q}_{t+h}} = (\underline{I} - \underline{A_t})^{-1} \cdot \underline{{}^DQ_{t+h}^{*}}$. This vector plus the vector of the above systematic disturbance term components explains the entire systematic variation in the directly estimated gross outputs:

$$(12)\quad \underline{Q^{*}_{t+h}} = \underline{\overline{Q}_{t+h}} + \underline{\overline{\varphi}_{t+h}} .$$

Since $\underline{\overline{\varphi}_{t+h}}$ - the systematic variation in gross outputs not explained by the final demands - is attributed to changes in the input-output matrix, we now proceed to change the coefficients of this matrix in such a way that the entire systematic variation in gross outputs is explained. The updating method is analogous to that of Stone and Brown. Construct a diagonal matrix $\underline{T_{t+h}}$ (of order m x m), whose diagonal elements are the solutions of the following system:

$$(13)\quad \underline{\overline{Q}_{t+h}} + \underline{\overline{\varphi}_{t+h}} = (\underline{I} - \underline{T_{t+h}} \cdot \underline{A_t})^{-1} \cdot \underline{{}^{D}Q^{*}_{t+h}}.$$

$\underline{{}^{D}Q^{*}_{t+h}}$ and $\underline{A_t}$ are directly estimated, $\underline{\overline{Q}_{t+h}}$ is derived from system (9), $\underline{\overline{\varphi}_{t+h}}$ is found through equation (11), and $\underline{T_{t+h}}$ remains to be calculated. System (13) may be rewritten as follows:

$$\underline{T_{t+h}} \cdot \underline{A_t} \cdot \left[\underline{\overline{Q}_{t+h}} + \underline{\overline{\varphi}_{t+h}} \right] = \underline{\overline{Q}_{t+h}} + \underline{\overline{\varphi}_{t+h}} - \underline{{}^{D}Q^{*}_{t+h}}$$

from which the value of the i'th diagonal element of matrix $\underline{T_{t+h}}$ may be determined immediately:

$$(14)\quad T^{i}_{t+h} = \frac{\overline{Q}^{i}_{t+h} + \overline{\varphi}^{i}_{t+h} - {}^{D}Q^{*i}_{t+h}}{\sum_{j} a^{ij}_{t} \cdot (\overline{Q}^{j}_{t+h} + \overline{\varphi}^{j}_{t+h})}$$

The first updated estimate of the input-output matrix, then, is $\underline{T_{t+h}} \cdot \underline{A_t}$.

The economic interpretation of this procedure is apparent. If the systematic variation in gross output not explained by final demand is zero (i.e. $\overline{\varphi}^{i}_{t+h} = 0$), then the corresponding element of the $\underline{T}$ matrix is unity (i.e. $T^{i}_{t+h} = 1$). (Intermediate good purchases of the i'th good = $\overline{Q}^{i}_{t+h} - {}^{D}Q^{*i}_{t+h} = \sum_{j} a^{ij}_{t+h} \cdot \overline{Q}^{j}_{t+h}$.) Any unexplained nonzero systematic variation in the i'th gross output is attributed to the technology embodied in that output.

Thus, every input-output coefficient in which the i'th good appears as input is adjusted by the same proportionality constant, T^{i}_{t+h}.

Naturally, there are many other ways of attributing the unexplained systematic gross output variation to the input-output coefficients. For example, if the $\underline{T_{t+h}}$ matrix postmultiplies A_t, then systematic variation in the i'th gross output is attributed to the technologies embodied in the buyers of the i'th good. (The input-output coefficients of the j'th buyer are all adjusted by the constant T^{j}_{t+h}). Which approach is to be adopted must depend on the special properties of the industrial technologies under consideration. As in the rAs method, we have many degrees of freedom. There are m^2 coefficients to be adjusted by means of one row control vector, $\underline{\overline{Q}_{t+h}} + \underline{\overline{\mathcal{Y}}_{t+h}} - \underline{{}^{D}Q^{*}_{t+h}}$, containing m elements. m degrees of freedom, however, may be eliminated by introducting a column control vector.

