IHS Economics Series
Working Paper 15
September 1995

# On the role of seasonal intercepts in seasonal cointegration

Philip Hans Franses
Robert M. Kunst

**Impressum**

**Author(s):**
Philip Hans Franses, Robert M. Kunst

**Title:**
On the role of seasonal intercepts in seasonal cointegration

**ISSN: Unspecified**

**1995 Institut für Höhere Studien - Institute for Advanced Studies (IHS)**
Josefstädter Straße 39, A-1080 Wien
E-Mail: office@ihs.ac.at
Web: www.ihs.ac.at

Institut für Höhere Studien (IHS), Wien
Institute for Advanced Studies, Vienna

Reihe Ökonomie / Economics Series No. 15

# ON THE ROLE OF SEASONAL INTERCEPTS IN SEASONAL COINTEGRATION

Philip Hans Franses, Robert M. Kunst

# On the role of seasonal intercepts in seasonal cointegration

**Philip Hans Franses, Robert M. Kunst**

Reihe Ökonomie / Economics Series No. 15

**September 1995**

Philip Hans Franses
Econometric Institute
Erasmus University Rotterdam
P.O.Box 1738
NL-3000 DR Rotterdam
Fax: +31/10/4527746
Phone: +31/10/4081273
e-mail: franses@ect.few.eur.nl

Robert M. Kunst
Institute for Advanced Studies
Stumpergasse 56, A-1060 Vienna
Fax: +43/1/599 91-163
Phone: +43/1/599 91-255
e-mail: kunst@ihssv.wsr.ac.at

**Institut für Höhere Studien (IHS), Wien**
**Institute for Advanced Studies, Vienna**

The Institute for Advanced Studies in Vienna is an independent center of postgraduate training and research in the social sciences. The **Economics Series** presents research done at the Economics Department of the Institute for Advanced Studies. Department members, guests, visitors, and other researchers are invited to contribute and to submit manuscripts to the editors. All papers are subjected to an internal refereeing process.

# Abstract

In the paper we consider the role of seasonal intercepts in seasonal cointegration analysis. For the nonseasonal unit root, such intercepts can generate a stochastic trend with a drift common to all observations. For the seasonal unit roots, however, we show that unrestricted seasonal intercepts generate trends that are different across the seasons. Since such seasonal trends may not appear in economic data, we propose a modified empirical method to test for seasonal cointegration. This method is illustrated on German consumption and income data.

# Zusammenfassung

Im Papier erwägen wir die Rolle saisonaler Regressionskonstanten in der Analyse saisonaler Kointegration. Für die nichtsaisonale Einheitswurzel können solche Konstanten stochastische Trends mit gemeinsamer Drift generieren. Für saisonale Einheitswurzeln zeigen wir jedoch, dass unrestringierte Saisonkonstanten Trends erzeugen, die über die Jahreszeiten unterschiedlich sind. Da solche wohl in ökonomischen Daten nicht auftreten, schlagen wir eine veränderte empirische Methode vor, um auf saisonale Kointegration zu testen. Diese Methode wird anhand deutscher Konsum- und Investitionsdaten dargebracht.



## Keywords
Deterministic and stochastic seasonality; Seasonal cointegration; Cointegration.

## JEL Classifications
C32

**Acknowledgments**
This work was initiated when the second author visited the Tinbergen Institute Rotterdam (February 1995). Its hospitality is gratefully acknowledged. The first author thanks the Royal Netherlands Academy of Arts and Sciences for its financial support.

## 1. Introduction

Many quarterly economic time series display trending and seasonal patterns which do not appear to be constant over time. A representation of time series that accounts for time-varying trends and seasonals assumes the presence of stochastic trends at the zero and seasonal frequencies. Formulated otherwise, for several economic time series one can assume the presence of nonseasonal and seasonal unit roots. Hylleberg et *al.* (1990) propose formal test statistics to investigate these roots in univariate time series. Given a set of economic time series, it is of interest to study whether these have stochastic trends at certain frequencies in common. Hence, a usual next step in analyzing a set of quarterly time series involves testing for cointegration at the nonseasonal and seasonal frequencies. Engle *et al.* (1993) propose so-called residual-based tests for seasonal cointegration, while Lee (1992) proposes tests for similar purposes based on a fully specified multivariate time series model.

Inference on cointegration and on common stochastic trends can be shown to depend critically on the presumed empirical model and on the deterministic regressors included in the auxiliary test regressions. See Johansen (1994) for a detailed treatment of the role of the constant and linear terms in analyzing cointegration at the nonseasonal frequency. This role is important since under the null and alternative hypotheses the constant and linear variables may have different implications. For example, an unrestricted constant in a model with imposed cointegration among two variables implies that the driving stochastic trend contains a drift. In the present paper, we consider the role of four seasonal intercepts in the seasonal cointegration model for quarterly data. Although the seasonal cointegration model with seasonal dummies is analyzed in Lee and Siklos (1992), the role of the four intercepts is not discussed. We show that the inclusion of unrestricted seasonal intercept parameters can lead to an undesirable feature of the data, and hence, that one may obtain inappropriate empirical results. This feature is that in case of cointegration at a seasonal frequency, the data are assumed to have deterministic trends that vary with the season. To overcome this, we propose a simple modification of the standard seasonal cointegration analysis.

The outline of this paper is as follows. In Section 2, we start with a discussion of the representation of seasonal cointegration with the inclusion of seasonal intercepts. We examine the role of these intercepts and summarize the main results in a proposition. In the second part of Section 2, we propose an alternative empirical strategy to test for seasonal cointegration. We also provide new tables with critical values for the relevant cases. In a sense, our strategy simply amounts to a partial cointegration analysis per frequency, where the intercept (with a specific form for each frequency) is restricted under the null hypothesis of cointegration. In Section 3, we illustrate our approach for quarterly consumption and income data for Germany, and we compare our results with those obtained using the Lee (1992) method. In Section 4, we conclude our paper with some remarks.

