THE THEORY OF NORMAL FORM GAMES FROM THE DIFFERENTIABLE VIEWPOINT

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Zusammenfassung

Zuerst wird eine alternative Definition regulärer Nash-Gleichgewichte eingeführt und gezeigt, daβ diese Definition dieselben Eigenschaften besitzt wie die bereits aus der Literatur bekannten Definitionen. Das Gleichungssystem, das zur Definition regulärer Gleichgewichte herangezogen wird, induziert dann eine global differenzierbare Struktur auf dem Raum gemischter Strategien. Wenn man diese Struktur als Vektorfeld auffaβt, kann man eine Reihe klassischer Resultate aus differenzierbarer Sicht beweisen. Darüberhinaus kann man Approximationen an dieses Vektorfeld benutzen, um Indizes für zusammenhängende Komponenten von Nash-Gleichgewichten zu definieren. Mit Hilfe dieser Indizes lassen sich Gleichgewichtskomponenten identifizieren die robust sind, in dem Sinne daβ ähnliche Spiele Gleichgewichte nahe der Komponente haben.

The Theory of Normal Form Games from the Differentiable Viewpoint

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Abstract. An alternative definition of regular equilibria is introduced and shown to have the same properties as those definitions already known from the literature. The system of equations used to define regular equilibria induces a globally differentiable structure on the space of mixed strategies. Interpreting this structure as a vector field, called the Nash field, allows for a reproduction of a number of classical results from a differentiable viewpoint. Moreover, approximations of the Nash field can be used to suitably define indices of connected components of equilibria and to identify equilibrium components which are robust against small payoff perturbations.

1. INTRODUCTION

From one point of view the material to be presented is simply a somewhat deviant approach to the theory of finite normal form games. From another it is a discussion of Nash refinements and robustness properties of Nash equilibria. Nash refinements were often motivated by initiating examples in which common sense could identify what went wrong with particular equilibria. This is *not* the approach taken here (that is: examples come at the end rather than at the beginning). Rather this paper is concerned with formal properties of a particular structure on the space of mixed strategies which will be viewed as a representation of the interaction in a normal form game. The discussion of various implied refinement notions is a byproduct. Still it is a very informative byproduct and may help to gain a more unified understanding of Nash refinements.

The "intuitive" approach to refinements which attempts to rule out undesirable properties discovered in examples, of course, has its merits. It is particularily transparent, because it often derives from extensive form considerations. As a consequence the most popular - at least

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within the economics profession - refinement concepts are in fact extensive form concepts, e.g. subgame perfection [**Selten**, 1965, 1975] or sequential equilibrium [**Kreps and Wilson**, 1982]. This is despite the view expressed by **von Neumann and Morgenstern** [1972, e.g. p.85] that the normal form and the extensive form are essentially equivalent and despite the fact that the normal form is mathematically somewhat more handy. It is even despite the strongly supported view, expressed by **Kohlberg and Mertens** [1986, p.1010], that the set of "strategically stable" equilibria should depend only on the reduced normal form of the game.

A preference for extensive form analysis may derive from the feeling that normal form analysis seemed to be unable to capture the essence of backward induction. With respect to this problem it is known, however, that a proper equilibrium [**Myerson**, 1978] of the normal form is sequential in any tree with that normal form [**van Damme**, 1984; **Kohlberg and Mertens**, 1986, p.1009]. Last, but not least, recent results obtained by **Mailath, Samuelson, and Swinkels** [1990] show that the normal form can even reproduce extensive-form-type reasoning and a notion like "subgame perfect in any extensive form with this particular normal form" can be given precise meaning. This is one of the reasons, why it seems worthwile to attempt a mathematical description of the structure of interaction within the complete *normal form* game as additional information to the predictions of the Nash equilibrium.

The reader may wonder what is meant by the vague phrase "structure of interaction in the whole game" and whether there is not already such a thing, induced by the best-reply correspondences, such that there remains nothing to be studied. In fact, the correspondence, manufactured from individual best-reply correspondences by taking the product, induces a structure on the space of mixed strategies which may well be viewed as a representation of interactions in the complete game [a programme carried through by **Kalai and Samet**, 1984, and **Balkenborg**, 1991, and extended to perturbations of the best-reply structure by **Hillas**, 1990]. But best-reply correspondences drop some information: Since they assign the maximizers, they drop the information on the ordering of the remaining pure strategies. Moreover, the structure induced by best-reply correspondences is not a very smooth one. Would it not be nicer to have a structure which contains even more information than the one induced by best-reply correspondences but is, moreover, differentiable? The latter is what the present paper is devoted to.

The present paper will start with a somewhat "naive" consideration of necessary conditions for a Nash equilibrium which will result in a system of equations that can be used to give an alternative definition of *regular* equilibria. This modified definition has the same properties as those already known for the standard definitions. But beyond these the system of equations induces a globally differentiable structure on the space of mixed strategy combinations. One way to think about this structure is to interpret it as a vector field and call it the "Nash field". Each equilibrium of a given game then corresponds to a zero of this vector field. Machinery from Differential Topology can now be used to study the Nash field. Well known results can be reproved in a different way. But, moreover, by approximations of the Nash field one can identify connected components of equilibria which are robust against slight payoff disturbances. Since the latter exercise only requires knowledge of the Nash field, but does *not* require the computation of payoff perturbations, it is an instance of what a differential approach is designed to do: To extract global information from local properties. In effect the tool to identify such robust equilibrium components ("essential components") is even of a somewhat independent interest. It consists of an assignement of indices to components such that the indices add up to the Euler characteristic of the space of mixed strategies across components. An equilibrium component with non-zero index can be shown to satisfy a number of desirable properties. On the other hand there are Stable Sets [**Kohlberg and Mertens**, 1986] which do *not* get assigned a non-zero index (see Section 4).

The appeal of the Nash field, however, also derives from its potential interpretation. The system of equations defining the Nash field *is* precisely the replicator dynamics for asymmetric games [as introduced for symmetric games by **Taylor and Jonker**, 1978]. Thus it does reflect the interaction of players in the whole normal form game, though possibly the interaction of large populations of "boundedly rational" players who revise their strategy choices in an evolutionary way. On the other hand the inertia of strategy choices by evolution has its advantages: It preserves the information on the ordering of all strategies, rather than concentrating on the best response (as the best-reply correspondences would do).

This suggests that there is an intimate relationship between self-enforcement properties of Nash equilibria and stability properties of the evolution described by the replicator dynamics, viz. the Nash field. In fact in a companion paper [**Ritzberger and Weibull**, forthcoming] it is shown that the range against which a given equilibrium is self-enforcing is indeed reflected in the qualitative behavior of the evolutionary dynamics around the equilibrium.

For the context of human players the evolutionary interpretation of the Nash field, of course, would have to specify some kind of learning process [cf. **Canning**, 1987, 1989; **Eichberger, Haller, and Milne**, 1990; **Kandori, Mailath, and Rob**, 1991]. This is, however, not the purpose of the present inquiry. Still it is worth to conjecture that for many simple learning models stability properties of the induced dynamics, as far as they are not governed by stochastic elements, will most likely be guided by the properties of the Nash field.

For the present purposes, however, the Nash field can be thought of as a purely formal object - one way to represent a game. The paper is organized as follows: Section 2 introduces notation and the modified definition of regular equilibria plus their properties. Section 3 studies the Nash field, reproves the well known result on existence of equilibria and oddness in the case of regularity and shows how to define indices of equilibrium components and how to exploit these indices. Section 4 is an illustrative chapter in that it shows how the Nash field can quickly inform the analyst on which equilibrium is risk dominant in the sense of **Harsanyi and Selten** [1988]. This section also contains examples illustrating applications of index theory. Section 5 summarizes.

2. Regular Equilibria

2.1 Notation. A *finite n-person normal form game* is a $2n$-tuple $\Gamma = (S_1, \dots, S_n, u_1, \dots, u_n)$, where S_i is a finite non-empty set, referred to as the set of *pure strategies* of player $i \in \mathcal{N} = \{1, \dots, n\}$. Denoting $S = \prod_{i \in \mathcal{N}} S_i$, the set of all pure strategy combinations, each u_i is a mapping $u_i : S \rightarrow \Re$, for each $i \in \mathcal{N}$, and is called player i's *payoff function*. A typical element of the space of pure strategies will be written $s = (s_1, \dots, s_n) \in S$. The cardinality of player i's pure strategy set S_i is denoted by $K_i = |S_i|$ and the cardinality of S is denoted by $K = \prod_{i \in \mathcal{N}} K_i$. For most of what follows it will be convenient to index player i's pure strategies by $k \in \{1, \dots, K_i\}$, such that $s_i^k \in S_i$ denotes the k'th pure strategy of player $i \in \mathcal{N}$. The set of *mixed strategies* of player $i \in \mathcal{N}$ is the set of probability distributions on S_i. The probability which player i assigns to his k-th pure strategy s_i^k will be denoted by $\sigma_i^k = \sigma(s_i^k)$ and the space of all mixed strategies of player $i \in \mathcal{N}$ will be denoted by Σ_i. Since by deciding upon $(K_i - 1)$ probabilities assigned to his pure strategies, player i already has decided on the probability of the remaining pure strategy (because the probabilities have to add up to unity), the space Σ_i can taken to be $(K_i - 1)$-dimensional, i.e.

$$\Sigma_i = \{\sigma_i : S_i \rightarrow \Re_+ \mid \sum_{k=1}^{K_i - 1} \sigma_i^k \leq 1\}$$

The set of mixed strategy combinations, Σ, is the product $\Sigma = \prod_{i \in \mathcal{N}} \Sigma_i$. A *completely mixed strategy* for player $i \in \mathcal{N}$ is a probability vector $\sigma_i \in$

$int\, \Sigma_i = \{\sigma_i : S_i \to \Re_{++} | \sum_{k=1}^{K_i-1} \sigma_i^k < 1\}$ and a completely mixed strategy combination is a mixed strategy combination $\sigma = (\sigma_1, \dots, \sigma_n) \in int\, \Sigma = \prod_{i \in \mathcal{N}} int\, \Sigma_i$. The strategy combination resulting from $\sigma \in \Sigma$, when σ_i is replaced by $\hat{\sigma}_i$, is denoted by $(\sigma_{-i}, \hat{\sigma}_i) = (\sigma_1, \dots, \sigma_{i-1}, \hat{\sigma}_i, \sigma_{i+1}, \dots, \sigma_n) \in \Sigma$. A pure strategy of player $i \in \mathcal{N}$ is identified with the degenerate probability distribution which assigns 1 to the pure strategy selected and zero to all other pure strategies. When in $\sigma \in \Sigma$ player i's strategy $\sigma_i \in \Sigma_i$ is replaced by such a degenerate distribution assigning all the weight to pure strategy $s_i \in S_i$, a shorthand notation frequently used will be $(\sigma_{-i}, s_i) \in \Sigma \setminus int\, \Sigma = \partial\Sigma$. For a given $\sigma_i \in \Sigma_i$ the subset of pure strategies to which σ_i assigns positive probability is called the support of σ_i, denoted

$$supp(\sigma_i) = \{s_i^k \in S_i \,|\, \sigma_i^k = \sigma_i(s_i^k) > 0\}.$$

Analogously, $supp(\sigma) = \prod_{i \in \mathcal{N}} supp(\sigma_i)$.