This possibility brings us to the second part of our analysis. Analogously to the rAs procedure, the premultiplication of the input-output matrix by the diagonal matrix $\underline{T}$ constitutes only half of the iterative updating method. The other half comprises postmultiplication of the input-output matrix by another diagonal matrix, $\underline{U}$. Just as the rAs method makes the base year input-output matrix consistent with respect to the relationship between the vectors $\underline{Q}$ and $\underline{QT}$ and between the vectors $\underline{Q}$ and $\underline{QB}$ in other years, so the $\underline{TAU}$ method makes the base-year input-output matrix consistent with respect to the relationship between the vectors $\underline{Q^{*}}$ and $\underline{{}^{D}Q^{*}}$ (as shown above) and between the vectors $\underline{P}$ and $\underline{VA}$ -- the price and value added vectors -- (to be shown below). The diagonal matrix $\underline{T}$ was derived from the Leontief output framework. The diagonal matrix $\underline{U}$ will now be derived from the corresponding Leontief price framework.

The price framework corresponding to system (5) may be expressed as follows:

$$(15) \quad \underline{P_t} = \underline{A'_t} \cdot \underline{P_t} + \underline{VA_t} ,$$

where $\underline{P_t}$ is a vector of prices for industrial products, $\underline{A'_t}$ is the transpose of the input-output matrix of system (5), and $\underline{VA_t}$ is a vector of industrial values added. In other words, the price of good i is equal to the total intermediate good costs (per unit of i) plus the value added of the i'th industry (per unit of i). Analogously to the output framework, we assume that the vectors $\underline{P}$ and $\underline{VA}$ are estimated year by year -- we call these estimates $\underline{P^*}$ and $\underline{VA^*}$, respectively -- whereas the matrix $\underline{A}$ has been calculated only for year t. The problem is to update $\underline{A}$ in accordance with the information given by $\underline{P^*}$ and $\underline{VA^*}$. Let $\underline{P^*}$ and $\underline{VA^*}$ be associated with vectors of disturbance terms which satisfy conditions (6). Given an estimate of $\underline{VA}$ for year t+h and the input-output matrix for the base year, system (15) permits an approximation of the price vector to be derived:

$$(16) \quad \underline{\bar{P}_{t+h}} = (\underline{I} - \underline{A'_t})^{-1} \cdot \underline{VA^*_{t+h}}.$$

We now compare the directly estimated price vector $\underline{P^*_{t+h}}$ with the derived price vector $\underline{\bar{P}_{t+h}}$. If the input-output matrix actually remains unchanged from year t to year $t+\tau$, then $\underline{P^*_{t+h}} = \underline{\bar{P}_{t+h}} + \underline{\bar{\varepsilon}_{t+h}}$, for h = 1, ..., τ , where $\underline{\bar{\varepsilon}_{t+h}}$ (not identical with the disturbance term vector of the Leontief output framework) satisfies conditions (6). However, given our direct estimate of $\underline{P^*_{t+h}}$ and our derived value of $\underline{\bar{P}_{t+h}}$, we may find that $\bar{\varepsilon}_{t+h}$ (= $\underline{P^*_{t+h}} - \underline{\bar{P}_{t+h}}$) varies systematically through time. In other words, the regressions

$$\begin{aligned} \bar{\varepsilon}^i_{t+h} &= f^i(t+h) + g^i \, (\bar{\varepsilon}^i_{t+h-1}, \bar{\varepsilon}^i_{t+h-2}, \ldots, \bar{\varepsilon}^i_{t+h-l}) + \bar{u}^i_{t+h} \\ &= \bar{\psi}^i_{t+h} + \bar{u}^i_{t+h} \end{aligned}$$

(for all i=1, ..., m, where $\bar{u}^i_{t+h}$ -- not identical with the disturbance term of regression (10) -- satisfies conditions (6)) reveal statistically significant nonzero systematic components, $\bar{\psi}^{i}_{t+h}$. Under the supposition that $\underline{VA}^{*}_{t+h}$ and $\underline{P}^{*}_{t+h}$ are associated with "well-behaved" disturbance terms, these systematic components must be attributed to changes in the coefficients of the input-output matrix. To explain the systematic variations in $\underline{P}^{*}_{t+h}$ which are not captured by system (16), we calculate the diagonal matrix $\underline{U}$ by means of the following system:

$$(17)\quad \underline{\bar{P}_{t+h}} + \underline{\bar{\psi}_{t+h}} = (\underline{I} - \underline{U_{t+h}} \cdot \underline{A'_t})^{-1} \cdot \underline{VA^{*}_{t+h}}.$$