## 2. Seasonal cointegration and seasonal intercepts

In this section we discuss the representation of a seasonal cointegration model. The key results on the impact of unrestricted and restricted seasonal intercept terms are summarized in a proposition. Our results clearly indicate a useful empirical modeling strategy for seasonal cointegration. In Section 2.2, we propose this strategy and we provide the relevant tables with critical values.

### *2.1 Representation*

A general representation of an autoregressive process for an $n\times 1$ vector time series $X_t$ ($t$=1,...,$N$), which allows for cointegration at seasonal and nonseasonal frequencies is:

$$\Delta_4 X_t = \alpha_1\beta_1' S(B)X_{t-1} + \alpha_2\beta_2' A(B)X_{t-1} + \alpha_3\beta_3'\Delta_2 X_{t-2} + \sum_{i=1}^{4} d_i\delta^i_{t-4[(t-1)/4]} + \varepsilon_t, \qquad (1)$$

where $\varepsilon_t$ is an $n\times 1$ vector white noise process. The $\delta^i_{t-4[(t-1)/4]}$ in (1) concern the conventional seasonal dummy variables. The Kronecker symbol expresses the structure of the deterministic seasonal dummies that can be equated to $t$ mod 4. [.] is used to denote the largest integer or entier function. The differencing filter $\Delta_k$, $k$=1,2,... is defined by $\Delta_k \equiv (1-B^k)$, where $B$ is the usual backward shift operator defined by $B^k X_t = X_{t-k}$, $k$=1,2,... Model (1) is more special than the most general form that should allow for non-synchronous seasonality at $\pi/2$ and it has the form used by Lee (1992). For ease of exposition, a possible short-run autoregressive influence has been excluded that would allow for VARs of any finite order $p$. An unrestricted vector autoregression for $X_t$ that corresponds to (1) is of order 4. The matrices $\alpha_i$ and $\beta_i$ are assumed to have full column ranks $r_i$ with $0 \le r_i < n$. The operators $S(B)$ and $A(B)$ are defined as follows:

$$S(B) = 1+B+B^2+B^3 \qquad\qquad A(B) = 1-B+B^2-B^3$$

$S(B)$ can be interpreted as the seasonal moving average smoothing operator and $A(B)$ as the alternating signs summing operator, hence "$S$" and "$A$".

The matrix $\alpha_1\beta_1'$ corresponds to nonseasonal cointegration at the zero frequency. The matrix $\alpha_2\beta_2'$ concerns seasonal cointegration at the bi-annual frequency, whereas the matrix $\alpha_3\beta_3'$ relates to seasonal cointegration at the annual frequency. See Engle *et al.* (1993) and Lee (1992) for additional discussions of model (1).

Given (1) and fixed starting values for the $X_t$ vector process, $X_t$ has a representation in starting values, innovations $\varepsilon_s$, and deterministic contributions $D_s$ for $s \le t$. Formally, this representation is achieved by inverting the seasonal operator $\Delta_4$ in (1). For the now classical case of zero-frequency cointegration, the mathematical derivation of such a representation is summarized in the Granger Representation Theorem (cf. Engle and Granger, 1987). In the present case of a seasonal cointegration model as in (1), the influence of deterministics is however more involved, and hence a representation theorem for seasonal cointegration contains complex structures. To highlight this phenomenon, we decompose the deterministic part $D_t$ of (1) into three components, i.e., $D_t = \mu + a_t + r_t$. Averaging over the seasonal cycle yields the time-constant drift $\mu = (d_1+d_2+d_3+d_4)/4$. The sum of the remaining components is then 0 over the four-quarter cycle. Similarly, $a_t$ is found by applying the alternating operator $A(B)$ to the sequence of seasonal constants. This results in $a_t = a \cos \pi(t-1)$ with $a = (d_1-d_2+d_3-d_4)/4$. The remainder $r_t$ has a distinct pattern of alternating constants of type $(b,c,-b,-c,b,c,\ldots)$, with $b = (d_1-d_3)/2$ and $c = (d_2-d_4)/2$. Then,

$$r_t = b\cos\frac{\pi}{2}(t-1) + c\cos\frac{\pi}{2}(t-2).$$

From the definition of $D_t = \mu + a_t + r_t$, where $\mu$, $a_t$, and $r_t$ are defined above, it is clear that formal application of the inverse operator $\Delta_4^{-1}$ has different effects on the three deterministic components. For example, $\Delta_4^{-1}\mu$ yields four parallel linear time trends that perpetuate the original seasonal starting pattern. Such a pattern does not seem unusual for seasonal and trending series observed in practice. In striking contrast, $a_t$ and $r_t$ generate divergent linear trends. In the case of $a_t$ and, e.g., $a>0$, parallel positive trends appear for even $t$ and parallel negative mirror images for odd $t$. In the case of $r_t$, the patterns look even more strikingly counterintuitive. However, in the multivariate model such divergent trends do not appear necessarily. Hence, the key problem addressed in this paper is how one should accommodate for and possibly restrict the deterministic part of seasonality such that the aforementioned implausible features are avoided.