The space of all mixed strategy combinations Σ is a compact and convex polyhedron in $\Re^M$, with dimension $M = \sum_{i \in \mathcal{N}} K_i - n$. Since players in a non-cooperative game decide independently on their mixed strategies, the joint probability that the pure strategy combination $s = (s_1^{k_1}, \dots, s_n^{k_n}) \in S,\ k_i \in \{1, \dots, K_i\}, \forall i \in \mathcal{N}$, will be played, given that player $i \in \mathcal{N}$ chooses $\sigma_i \in \Sigma_i$, is given by

$$\sigma(s) = \sigma(s_1^{k_1}, \dots, s_n^{k_n}) = \prod_{i \in \mathcal{N}} \sigma_i^{k_i}.$$

The *expected payoff* to player $i \in \mathcal{N}$, given that $\sigma \in \Sigma$ is played, is

$$U_i(\sigma) = \sum_{s \in S} u_i(s)\sigma(s),$$

i.e. is a multilinear function $U_i : \Sigma \to \Re$.

Since S_i is a finite set for each $i \in \mathcal{N}$, the payoff functions $u_i : S \to \Re$ can only take finitely many values. Collecting these values $u_i(s), s \in S$, in a K-dimensional vector for each $i \in \mathcal{N}$ and collecting these vectors $u_i = (u_i(s))_{s \in S}$ in a nK-dimensional vector $u = (u_i)_{i \in \mathcal{N}}$ makes it possible to identify Γ, for fixed player set and fixed pure strategy sets, with a point $u \in \Re^{nK}$. Writing $G(S_1, \dots, S_n)$ for the set of all normal form games with pure strategy sets $(S_1, \dots, S_n)$, there is, consequently, a one-to-one correspondence between $\Re^{nK}$ and $G(S_1, \dots, S_n)$. The notation for a game $\Gamma \in G(S_1, \dots, S_n)$ with *payoff vector* $u = (u_i)_{i \in \mathcal{N}} = ((u_i(s))_{s \in S})_{i \in \mathcal{N}} \in \Re^{nK}$ will frequently read $\Gamma = \Gamma(u)$. Within $G(S_1, \dots, S_n)$ there is, therefore, a natural way to measure distances between games by measuring the euclidean distance between their payoff vectors in $\Re^{nK}$. Accordingly, the measure of a subset of games in $G(S_1, \dots, S_n)$ is determined by the Lebesgue-measure of the corresponding subset of payoff vectors in $\Re^{nK}$. If it is necessary to stress the dependence of some mapping b on the payoff vector $u \in \Re^{nK}$, subscripts u will be used, i.e. b will be written as b_u.

For a fixed game $\Gamma \in G(S_1, \dots, S_n)$ define the set of *best replies* of player $i \in \mathcal{N}$ against a stategy combination $\sigma \in \Sigma$ as the correspondence $BR_i : \Sigma \to \Sigma_i$ defined by

$$BR_i(\sigma) = \arg \max_{\hat{\sigma}_i \in \Sigma_i} U_i(\sigma_{-i}, \hat{\sigma}_i).$$

Where this is necessary, the set of pure best replies of player $i \in \mathcal{N}$ against $\sigma \in \Sigma$ will be written as $\overline{BR}_i(\sigma)$. Let the correspondece $BR = \prod_{i \in \mathcal{N}} BR_i$.

A *Nash equilibrium* (or, an equilibrium) of a game $\Gamma \in G(S_1, \dots, S_n)$ is a strategy combination $\sigma \in \Sigma$ such that $\sigma \in BR(\sigma)$. Such an equilibrium always exists for any game $\Gamma \in G(S_1, \dots, S_n)$ [**Nash**, 1951]. The set of equilibria of a game Γ will be denoted by $\mathrm{E}(\Gamma)$. The set $\mathrm{E}(\Gamma)$ can also be viewed as a correspondence mapping $G(S_1, \dots, S_n)$ into Σ.

A *strict* equilibrium is a $\sigma \in \mathrm{E}(\Gamma)$ which satisfies $\overline{BR}(\sigma) = \{\sigma\}$ [**Harsanyi**, 1973]. A *quasi-strict* equilibrium is a $\sigma \in \mathrm{E}(\Gamma)$ which satisfies $\overline{BR}(\sigma) = supp(\sigma)$ [**Harsanyi**, 1973; the terminology is from **van Damme**, 1987].

2.2 Regular Equilibria. In this subsection a modified definition of *regular* equilibrium is introduced and it is shown that a regular equilibrium possesses all robustness properties one can reasonably hope for. In particular, the present definition of regularity has the same implication as the definition introduced in **van Damme** [1987, chp. 2.5]. Moreover, in the example by which van Damme motivates his own deviation from Harsanyi's original definition [**van Damme**, 1987, Fig.2.5.1, p.39], the present definition selects the same equilibrium as van Damme's definition and does not rule out both equilibria, as **Harsanyi's** [1973] definition would do.[1]

At an equilibrium $\sigma \in \mathrm{E}(\Gamma)$ each $\sigma_i \in \Sigma_i$ must maximize the expected payoff $U_i(\sigma)$ for each $i \in \mathcal{N}$ subject to the constraint that $\sigma_i \in \Sigma_i$. Since $U_i(\sigma)$ is linear in $\sigma_i \in \Sigma_i$ this boils down to a problem of constrained,

[1]In fact I believe that the present definition of regular equilibria is equivalent to van Damme's, but I have not yet succeeded in proving this claim.

non-negative linear programming:

$$\max_{(\sigma_i^1,\dots,\sigma_i^{K_i-1})} [\sum_{k=1}^{K_i-1} \sigma_i^k U_i(\sigma_{-i},\, s_i^k) + (1-\sum_{k=1}^{K_i-1} \sigma_i^k) U_i(\sigma_{-i},\, s_i^{K_i})]$$

$$s.t. \sum_{k=1}^{K_i-1} \sigma_i^k - 1 \leq 0, \quad \sigma_i^k \geq 0, \quad \forall k = 1,\dots,K_i-1.$$

By linearity the Kuhn-Tucker complementary-slackness conditions are necessary and sufficient, such that the optimum $\bar{\sigma}_i \in \Sigma_i$ can - after eliminating the Lagrange multiplier - be characterized by

$$(1) \qquad \begin{array}{c} U_i(\sigma_{-i},\, s_i^k) \leq U_i(\sigma_{-i}, \bar{\sigma}_i),\ \forall s_i^k \in S_i, \\ \bar{\sigma}_i^k[U_i(\sigma_{-i},\, s_i^k) - U_i(\sigma_{-i}, \bar{\sigma}_i)] = 0,\ \forall k = 1,\dots,K_i-1. \end{array}$$

At an equilibrium, however, $\bar{\sigma}_i = \sigma_i$ must hold, such that the second part of (1) reads

$$(2) \qquad \sigma_i^k[U_i(\sigma_{-i},\, s_i^k) - U_i(\sigma)] = 0,\ \forall k = 1,\dots,K_i-1,$$

for all $i \in \mathcal{N}$. This already is the modification to be introduced: While van Damme's definition uses pure strategy combinations jointly with the mix of the other players $U_i(\sigma_{-i},\, s_i)$, $s_i \in supp(\sigma_i)$, and Harsanyi's definition uses an equal mix across all pure strategies (the so-called centroid-strategy) $U_i(\sigma_{-i},\, (1/K_i, \dots,\, 1/K_i))$, as the reference point, instead of $U_i(\sigma)$ in (2), the present definition of regularity uses for each player
(a) a mixed strategy space reduced by one dimension and
(b) the equilibrium itself as the reference point.

Formally, let the function $b : \Sigma \to \Re^M$ be defined by

$$(3) \qquad b_i^k(\sigma) = \sigma_i^k[U_i(\sigma_{-i},\, s_i^k) - U_i(\sigma)],\ \forall k = 1,\dots,K_i-1,\ \forall i \in \mathcal{N}.$$

The way it is defined, b is a polynomial function and, therefore, infinitely often continuously differentiable on a neighbourhood of $\Sigma \subset \Re^M$. (The definition by van Damme, by comparison, has to switch to another pure strategy combination as the reference point, when two equilibria with disjoint supports are studied. This induces "kinks" of the corresponding mapping at indifference surfaces off the equilibrium.) Let $D_\sigma b(\bar{\sigma})$ denote the Jacobian matrix of the mapping b at a point $\bar{\sigma} \in \mathrm{E}(\Gamma)$ and denote by $|D_\sigma b(\bar{\sigma})|$ its determinant [the definition to follow was first introduced by **Ritzberger and Vogelsberger**, 1990].

Definition: An equilibrium $\bar{\sigma} \in \mathrm{E}(\Gamma)$ is said to be *regular*, if and only if $|D_\sigma b(\bar{\sigma})| \neq 0$.

The steps to follow are intended to evaluate the properties of regular equilibria. Since these turn out to be the same as those of standard definitions, most proofs are gathered in the Appendix and the text only contains the statements [for the parallel results see: **van Damme**, 1987, chp.2.5.].

LEMMA 1. *If $\bar{s}_i \notin supp(\sigma_i)$ and $b(\sigma) = 0$, then $[U_i(\sigma_{-i}, \bar{s}_i) - U_i(\sigma)] \in \Re$ is an eigenvalue of the Jacobian matrix $D_\sigma b(\sigma)$.*

(PROOF: see Appendix)

COROLLARY 1. *Every regular equilibrium is quasi-strict.*

PROOF: If $\sigma \in \mathrm{E}(\Gamma)$ is not quasi-strict, then for some $i \in \mathcal{N}$ there exists $s_i^k \notin supp(\sigma_i)$ such that $U_i(\sigma_{-i}, s_i^k) - U_i(\sigma) = 0$ which implies that $|D_\sigma b(\sigma)| = 0$. ∎

COROLLARY 2. *Every strict equilibrium is regular.*

PROOF: Since every strict equilibrium is in pure strategies, for each $i \in \mathcal{N}$ there are $(K_i - 1)$ pure strategies not used at $\sigma \in \mathrm{E}(\Gamma)$ which give the $(K_i - 1)$ corresponding eigenvalues $[U_i(\sigma_{-i}, s_i^k) - U_i(\sigma)] < 0, s_i^k \notin supp(\sigma_i)$. This determines $\sum_{i \in \mathcal{N}} K_i - n = M$ real and negative eigenvalues which are all eigenvalues of $D_\sigma b(\sigma)$, such that $|D_\sigma b(\sigma)| \neq 0$. ∎

The latter result can be sharpened to a rather obvious conclusion:

COROLLARY 3. *A pure strategy equilibrium is regular, if and only if it is strict.*

PROOF: Corollary 2 covers the if part. Since at a pure strategy equilibrium all eigenvalues are known and regularity of the equilibrium implies that there is no zero eigenvalue, one must have $[U_i(\sigma_{-i}, s_i^k) - U_i(\sigma)] < 0,\ \forall s_i^k \notin supp(\sigma_i),\ \forall i \in \mathcal{N}$. ∎

The next step is to show that regular equilibria are *strongly stable* in the sense of **Kojima, Okada, and Shindoh** [1985].

