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PRIOR INFORMATION IN FORECASTING
WITH
ECONOMETRIC MODELS

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1. Introduction

Judging the forecasting performance of an econometric model we experience the need to distinguish between testing an econometric model, and testing a forecaster who forecasts by means of an econometric model. This reflects the fact that most model proprietors agree that they can improve their model forecasts by adjusting the model's constant terms, slopes, or exogenous variables in a way that on the basis of all their available information the forecast is likely to be improved. At least the following three arguments are used to justify corrections by the model proprietor in the forecasting process.

Firstly, at the time a forecast is made the forecaster faces uncertainty about many exogenous variables of a model on the one hand, but is not ignorant about at least some endogenous variables on the other hand. Important sources of information like monthly indices of production or anticipatory data from sample surveys are usually not contained in traditional structural econometric models but convey information about future values of endogenous model variables.

Secondly, additional uncertainty is introduced into the model forecast by the preliminary character of the most recent reported data which are needed for lagged endogenous variables and for updating the parameter estimates. Discrepancies between the model forecast and latest reported figures may be either attributed to the preliminary character of the latter or to obsolete parameter estimates. In both cases corrections by the forecaster seem to be justified.

Thirdly, empirical and theoretical evidence suggests that socio-economic systems are not time-invariant, therefore requiring permanent updating of the parameter estimates as soon as new sample information becomes available to check the validity of the parameter specification. Aware of this fact but unable to reestimate all parameters of a model for each forecasting exercise because of the amount of work involved, most forecasters take a shortcut and adjust only a few parameters which seem crucial to them.

The main objection against the above indicated adjustments by the model proprietor is the lack of systematic methodology drawing forth criticism about too much subjectivity involved in forecasting with econometric models. Anybody, however, who is advocating a purely mechanical use of econometric models in the forecasting process should be reminded that subjective elements are involved at all stages of econometric model building, starting with the selection of the data base up to the specification of all model equations and the choice of the forecasts for the exogenous variables. Nevertheless there remains the necessity to increase the transparency of the interaction between an econometric model and the model proprietor in the forecasting process.

This paper is an extension of an earlier paper by R.Mariano and S.Schleicher [1972]. It attempts to develop a unified approach to the above indicated problem areas in a forecasting exercise, namely, the use of prior information about endogenous variables, the treatment of preliminary data, and the updating of estimated parameter values when new sample information becomes available. The statistical instrument in all three cases is a sequential estimator for a general dynamic system, better known as Kalman filter, with different interpretations of the system state.

2. Kalman Filter Algorithm

The statistical analysis in the following sections will be based on a general discrete stochastic dynamical system described by a stochastic vector difference equation

$$s_t = A_t s_{t-1} + f_t + v_t \tag{2.1a}$$

which is called the system state model as the p-vector s_t is the system state vector at time t. Further f_t is a deterministic vector and v_t a disturbance term characterized by first and second moments such that

$$v_t \sim (o, V_t) \; . \tag{2.1b}$$

With the system state model connected is the so-called measurement or observation model

$$m_t = H_t s_t + g_t + w_t \tag{2.1c}$$

as the q-vector m_t contains the measurements or observations from which we draw conclusions about the unobservable system state s_t. Again g_t is a deterministic vector and w_t a disturbance term described by

$$w_t \sim (o, W_t) \; . \tag{2.1d}$$

For this system (1) there exists a very elegant sequential estimator for the unobservable system state s_t which after its author is called Kalman filter.

<u>Theorem:</u> (Kalman Filter)

Given measurements m_t, known matrices A_t and H_t, known vectors f_t and g_t, known covariance matrices V_t and W_t, and initial conditions for the system state estimate $\hat{s}_{o|o}$ and the corresponding error covariance matrix $S_{o|o}$, a best linear unbiased estimator for the above specified system (1) is defined by the following recursive relationships:

state estimate

$$s_{t|t} = A_t s_{t-1|t-1} + f_t + K_t(m_t - H_t A_t s_{t-1|t-1} - g_t) \, , \tag{2.2a}$$

prior state error covariance matrix

$$S_{t|t-1} = A_t S_{t-1|t-1} A_t' + V_t \, , \tag{2.2b}$$

filter gain matrix

$$K_t = S_{t|t-1} H_t' (H_t S_{t|t-1} H_t' + W_t)^{-1} \, , \tag{2.2c}$$

posterior state error covariance matrix

$$S_{t|t} = (I - K_t H_t) S_{t|t-1} \, . \tag{2.2d}$$

Proof. Extensive discussions of the Kalman filter can be found e.g. in Aström [197o] or Jazwinski [197o].