For illustrative purposes, we start with a simple VAR(1) model with cointegration at the zero frequency. This model reads as

$$\Delta_1 X_t = \alpha\beta' X_{t-1} + \mu + \varepsilon_t \tag{2}$$

From integrating (2), it is clear that $X_t$ depends on a linear trend through $\alpha^{\perp}{}'\mu$ only, where $\alpha^{\perp}$ denotes the orthogonal complement to $\alpha$. Hence, even if $\mu$ is not 0 but $\alpha^{\perp}{}'\mu=0$, there is no linear trend in the multivariate process $X_t$. For a similar property in the seasonal cointegration model, let us first re-write (1) with decomposed deterministics, i.e., with $D_t=\mu+a_t+r_t$, as:

$$\Delta_4 X_t = \alpha_1\beta_1' S(B) X_{t-1} + \alpha_2\beta_2' A(B) X_{t-1} + \alpha_3\beta_3'\Delta_2 X_{t-2} + \mu + \\ + a\cos\pi(t-1) + (b,c)\left(\cos\frac{\pi}{2}(t-1), \cos\frac{\pi}{2}(t-2)\right)' + \varepsilon_t, \tag{3}$$

where $(b, c)$ is an $n\times 2$ matrix expressing the influence of the two annual dummies, each one of type (1,0,-1,0,1,0,-1,...) with one of them lagged one quarter. Obviously, (3) has exactly the same number of parameters as (1), since $(\mu,a,b,c)$ replaces $(d_1,\ldots,d_4)$. It is now possible to show that the unwanted and implausible divergent seasonal trends appear in the representation of $X_t$ through $\alpha_2^{\perp}{}'a$ and $\alpha_3^{\perp}{}'(b,c)$ only. We state this result in the following proposition.

*Proposition. If a vector autoregression is given by (1) or equivalently (3) and $X_t$ has four fixed consecutive starting values, then, in general, its deterministic part consists of an n-dimensional linear time trend proportional to $\alpha_1^{\perp}{}'\mu$, linear time trends diverging over the seasonal cycle proportional to $\alpha_2^{\perp}{}'a$ and $\alpha_3^{\perp}{}'(b,c)$, and a periodic pattern of constants. If $\alpha_2^{\perp}{}'a=0$ and $\alpha_3^{\perp}{}'(b,c)=0$, the deterministic part only contains a linear trend and four seasonal constants.*

*Proof of the proposition.* We use a result developed by Tsay and Tiao (1990) who build on previous work by Chan and Wei (1988). We re-write the VAR system in state-space form:

$$\begin{bmatrix} X_t \\ X_{t-1} \\ X_{t-2} \\ X_{t-3} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Gamma}_1 & \boldsymbol{\Gamma}_2 & \boldsymbol{\Gamma}_3 & \boldsymbol{\Gamma}_4 \\ \mathbf{I} & 0 & 0 & 0 \\ 0 & \mathbf{I} & 0 & 0 \\ 0 & 0 & \mathbf{I} & 0 \end{bmatrix} \begin{bmatrix} X_{t-1} \\ X_{t-2} \\ X_{t-3} \\ X_{t-4} \end{bmatrix} + \begin{bmatrix} \varepsilon_t \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{or} \quad X_t^* = \boldsymbol{\Gamma} X_{t-1}^* + \varepsilon_t^*, \tag{4}$$

where $\Gamma_j$, $j$=1,2,3,4 are matrices directly depending on $\alpha_i$ and $\beta_i$ of (1). With deterministic terms in (1), one must add additional terms to the right of (4), again with block zeros except in the first $n$-block as in $\varepsilon_t^*$. Tsay and Tiao (1990) use the Jordan canonical form of the state transition matrix $\mathbf{D}=\mathbf{T}^{-1}\boldsymbol{\Gamma}\mathbf{T}$ to rotate the system into $\mathbf{T}(X_t',X_{t-1}',X_{t-2}',X_{t-3}')'=\mathbf{T}X_t^*=Y_t$, where $\mathbf{T}$ denotes a transition matrix. The vector process $Y_t$ is a $4n$-dimensional process, naturally ordered according to the eigenvalues of the original transition matrix. One could, e.g., consider $Y_t$ with an $(n-r_1)$-dimensional subvector corresponding to the eigenvalue of +1, continue with an $(n-r_2)$-dimensional subvector corresponding to the eigenvalue of -1, which is the unit root that concerns the bi-annual frequency and a pair of $(n-r_3)$-dimensional vectors corresponding to $\pm i$, which are the complex unit roots that concern the annual frequency. The real and imaginary parts of these latter vectors can also be interpreted in the real numbers as real eigenvectors to the eigenvalue of -1 in the squared transition matrix. The remaining eigenvalues are less than 1 in modulus. The matrix $\mathbf{T}$ contains the eigenvectors, i.e., $\mathbf{T} = (\mathbf{T}_1,...,\mathbf{T}_4)'$ with the components $\mathbf{T}_i$, $i$=1,...,4 corresponding to +1,-1, $\pm i$, and the remainder. Hence, the first $(n-r_1)$-component of $Y_t$ is a random walk of the form

$$W_t = W_{t-1} + \mathbf{T}_1' \begin{pmatrix} \varepsilon_t \\ 0 \\ 0 \\ 0 \end{pmatrix} + \mathbf{T}_1' \begin{pmatrix} \mu, a_t, r_t \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{5}$$

This representation shows that it is only the first $n$-part of the eigenvectors comprising $\mathbf{T}_1$ that has any further influence on the deterministics in the rightmost term. We denote this first part as $\mathbf{T}_{11}$. If a starting value for $W_1$ is given, then (5) can be inverted. The generated random walk is superimposed with a time trend of the form $\mathbf{T}_{11}'\mu t$. The seasonal variables $a_t$ and $r_t$ generate an additional cycle of constants. The whole system can be transformed into the original $X_t$ by the inverse transformation matrix $\mathbf{T}^{-1}$. The contribution of the subsystem (5) is then a stochastic random walk component, a linear trend proportional to $\mathbf{T}_{11}'\mu$, and a basic repetitive pattern of constants. Hence, (5) yields a plausible impact of intercepts.