This system may be rewritten as

$$\underline{U_{t+h}} \cdot \underline{A'_t} \cdot (\underline{\bar{P}_{t+h}} + \underline{\bar{\psi}_{t+h}}) = \underline{\bar{P}_{t+h}} + \underline{\bar{\psi}_{t+h}} - \underline{VA^{*}_{t+h}},$$

and therefore the j'th diagonal element of matrix $\underline{U_{t+h}}$ may be specified:

$$(18)\quad U^{j}_{t+h} = \frac{\bar{P}^{j}_{t+h} + \bar{\psi}^{j}_{t+h} - VA^{*j}_{t+h}}{\sum_{i} a^{ij}_{t} \cdot (\bar{P}^{i}_{t+h} + \bar{\psi}^{i}_{t+h})}.$$

The updated estimate of the input-output matrix may now be written as $\underline{U_{t+h}} \cdot \underline{A'_t}$. Economically speaking, any systematic variation in the price of the i'th good is attributed to the technologies of the buyers of that good.

The diagonal matrices $\underline{T}$ and $\underline{U}$ may now be used in conjunction with one another to update the input-output matrix. If the matrix $\underline{A_t}$ is to be updated to year t + h, this matrix must first be formulated in terms of the relative prices obtaining in year t+h: $A^{*}_{t} = \underline{P_{t+h}} \cdot \underline{A_t} \cdot \underline{P^{-1}_{t+h}}$, where $\underline{P_{t+h}}$ is a diagonal matrix of the relative prices of years t+h and t. Second, calculate the first $\underline{T_{t+h}}$ matrix, $\underline{T^{1}_{t+h}}$, by means of equation (14) and derive the first approximation of the updated input-output matrix: $\underline{A^{1}_{t+h}} = \underline{T^{1}_{t+h}} \cdot \underline{A^{*}_{t}}$. Third, calculate the first $\underline{U_{t+h}}$ matrix, $\underline{U^{1}_{t+h}}$, from equation (18), using the coefficients of $\underline{A}^1_{t+h}$, and derive the second approximation of the input-output matrix: $\underline{A}^2_{t+h} = \underline{A}^1_{t+h} \cdot \underline{U}^1_{t+h}$. Now system (17), the Leontief price equation system, holds, but system (13), the Leontief output equation system, is violated. The latter system may be reinstated by recalculating the $\underline{T}_{t+h}$ matrix from equation (14), this time using the coefficients of $\underline{A}^2_{t+h}$, and deriving the third approximation of the input-output matrix: $\underline{A}^3_{t+h} = \underline{T}^2_{t+h} \cdot \underline{A}^2_{t+h}$. Then the columns must be scaled to the correct column sums through calculating a new matrix $\underline{U}^2_{t+h}$, and so on. This iterative process, like that of the rAs method, is convergent for a nonnegative input-output matrix. (This properly is apparent upon comparison of equations (3) and (4) with equations (14) and (18)).

In summary, the TAU method updates the base-year input-output matrix by making it consistent with the empirically observed relations between $\underline{^DQ}^*$ and $\underline{Q}^*$ and between $\underline{VA}^*$ and $\underline{P}^*$. All systematic variation in Q^{*i}_{t+h} which is not explained by $\underline{^DQ^*_{t+h}}$ and $\underline{A}_t$ (i.e. by the equation $\overline{Q}^i_{t+h} = \sum_j a^{ij}_t \cdot \overline{Q}^j_{t+h} + {}^DQ^{+i}_{t+h}$) is attributed to a change in the technology embodied in the i'th good. Similarly, all systematic variation in P^{*j}_{t+h} which is not explained by $\underline{VA}^*_{t+h}$ and $\underline{A}_t$ (i.e. by the equation $\overline{P}^j_{t+h} = \sum_i a^{ij}_t \cdot \overline{P}^i_{t+h} + \underline{VA^{*j}_{t+h}}$) is attributed to a change in the productive processes of the buyers of the j'th good. Beyond doubt, this is a somewhat arbitrary procedure. As we shall see, however, it is this arbitrariness which makes our updating method particularly useful.

Yet there may be some presumption in favor of attributing technological change in the manner prescribed by the TAU method. If the equation $\overline{Q}^i_{t+h} = \sum_j a^{ij}_t \cdot \overline{Q}^i_{t+h} + {}^DQ^{+i}_{t+h}$ does not capture certain systematic variations in the i'th gross output, then it may seem plausible that the variations are due to technological changes inherent in the i'th good itself (rather than equal proportionale changes in the technologies of all the buyers of the i'th good). Moreover, if the equation $\bar{P}^{j}_{t+h} = \sum_{i} a^{ij}_{t} \cdot \bar{P}^{i}_{t+h} + VA^{j*}_{t+h}$ leaves unexplained certain systematic variations in the price of the j'th good, then it may seem plausible that these variations are due to technological changes in the j'th industry (rather than equal proportionate changes in the technologies embodied in all of the intermediate goods purchased by that industry).