THEOREM 1. *Let $\bar{\Gamma} \in G(S_1, \ldots, S_n)$ and assume that $\bar{\sigma} \in E(\bar{\Gamma})$ is a regular equilibrium. Let $\bar{u} \in \Re^{nK}$ denote the payoff vector of $\bar{\Gamma}$. Then there exists a neighbourhood $\mathcal{U}$ of $\bar{u}$ in $\Re^{nK}$ and a neigbourhood $\mathcal{V}$ of $\bar{\sigma}$ in $\Re^M$, such that*

(i) $|E(\Gamma(u)) \cap \mathcal{V}| = 1, \quad \forall u \in \mathcal{U}$, and
(ii) *the mapping $\sigma : \mathcal{U} \to \mathcal{V}$, defined by $\{\sigma(u)\} = E(\Gamma(u)) \cap \mathcal{V}$, is continuous.*

(PROOF: see Appendix)

An equilibrium $\sigma \in \mathrm{E}(\Gamma)$ is said to be *isolated*, if and only if there exists a neigbourhood $\mathcal{V}$ of σ in $\Re^M$, such that $\mathrm{E}(\Gamma) \cap \mathcal{V} = \{\sigma\}$. An equilibrium $\sigma \in \mathrm{E}(\Gamma)$ is said to be *essential*, if and only if every game Γ' in a neighbourhood of Γ (in $\Re^{nK}$) has an equilibrium $\sigma' \in E(\Gamma')$ in a neighbourhood of $\sigma \in \mathrm{E}(\Gamma)$ [**Wu Wen-Tsün and Jiang Jia-He**, 1962].

COROLLARY 4. *Every regular equilibrium is essential and isolated.*

COROLLARY 5. *Every regular equilibrium is strictly perfect [***Okada**, *1981] and proper [***Myerson**, *1978].*

PROOF: The first part follows from Theorem 2.4.3 in **van Damme** [1987, p.34], where it is proved that every essential equilibrium is strictly perfect. Theorems 2.4.7 and 2.3.8 in **van Damme** [1987, p.36 and p.32] ensure that every strongly stable equilibrium is strictly proper and every strictly proper equilibrium is proper. ∎

Except for the rather obvious Corollary 3 all these results are known from **van Damme** [1987, chp.2.5]. The only reason, these results are listed here, is to show that the present definition does not change the properties of regular equilibria. The methods of proofs use eigenvalues of the Jacobian matrix, rather than the more straightforward methods of van Damme, because this is a natural approach in the present setting.

An equilibrium of a game Γ is called *near strict* [**Fudenberg, Kreps, and Levine**, 1988, p.357] in the normal form, if there exists a sequence of games converging to Γ for which this equilibrium is a strict equilibrium.

COROLLARY 6. *Every pure strategy equilibrium is near strict in the normal form.*

PROOF: At a pure strategy equilibrium $\bar{\sigma} \in \mathrm{E}(\Gamma)$

$$[U_i(\bar{\sigma}_{-i}, s_i) - U_i(\bar{\sigma})] \leq 0, \ \forall s_i \notin supp(\bar{\sigma}_i), \ \forall i \in \mathcal{N}.$$

Since these eigenvalues of $D_\sigma b(\bar{\sigma})$ (by Lemma 1) are linear in $u \in \Re^{nK}$, the required sequence $\{\Gamma^m\}_{m=1}^{\infty}$, $\Gamma^m \to \Gamma$, can be constructed in such a way that all payoffs to a player $i \in \mathcal{N}$ to his pure strategy $s_i \notin supp(\bar{\sigma}_i)$, for which equality holds (instead of a strict inequality), are disturbed downwards. ∎

Corollary 6 is merely a restatement of Proposition 1 in **Fudenberg, Kreps, and Levine** [1988]. In the present context, however, this result emerges from the particularily transparent behavior of the vector field b around pure strategy combinations (which are always zeros of b).

Finally another well known result [**van Damme**, 1987, chp.2.6; **Harsanyi**, 1973; **Wilson**, 1971] is stated.

THEOREM 2. *For almost all games* $\Gamma \in G(S_1, \dots, S_n)$ *all equilibria are regular.*

(PROOF: see Appendix)

The phrase "almost all" in Theorem 2 refers to $G(S_1, \dots, S_n)$, i.e. to $\Re^{nK}$ (that is: to a dense open subset of $\Re^{nK}$). On the other hand, nearly any non-trivial extensive form will impose certain indifference relations upon the corresponding normal form and will, thereby, give rise to a degenerate normal form game in $G(S_1, \dots, S_n)$. In fact a given extensive form specifies a linear subspace of $G(S_1, \dots, S_n)$, as can be seen from writing the ties emerging from strategy combinations that lead to the same terminal nodes as a system of linear equations in $u \in \Re^{nK}$. There is no guarantee whatsoever that in such a subspace (corresponding to a given tree) games without regular equilibria will not cover a (relatively) open subset. On the other hand most of the refinement literature is motivated by extensive form arguments. From this point of view, therefore, the statement of Theorem 2 should be interpreted with due care.

3. THE NASH FIELD

An advantage of the system of equations defined by (3) is that it induces a globally differentiable structure on a neighbourhood of the polyhedron Σ. This structure contains, in my view, valuable information about the underlying game Γ. One particularily attractive way to view the structure induced by b on Σ is to think of it as a vector field on Σ. If it is appropriate to stress the interpretation of the mapping b defined in (3) as a vector field on Σ, the notation $\vec{b}$ will be used. The vector field $\vec{b}$ (or $\vec{b}_u$, if the dependence on the payoff vector $u \in \Re^{nK}$ is to be stressed) will be called the *Nash field* .

The reason to assign such a prominent name to $\vec{b}$ is that it indeed captures the spirit of the Nash equilibrium as a solution concept. Suppose that for some reason the players consider to play an arbitrary strategy combination $\sigma \in \Sigma$. If now one of the players $i \in \mathcal{N}$ considers to deviate from $\sigma \in \Sigma$, her coordinates of $\vec{b}$, denoted $\vec{b}_i(\sigma) = (b_i^1(\sigma), \dots, b_i^{K_i-1}(\sigma))$, tell her one direction in which she could unilaterally change her mix in order to improve her expected payoff. This can be seen from the following direct calculation:

$$U_i(\sigma_{-i}, \sigma_i + \vec{b}_i(\sigma)) = U_i(\sigma) + \sum_{k=1}^{K_i-1} \sigma_i^k [U_i(\sigma_{-i}, s_i^k)]^2 +$$

$$+ (1 - \sum_{k=1}^{K_i-1} \sigma_i^k)[U_i(\sigma_{-i}, s_i^{K_i})]^2 - [U_i(\sigma)]^2 =$$

$$= U_i(\sigma) + Var(U_i(\sigma_{-i}, s_i) \,|\, \sigma_i) \geq U_i(\sigma)\,, \ \forall i \in \mathcal{N}.$$

If no such direction $\vec{b}_i(\sigma)$ exists, along which the player under consideration can unilaterally improve, for any player, then $\sigma \in \Sigma$ must be a zero of the vector field $\vec{b}$.

Very vaguely speaking, $\vec{b}$ could be viewed as the direction in which players with only bounded rationality, who change their mix only in very small steps (i.e. in a differentiable manner), but still try to improve unilaterally (i.e. assuming that other players stay with their present mix), would take the "state" of the game. In fact the differential equation $d\sigma = \vec{b}(\sigma)\, dt$ *is* precisely the (deterministic and continuous) *replicator dynamics* for asymmetric games [as introduced for symmetric contests by **Taylor and Jonker**, 1978].

Samuelson and Zhang [1990, Theorem 3] show that any evolutionary selection dynamics which is regular and aggregate monotonic *is* the replicator dynamics up to a player-specific (positive) scale factor. This motivates why the Nash field can be viewed as a representation of interaction in the whole normal form game. But the analogy carries further. In a companion paper [**Ritzberger and Weibull**, forthcoming] the relation between the replicator dynamics, viz. the Nash field, and self-enforcement properties of equilibria is explored in more detail. It turns out that for arbitrary n-person finite normal form games
(i) pure strategies which do not survive iterated elimination of strictly dominated strategies are eliminated by the dynamics in the long run;
(ii) a rest point of the replicator dynamics is *locally asymptotically stable* [**Hirsch and Smale**, 1974, p.186], *if and only if* it is a strict equilibrium [this result was first proved by **Ritzberger and Vogelsberger**, 1990];
(iii) every *robust* equilibrium [**Okada**, 1983] is *weakly stable* [i.e. for every neighbourhood there exists another one, contained in the first, such that any trajectory starting in the second neighbourhood will forever remain in the first, cf. **Hirsch and Smale**, 1974, p.185];
(iv) a rest point of the replicator dynamics which is not a Nash equilibrium cannot even be weakly stable.
Result *(ii)* can be rephrased as saying that the property of a Nash equi-

librium to be unambiguously self-enforcing with respect to a neighbourhood (i.e. strictness) is equivalent to local asymptotic stability of the replicator dynamics. Result *(iii)* says that any Nash equilibrium which is self-enforcing with respect to a neighbourhood, though not necessarily unambiguously so, (i.e. robust) is weakly stable. These results illustrate the intimate relation between maximizing behavior and evolutionary selection and justify the interest in the Nash field.

For the present purposes, however, only very basic properties of the Nash field are relevant. To be able to interpret b as a vector field $\vec{b}$ on Σ, it is necessary to show that b indeed maps into the tangent space of Σ which, because of the simple structure of Σ, is just $\Re^M$. But beyond this trivial step a much stronger result is available: Let $\sigma(t, \sigma^0)$, $t \in \Re_+$, $\sigma^0 \in \Sigma$, be a solution to the system of differential equations $d\sigma = \vec{b}(\sigma)\, dt$, with $\sigma(0, \sigma^0) = \sigma^0$. In the following Lemma it is shown that no solution $\sigma(t, \sigma^0)$ ever leaves the boundary face of Σ in which it starts. In other words, Σ and each of its boundary faces (the boundary of Σ will be denoted $\partial\Sigma$) are invariant under the operation of $\vec{b}$.

LEMMA 2. *If $\sigma^0 \in \Sigma$, then $\sigma(t, \sigma^0) \in \Sigma$, $\forall t \in \Re_+$ and, moreover, if $\sigma^0 \in \partial\Sigma$, then $\sigma(t, \sigma^0) \in \partial\Sigma$, $\forall t \in \Re_+$. Finally, no trajectory ever leaves the boundary face in which it starts.*

PROOF: It suffices to demonstrate the second part of the statement, because, if the latter is true, by continuity no solution path can ever leave Σ, once it starts in Σ. With the understanding that $\sigma(0, \sigma^0) = \bar{\sigma} \in \partial\Sigma$ abbreviate $\sigma(t, \bar{\sigma}) = \sigma(t)$, $\forall t \in \Re_+$. Since $\vec{b}(\sigma)$ is continuously differentiable the differential equation has a unique solution $\sigma(t, \bar{\sigma}) = \sigma(t)$ which satifies the initial condition $\sigma(0) = \bar{\sigma}$ [**Hirsch and Smale**, 1974, pp.162]. By uniqueness of the solution satisfying the initial condition it is possible to write $\sigma_i^k(t)$ as

$$\begin{aligned}\sigma_i^k(t) =&\bar{\sigma}_i^k \exp\{\int_0^t U_i(\sigma_{-i}(\tau),\, s_i^k)\, d\tau\}\times \\ &\times \left\{(1 - \sum_{h=1}^{K_i-1} \bar{\sigma}_i^h) \exp\{\int_0^t U_i(\sigma_{-i}(\tau),\, s_i^{K_i})\, d\tau\} + \right. \\ &\left. + \sum_{h=1}^{K_i-1} \bar{\sigma}_i^h \exp\{\int_0^t U_i(\sigma_{-i}(\tau),\, s_i^h)\, d\tau\}\right\}^{-1}\end{aligned}$$

If now $\bar{\sigma} \in \partial\Sigma$, then there exists some $i \in \mathcal{N}$ and $s_i^k \in S_i$, such that either (i) $\bar{\sigma}_i^k = 0$, or (ii) $\bar{\sigma}_i^k = 1$. In case (i) $\bar{\sigma}_i^k = 0$ implies by the above solution $\sigma_i^k(t) = 0$, $\forall t \in \Re_+$, and in case (ii) $\bar{\sigma}_i^k = 1 \implies \bar{\sigma}_i^h = 0$, $\forall h \neq k$, implies $\sigma_i^h(t) = 0$, $\forall t \in \Re_+$, $\forall h \neq k$, such that $\sigma_i^k(t) = 1$ for all $t \in \Re_+$. ∎

REMARK. *The reason why it is possible to solve explicitly for one component $\sigma_i^k(t)$ in the preceeding proof is that in each component one obtains a Riccati-equation for which explicit solutions are known.*

Lemma 2 implies that the restriction of the Nash field to some boundary face of Σ is precisely the Nash field of the reduced game obtained by deleting all strategies which are not used in this boundary face. On the one hand this is a nice "slicing" property of the Nash field. On the other hand this property is responsible for the emergence of "artificial" zeros of the Nash field on the boundary of Σ which are not Nash equilibria. In particular every pure strategy combination will be a zero of the Nash field. As a consequence not all zeros of the Nash field are Nash equilibria. The next result shows, how easy it is to distinguish between zeros of the Nash field which are equilibria and those which are not.