The second component has dimension $n$-$r_2$ and looks like

$$\tilde{W}_t = -\tilde{W}_{t-1} + \mathbf{T}_2' \begin{pmatrix} \varepsilon_t \\ 0 \\ 0 \\ 0 \end{pmatrix} + \mathbf{T}_2' \begin{pmatrix} \mu, a_t, r_t \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{6}$$

This is an ($n$-$r_2$)-dimensional random jump process. We again denote the first section of $\mathrm{T}_2$ by $\mathrm{T}_{21}$. Inverting the representation (6) using one starting value leads to five parts. Firstly, the purely stochastic alternating sum of white noise. Secondly, an alternating influence of the form ($\mathbf{T}_{21}'\mu$,0,$\mathbf{T}_{21}'\mu$,0,$\mathbf{T}_{21}'\mu$,0,...). Thirdly, two diverging trends at odd and even indices $t$ corresponding to $\mathrm{T}_{21}'a_t$, i.e. different trends for different seasons. Fourthly, a cyclical pattern of constants corresponding to the $r_t$ influence. Fifthly, an alternating additional term deriving from the starting conditions. The diverging seasonal trends in the third influences deserve further attention, since it is this effect that may not be present in empirical data. That influence, however, is strictly rooted in $\mathrm{T}_{21}'a$. Re-transforming with $\mathrm{T}^{-1}$, it can be shown that there is a stochastic seasonal cycle of periodicity $\pi$ in the original process depending on $\mathbf{T}_{21}$ and an additional deterministic feature proportional to $\mathrm{T}_{21}'a$.

In order to continue with our proof of the proposition, we need the first sections of the eigenvectors of the transition matrix $\mathbf{\Gamma}$ in (4) with respect to the unit eigenvalues. To this aim, we directly express $\mathbf{\Gamma}$ as

$$\mathbf{\Gamma} = \begin{bmatrix} \alpha_1\beta_1'+\alpha_2\beta_2' & \alpha_1\beta_1'-\alpha_2\beta_2'+\alpha_3\beta_3' & \alpha_1\beta_1'+\alpha_2\beta_2' & \alpha_1\beta_1'-\alpha_2\beta_2'-\alpha_3\beta_3'+\mathbf{I} \\ \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix}.$$

An eigenvector for the eigenvalue of +1 is then defined by the property

$$\tilde{x}'\mathbf{\Gamma} = \tilde{x}'$$

Partitioning $\tilde{x}' = [\tilde{x}_1' \quad \tilde{x}_2' \quad \tilde{x}_3' \quad \tilde{x}_4']$, we obtain the equation system

$$\begin{aligned} &\tilde{x}_1'(\alpha_1\beta_1'+\alpha_2\beta_2') + \tilde{x}_2' = \tilde{x}_1' \\ &\tilde{x}_1'(\alpha_1\beta_1'-\alpha_2\beta_2'+\alpha_3\beta_3') + \tilde{x}_3' = \tilde{x}_2' \\ &\tilde{x}_1'(\alpha_1\beta_1'+\alpha_2\beta_2') + \tilde{x}_4' = \tilde{x}_3' \\ &\tilde{x}_1'(\alpha_1\beta_1'-\alpha_2\beta_2'-\alpha_3\beta_3'+\mathbf{I}) = \tilde{x}_4' \end{aligned}$$

which, summed up, yields directly

$$4\tilde{x}_1'\alpha_1\beta_1' = 0 \quad \Rightarrow \quad \tilde{x}_1 \propto \alpha_1^{\perp}.$$

A similar technique for the seasonal root of -1 confirms the conjecture that the first part of the corresponding eigenvector is proportional to $\alpha_2^{\perp}$. It follows directly that the other part of the $n\times n$-dimensional space, $\alpha_2$, does not influence the possibly undesirable feature of expanding trends, as it disappears after the transformation into the Jordan coordinates.

For the complex root pair of $\pm i$, we have the basic condition $\tilde{x}'\mathbf{\Gamma} = i\tilde{x}'$, i.e.,

$$\begin{aligned} &\tilde{x}_1'(\alpha_1\beta_1'+\alpha_2\beta_2') + \tilde{x}_2' = i\tilde{x}_1' \\ &\tilde{x}_1'(\alpha_1\beta_1'-\alpha_2\beta_2'+\alpha_3\beta_3') + \tilde{x}_3' = i\tilde{x}_2' \\ &\tilde{x}_1'(\alpha_1\beta_1'+\alpha_2\beta_2') + \tilde{x}_4' = i\tilde{x}_3' \\ &\tilde{x}_1'(\alpha_1\beta_1'-\alpha_2\beta_2'-\alpha_3\beta_3'+\mathbf{I}) = i\tilde{x}_4' \end{aligned}$$

Subtracting the third from the first equation and the last from the second yields

$$\left.\begin{aligned} \tilde{x}_2'-\tilde{x}_4' &= i(\tilde{x}_1'-\tilde{x}_3') \\ 2\tilde{x}_1'\alpha_3\beta_3'+\tilde{x}_3'-\tilde{x}_1' &= i(\tilde{x}_2'-\tilde{x}_4') \end{aligned}\right\} \Rightarrow \tilde{x}_1'\alpha_3\beta_3' = 0$$

and hence the proposed condition also holds for the complex pair. Note that this proof gets slightly more involved if the general seasonal model instead of specification (1) is considered.

Due to arguments entirely analogous to the previous cases, only $\alpha_3^{\perp}{}' r_t$ is able to generate the possibly implausible feature of seasonally expanding trends. Hence, if $\alpha_3^{\perp}{}'(b,c)=0$, model (3) is free of that undesirable feature. This completes the proof of our proposition. □

Some additional remarks can be made. The first is that one should note that the remaining $3n$ components of the eigenvectors - which are not used in the proof - are not trivial. For example, for the root +1, we obtain

$$\tilde{x}'=\left[\alpha_1^{\perp}{}' \quad \alpha_1^{\perp}{}'(\mathbf{I}-\alpha_2\beta_2') \quad \alpha_1^{\perp}{}'(\mathbf{I}-\alpha_3\beta_3') \quad \alpha_1^{\perp}{}'(\mathbf{I}-\alpha_2\beta_2'-\alpha_3\beta_3')\right]$$

This means that, if one wants to extend Granger's definition of a common trend in zero-frequency cointegrated systems to this seasonal case, $\tilde{x}'\left(X_t', X_{t-1}', X_{t-2}', X_{t-3}'\right)'$ may be preferable to simple $\alpha_1^{\perp}{}' X_t$, as the former is a multivariate random walk while the latter is not. The second remark is that our proposition can easily be generalized to higher-order systems of the form

$$\Delta_4 X_t = \alpha_1\beta_1' S(B) X_{t-1} + \alpha_2\beta_2' A(B) X_{t-1} + \alpha_3\beta_3' \Delta_2 X_{t-2} + \sum_{i=1}^{p} \varphi_i \Delta_4 X_{t-i} +$$
$$+ \sum_{i=1}^{4} d_i \delta^i_{t-4[(t-1)/4]} + \varepsilon_t ,$$

which extends (1) by the inclusion of $p$ lags of $\Delta_4 X_t$ variables, by a straightforward extension of the proof.