However, this is not a compelling argument in general and different ways of attributing technological changes may be more reasonable under certain circumstances. (For example, if a^{ij} stands for the input of cotton per unit of apparel, it seems sensible to attribute most of the technological change to the apparel industry. A change in the quality of cotton that permits lower input of cotton per unit of each output of the cotton buyers, even if the productive processes of these buyers remain unchanged, strikes one as an unlikely occurence.) Let us examine what our flexibility in attribution schemes implies.

4. Variations of the New Updating Method

A straight-forward alternative to the to our updating method presents itself: the UAT method. In this method, the $\underline{T}$ matrix is calculated from the following system:

$$(19)\quad \underline{\bar{Q}_{t+h}} + \underline{\varphi_{t+h}} = (\underline{I} - \underline{A_t} \cdot \underline{T_{t+h}})^{-1} \cdot \underline{{}^{D}Q^{*}_{t+h}},$$

and the $\underline{U}$ matrix is computed from the following system:

$$(20)\quad \underline{\bar{P}_{t+h}} + \underline{\psi_{t+h}} = (\underline{I} - \underline{A'_t} \cdot \underline{U_{t+h}})^{-1} \cdot \underline{VA^{*}_{t+h}}.$$

From these two systems, it is apparent that the i'th diagonal element of the $\underline{T}$ matrix is

$$(21)\quad T^{i}_{t+h} = \frac{\sum_j b^{ij}_t \cdot (\bar{Q}^{j}_{t+h} + \varphi^{j}_{t+h} - {}^{D}Q^{j*}_{t+h})}{\bar{Q}^{i}_{t+h} + \bar{\varphi}^{i}_{t+h}},$$

and the j'th diagonal element of the $\underline{U}$ matrix is

$$(22)\quad U^{j}_{t+h} = \frac{\sum_i b^{ij}_t \cdot (\bar{P}^{i}_{t+h} + \psi^{i}_{t+h} - VA^{i*}_{t+h})}{\bar{P}^{j}_{t+h} + \bar{\psi}^{j}_{t+h}},$$

where $\left[b^{ij}\right] = \underline{A_t^{-1}}$.

In other words, $\underline{\varphi_{t+h}}$ is attributed to the columns of the base-year input-output matrix and $\underline{\psi_{t+h}}$ is attributed to the rows -- the obverse of the attribution scheme above. We are now in a position to derive three separate estimates of an input-output matrix updated to year t+h from year t: rAs, TAU, and UAT. As we have noted above, each of these methods contains $(n^2 - 2m)$ degrees of freedom and each degree of freedom implies an inability to determine the precise values of the input-output coefficients from the given empirical data. A comparison of input-output matrices updated through different methods provides an indication of the range over which input-output coefficients may vary and still be consistent with available empirical observations. Variety in updating methods is important as a way of assessing the reliability of the updated coefficient estimates. Also, as we have noted above, a choice of compromise between the various estimated input-output tables may be possible on the basis of independent industrial evidence.

Another advantage of the flexibility of the TAU method is that it may be used in conjunction with the rAs method to permit the derivation of updated input-output coefficients in case where the data demanded by the rAs method is not available, or to improve the estimates of the coefficients whenever the above data is available. Let us briefly examine each of these possibilities.

In order to implement the rAs method, data on $\underline{Q}$ (gross outputs by industry), $\underline{QB}$ (intermediate purchases by buyer), and $\underline{QT}$ (intermediate purchases by seller) are required. While data on $\underline{Q}$ and $\underline{QB}$ are available on a yearly basis for most advanced, capitalist economies, yearly data on $\underline{QT}$ are often much harder, if not impossible, to come by. Without these data, however, the rows of an input-output matrix cannot be adjusted. To proceed by adjusting only the columns (i.e. $\underline{As}$) is tantamount to assuming that all technological change occuring in the economy is attributable to the productive processes of the buyers of intermediate goods rather than to the types of goods -- a blatantly unrealistic assumption. Yet, when data on $\underline{QT}$ is absent, data on prices and values added (by industry) may nevertheless be available. Empirical estimates of $\underline{P}$ and $\underline{VA}$ are all that is needed to implement the row adjustments necessary to ensure consistency between these two vectors in the Leontief price framework. The $\underline{U}$ matrix may be calculated through system (20), i.e. the matrix $\overline{A_t}$ is premultiplied by $\overline{U_{t+h}}$. The resulting updating method may be identified as the $\overline{UAs}$ method.