In the above notation in case of double subscripts the first refers to the system state and the second to the latest available measurement. Therefore, e.g., $S_{t|t-1}$ is the state error covariance matrix for the system state at time t, given t-1 measurements.

3. Problem 1: Prior Information about Endogenous Variables

Every econometric model can be described by the following usually nonlinear vector function

$$F(y_t, y_{t-1}, \ldots, x_t, x_{t-1}, \ldots, b, u_t) = o \tag{3.1}$$

where y_t denotes the vector of endogenous variables at time t, x_t the vector of exogenous variables, b the vector of parameters to be estimated, and u_t the vector of disturbances. For a given model specification which is described by the list of endogenous and exogenous variables, the functional form of the equations constituting the model, and all restrictions on the parameter space, we obtain parameter estimates b on the basis of the available sample information. For given parameter estimates, given exogenous, and given lagged endogenous variables follows the probability distribution of the endogenous variables

$$f(y_t \mid y_{t-1}, \ldots, x_t, x_{t-1}, \ldots, \hat{b}) \qquad (3.2)$$

which represents the conditional forecast of an econometric model. In the sequel of our analysis we assume that we can approximate this distribution by first and second moments. Thus the stochastic vector of our endogenous model variables to be projected is described by

$$y_t \sim (\bar{y}_t, S_t^y) \qquad (3.3)$$

with $\bar{y}_t$ being the expected value of the endogenous variables and S_t^y the corresponding covariance matrix. We assume these parameters to be known, either from the reduced form in case of a linear system or from sample simulations in case of a nonlinear system. Thus we may write our structural econometric model alternatively

$$y_t = \bar{y}_t + v_t \ , \qquad (3.4a)$$

the disturbance term characterized by

$$v_t \sim (o, S_t^y) \ . \qquad (3.4b)$$

Without any further information the best forecast of y_t, that is the best estimate of y_t for a wide class of loss functions (c.f. Aström [197o], p.213) is the expectation of y_t:

$$y_t = E(y_t) = \bar{y}_t \ . \qquad (3.5)$$

This means, that having no other information available but that from the econometric model, the expected value of the model forecast will produce the optimal result.

Ignoring for the moment problems caused by incorrect model specification and parameter estimation, this textbook representation of a forecast with an econometric model is based on two crucial assumptions, namely, (i) that the forecaster has no prior information about the endogenous variables to be projected, and (ii) that the exogenous variables are known with certainty.But both assumptions do not fit to a typical forecasting situation. It is more realistic to concede that the forecaster faces uncertainty about many exogenous variables, but that on the other hand he is not ignorant of at least some endogenous variables. It remains to adjust the forecasting procedure with an econometric model to these more realistic information aspects.

At the time a forecast is made even many policy instruments like public expenditures, effective tax rates, and other exogenous variables reflecting e.g. foreign economic activity are hardly precisely known. The forecaster is dependent on various sources of information and his own judgement in projecting these exogenous variables, having at least a vague idea about the precision of the values used.

Information about endogenous variables in addition to the model projection may originate from different sources. Time series analysis on monthly reported data like industrial production, trade balance, and consumer prices may provide useful information about the corresponding model variables at least for the subsequent quarters. Consumer and investment survey data may shed, after a correction for reporting biases, some light on the actual level of private consumption and investment expenditures. In addition the forecaster's intuition and experience might make him believe that some results are more realistic than others. It is intuitively plausible and it will be shown immediately analytically that the introduction of this additional information will increase the precision of the forecast.