### *2.2 An alternative empirical method*

In practice one may want to test for seasonal and nonseasonal cointegration in a model framework which does not allow for diverging seasonal trends in the data. For that purpose, we use our proposition to re-write (3) in a simple form that permits exclusion of such diverging trends, i.e.,

$$\Delta_4 X_t = \alpha_1\beta_1' S(B) X_{t-1} + \alpha_2\left(\beta_2' A(B) X_{t-1} + a^* \cos\pi(t-1)\right) +$$
$$+ \alpha_3\left\{\beta_3' \Delta_2 X_{t-2} + (b^*, c^*)\left(\cos\tfrac{\pi}{2}(t-1), \cos\tfrac{\pi}{2}(t-2)\right)\right\} + \mu + \varepsilon_t \qquad (7)$$

for $t$=1,2,...,$N$. Note that in most empirically relevant cases a linear time trend generated by $\mu$ is perfectly admissible, as most economic time series are trending. Further note the change in dimensionality between $a$, $b$, $c$ in (3) and $a^*$, $b^*$, $c^*$ in (7). In (7), the row dimensions of the vectors are only $r_2$ and $r_3$ whereas in (3) they have row dimension $n$. Maximum-likelihood estimation of (7) is straightforward and it amounts to a reduced-rank system regression of $\Delta_4 X_t$, in analogy to the traditional estimation of frequency-zero cointegrated models. Johansen (1994) points out that, in such cases, the rank of e.g. $\alpha_2\beta_2'$ is determined by the canonical correlations between $\Delta_4 X_t$ and $(A(B)X_{t-1}, \cos \pi(t-1))$, conditional on remaining influences. Hence, the right-hand side set of variables is to be extended by the deterministic influences. For the nonseasonal case this deterministic term is 1, in our case of seasonal cointegration at the bi-annual frequency it is $\cos \pi(t-1)$. Lee (1992) outlines that the three terms $S(B)X_t$, $A(B)X_t$, and $\Delta_2 X_t$ are asymptotically independent, hence tests on the various ranks can be conducted by conditioning on the complete set of remaining variables. In the absence of rank restrictions at other frequencies, conditioning can be conducted efficiently by auxiliary least squares regressions preliminary to the canonical analysis. A certain loss in efficiency may occur due to the fact that the rank at different frequencies may not be full but also restricted. Lee (1992) argues that this loss of efficiency is negligibly small.

Because of the independence of the three terms $S(B)X_t$, $A(B)X_t$, and $\Delta_2 X_t$, the asymptotic distribution of the likelihood-ratio test statistic for testing hypotheses on the rank of the matrices does not depend on the remaining frequencies. As a consequence, tests for the rank of $\alpha_2\beta_2'$ can be based on

$$\int_0^1 (dB)F'\left[\int_0^1 FF'\right]^{-1}\int_0^1 F(dB)',$$

where $F$ denotes the extended $(n-r_2)$-dimensional limit process of a process of the type

$$X_t = -X_{t-1}+\varepsilon_t .$$

The limit process of this $X_t$ is, however, standard Brownian motion again. Replacing every other $\varepsilon_t$ by $-\varepsilon_t$, this is obvious from symmetry arguments. Using the same symmetry argument, we can show that extending such a process by the alternating $a_t$ variable is tantamount to extending the usual random walk by 1. Hence, for the frequency $\pi$, the standard table T.III of

Johansen (1989) can be used. For the frequency $\pi/2$, however, we need a slightly different table.

Based on 10,000 Monte Carlo replications, significance points for the trace test statistic

$$\xi_i = -N \sum_{j=n-i+1}^{n} \log(1-\rho_j)$$

for the $i$ smallest squared canonical correlations at the respective frequencies are given in Table 1a and 1b for the cases $i$=1 and $i$=2. The distribution depends on $i$ only. If there are only two variables, i.e., $n$=2, the situation is as follows. The statistic $\xi_1$ tests for the null hypothesis of integration at the respective frequency against the alternative of no integration at that frequency, while $\xi_2$ tests the null hypothesis of no cointegration against the alternative of cointegration *or* no integration. If, in the second case, the maintained hypothesis is integration at that frequency, a variant called the $\lambda_{max}$ test should be used. We tabulate significance points for this $\lambda_{max}$ test statistic in Table 1c.

The critical values in Tables 1a-c lead to a few remarks. The first is that spuriously augmenting lags (i.e., the cases where $p$=1) tends to decrease critical values slightly at $\omega=\pi$ and $\omega=\pi/2$ but not at $\omega=0$. The second remark is that all cases (drift or no drift, spurious augmenting lags present or absent) produce the same asymptotic distribution for $\omega=\pi$ and $\omega=\pi/2$. The third remark is that the asymptotic distribution for $\omega=0$ depends on the presence or absence of a drift only. Lag augmentation does not seem to have much effect. Fourthly, the differences between $N$=100 and $N$=200 are not very pronounced. Smaller $N$ are probably uninteresting because of the low power of the tests, larger $N$ are unlikely to occur in current econometric practice. Finally, for $\omega=0$ and $\omega=\pi$, correspondence to existing tables of simulated significance points (see Johansen, 1989, and Osterwald-Lenum, 1992) is close. Exception is the no-drift design for $\omega=0$, where the statistics are considerably larger even for $N$=200. Correspondence to published tables is a good indicator of the strength of finite-sample cross effects between frequencies. Our results show that such cross effects play little role, confirming the conjectures reported by Lee (1992) and Lee and Siklos (1992).