If all the empirical prerequisites of the rAs method are met, thus permitting the matrix $\underline{A_t}$ to be updated, the $\underline{U}$ matrix may be employed to reduce the $(m^2 - 2m)$ degrees of freedom allowed by the updating procedure. There are several ways in which this problem of underdetermined input-output coefficients may be ameliorated. One method may be designated as the $rA\begin{Bmatrix} s \\ u \end{Bmatrix}$ method. Using the base-year input-output matrix, first compute $\underline{r}$ through equation (3) and derive the first approximation of the updated input-output matrix: $\underline{A^1_{t+h}} = \underline{r^1_{t+h}} \cdot \underline{A_t}$. Second, using the coefficients of $\underline{A^1_{t+h}}$, compute $\underline{s}$ through equation (4) and derive the second approximation: $\underline{A^2_{t+h}} = \underline{A^1_{t+h}} \cdot \underline{s^1_{t+h}}$. Third, adjust the rows of the resulting matrix: $\underline{A^3_{t+h}} = \underline{r^2_{t+h}} \cdot \underline{A^2_{t+h}}$. Fourth, compute $\underline{U}$ through equation (18) and derive the next approximation of the updated input-output matrix: $\underline{A^4_{t+h}} = \underline{A^3_{t+h}} \cdot \underline{U^3_{t+h}}$. Then recompute $\underline{r}$ to derive another approximation, thereby beginning the second cycle of this iterative procedure. The procedure is continued until each of the three underlying equation systems is satisfied to a given degree of accuracy :

$$\underline{\tilde{A}_{t+h}} \cdot \underline{Q_{t+h}} = \underline{QT_{t+h}}$$

$$\underline{Q'_{t+h}} \cdot \underline{\tilde{A}_{t+h}} = \underline{QB_{t+h}}$$

$$\underline{\tilde{A}'_{t+h}} \cdot (\underline{\overline{P}_{t+h}} + \underline{\overline{\varphi}_{t+h}}) = (\underline{\overline{P}_{t+h}} + \underline{\overline{\varphi}_{t+h}} - \underline{VA^*_{t+h}}) ,$$

where $\underline{\tilde{A}_{t+h}}$ is the updated input-output matrix. This updating method implies $(m^2 - 3m)$ degrees of freedom in the coefficient determination. There are several other, but analogous, methods of updating $\underline{A_t}$. Each method yields, presumably, different estimates of the input-output coefficients for year t+h. These alternate methods may be summarized by the following designations: $\begin{Bmatrix} r \\ u \end{Bmatrix}$ As, $\begin{Bmatrix} T \\ U \end{Bmatrix}$ As, and TA $\begin{Bmatrix} s \\ u \end{Bmatrix}$. As we have noted above, the possibility of examining a variety of input-output matrix estimates is to be welcomed.(Matrix $\underline{T}$ cannot be used like matrix $\underline{U}$ to supplement the rAs method. The reason is that, whereas the vector pairs $\left[\underline{Q_{t+h}} \vdots \underline{QT_{t+h}}\right]$, $\left[\underline{Q_{t+h}} \vdots \underline{QB_{t+h}}\right]$, and $\left[(\underline{\bar{P}_{t+h}} + \underline{\bar{\Psi}_{t+h}}) \vdots (\underline{\bar{P}_{t+h}} + \underline{\bar{\varphi}_{t+h}} - \underline{VA^*_{t+h}})\right]$ may be expected to be linearly independent of one another, the first two vector pairs may not be linearly independent of $(\underline{\bar{Q}_{t+h}} + \underline{\bar{\mathcal{G}}_{t+h}}) \vdots (\underline{\bar{Q}_{t+h}} + \underline{\bar{\mathcal{G}}_{t+h}} - \underline{{}^DQ^*_{t+h}})$. In fact, $\underline{Q_{t+h}} = \underline{\bar{Q}_{t+h}} + \underline{\bar{\mathcal{G}}_{t+h}}$ and $\underline{QT_{t+h}} = \underline{\bar{Q}_{t+h}} + \underline{\bar{\mathcal{G}}_{t+h}} - \underline{{}^DQ^*_{t+h}}$ on the average -- if the demand functions are specified and estimated properly. Thus, matrices $\underline{r}$ and $\underline{T}$ cannot be used in conjunction).