LEMMA 3. *$\bar\sigma \in E(\Gamma) \iff \vec{b}(\bar\sigma) = 0$ and $U_i(\bar\sigma_{-i}, s_i) \leq U_i(\bar\sigma)$, $\forall s_i \notin supp(\bar\sigma_i)$, $\forall i \in \mathcal{N}$.*

PROOF: If $\bar\sigma \in \mathrm{E}(\Gamma)$, then clearly from (1) and (2) one must have $\vec{b}(\bar\sigma) = 0$ and $U_i(\bar\sigma_{-i}, s_i) \leq U_i(\bar\sigma)$, $\forall s_i \notin supp(\bar\sigma_i)$, $\forall i \in \mathcal{N}$. If the latter is true, then $\vec{b}(\bar\sigma) = 0$ implies $U_i(\bar\sigma_{-i}, s_i) = U_i(\bar\sigma)$, $\forall s_i \in supp(\bar\sigma_i)$, such that $U_i(\bar\sigma_{-i}, \hat{s}_i) \leq U_i(\bar\sigma)$, $\forall \hat{s}_i \notin supp(\bar\sigma_i)$, $\forall i \in \mathcal{N}$, is sufficient for $\bar\sigma \in \mathrm{E}(\Gamma)$. ∎

To check whether a zero of $\vec{b}$ is an equilibrium it, therefore, suffices to check the eigenvalues of the Jacobian matrix corresponding to unused pure strategies. Indeed, therefore, the property of a zero of the Nash field to form an equilibrium boils down to a simple eigenvalue condition on the Jacobian matrix $D_\sigma \vec{b}(\bar\sigma)$. In this sense the problem of determining Nash equilibria reduces to a problem of solving a system of equations and checking the solutions.

Lemmas 2 and 3 suggest that the analysis can be made more transparent by slightly perturbing the Nash field such that the perturbed vector field points inwards at the boundary. To do so some extra definitions are required. Let $\mathcal{B}^o$ denote the set of all smooth vector fields that point inward at the boundary of Σ,

$$\begin{aligned}\mathcal{B}^o = \{\beta\colon \Sigma \to \Re^M \mid \beta \in C^\infty,\ & \sigma_i^k = 0 \implies \beta_i^k(\sigma) > 0,\\ & \sigma_i^k = 1 \implies \beta_i^k(\sigma) < 0,\ \forall k = 1, \ldots, K_i - 1,\\ & \sum_{h=1}^{K_i-1} \sigma_i^h = 1 \implies \sum_{h=1}^{K_i-1} \beta_i^h(\sigma) < 0,\ \forall i \in \mathcal{N}\}.\end{aligned}$$

A mapping $F\colon \Sigma \times \Re_+ \to \Re^M$ is called an *interior approximation* of the Nash field $\vec{b}$, if and only if
(i) $F(\sigma, 0) = \vec{b}(\sigma)$,
(ii) $f_\lambda\colon \Sigma \to \Re^M$ defined by $f_\lambda(\sigma) = F(\sigma, \lambda)$ satisfies $f_\lambda \in \mathcal{B}^o$, for any fixed $\lambda > 0$, and
(iii) F is continuously differentiable on (a neighbourhood of) $\Sigma \times \Re_+$.
An interior approximation F is called *regular*, if there exists some $\bar{\lambda} > 0$ such that $f_\lambda(\sigma) \equiv F(\sigma, \lambda)$ has only finitely many zeros on Σ all of which are isolated points for any fixed $\lambda \in (0, \bar{\lambda})$.

Since clearly such interior approximations always exist, a new method of proof for another well known result is obtained. [**Hofbauer and Sigmund**, 1988, pp.166, use a similar but more special method of proof for similar replicator equations.] Only the last part, (iii), of Theorem 3 may be not so widely known. It has recently been arrived at, via an alternative method, by **Gul, Pearce, and Stacchetti** [1990].

THEOREM 3. *(i) Every game $\Gamma \in G(S_1, \ldots, S_n)$ has at least one Nash equilibrium. If all Nash equilibria of Γ are regular (as they are for almost all games), then (ii) their number is finite and odd and (iii), if Γ has $m \geq 1$ pure Nash equilibria, then it has at least $m - 1$ mixed Nash equilibria.*

PROOF: (i) Consider some interior approximation $F\colon \Sigma \times \Re_+ \to \Re^M$ of the Nash field $\vec{b}$. By definition $f_\lambda\colon \Sigma \to \Re^M$, defined by $f_\lambda(\sigma) = F(\sigma, \lambda)$, points inward at the boundary $\partial\Sigma$ of Σ, for all $\lambda > 0$, since $f_\lambda \in \mathcal{B}^o$. An application of Brouwer's Fixed Point Theorem to the solution curves of $d\sigma = f_\lambda(\sigma)\, dt$, $\lambda > 0$, for some fixed $t > 0$ shows that the vector field f_λ must have a zero in $int\,\Sigma$. Let such a zero of f_λ be denoted by $\sigma(\lambda) \in int\,\Sigma$. Any sequence $\{\sigma(\lambda)\}_{\lambda\downarrow 0}$ must have a cluster point in Σ, because Σ is compact.

The crucial step is to show that any such cluster point $\bar{\sigma} \in \Sigma$ must be a Nash equilibrium: Consider any $s_i^k \notin supp(\bar{\sigma}_i)$. From $f_\lambda \in \mathcal{B}^o, \forall\, \lambda > 0$, it follows that $\bar{\sigma}_i^k = 0$ implies $\partial F_i^k(\bar{\sigma}, 0)/\partial\lambda > 0$. Since F is continuously differentiable on $\Sigma \times \Re_+$, there exists a neighbourhood $\mathcal{O}$ of $(\bar{\sigma}, 0)$ in $\Re^M \times \Re$ such that $\partial F_i^k(\sigma, \lambda)/\partial\lambda > 0$, $\forall\,(\sigma, \lambda) \in \mathcal{O} \cap (\Sigma \times \Re_+)$. Since $\bar{\sigma}$ is a cluster point of the sequence $\{\sigma(\lambda)\}_{\lambda\downarrow 0}$, for every neighbourhood $\mathcal{U} \subset \mathcal{O} \cap (\Sigma \times \Re_+)$ of $(\bar{\sigma}, 0)$ there exists $\lambda > 0$ such that $(\sigma(\lambda), \lambda) \in \mathcal{U}$. By Taylor expansion at $\lambda = 0$ one has

$$0 = f_{\lambda,\,i}^k(\sigma(\lambda)) = b_i^k(\sigma(\lambda)) + \frac{\partial F_i^k(\sigma(\lambda), \theta)}{\partial\lambda}\theta, \quad \theta \in (0, \lambda),$$

for every $\lambda > 0$ sufficiently small, such that $(\sigma(\lambda), \lambda) \in \mathcal{O} \cap (\Sigma \times \Re_+)$

implies

$$U_i(\sigma_{-i}(\lambda),\, s_i^k) - U_i(\sigma(\lambda)) = -\frac{\partial F_i^k(\sigma(\lambda),\, \theta)}{\partial \lambda} \frac{\theta}{\sigma_i^k(\lambda)} < 0\,.$$

By continuity the weak inequality

$$U_i(\bar{\sigma}_{-i},\, s_i^k) - U_i(\bar{\sigma}) \leq 0$$

obtains as $\lambda \downarrow 0$ for all $s_i^k \notin supp(\bar{\sigma}_i)$. Since a cluster point $\bar{\sigma}$ must also satisfy $\vec{b}(\bar{\sigma}) = 0$, it follows from Lemma 3 that $\bar{\sigma} \in E(\Gamma)$. (If $s_i^k = s_i^{K_i} \notin supp(\bar{\sigma}_i)$, then the same argument with $\sum_{h=1}^{K_i-1} f_{\lambda,\,i}^h(\sigma(\lambda))$ instead of $f_{\lambda,\,i}^k(\sigma(\lambda))$ yields $U_i(\bar{\sigma}) - U_i(\bar{\sigma}_{-i},\, s_i^{K_i}) \geq 0$.) This completes the first part.

(ii) Define the smooth functions $\pi_\delta\colon int\,\Sigma \to \Re$, $\pi_\delta \in \mathcal{C}^\infty(int\,\Sigma)$, for any $\delta \in (0,\, \prod_{i\in\mathcal{N}} K_i^{-K_i})$, by

$$\pi_\delta(\sigma) = [\prod_{i\in\mathcal{N}}(1 - \sum_{k=1}^{K_i-1} \sigma_i^k) \prod_{k=1}^{K_i-1} \sigma_i^k] - \delta\,.$$

The gradients of these functions are given by

$$\frac{\partial \pi_\delta(\sigma)}{\partial \sigma_i^k} = \frac{(1 - \sum_{h=1}^{K_i-1} \sigma_i^h - \sigma_i^k)(\pi_\delta(\sigma) + \delta)}{\sigma_i^k(1 - \sum_{h=1}^{K_i-1} \sigma_i^h)}\,,$$

for all $k = 1, \ldots, K_i - 1$, $\forall\, i \in \mathcal{N}$. If $\forall\, i \in \mathcal{N}$, $\forall\, k = 1, \ldots, K_i - 1$, one would have $\sigma_i^k = 1 - \sum_{h=1}^{K_i-1} \sigma_i^h$, then $\sigma_i^k = 1 - \sum_h \sigma_i^h = 1/K_i$, $\forall\, k = 1, \ldots, K_i - 1$, $\forall\, i \in \mathcal{N}$, such that

$$\pi_\delta(\sigma) = \prod_{i\in\mathcal{N}} K_i^{-K_i} - \delta > 0 \implies \sigma \notin \pi_\delta^{-1}(0)\,.$$

Thus $\forall\, \sigma \in \pi_\delta^{-1}(0)$ there exists $i \in \mathcal{N}$ and some $k \in \{1, \ldots, K_i - 1\}$ such that $\sigma_i^k \neq 1 - \sum_{h=1}^{K_i-1} \sigma_i^h$. Consequently $0 \in \Re$ is a regular value of π_δ for all $\delta \in (0,\, \prod_{i\in\mathcal{N}} K_i^{-K_i})$. It follows that $\Pi_\delta = \{\sigma \in int\,\Sigma \mid \pi_\delta(\sigma) \geq 0\}$ is an M-dimensional manifold with boundary, and the boundary is $\pi_\delta^{-1}(0)$ [**Milnor**, 1965, p.12; **Guillemin and Pollack**, 1974, p.62]. Finally, it is easy to see that as $\delta \downarrow 0$ the compact manifold Π_δ converges to Σ and $\partial\Pi_\delta$ converges to $\partial\Sigma$.