In the next step of our attempt to make systematic and efficient use of all information which is available at the time a forecast is prepared we assume that we are able to describe this additional information by the stochastic vector y_t^a, the superscript indicating the anticipatory character of this variable, with mean vector $\bar{y}_t^a$ and covariance matrix S_t^a:

$$y_t^a \sim (\bar{y}_t^a, S_t^a) . \qquad (3.6)$$

Let us further assume that the vector y_t^a containing our additional or prior information for the forecast period t is related to the vector of projected endogenous variables y_t via the following linear stochastic relationship

$$y_t^a = H_t y_t + w_t , \qquad (3.7a)$$

the stochastic disturbance vector w_t being characterized by

$$w_t \sim (o, S_t^a) . \qquad (3.7b)$$

The two vectors y_t and y_t^a need not be of the same dimension. H_t is a conformable matrix with known coefficients.

We thus have arrived at what we call an <u>extended model</u> as we added the anticipations model (3.7) to our structural econometric model (3.4). This extended model can be considered a static version of the general dynamic system proposed in section 2, the structural econometric model interpreted as state model and the anticipations model interpreted as measurement or observation model. We identify the vector of endogenous variables y_t as the system state variables s_t and the vector y_t^a containing all prior information about the endogenous variables as the measurement variables m_t.

With $\bar{y}_t$ as the expected value of the system state and S_t^y as corresponding error covariance matrix, both prior to the measurement y_t^a, we get as a best linear unbiased estimator for the projected endogenous variables of the extended model from (2.2a)

$$y_{t|y_t^a} = \bar{y}_t + K_t(y_t^a - H_t\bar{y}_t) \ , \qquad (3.8a)$$

the gain matrix being defined according to (2.2c) as

$$K_t = S_t^y H_t'(H_t S_t^y H_t' + S_t^a)^{-1} \ , \qquad (3.8b)$$

and the covariance matrix of the forecast after having made use of the prior information y_t^a as in (2.2d)

$$S^y_{t|y_t^a} = (I - K_t H_t) S_t^y \ . \qquad (3.8c)$$

The interpretation of the above results is very appealing. (3.8a) indicates that the optimal forecast which makes both use of the structural econometric model and all additional available information is based on the model projection $\bar{y}_t$ plus an additive correction term. This correction term is proportional to the discrepancy between prior information y_t^a and the model projection $H_t\bar{y}_t$ for the measurement variable. The extent to which this discrepancy is used for the correction is determined by the weighting matrix K_t, which in turn according to (3.8b) depends on the relative precision of the two models, namely the covariance matrix of the structural model S_t^y and the covariance matrix of the measurement model S_t^a. As the term $K_t H_t$ is positive definite we can judge from (3.8c) the gain in precision by making use of the prior information.

The above analysis reveals that whenever prior information about some endogenous variables becomes available the optimal forecast is no longer the expected value of the structural model solution alone but the result we get by adding a correction term. This seems to be a justification of the widespread constant-term adjustment practice among model proprietors. Whereas, however, ad hoc adjustments very often give the impression of arbitrariness and subjectivity the proposed forecasting procedure based on an extended model meets statistical optimality properties and makes the forecasting process more transparent as the model proprietor is induced to specify explicitely all his information used to produce a forecast.

The additional computations required for the extended model are rather modest. An error analysis both of the structural model and the anticipations model is needed to estimate the error covariance matrices S_t^y and S_t^a which enter according to (3.8b) the calculations of the gain matrix K_t. Then the structural model is solved to get the model forecast $\bar{y}_t$. The prior information y_t^a is taken into account by adding to $\bar{y}_t$ the correction term $K_t(y_t^a - H_t\bar{y}_t)$. There are no restrictions for the dimension of the vector y_t^a with respect to y_t. The vector y_t^a may contain only one element, that means all prior information is described by one variable, but yet corrections will occur for all elements of y_t because of the covariance relationships involved. The vector y_t^a may become larger than y_t if a variety of sources of prior information is available.

The lack of anticipatory data in structural econometric models which is mainly due to a lack of sound economic theory explaining the formation of expectations in the context of a macro model led to a number of models using anticipatory variables for the explanation of related endogenous variables. Typical examples are the model built by R.C.Fair [1970] and the Anticipations Version of Wharton Mark III. For the Wharton model a comparison of the forecasting accuracy of the Anticipations Version with the Standard Version is available (Adams and Duggal [1974]) which indicates substantial gains in forecasting performance when anticipatory data are used. The same article, however, reports considerable differences in the multiplier behavior in as far, as the Anticipations Version shows a lower multiplier sensitivity than the Standard Version.