In conclusion, it is important to take note of the analogy between the rAs and the TAU methods. In essence, these methods are simply two alternative ways of specifying row and columns controls whereby an input-output table may be updated. The row controls of the rAs method are specified through the system $\underline{r_{t+h}} \cdot \underline{A_t} \cdot \underline{Q_{t+h}} = \underline{QT_{t+h}}$, whereas the row controls of the TAU method are determined by the system $\underline{T_{t+h}} \cdot \underline{A_t} \cdot (\underline{\bar{Q}_{t+h}} + \underline{\bar{\mathcal{G}}_{t+h}}) = (\underline{\bar{Q}_{t+h}} + \underline{\bar{\mathcal{G}}_{t+h}}) - \underline{{}^DQ^*_{t+h}}$. $\underline{Q_{t+h}}$ differs from $(\underline{\bar{Q}_{t+h}} + \underline{\bar{\mathcal{G}}_{t+h}})$ only by a vector of disturbance terms which have no systematic components. $\underline{QT_{t+h}}$ is equal to $(\underline{Q_{t+h}} - \underline{{}^DQ_{t+h}})$ and thus differs from $(\underline{\bar{Q}_{t+h}} + \underline{\bar{\mathcal{G}}_{t+h}} - \underline{{}^DQ^*_{t+h}})$ only by such a disturbance terms vector as well. Hence, in principle, there should be no systematic difference between the row controls of the rAs method and those of the TAU method. In practice, a systematic difference may occur, however. This question hinges on how the demand functions are estimated. If the demand functions accurately mimic demand behavior in the real world, then the two updating methods are, for all practical purposes, identical. Given the present state of economic theory and econometrics, it is safe to say that we are unreachably far away from this ideal. Since both simulation analysis and forecasting dictate the use of input-output matrices in conjunction with estimated demand functions, it may be wise to update the input-output matrices with a view of the imperfections inherent in the demand functions. Successful prediction of industrial outputs and meaningful sensitivity analysis hinges crucially on the consistency between final demands and gross outputs as prescribed by the Leontief output system; consequently, it may be useful to use an updating method that achieves this consistency.

The analogy between the row controls of the rAs method and those of the TAU method cannot be extended to the column controls as well. The column controls of the rAs method are determined through the system $\underline{Q'_{t+h}} \cdot \underline{A_t} \cdot \underline{s_{t+h}} = \underline{QB'_{t+h}}$, while the column controls of the TAU method are given by the system

$$(\underline{\overline{P}'_{t+h}} + \underline{\overline{\psi}'_{t+h}}) \cdot \underline{A_t} \cdot \underline{U_{t+h}} = \underline{\overline{P}'_{t+h}} + \underline{\overline{\psi}'_{t+h}} - \underline{VA^{*'}_{t+h}} .$$

Since the vectors $\underline{Q'_{t+h}}$ and $\underline{QB'_{t+h}}$ are not linearly dependent on the vectors $(\underline{\overline{P}'_{t+h}} + \underline{\overline{\psi}'_{t+h}})$ and $(\underline{\overline{P}'_{t+h}} + \underline{\overline{\psi}'_{t+h}} - \underline{VA^{*'}_{t+h}})$, it is possible, as we have seen, to use this system to reduce the degrees of freedom in the determination of the input-output coefficients. The column controls of the TAU method ensure the consistency between values added and prices, and thus they also may prove valuable in forecasting and simulation analysis.

Footnotes

1. "Production coefficients" are to be interpreted broadly here, e.g. they include what are sometimes called "allocation coefficients".

2. Richard Stone and J.A.C.Brown, "a Long-term growth Model for the British Economy," in Europe's Future in Figures, ed. R.C. Geary, North-Holland Publishing Company, Amsterdam, 1962.

3. In most applications, it is unnecessary to update every element of the input-output matrix; some elements may be estimated directly. For such cases, the rAs method may be implemented exclusively on the remaining elements. The modification of the updating algorithm which this situation necessitates is too trivial to deserve description.

4. See W.Edwards Deming, Statistical Adjustment of Data. Wiley, New York, 1943, pp.115-117.