## 3. An application

As an empirical example, we use the German real disposable income and private consumer expenditure series contained in Lütkepohl (1991, Appendix E) that are measured for 1960.1-1987.4.

### *3.1 Results of standard method*

An application of the standard seasonal cointegration tests in the spirit of Lee (1992) and Lee and Siklos (1992) results in the evidence displayed in Table 2. The number of lags $p$ of $\Delta_4 X_t$ ranks between 0 and 4. These values correspond to VARs for $X_t$ of orders 4 to 8. The optimum lag order is unclear and we abstain from selecting such a lag. For each of the four different specifications of added deterministics, we give the test statistic for the two eigenvalues ($\xi_2$) and the one for the smallest eigenvalue ($\xi_1$). If $\xi_2$ leads to statistical rejection, we have evidence of cointegration. If additionally $\xi_1$ is significant, the corresponding unit root at the relevant frequency is absent. If none of the two tests rejects, there is no cointegration and there are two independent sources of nonstationarity in the two time series.

From the results in Table 2, we observe that there is no evidence on frequency-zero cointegration, except for one poorly specified case (constant, $p$=0). With respect to the other frequencies, evidence hinges critically on the deterministic specification and on $p$. We find cointegration at frequency $\pi$ only if seasonal dummies are used. This may be explained as follows. Without inserting seasonal dummies, the procedure finds two independent sources of seasonal cycles and only after inserting dummies, one of the two sources is identified as identical to the deterministic sine wave. The effect is even stronger at $\pi/2$. Here, after introducing seasonal dummies, a unit root is rejected for low $p$, hence both previously detected sources of seasonal variation are then identified as being deterministic manifestations. Although this latter result is not backed strongly in all specifications, we tend to rely on it as it is confirmed by (unreported) univariate testing of unit roots for the single series, which suggests that unit roots are found at frequencies 0 and $\pi$ but not at $\pi/2$. In sum, the standard method to test for seasonal cointegration yields evidence in favor of cointegration at the frequency $\pi$.

### *3.2 Results of restricted method based on (7)*

When we apply the maximum likelihood estimator outlined in Section 2.2 for model (7), we obtain trace statistic values, which are summarized in Table 3. Comparing the results with the simulated fractiles in Tables 1a and 1b shows that the overall evidence on seasonal cointegration at frequency $\pi$ weakens and is now restricted to the auxiliary regressions with $p$=0 and $p$=1. In other words, the standard method tends to enhance the constancy or co-constancy of seasonal patterns whereas the restricted method finds more evidence of changing seasonal patterns as it does not permit fixed seasonal effects that would generate divergent trends.

It should be mentioned, though, that it is yet unknown whether decisions based on our restricted method are more accurate than those based on the standard method of Lee (1992). In large samples, imposing a valid restriction certainly increases test power but in smaller samples the answer to this question may depend on a variety of details in the overall specification. Our proposed restricted procedure can be valuable in case one wants to abstain from inadmissible long-run behavior from the outset. Notice that this implies that considering linear trends as additional deterministic regressors is not attractive. A slight extension of our proposition shows that the coefficient vector of such trends would have to be orthogonal to $\alpha_1^{\perp}, \alpha_2^{\perp}, \alpha_3^{\perp}$ simultaneously in order to avoid the appearance of quadratic trends or seasonally diverging trends in the resulting model.

## 4. Concluding remarks

In this paper we have shown that unrestricted seasonal intercepts in a seasonal cointegration model can lead to diverging seasonal trends. Since such seasonal trends may be implausible in practical occasions, we proposed an alternative empirical method to investigate seasonal cointegration where we impose restrictions on the seasonal intercept parameters. We tabulated critical values for the various test statistics in the case of two variables. A comparison of the standard method and our proposed restricted method to quarterly consumption and income data for Germany resulted in a dramatic weakening of the evidence for seasonal cointegration when we restrict the seasonal dummy parameters.

The analysis in the present paper can be extended in at least two ways. Firstly, it should be investigated using an extensive simulation study whether our restricted method consistently outperforms the standard method in the case of, e.g., no diverging trends or no deterministics at all. In other words, it seems useful to study the relative loss of using the standard unrestricted method in a variety of occasions. Secondly, and very much related to the first topic, it seems interesting to re-evaluate earlier empirical studies of seasonal cointegration to investigate the robustness of the findings to the restriction of seasonal dummy parameters. Indeed, the finding of more or less similar results across the two methods may yields additional confidence in the reported outcomes.

**References**

Chan, N.H. and C.Z. Wei, 1988, Limiting distributions of least squares estimates of unstable autoregressive processes, Annals of Statistics 16, 367-401.

Engle, R.F. and C.W.J. Granger, 1987, Co-integration and error correction: Representation, estimation and testing, Econometrica 55, 251-276.

Engle, R.F., C.W.J. Granger, S. Hylleberg, and H.S. Lee, 1993, Seasonal cointegration: The Japanese consumption function, Journal of Econometrics 55, 275-298.

Hylleberg, S., R.F. Engle, C.W.J. Granger, and B.S. Yoo, 1990, Seasonal integration and cointegration, Journal of Econometrics 44, 215-238.

Johansen, S., 1989, Likelihood Based Inference on Cointegration, Theory and Applications (CIDE Lecture Notes, Bagni di Lucca, Italy).

Johansen, S., 1994, The role of the constant and linear terms in cointegration analysis of nonstationary variables, Econometric Reviews 13, 205-229.