Our proposal suggests instead of switching between an anticipations version for forecasting purposes and a standard version for policy analysis to combine both models to one extended model where the standard structural model serves as state model and the anticipations-realizations equations form the measurement or observation model. This procedure makes optimal use of both the structural model and the anticipatory information for forecasting purposes while leaving unchanged the multiplier properties of the structural model for policy analysis.

4. Problem 2: Treatment of Preliminary Data

Economic observations in general and the most recent reported data in particular because of their preliminary character contain considerable measurement errors. Unfortunately these latest reported figures are of special importance in the forecasting process as they are needed as model input for lagged endogenous variables and as one might want to use them for updating the parameter estimates.

As far as the treatment of preliminary data in a forecasting exercise is concerned we observe three modes of behavior of the model proprietors. Firstly, they accept preliminary data as they are and make revisions as soon as they become available. Secondly, they discard the most recent figures and up to a certain time lag trust only the model projections and use them as input for lagged endogenous variables. Thirdly, they form a weighted average of both model forecast and published data and use these numbers for further calculations. This mode of behavior comes close to our proposal for the treatment of preliminary data.

A comparison of time series containing preliminary data y_t^p with the corresponding revised values y_t enables us to estimate reporting biases and error variances S_t^p due to the preliminary character of the figures. This information can be cast into a measurement model which indicates the relationship between preliminary data and their final values,

$$y_t^p = H_t y_t + w_t \ , \qquad (4.1a)$$

the disturbance term described by

$$w_t \sim (o, \ S_t^p) \ . \qquad (4.1b)$$

The matrix H_t contains corrections for reporting biases.

The second source of information about the true but unobservable state of the economy is the latest available forecast, based both on a structural model and all additional sources of information. Denoting the expected value of the forecast of endogenous variables by $\bar{y}_t^f$, which may be identical with (3.8a), and the corresponding error covariance by S_t^y, which we may get from (3.8c), the information stemming from the forecast can be written as

$$y_t = \bar{y}_t^f + v_t \ , \qquad (4.2a)$$

the error term characterized by

$$v_t \sim (o, \ S_t^y) \ . \qquad (4.2b)$$

Again we may consider (4.1) and (4.2) a static version of the general dynamic system. The unobservable vector y_t is the system state with corresponding state model (4.2). Measurements of preliminary character y_t^p convey information about the true state y_t in a way which is specified by the measurement model (4.1).

The analysis of section 3 suggests that instead of choosing the preliminary data or the forecast data as information about the state of the economy, we should make optimal use of both sources of information. This is again performed by a Kalman filter.

A best linear unbiased estimate of the state of the economy based both on forecast and preliminary data is calculated according to (2.2a) as

$$\hat{y}_{t|y_t^p} = \bar{y}_t^f + K_t(y_t^p - H_t\bar{y}_t^f) \ , \tag{4.3a}$$

with gain matrix

$$K_t = S_t^y H_t'(H_t S_t^y H_t' + S_t^p)^{-1} \ , \tag{4.3b}$$

and error covariance matrix of the system state estimate after having made use of the preliminary data

$$S_{t|y_t^p}^y = (I - K_t H_t) S_t^y \ . \tag{4.3c}$$

Estimate (4.3a) for the state of the economy is a weighted average of preliminary reported data and forecast data with weights being determined according to the relative precision of the two sources of information for the state of the economy. This estimate is recommended both as input for lagged model variables in the next forecasting periods and for updating the parameter estimates.

5. Problem 3: Updating Parameter Estimates

The typical small data base from which the parameters of econometric models are estimated on the one hand, and both empirical and theoretical evidence for structural shifts on the other hand, are strong arguments for using any new sample information to update the parameter estimates. The time series of parameter estimates contain valuable information about the influence of individual data points and indicate whether any prior specification about model parameters, e.g. constancy over time, is not in contradiction to sample results.