Define on Π_δ, for any $\delta \in (0,\, \prod_i K_i^{-K_i})$, the vector field $\vec{p}\colon \Pi_\delta \to \Re^M$ by $p_i^k(\sigma) = \sigma_i^k\, K_i - 1$, $\forall\, k = 1, \ldots, K_i - 1$, $\forall\, i \in \mathcal{N}$. The vector field

$\vec{p}$ has only a single zero on Π_δ, namely the combination in the interior $\sigma_i^k = 1/K_i$, $k = 1, \ldots, K_i - 1$, $\forall i \in \mathcal{N}$. At the boundary, $\pi_\delta(\sigma) = 0$, the vector field $\vec{p}$ points outward, because

$$\frac{d\pi_\delta(\sigma)}{dt}\Big|_{\pi_\delta=0} = \delta \sum_{i\in\mathcal{N}} \sum_{k=1}^{K_i-1} [\frac{1}{\sigma_i^k} - \frac{1}{1-\sum_h \sigma_i^h}](\sigma_i^k K_i - 1) =$$

$$= \delta \sum_{i\in\mathcal{N}} [K_i^2 - \sum_{k=1}^{K_i-1} \frac{1}{\sigma_i^k} - \frac{1}{1-\sum_h \sigma_i^h}] < 0\,,$$

using the differential equation $d\sigma = \vec{p}(\sigma)\,dt$. According to the Poincaré-Hopf Theorem for manifolds with boundary [**Milnor**, 1965, p.35] the sum of indices at the zeros of $\vec{p}$ is equal to the Euler characteristic, $\chi(\Pi_\delta)$, of Π_δ. The index of $((1/K_i, \ldots, 1/K_i)_{k=1}^{K_i-1})_{i\in\mathcal{N}}$ is $+1$ [see **Milnor**, 1965, p.37], such that (Π_δ is effectively contractible to a point and) the Euler characteristic $\chi(\Pi_\delta)$ is $+1$, for all $\delta \in (0, \prod_{i\in\mathcal{N}} K_i^{-K_i})$.

If all equilibria are regular, then there are only finitely many, because Σ is compact. Let the equilibria be denoted by $\sigma^1, \ldots, \sigma^Q$. By the implicit function theorem each equilibrium $\sigma^q \in E(\Gamma)$, $q = 1, \ldots, Q$, is continuously approximated by a unique family $\{\sigma^q(\lambda)\}_{\lambda>0}$ of zeros of the vector fields f_λ derived from the interior approximation F. Since the determinant (of the Jacobian matrices) is a continuous function, there exists some $\bar{\lambda} > 0$ such that $|D_\sigma f_\lambda(\sigma^q(\lambda))| \neq 0$, $\forall q = 1, \ldots, Q$, $\forall \lambda \in [0, \bar{\lambda})$. It is easy to see that there must be some $\lambda_o \in (0, \bar{\lambda})$ such that $\{\sigma^1(\lambda_o), \ldots, \sigma^Q(\lambda_o)\} = f_{\lambda_o}^{-1}(0)$: Otherwise there would be a sequence $\{\hat{\sigma}(\lambda)\}_{\lambda\downarrow 0}$, $\hat{\sigma}(\lambda) \in f_\lambda^{-1}(0)$, but $\hat{\sigma}(\lambda) \notin \{\sigma^1(\lambda), \ldots, \sigma^Q(\lambda)\}$, with a cluster point $\sigma^o \in \Sigma$ which must be a Nash equilibrium (by the argument in (i)), but must satisfy $|D_\sigma \vec{b}(\sigma^o)| = 0$. But $\sigma^o \in E(\Gamma)$ and $|D_\sigma \vec{b}(\sigma^o)| = 0$ would contradict the hypothesis that all equilibria are regular.

Now choose $\delta > 0$ sufficiently small such that $f_{\lambda_o}^{-1}(0) \subset int\,\Pi_\delta$ and the vector field f_{λ_o} points inward at $\partial\Pi_\delta$. This is always possible, because Π_δ converges to Σ as $\delta \downarrow 0$ and f_{λ_o} points inward at $\partial\Sigma$. The set $f_{\lambda_o}^{-1}(0)$ must coincide with the set of zeros of the vector field $-f_{\lambda_o}$ which points outward at the boundary of Π_δ by construction. Applying the Poincaré-Hopf Theorem for manifolds with boundary [**Milnor**, 1965, p.35] to the restriction of $-f_{\lambda_o}$ to Π_δ, the indices of zeros of $-f_{\lambda_o}$ must sum to $+1 = \chi(\Pi_\delta)$. Since the indices are the signs of $|-D_\sigma f_{\lambda_o}(\sigma^q(\lambda_o))|$, $q = 1, \ldots, Q$, and therefore only take values in $\{-1, +1\}$, the number of zeros, Q, must be odd.

Since the index of a zero of $-f_\lambda$ does not change as $\lambda \downarrow 0$ by regularity, the regular equilibria inherit the indices of their continuous approximations and their number, Q, must also be odd. Briefly, the above argument shows that, if all equilibria are regular, then their indices $sign\,| - D_\sigma \vec{b}(\sigma^q)|$ must sum to $+1$, the Euler characteristic of Σ. [The above argument in fact can be viewed as a variation of Theorem 1 in **Dierker**, 1972.]

(iii) Finally observe that, if $\bar{\sigma} \in E(\Gamma)$ is a pure equilibrium, then by Lemma 1 and the fact that the determinant is the product of the eigenvalues its index must be $+1$, because $| - D_\sigma b(\bar{\sigma})| > 0$. Since the indices sum to $+1$ over $E(\Gamma)$, if Γ has $m \geq 1$ pure equilibria, then there must be at least $m - 1$ equilibria with index -1. And equilibria with index -1 must be mixed. ∎

Incidentally the method of proof for Theorem 3, which uses interior approximations of the Nash field, turns out to be of more general use. Note that one step in the proof shows that *all* cluster points (for $\lambda \downarrow 0$) of zeros of *any* interior approximation of the Nash field form Nash equilibria. Thus interior approximations eliminate the artificial zeros of $\vec{b}$ which are not Nash equilibria. In particular the consideration of *regular* interior approximations can make the behavior of the Nash field very transparent.

Recall from **Kohlberg and Mertens** [1986, Proposition 1] that the set of Nash equilibria for all games consists of finitely many connected components, at least one of which is such that all close games have an equilibrium close to the component. Using *regular* interior approximations index theory like in Theorem 3 can be extended to define indices for connected components $C \subset E(\Gamma)$.

For some regular interior approximation F of the Nash field define the family of mappings $f_\lambda\colon \Sigma \to \Re^M$ by $f_\lambda(\sigma) = F(\sigma, \lambda)$. Thus $f_0(\sigma) = \vec{b}(\sigma)$, $f_\lambda \in \mathcal{B}^o$, $\forall\, \lambda > 0$, and $\exists\, \bar{\lambda} > 0$ such that $f_\lambda^{-1}(0)$ consists of finitely many isolated points, $\forall\, \lambda \in (0, \bar{\lambda})$. For any given regular interior approximation F of the Nash field and fixed $\lambda \in (0, \bar{\lambda})$ let, for $\bar{\sigma} \in f_\lambda^{-1}(0)$, the index of $\bar{\sigma}$ be denoted by $ind(\bar{\sigma})$. Note that, if $\bar{\sigma} \in f_\lambda^{-1}(0)$ is a regular zero, then $ind(\bar{\sigma}) = sign| - D_\sigma f_\lambda(\bar{\sigma})|$.

For a connected component C, $C \subset E(\Gamma)$, denote by $\mathcal{U}_C$ a neighbourhood of C which is sufficiently small such that all the $\mathcal{U}_C$'s are pairwise disjoint across components. For each component $C \subset E(\Gamma)$ define the *index* of C *with respect to* the regular interior approximation F, denoted $I(C, F)$, by

$$I(C, F) = \lim_{\lambda \downarrow 0} \sum_{\bar{\sigma} \in f_\lambda^{-1}(0) \cap \mathcal{U}_C} ind(\bar{\sigma})\,.$$

The definition of $I(C, F)$ has two consequences. First, if all equilibria of Γ are regular, then the index of each equilibrium $\bar{\sigma} \in E(\Gamma)$

is simply given by $sign| - D_\sigma \vec{b}(\bar{\sigma})|$ and, therefore, coincides with the straightforward definition of an index, by the implicit function theorem. Second, because for each f_λ, with $\lambda > 0$ sufficiently small, there exists some $\delta > 0$ such that $f_\lambda^{-1}(0)$ is contained in the interior of $\Pi_\delta = \{\sigma \in int\Sigma \mid \pi_\delta(\sigma) \geq 0\}$ (because Π_δ converges to Σ as $\delta \downarrow 0$ and $f_\lambda \in \mathcal{B}^o$, $\forall\, \lambda > 0$, as in the proof of Theorem 3, (ii)), and f_λ points inward at $\partial\Pi_\delta$, the sum of indices $I(C,\, F)$ across components of equilibria equals $+1$, the Euler characteristic of Σ, by the Poincaré-Hopf Theorem. Note that by convention $f_\lambda^{-1}(0) \cap \mathcal{U}_C = \emptyset$, $\forall\, \lambda \in (0,\, \bar{\lambda})$ implies that $I(C,\, F) = 0$. The strength of the above definition of an index derives from the fact that the index only depends on Γ resp. $\vec{b}$, but not on the particular interior approximation chosen to calculate it.

LEMMA 4. *(i) If G is a regular interior approximations of the Nash field and $C \subset E(\Gamma)$ is a connected component of Nash equilibria, then $I(C,\, G)$ is well defined. (ii) If F is some alternative regular interior approximation, then $I(C,\, F) = I(C,\, G)$.*

PROOF: (i) Any vector field g_λ, $\lambda \in (0,\, \bar{\lambda})$, with isolated zeros can be replaced by a nondegenerate vector field $\tilde{g}_\lambda$ with $|D_\sigma \tilde{g}_\lambda(\bar{\sigma})| \neq 0$, $\forall\, \bar{\sigma} \in \tilde{g}_\lambda^{-1}(0)$, without altering the index sums within arbitrary small neighbourhoods of $\hat{\sigma} \in g_\lambda^{-1}(0)$ and leaving the vector field g_λ outside slightly larger neighbourhoods unaltered [**Milnor**, 1965, p.40]. Choosing these neighbourhoods sufficiently small (such that they are all contained in the union of the neighbourhoods $\mathcal{U}_C$ across components) the regular interior approximation G can be replaced by the regular interior approximation $\tilde{G}$ defined by $\tilde{G}(\sigma,\, \lambda) = \tilde{g}_\lambda(\sigma)$. From the implicit function theorem it follows that the limit in the definition of $I(C,\, \tilde{G})$ is well defined. By construction $I(C,\, G) = I(C,\, \tilde{G})$ for every connected component $C \subset E(\Gamma)$.