Lee, H.S., 1992, Maximum likelihood inference on cointegration and seasonal cointegration, Journal of Econometrics 54, 1-47.

Lee, H.S., and P. Siklos, 1992, Seasonality in time series: Money-income causality in U.S. data revisited, Unpublished manuscript. (Tulane University, New Orleans).

Lütkepohl, H., 1991, Introduction to multiple time series analysis (Springer Verlag, Berlin).

Osterwald-Lenum, M., 1992, A note with quantiles of the asymptotic distribution of the maximum likelihood cointegration rank statistics: Four cases, Oxford Bulletin of Economics and Statistics 54, 461-472.

Tsay, R.S. and G.C. Tiao, 1990, Asymptotic properties of multivariate nonstationary processes with applications to autoregressions, Annals of Statistics 18, 220-250.

Table 1a

Simulated significance points (based on 10,000 Monte Carlo replications) for trace test statistics $\xi_1$ under the restriction of no diverging seasonal trends, i.e., model (7) in text. The data generating process is $\Delta_4 X_t = \mu + \varepsilon_t$, where $\varepsilon_t$ is univariate Gaussian with $\sigma^2=1$.

| | | | $\omega=0$ | | | $\omega=\pi$ | | | $\omega=\pi/2$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $N$ | ρ | μ | .90 | .95 | .99 | .90 | .95 | .99 | .90 | .95 | .99 |
| 100 | 0 | 0 | 6.5 | 8.1 | 11.3 | 7.4 | 8.9 | 12.4 | 10.8 | 12.8 | 16.5 |
| 100 | 1 | 0 | 6.6 | 8.3 | 11.9 | 7.2 | 8.6 | 11.9 | 11.1 | 12.9 | 17.5 |
| 100 | 0 | 1 | 2.8 | 3.8 | 7.1 | 7.3 | 8.8 | 12.1 | 10.8 | 12.6 | 16.4 |
| 100 | 1 | 1 | 2.9 | 4.1 | 7.3 | 7.2 | 8.7 | 12.1 | 11.0 | 12.9 | 17.3 |
| 200 | 0 | 0 | 6.7 | 8.2 | 11.5 | 7.4 | 9.0 | 12.4 | 11.0 | 12.9 | 17.0 |
| 200 | 1 | 0 | 6.7 | 8.4 | 11.8 | 7.3 | 8.8 | 12.3 | 11.2 | 13.0 | 17.0 |
| 200 | 0 | 1 | 2.7 | 3.9 | 6.4 | 7.4 | 9.0 | 12.4 | 11.0 | 12.8 | 17.0 |
| 200 | 1 | 1 | 2.8 | 4.0 | 6.7 | 7.4 | 8.9 | 12.4 | 11.1 | 12.9 | 17.1 |

Notes: The $\xi_1$ test statistic considers the hypothesis of integration at frequency $\omega$ against the alternative of no integration at that frequency. $N$ is the number of observations, $p$ denotes whether the auxiliary regression (7) includes an additional lag of $\Delta_4 X_t$ (1) or not (0), and $\mu$ is the constant term in the DGP.

Table 1b

Simulated significance points (based on 10,000 Monte Carlo replications) for trace test statistics $\xi_2$ under the restriction of no diverging seasonal trends, i.e., model (7) in text. The data generating process is $\Delta_4 X_t = \mu(1,1)' + \varepsilon_t$, where $\varepsilon_t$ is bivariate Gaussian with $\Sigma = I$.

| | | | ω=0 | | | ω=π | | | ω=π/2 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| *N* | ρ | μ | .90 | .95 | .99 | .90 | .95 | .99 | .90 | .95 | .99 |
| 100 | 0 | 0 | 16.0 | 18.2 | 23.1 | 17.9 | 20.1 | 24.8 | 23.9 | 26.5 | 31.9 |
| 100 | 1 | 0 | 16.4 | 18.8 | 23.8 | 17.0 | 19.2 | 23.4 | 23.3 | 25.8 | 31.0 |
| 100 | 0 | 1 | 13.6 | 15.5 | 20.1 | 17.8 | 20.1 | 24.8 | 23.9 | 26.4 | 31.7 |
| 100 | 1 | 1 | 13.9 | 16.3 | 21.1 | 17.0 | 19.2 | 23.7 | 23.3 | 25.8 | 30.8 |
| 200 | 0 | 0 | 16.0 | 18.3 | 22.8 | 17.9 | 20.1 | 24.9 | 24.0 | 26.3 | 31.8 |
| 200 | 1 | 0 | 16.1 | 18.4 | 23.2 | 17.2 | 19.4 | 23.6 | 23.7 | 25.9 | 30.9 |
| 200 | 0 | 1 | 13.3 | 15.5 | 19.8 | 17.9 | 20.1 | 24.8 | 23.9 | 26.3 | 31.8 |
| 200 | 1 | 1 | 13.7 | 15.7 | 20.3 | 17.2 | 19.3 | 23.8 | 23.6 | 25.9 | 31.0 |

Notes: The $\xi_2$ test statistic considers the hypothesis of no cointegration at frequency ω against the alternative of cointegration or no integration at that frequency. $N$ is the number of observations, ρ denotes whether the auxiliary regression (7) includes an additional lag of $\Delta_4 X_t$ (1) or not (0), and μ is the constant term in the DGP.