One reason why in most econometric models parameter estimates are not revised more than once every two or three years lies in the considerable computational burden involved in reestimating the model coefficients. In contrast to the traditional batch type estimators which need for each reestimation explicitely the whole data base, sequential estimatiors as f.i. the Kalman filter make only implicite use of past data by referring to the most recent estimate and the corresponding error covariance matrix. Without going into details we just want to indicate that the Kalman filter technique offers both computational advantages and great flexibility in utilizing prior information about the parameter space. All classical econometric single equation estimators and many simultaneous equations estimators can easily be put into a sequential form by using the Kalman filter of the general dynamic system of section 2. Identifying the parameters to be estimated as the unobservable system state and the regression equation as measurement or observation model our estimation problem is described by

$$b_t = A_t b_{t-1} + f_t + v_t \,, \tag{5.1a}$$

$$v_t \sim (o,\, S_t^v) \,, \tag{5.1b}$$

$$y_t = X_t b_t + w_t \ , \qquad (5.1c)$$

$$w_t \sim (o, S_t^w) \ . \qquad (5.1d)$$

We recognize in (5.1) again the general linear dynamic model of section 2. Thus a best linear unbiased estimator of b_t based on observations of y up to time t, known X_t and A_t, and known covariances of the error terms is given according to (2.2) by the following recursive relationships:

$$\hat{b}_{t|t} = A_t \hat{b}_{t-1|t-1} + f_t + K_t(y_t - X_t \hat{b}_{t-1|t-1}) \ , \qquad (5.2a)$$

$$S_{t|t-1} = A_t S_{t-1} A_t' + S_t^v \ , \qquad (5.2b)$$

$$K_t = S_{t|t-1} X_t'(X_t S_{t|t-1} X_t' + S_t^w)^{-1} \ , \qquad (5.2c)$$

$$S_{t|t} = (I - K_t X_t) S_{t|t-1} \ , \qquad (5.2d)$$

were $\hat{b}_{t|t}$ denotes the best linear unbiased estimate of the parameter vector b at time t given t observations of y, $S_{t|t-1}$ is the corresponding error covariance matrix before, and $S_{t|t}$ the corresponding error covariance matrix after the occurence of measurement y_t.

The familiar classical econometric estimators in sequential representation follow as special cases after appropriate modification of the state model.

A state model of the form

$$b_t = b_{t-1} \qquad (5.3)$$

leads to the <u>sequential generalized least squares estimator</u> with

$$\hat{b}_{t|t} = \hat{b}_{t-1|t-1} + K_t(y_t - X_t \hat{b}_{t-1|t-1}) \ , \qquad (5.4a)$$

$$S_{t|t-1} = S_{t-1|t-1} \ , \qquad (5.4b)$$

and the remaining recursive relationships identical with (5.2).

We derive the sequential Bayesian estimator for a defined prior distribution on the parameter space for b_t with mean $\bar{b}_t$ and covariance matrix S_t^v after putting the state model into the form

$$b_t = \bar{b}_t + v_t \; , \tag{5.5}$$

as

$$\hat{b}_{t|t} = \bar{b}_t + K_t(y_t - X_t\bar{b}_t) \; , \tag{5.6a}$$

$$S_{t|t-1} = S_t^v \; , \tag{5.6b}$$

and the remaining recursive relationships again identical with (5.2).

The computations become extremely simple if we update the parameter estimates immediately after a new sample observation becomes available. Then the inversion in the equation for the filter gain (5.2c) reduces to a division by a constant and only a few matrix multiplications yield the updated estimate. Thus sequential estimators seem to be an extremely useful tool to keep the parameter estimates of an econometric model up to date and to improve therefore the model forecasting performance.

6. Summary

In this paper we suggested on the basis of the Kalman filter theory systematic procedures to incorporate prior information in the forecasting process. According to our proposals a forecasting exercise involves the following three steps.

Step 1. Whenever a new observation becomes available a correction for the measurement error is made by utilizing the information contained in the latest available forecast for this variable. The corrected value is used as model input for lagged variables and eventually for updating parameter estimates.

Step 2. Parameter estimates are updated advantageously by using sequential estimators. The time series of the parameter estimates indicate the validity of the parameter specification.

Step 3. All available prior information about the endogenous model variables is cast into an anticipations model which together with the structural model yields the optimal forecast.

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