(ii) Choose now $\lambda^* > 0$ such that both $f_\lambda^{-1}(0)$ and $g_\lambda^{-1}(0)$ consist of finitely many isolated points for all $\lambda \in (0,\, \lambda^*)$. Let $\mathcal{U}_B$ denote the union of all neighbourhoods of equilibrium components other than C such that $closure(\mathcal{U}_B) \cap closure(\mathcal{U}_C) = \emptyset$. By Urysohn's Lemma there exists a smooth real-valued function ϕ on (a neighbourhood of) Σ such that $\phi(\sigma) = 0$, $\forall\, \sigma \in closure(\mathcal{U}_B \cap \Sigma)$, and $\phi(\sigma) = 1$, $\forall\, \sigma \in closure(\mathcal{U}_C \cap \Sigma)$, and $0 \leq \phi(\sigma) \leq 1$, $\forall\, \sigma \in \Sigma$. Define $H\colon \Sigma \times \Re_+ \to \Re^M$ by

$$H(\sigma,\, \lambda) = \phi(\sigma)\, f_\lambda(\sigma) + [1 - \phi(\sigma)]\, g_\lambda(\sigma)\ ,$$

and let $h_\lambda(\sigma) = H(\sigma,\, \lambda)$.

Clearly H is an interior approximation of the Nash field, because $H(\sigma,\, 0) = \vec{b}(\sigma)$, $h_\lambda \in \mathcal{B}^o$, $\forall\, \lambda > 0$, and H is continuously differentiable on $\Sigma \times \Re_+$. It remains to show that H is a regular interior approximation: If for all $\bar{\lambda} \in (0, \lambda^*)$ there exists some $\lambda \in (0, \bar{\lambda})$ such that

$$h_\lambda^{-1}(0) \cap int\Sigma \setminus [closure(\mathcal{U}_B \cap \Sigma) \cup closure(\mathcal{U}_C \cap \Sigma)] \neq \emptyset,$$

then there exists a sequence $\{(\lambda^l, \sigma^l)\}_{l=1}^{\infty}$, with

$$(\lambda^l, \sigma^l) \in (0, \lambda^*) \times [int\Sigma \setminus (closure(\mathcal{U}_B \cap \Sigma) \cup closure(\mathcal{U}_C \cap \Sigma))],$$

such that $\lambda^l \longrightarrow_{l\to\infty} 0$ and $h_{\lambda^l}(\sigma^l) = 0$, $\forall l = 1, 2, \ldots$. By compactness there exists a cluster point $\sigma^o \in \Sigma \setminus [(\mathcal{U}_B \cap \Sigma) \cup (\mathcal{U}_C \cap \Sigma)]$ which must be a zero of the Nash field, $\vec{b}(\sigma^o) = 0$. By the same argument as in the proof of Theorem 3, (i), it can be shown that such a cluster point σ^o must satisfy $\sigma^o \in E(\Gamma)$. But $E(\Gamma) \subset (\mathcal{U}_B \cup \mathcal{U}_C) \cap \Sigma$, such that a contradiction is obtained. Consequently, there exists some $\bar{\lambda} \in (0, \lambda^*)$ such that

$$h_\lambda^{-1}(0) \subset (\mathcal{U}_B \cup \mathcal{U}_C) \cap \Sigma, \quad \forall \lambda \in (0, \bar{\lambda}).$$

Since on $(\mathcal{U}_B \cup \mathcal{U}_C) \cap \Sigma$ the vector field h_λ equals either f_λ (on $\mathcal{U}_C$) or g_λ (on $\mathcal{U}_B$), it has only finitely many isolated zeros and is, therefore, regular.

Let z be the index sum over $\mathcal{U}_B \cap \Sigma$, i.e.

$$z = \lim_{\lambda \downarrow 0} \sum_{\bar{\sigma} \in h_\lambda^{-1}(0) \cap \mathcal{U}_B} ind(\bar{\sigma}) = \lim_{\lambda \downarrow 0} \sum_{\bar{\sigma} \in g_\lambda^{-1}(0) \cap \mathcal{U}_B} ind(\bar{\sigma}).$$

Since $h_\lambda \in \mathcal{B}^o$, $\forall \lambda > 0$, there exists some $\delta > 0$ such that $h_\lambda^{-1}(0) \subset int\,\Pi_\delta$. Applying the Poincaré-Hopf Theorem for manifolds with boundary to the restriction $-h_\lambda \colon \Pi_\delta \to \Re^M$ yields for all $\lambda \in (0, \bar{\lambda})$ that

$$\sum_{\bar{\sigma} \in h_\lambda^{-1}(0)} ind(\bar{\sigma}) = z + \sum_{\bar{\sigma} \in h_\lambda^{-1}(0) \cap \mathcal{U}_C} ind(\bar{\sigma}) = 1 \implies$$
$$\implies I(C, H) = 1 - z = I(C, F).$$

On the other hand

$$\sum_{\bar{\sigma} \in g_\lambda^{-1}(0)} ind(\bar{\sigma}) = z + \sum_{\bar{\sigma} \in g_\lambda^{-1}(0) \cap \mathcal{U}_C} ind(\bar{\sigma}) = 1$$

yields $I(C, G) = 1 - z = I(C, F)$. ∎

The Lemma implies that the index assigned to a connected component $C \subset E(\Gamma)$ is independent of the particular regular interior approximation chosen to calculate it. One can, therefore, drop the argument referring to the approximation and call it *the index* of the component C, denoted $Ind(C)$. The index $Ind(C)$ depends only on the particular game $\Gamma \in G(S_1, \ldots, S_n)$ under consideration and does not require the computation of payoff perturbations.

The whole point of the exercise is to show that $Ind(C)$ is an appropriate generalization of the notion of regularity in the sense that it allows to extract global properties (in the space of normal form games) of equilibrium components from purely local information. The power of regularity rests with Theorem 1. The Theorem to follow provides a set-valued analogue to the essentiality of regular equilibrium points. In analogy to the notion of an essential equilibrium point, call a connected component $C \subset E(\Gamma)$ an *essential component*, if for all $\varepsilon > 0$ there is some $\delta > 0$ such that every game Γ' which satisfies $\|\Gamma, \Gamma'\| < \delta$ has a Nash equilibrium within (Hausdorff-) distance ε from C. The "local" information summarized in $Ind(C)$ turns out to provide the "global" information on $C \subset E(\Gamma)$ required to identify essential components.

THEOREM 4. *Let $C \subset E(\Gamma)$ be a connected component of Nash equilibria. If $Ind(C) \neq 0$, then C is an essential component.*

PROOF: The claim is trivially true for $M = 1$, because then the game is a 1-person game with two pure strategies. Thus from now on assume $M > 1$. Let $B_\varepsilon(C) = \{\sigma \in \Re^M \mid \inf_{\bar\sigma \in C}\|\bar\sigma, \sigma\| < \varepsilon\}$ be an open ε-neighbourhood of the connected component $C \subset E(\Gamma)$, for any $\varepsilon > 0$, and let $B^o_\varepsilon(C) = B_\varepsilon(C) \cap \Sigma$. The payoff vector of the game Γ under consideration will be denoted $\bar u \in \Re^{nK}$, $\Gamma = \Gamma(\bar u)$.

If $C \subset E(\Gamma)$ is not essential, then there exists $\hat u \in \Re^{nK} \setminus \{\bar u\}$ such that for all $\alpha \in (0, 1]$

$$E(\Gamma(\alpha\hat u + (1-\alpha)\bar u)) \cap B_\varepsilon(C) = \emptyset\,,$$

for all $\varepsilon > 0$ sufficiently small. Denote by $R = \{u \in \Re^{nK} \mid u = \alpha\hat u + (1-\alpha)\bar u,\ \alpha \in [0, 1)\}$ the relevant space of games (more generally R can be any one-dimensional set diffeomorphic to a half-open interval). As a one-dimensional manifold with boundary R is oriented along increasing $\alpha \in [0, 1)$.

Now choose a smooth map $G\colon R\times\Sigma\times\Re_+ \to \Re^M$ such that $F_u(\sigma, \lambda) \equiv G(u, \sigma, \lambda)$ for fixed $u \in R$ is an interior approximation of the Nash field $\vec b_u$ for the game $\Gamma(u)$, i.e. $F_u(\sigma, 0) = \vec b_u(\sigma)$, $f_{u,\lambda} \equiv F_u(\,.\,, \lambda) \in \mathcal{B}^o$, $\forall\,\lambda > 0$, and F_u is continuously differentiable on $\Sigma \times \Re_+$. Define the family of mappings $G_\lambda\colon R \times int\Sigma \to \Re^M$, for all $\lambda > 0$, by $G_\lambda(u, \sigma) = G(u, \sigma, \lambda)$. For $M > 1$ the mapping G can be chosen such that there exists some $\bar{\lambda} > 0$ such that for all $\lambda \in (0, \bar{\lambda})$ the origin $0 \in \Re^M$ is a regular value both for G_λ and for $f_{\bar{u},\lambda} = G(\bar{u}, ., \lambda) = G_\lambda|\partial(R \times int\Sigma)$ by the Transversality Theorem [**Hirsch**, 1976, p.74].

Since for each $\lambda \in (0, \bar{\lambda})$ the mapping G_λ is one from an $(M+1)$-dimensional manifold with boundary to the M-dimensional euclidian space $\Re^M$, and since $0 \in \Re^M$ is a regular value for both G_λ and $f_{\bar{u},\lambda} = G_\lambda|\partial(R \times int\Sigma) = G_\lambda|\{\bar{u}\} \times int\Sigma$, the generalized preimage theorem [**Guillemin and Pollack**, 1974, p.60] implies that $G_\lambda^{-1}(0)$ is a smooth one-dimensional manifold with boundary $\partial\{G_\lambda^{-1}(0)\} = G_\lambda^{-1}(0) \cap \{\bar{u}\} \times int\Sigma$.

Define the set $G_o^{-1}(0)$ as the set of all pairs $(u, \sigma) \in R \times \Sigma$ such that $\vec{b}_u(\sigma) = 0$ and for all neighbourhoods $\mathcal{O}$ of (u, σ) in $R \times \Sigma$ there exists $\lambda' > 0$ such that $\mathcal{O} \cap G_\lambda^{-1}(0) \neq \emptyset$, $\forall \lambda \in (0, \lambda']$. This "limit" set is the graph of the correspondence $\Psi\colon R \to \Sigma$ defined by

$$\Psi(u) = \{\sigma \in \Sigma \mid (u, \sigma) \in G_o^{-1}(0)\}.$$

Claim 1: The next step is to show that $\Psi(u) \subset E(\Gamma(u))$, $\forall u \in R$: Consider some $\bar{\sigma} \notin E(\Gamma(u))$, $\vec{b}_u(\bar{\sigma}) = 0$, with $\vec{b}_u$ the Nash field for $\Gamma(u)$. By Lemma 3 there exists some $i \in \mathcal{N}$ and some $s_i^k \notin supp(\bar{\sigma}_i)$ such that $U_i(\bar{\sigma}_{-i}, s_i^k) - U_i(\bar{\sigma}) > 0$. By continuity there exists a neighbourhood $\mathcal{U}$ of $\bar{\sigma}$ in $\Re^M$ and a neighbourhood $\mathcal{V}$ of u in R such that $b^k_{u',i}(\sigma) > 0$, $\forall (u', \sigma) \in \mathcal{V} \times (\mathcal{U} \cap int\Sigma)$. Since G is smooth, there exists a neighbourhood $\mathcal{O} \subset \mathcal{V} \times (\mathcal{U} \cap int\Sigma)$ of $(u, \bar{\sigma})$ in $R \times \Sigma$ such that $G_i^k(u', \sigma, \lambda) > 0$, $\forall (u', \sigma) \in \mathcal{O}$, and for all $\lambda > 0$ sufficiently small. Thus $G_\lambda^{-1}(0) \cap \mathcal{O} = \emptyset$, for all $\lambda > 0$ sufficiently small, such that $\bar{\sigma} \notin \Psi(u)$. This establishes Claim 1 that $\Psi(u) \subset E(\Gamma(u))$, $\forall u \in R$.