Table 1c

Simulated significance points (based on 10,000 Monte Carlo replications) for eigenvalue test statistic $\lambda_{max}$ under the restriction of no diverging seasonal trends, i.e., model (7) in text. The data generating process is $\Delta_4 X_t = \mu(1,1)'+\varepsilon_t$, where $\varepsilon_t$ is bivariate Gaussian with $\Sigma = I$.

| | | | $\omega=0$ | | | $\omega=\pi$ | | | $\omega=\pi/2$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| N | ρ | μ | .90 | .95 | .99 | .90 | .95 | .99 | .90 | .95 | .99 |
| 100 | 0 | 0 | 13.3 | 15.1 | 18.8 | 13.7 | 15.7 | 19.5 | 17.3 | 19.4 | 23.8 |
| 100 | 1 | 0 | 13.4 | 15.5 | 20.9 | 13.1 | 15.1 | 18.8 | 16.9 | 18.9 | 23.4 |
| 100 | 0 | 1 | 12.4 | 14.3 | 18.6 | 13.8 | 15.7 | 19.6 | 17.3 | 19.3 | 23.8 |
| 100 | 1 | 1 | 12.6 | 14.8 | 19.3 | 13.2 | 15.1 | 18.7 | 16.8 | 18.9 | 23.4 |
| 200 | 0 | 0 | 13.1 | 15.0 | 19.5 | 13.8 | 15.8 | 20.0 | 17.4 | 19.5 | 23.9 |
| 200 | 1 | 0 | 13.2 | 15.2 | 19.7 | 13.3 | 15.2 | 18.9 | 17.1 | 19.1 | 23.3 |
| 200 | 0 | 1 | 12.1 | 14.1 | 18.3 | 13.7 | 15.8 | 20.0 | 17.4 | 19.5 | 23.9 |
| 200 | 1 | 1 | 12.3 | 14.3 | 18.9 | 13.3 | 15.2 | 18.9 | 17.1 | 19.2 | 23.2 |

Notes: The $\lambda_{max}$ test statistic considers the hypothesis of integration at frequency $\omega$. $N$ is the number of observations, $\rho$ denotes whether the auxiliary regression (7) includes an additional lag of $\Delta_4 X_t$ (1) or not (0), and $\mu$ is the constant term in the DGP.

Table 2

Results of standard seasonal cointegration tests for German consumption and income.

| | constant | | seasonals | | trend | | seasonals & trend | |
|---|---|---|---|---|---|---|---|---|
| $p$ | $\xi_1$ | $\xi_2$ | $\xi_1$ | $\xi_2$ | $\xi_1$ | $\xi_2$ | $\xi_1$ | $\xi_2$ |
| Frequency 0 | | | | | | | | |
| 0 | 4.71 | 20.45* | 3.29 | 9.18 | 0.93 | 6.87 | 1.47 | 6.88 |
| 1 | 1.49 | 5.52 | 1.60 | 6.38 | 1.73 | 9.10 | 1.94 | 11.97 |
| 2 | 1.07 | 5.83 | 0.97 | 5.50 | 3.00 | 12.40 | 3.27 | 13.03 |
| 3 | 1.28 | 8.03 | 1.47 | 8.36 | 3.48 | 16.29 | 2.57 | 13.39 |
| 4 | 2.10 | 10.86 | 0.94 | 9.71 | 3.11 | 10.02 | 3.99 | 11.17 |
| Frequency $\pi$ | | | | | | | | |
| 0 | 1.47 | 6.88 | 6.47 | 62.51** | 1.09 | 10.28 | 6.56 | 62.78** |
| 1 | 1.94 | 11.97 | 4.65 | 27.67** | 0.340 | 6.58 | 4.44 | 27.66** |
| 2 | 3.27 | 13.03* | 5.01 | 38.23** | 0.27 | 7.71 | 5.09 | 39.11** |
| 3 | 2.57 | 13.39* | 2.60 | 41.78** | 0.42 | 4.98 | 2.48 | 40.87** |
| 4 | 3.99 | 11.17 | 2.95 | 34.55** | 0.41 | 6.00 | 2.97 | 35.09** |
| Frequency $\pi/2$ | | | | | | | | |
| 0 | 1.68 | 4.84 | 22.19** | 50.58** | 1.66 | 4.93 | 22.31** | 51.22** |
| 1 | 0.14 | 4.15 | 13.19* | 35.14** | 0.14 | 4.40 | 13.11* | 35.96** |
| 2 | 0.32 | 3.38 | 7.77 | 24.55* | 0.36 | 3.48 | 7.84 | 23.77 |
| 3 | 0.31 | 3.68 | 5.63 | 22.03 | 0.37 | 3.54 | 5.72 | 21.17 |
| 4 | 0.00 | 3.20 | 4.56 | 14.68 | 0.00 | 3.20 | 4.45 | 15.09 |

Notes: The auxiliary regression is (1) with the inclusion of $p$ lags of $\Delta_4 X_t$. The regression can contain a constant (constant), four seasonal dummy variables (seasonals), a constant and a trend (trend), or four seasonal dummy variables and a trend (seasonals & trend). ** denotes significance at the 5% level, * denotes significance at the 10% level.

Table 3

Results of seasonal cointegration tests restricted at seasonals based on regression (7).

| | $\omega=0$ | | $\omega=\pi$ | | $\omega=\pi/2$ | |
|---|---|---|---|---|---|---|
| $p$ | $\xi_1$ | $\xi_2$ | $\xi_1$ | $\xi_2$ | $\xi_1$ | $\xi_2$ |
| 0 | 3.29* | 9.18 | 6.74 | 57.88** | 22.80** | 50.08** |
| 1 | 1.60 | 6.37 | 5.26 | 21.30** | 12.72 | 34.00** |
| 2 | 0.97 | 5.50 | 5.38 | 18.48* | 7.09 | 23.45 |
| 3 | 1.47 | 8.36 | 2.95 | 11.25 | 4.74 | 20.29 |
| 4 | 0.94 | 9.71 | 2.92 | 8.94 | 3.90 | 13.83 |

Notes: The auxiliary regression is (7) and contains $p$ lags of $\Delta_4 X_t$. ** denotes significance at the 5% level, * denotes significance at the 10% level.

**Institut für Höhere Studien**
**Institute for Advanced Studies**
Stumpergasse 56
A-1060 Vienna
Austria

Phone: +43-1-599 91-145
Fax: +43-1-599 91-163