As a consequence the set $G_o^{-1}(0)$ is contained in the graph of the Nash equilibrium correspondence over R, i.e.

$$G_o^{-1}(0) \subset \mathcal{G}_R(E) = \{(u, \sigma) \in R \times \Sigma \mid \sigma \in E(\Gamma(u))\}.$$

If $C \subset E(\Gamma(\bar{u}))$ is not essential, then for all $u \in int\,R = R \setminus \{\bar{u}\}$ one must, therefore, have that $G_o^{-1}(0) \cap \{u\} \times B_\varepsilon^o(C) = \emptyset$, for all $\varepsilon > 0$ sufficiently small, viz.

$$\exists \bar{\varepsilon} > 0\colon\ G_o^{-1}(0) \cap (int\,R \times B_\varepsilon^o(C)) = \emptyset,\ \forall \varepsilon \in (0, \bar{\varepsilon}).$$

There are two possibilities: Either $G_o^{-1}(0) \cap \{\bar{u}\} \times B_\varepsilon^o(C) = \emptyset$, in which case the interior approximation to $\vec{b}_{\bar{u}}$ has no zero close to C such that $Ind(C) = 0$, or $G_o^{-1}(0) \cap \{\bar{u}\} \times B_\varepsilon^o(C) \neq \emptyset$. Since in the first case there remains nothing to be shown, assume the second case.

Recall that for any $\lambda \in (0, \bar{\lambda})$ the set $G_\lambda^{-1}(0)$ consists of finitely many closed or half-open intervals, and circles [by the classification of 1-manifolds: **Milnor**, 1965, p.55; **Guillemin and Pollack**, 1974, p.64]. Since by construction $0 \in \Re^M$ is a regular value for G_λ, it is also a regular value for G, such that the preimage of $0 \in \Re^M$ under the restriction of G to $R \times int\Sigma \times \Re_{++}$ is a 2-dimensional manifold. As a consequence a property demonstrated for some $\lambda' \in (0, \bar{\lambda})$ will also hold true for all $\lambda \in (0, \lambda')$. The property which is of relevance here is the following:

Claim 2: For any $\lambda \in (0, \bar{\lambda})$ sufficiently small the set $G_\lambda^{-1} \cap R \times B_\varepsilon^o(C)$, $\varepsilon > 0$ sufficiently small, consists of finitely many closed intervals ("arcs") and circles (the latter disjoint from $\{\bar{u}\} \times B_\varepsilon^o(C)$) such that *all* boundary points of the arcs are contained in $\{\bar{u}\} \times B_\varepsilon^o(C)$. In particular, $G_\lambda^{-1}(0) \cap R \times B_\varepsilon^o(C)$ does not contain any half-open intervals.

Suppose this is not true. Then it is possible to find an arc or a half-open interval in $G_\lambda^{-1}(0)$ which starts in $\{\bar{u}\} \times B_\varepsilon^o(C)$, but leaves $R \times B_\varepsilon^o(C)$ at some point. Since $G^{-1}(0)$ is a 2-manifold, this must also hold for any smaller λ. But $G_o^{-1}(0) \cap int\, R \times B_\varepsilon^o(C) = \emptyset$ implies that with decreasing λ this particular piece of $G_\lambda^{-1}(0)$ must leave $R \times B_\varepsilon^o(C)$ above successively "smaller" values of $u \in R$ (i.e. closer and closer to $\bar{u}$). Hence $G_o^{-1}(0) \cap \{\bar{u}\} \times B_\varepsilon^o(C)$ will *not* be contained in $\{\bar{u}\} \times C$. Rather there will be a connected piece of $G_o^{-1}(0)$, contained in $\{\bar{u}\} \times \Sigma$, which begins in $\{\bar{u}\} \times C$ and ends outside of $\{\bar{u}\} \times C$. But this contradicts the hypothesis that C is a full connected component of Nash equilibria (by Claim 1). The conclusion is that any arc starting in $\{\bar{u}\} \times B_\varepsilon^o(C)$ must also end in this set. This establishes Claim 2.

Determining an orientation for each arc in $G_\lambda^{-1}(0)$ from the standard orientations of $R \times \Sigma$ and $\Re^M$ [see: **Milnor**, 1965, p.28] shows that a positively oriented unit vector tangent to an arc will point inward at one boundary point and outward at the other. Since by construction $f_{\bar{u},\lambda}$ has only regular zeros, the orientations of the boundary points of arcs coincide with the indices of the corresponding zeros of $f_{\bar{u},\lambda}$. Thus all the zeros of $f_{\bar{u},\lambda}$ relevant for computing $Ind(C)$ come in pairs of indices $+1$ and -1. Summing over all these zeros, consequently, yields $Ind(C) = 0$. ∎

Theorem 4 shows that $Ind(C)$ is indeed an appropriate generalization of what the determinant of the Jacobian at a regular equilibrium provides as local information. A component $C \subset E(\Gamma)$ with $Ind(C) \neq 0$ is robust against payoff perturbations, despite the fact that payoff perturbations need not be considered to calculate $Ind(C)$. As a simple consequence of Theorem 4 $Ind(C) \neq 0$ implies several other desirable properties.

PROPOSITION 1. *For all games $\Gamma \in G(S_1, \ldots, S_n)$, if a connected component $C \subset E(\Gamma)$ satisfies $Ind(C) \neq 0$, then (i) C contains a Stable Set in the sense of* **Kohlberg and Mertens** *[1986], and (ii) it contains an equilibrium which induces a sequential equilibrium [***Kreps and Wilson***, 1982] in any extensive form game with the normal form Γ. Moreover, for almost all games in the space of extensive form games (iii) the (sequential) outcome induced by equilibria in C is constant across C.*

PROOF: (i) If $Ind(C) \neq 0$, then C is essential. By translating strategy perturbations into payoff perturbations [as in the proof of Theorem 2.4.3. in **van Damme**, 1987, p.34] it can be shown that C satisfies the defining property (S) of a Stable Set [**Kohlberg and Mertens**, 1986, p.1027]:

"*Property (S)*: $S \subset E(\Gamma)$ is closed and $\forall \varepsilon > 0$ there exists $\delta_o > 0$ such that for any $\hat{\sigma} \in int\Sigma$ and any $\delta = (\delta_i)_{i \in \mathcal{N}}$, $\delta_i \in (0, \delta_o)$, the perturbed game where every strategy σ_i is replaced by $(1 - \delta_i)\sigma_i + \delta_i \hat{\sigma}_i$, $\forall i \in \mathcal{N}$, has an equilibrium ε-close to S."

Consider the collection of closed subsets of C which satisfy property (S), ordered by set inclusion. By compactness the intersection of any ordered chain in this collection is non-empty and belongs to the collection. Thus the existence of a minimal element follows from Zorn's Lemma. Such a minimal element must be a Stable Set.

(ii) An essential component C always contains a hyperstable set [**Kohlberg and Mertens**, 1986, p.1022] by the same argument as in (i) and, therefore, a proper equilibrium [**Kohlberg and Mertens**, 1986, Proposition 3]. A proper equilibrium induces a sequential equilibrium in any tree with normal form Γ [**van Damme**, 1984; **Kohlberg and Mertens**, 1986, p.1009].

(iii) Generic extensive form games have only a finite number of equilibrium outcomes [**Kreps and Wilson**, 1982, Theorem 2]. The set of Nash equilibria for all normal form games consists of finitely many connected components [**Kohlberg and Mertens**, 1986, Proposition 1]. The space of extensive form games is a linear subspace of the space of normal form games $G(S_1, \ldots, S_n)$, as can be seen from writing the ties, induced between strategy combinations that lead to the same terminal node, as a system of linear equations in $u \in \Re^{nK}$ [compare also: **Mailath, Samuelson, and Swinkels**, 1990]. As a consequence of these three facts for generic extensive form games the component C uniquely identifies a sequential outcome of any extensive form game with normal form Γ (because it contains a proper equilibrium). ∎

As a consequence of Proposition 1, (i), a component C that satisfies $Ind(C) \neq 0$ also satisfies "Iterated Dominance" and "Independence of Non-Best Responses" [**Kohlberg and Mertens**, 1986, Proposition 6;

the terminology is from **van Damme**, 1990]. Thus to identify a component $C \subset E(\Gamma)$ with $Ind(C) \neq 0$ may be as far as one has to go. In fact it will generically, in the space of extensive form games, suffice for a normative recommendation, if such a recommendation is on behavior along the equilibrium path. Off the equilibrium path still some seemingly desirable properties may be violated. In particular an essential component may contain inadmissible equilibria. This certainly is a drawback, although one may have doubts on the viability of an axiom requiring players to use only undominated strategies: After all, a given player's dominated strategy may only be inferior against a strategy combination of the opponents, where some other player uses a dominated strategy [cf. **Samuelson**, 1991]. In an evolutionary context an equilibrium which prescribes the use of an inadmissible strategy would simply be a composition of populations in which certain behavorial options are not tested against. That is: a behavorial pattern (pure strategy) may be used, despite being dominated, because it prescribes inferior actions only in circumstances that never occur, such that it will not be selected against. Similarily, in naive learning processes some insights may never be learned, because the circumstances, where they matter, never realise [this is why many learning models use some kind of exogeneous randomness, cf. **Canning**, 1987, 1989; **Fudenberg and Kreps**, 1988, for extensive form games]. Still it is undebatable that there are further desirable properties, beyond those already mentioned, which would require further selection (beyond essential components), yielding smaller solution sets (probably at the expense of generating more solutions) [cf. **Hillas**, 1990; **Mertens**, 1987, 1989].

But equilibrium refinements are only one aspect of Theorem 4. The proof of Theorem 4 in fact allows for another insight which concerns the structure of the Nash equilibrium correspondence [similar results are: Theorem 1 of **Kohlberg and Mertens**, 1986; Theorem 3.1. of **Blume and Zame**, 1989; and Theorem 1 of **Schanuel, Simon, and Zame**, 1990].

PROPOSITION 2. *The graph of the Nash equilibrium correspondence,* $\mathcal{G}(E) = \{(u, \sigma) \in \Re^{nK} \times \Sigma \mid \sigma \in E(\Gamma(u))\}$, *can be arbitrarily closely approximated by a differentiable manifold of dimension* nK.

PROOF: To produce a particular interior approximation for all Nash fields $\vec{b}_u$ consider the mapping $G\colon \Re^{nK} \times int\Sigma \times \Re_{++} \to \Re^M$ defined by

$$G_i^k(u, \sigma, \lambda) = b_i^k(\sigma) + \lambda(1 - \sigma_i^k K_i),$$

for $k = 1, \ldots, K_i - 1, \forall i \in \mathcal{N}$. Let $F_u(\sigma, \lambda) = G(u, \sigma, \lambda)$ and $f_{u,\lambda}(\sigma) =$

$F_u(\sigma, \lambda), \forall \lambda > 0$. Clearly F_u extended to $\Sigma \times \Re_+$ is an interior approximation of $\vec{b}_u$.

Since $\Re^{nK}$, $int\Sigma$, and $\Re_{++}$ are smooth manifolds, so is their product, such that G is a smooth mapping of manifolds. By definition no zero of $G_\lambda(u, \sigma) \equiv G(u, \sigma, \lambda)$ can emerge at the boundary of Σ as long as $\lambda > 0$. Then by Lemma A.2. (in the Appendix) the Jacobian matrix $D_{(u,\sigma,\lambda)}G(\bar{u}, \bar{\sigma}, \bar{\lambda})$ is surjective, i.e. has maximal rank ($= M$), at any point where $G(\bar{u}, \bar{\sigma}, \bar{\lambda}) = 0$. As a consequence $0 \in \Re^M$ is a regular value of G. The Preimage Theorem [**Guillemin and Pollack**, 1974, p.21] implies that $G^{-1}(0)$ is a smooth manifold of dimension $(nK + 1)$. The very same argument establishes that for each fixed $\lambda > 0$ the preimage of $0 \in \Re^M$ under G_λ is a differentiable manifold of dimension nK.

Like in the proof of Theorem 4 let $G_o^{-1}(0)$ be the set of all $(u, \sigma) \in \Re^{nK} \times \Sigma$ such that $\vec{b}_u(\sigma) = 0$ and for all neighbourhoods $\mathcal{O}$ of (u, σ) there exists $\bar{\lambda} > 0$ such that $G_\lambda^{-1}(0) \cap \mathcal{O} \neq \emptyset$, $\forall \lambda \in (0, \bar{\lambda}]$. Duplicating the argument in the demonstration of Claim 1 in the proof of Theorem 4 yields $G_o^{-1}(0) \subset \mathcal{G}(E)$.

Now observe that $G_o^{-1}(0)$ and $\mathcal{G}(E)$ agree on a dense open subset of $\Re^{nK}$ by Theorem 2 and the implicit function theorem. Since the differentiable and nK-dimensional manifolds $G_\lambda^{-1}(0)$ converge pointwise to at least some part of $\mathcal{G}(E)$ (namely to $G_o^{-1}(0)$), but by Theorem 1 of **Kohlberg and Mertens** [1986] the graph $\mathcal{G}(E)$ is itself homeomorphic to $\Re^{nK}$, each $G_\lambda^{-1}(0)$ must approximate all of $\mathcal{G}(E)$. ▮

In the sense of Proposition 2 the consideration of cluster points of zeros of interior approximations does not really drop any important information on the structure of the equilibrium correspondence. Although $\mathcal{G}(E)$ is not a smooth manifold, it is the "limit" of smooth nK-dimensional manifolds. Theorem 4 is merely the payoff to understanding this structure.

REMARK. *The particular interior approximations used in the proof of Proposition 2 are reminiscent from* **Harsanyi** *[1973]. Their zeros, for $\lambda > 0$, are equivalent to the necessary and sufficient conditions for an equilibrium of a game, where payoffs are given by the payoffs in Γ plus λ times the payoffs from the "logarithmic game". In fact the logarithmic tracing procedure as introduced by* **Harsanyi and Selten** *[1988, pp.165] is a particular way to produce interior approximations of the Nash field.*

To summarize on how the Nash field can help with equilibrium selection: If a game has strict equilibria, these seem preferable to any nonstrict ones (the Nash field may even help to compare strict equilibria:

see Section 4.1.). For games without strict, but with regular equilibria, the latter seem an arguable choice (note that regular equilibria always have a non-zero index). For games without regular equilibria the Nash field can help identifying components which will be robust in the sense that close games will have equilibria close to the component. The latter can be achieved by checking the indices of connected components of equilibria, without having to compute payoff perturbations. And such a robustness property ("essentiality") seems to be the least one could ask from a selection outcome, because the analyst can never be sure to have picked precicely the correct payoffs. On the other hand, if one is willing to go as far as requiring that a solution set is contained in a component with non-zero index, even Stable Sets [in the sense of **Kohlberg and Mertens**, 1986] may fail to satisfy this (as an example in Section 4.2. below will illustrate).

4. Applications

4.1. Risk Dominance in 2×2 games. In developing a complete theory of equilibrium selection **Harsanyi and Selten** [1988] introduce the concept of Risk Dominance. The intuitive argument for this criterion works as follows: Suppose all players in a given game are certain that one of two possible equilibria will be played, but they are uncertain as to which of the two. In this state of confusion the players enter a process of expectation formation. Starting from a prior distribution over the actions of other players each player tries to improve her forecast of the behavior of her rivals by taking into account what a given vector of prior distributions over the actions will lead her opponents to do. Once a player has figured out what the responses of the other players to the priors will be, she adjusts her estimate and again calculates the consequences of this new distribution over the other players' actions. Where this process ends, the risk dominant equilibrium is located.

For the class of 2×2 games with two strict equilibria **Harsanyi and Selten** [1988, chp. 3.9] have formalized the notion of Risk Dominance in three axioms and they have shown that the risk dominant equilibrium is fully characterized by possessing the larger Nash-product. (For other games Harsanyi and Selten formalize Risk Dominance by the tracing-procedure.)

Translating the Nash-product property into terms of the Nash field gives a nice illustration of the information contained in the Nash field. First, it is quite commonplace that all 2×2 games with two strict equilibria are more or less of the type illustrated in Figure 1 below. In Figure 1 the unit square is Σ, the bold lines are the graphs of the best-reply correspondences of the two players and the points A, B, and C are the three equilibria, two of which (A and B) are strict. The arrows portray the behavior of the Nash field: A and B are, as strict equilibria, locally asymptotically stable, while C is a saddle point. This already is sufficient to restrict attention to A and B. It is now tempting to argue that in Figure 1 knowledge of the Nash field already is sufficient to select A as the "better" equilibrium, because "A absorbs a larger part of Σ than B does". But can this be made more precise? The answer is in the affirmative, if one takes into account the Nash-products of the two equilibria A and B.

(Insert Figure 1 about here)

Letting A being associated with the pure strategy combination (s_1^1, s_2^1) and B being associated with the pure strategy combination (s_1^2, s_2^2), the Nash-products, $NP(.)$, are given by

$$NP(A) = [u_1(s_1^1, s_2^1) - u_1(s_1^2, s_2^1)][u_2(s_1^1, s_2^1) - u_2(s_1^1, s_2^2)],$$
$$NP(B) = [u_1(s_1^2, s_2^2) - u_1(s_1^1, s_2^2)][u_2(s_1^2, s_2^2) - u_2(s_1^2, s_2^1)].$$

Because the determinant is the product of the eigenvalues, Lemma 1 applied to the above shows that the Nash-products equal the determinants of the Jacobian matrix of the Nash field at the corresponding equilibria. But the determinant is just the volume of the image of the unit cube under the linear mapping $D_\sigma b(\bar{\sigma})$. In other words: The determinant of $D_\sigma b(\bar{\sigma})$ is the coefficient of contraction of (oriented) volume, in the sense that the volume of any figure is contracted by a factor of $-|D_\sigma b(\bar{\sigma})|$. This makes it precise, what was meant by the somewhat vague phrase "A absorbs more of Σ than B does". For the class of 2×2 games with two strict Nash equilibria it thus turns out that Risk Dominance is equivalent to the condition that the preferred equilibrium has the larger absolute value of the determinant of $D_\sigma b(\bar{\sigma})$ at the equilibrium.

It is worth mentioning that the above logic holds true for all games with a strict Nash equilibrium: The determinant of $D_\sigma b(\bar{\sigma})$ at the strict equilibrium $\sigma \in \mathrm{E}(\Gamma)$ always equals a somewhat generalized Nash-product,

$$|D_\sigma b(\bar{\sigma})| = \prod_{i \in \mathcal{N}} \prod_{s_i \notin supp(\bar{\sigma}_i)} [U_i(\bar{\sigma}_{-i}, s_i) - U_i(\bar{\sigma})] =$$
$$= (-1)^M \prod_{i \in \mathcal{N}} \prod_{s_i \notin supp(\bar{\sigma}_i)} [U_i(\bar{\sigma}) - U_i(\bar{\sigma}_{-i}, s_i)]$$

and always measures the coefficient by which the strict equilibrium locally "absorbs its neighbourhood" [in a very similar sense as in **Kalai and Samet**, 1984].

4.2. Indices of components. In simple games it is often very easy to determine which component has a non-zero index. An example of this is provided by the well-known "beer-quiche" signalling game, due to Kreps. The set of Nash equilibria for this game consists of two connected components: In the first a strong signal is sent by (both types of) the incumbent and the entrant retreats upon seeing a strong signal, while the entrant would fight (with probability $\geq 1/2$) upon seeing a weak signal. In the second component (both types of) the incumbent send a weak signal in response to which the entrant retreats, while the entrant fights (with probability $\geq 1/2$) if the signal is strong. **Kohlberg and Mertens** [1986, pp.1031] show that the second, "unintuitive" component does not contain a Stable Set. By Proposition 1 the index of the second component must, therefore, be zero. Since indices sum to $+1$ across components, the first, "intuitive" component must have a non-zero index, namely index $+1$.

Somewhat more interesting is the game of Figure 4 in **van Damme** [1989], reproduced as Figure 2 below. This game has two components of equilibria. The first consists of the strict equilibrium in which player 1 chooses to play the 2×3 subgame in which she chooses T, and player 2 responds with L.

(Insert Figure 2 about here)

Since this component is a strict equilibrium, it must have index $+1$. As a consequence, the other component in which player 1 chooses her outside option has index zero. Still **van Damme** [1989, p.487] shows that this second component also forms a Stable Set. To require a non-zero index would thus lead to the selection of the strict equilibrium which, as **van Damme** [1989] argues, is the only one in this example consistent with Forward Induction. The example also illustrates that there may be equilibrium components which contain a Stable Set, but still have index zero. It is, however, easy to see that every game has at least one component with non-zero index (because otherwise indices would not sum to $+1$) and, therefore, a Stable Set contained in such a component. Whether these Stable Sets (which are contained in components with non-zero index) *are* those that are consistent with Forward Induction remains to be seen.

5. Conclusions

The present paper has demonstrated that the structure of interaction of players in a given normal form game at any point in the space of mixed strategies can be represented by a smooth vector field, called the Nash field. This vector field can be exploited in a number of ways. Regular equilibria can be defined from this smooth structure in a straightforward way. Second, index theory can be generalized to connected components of equilibria, providing global information on robustness against payoff perturbations from purely local properties.

Beyond these insights the Nash field provides a convenient way of representing the type of strategic interaction presumably modelled by the game. When the Nash field is used to define a system of differential equations it results in the replicator dynamics for asymmetric games. It thus provides a natural link between the "classical" theory of games played by rational players and evolutionary game theory.

Finally one may at least hope that various refinement concepts have counterparts in the behavior of the Nash field. This is certainly true for strict and robust equilibrium points. Section 4.1. has also illustrated this for Risk Dominance in 2×2 games. But the applications of index theory to the examples in Section 4.2. also illustrate that the Nash field is capable of sheding doubts even on very strong refinement concepts.

APPENDIX

LEMMA 1. *If $\bar{s}_i \notin supp(\sigma_i)$ and $b(\sigma) = 0$, then $[U_i(\sigma_{-i}, \bar{s}_i) - U_i(\sigma)] \in \Re$ is an eigenvalue of the Jacobian matrix $D_\sigma b(\sigma)$.*

PROOF: First suppose $s_i^k \notin supp(\sigma_i)$ is such that $k < K_i$. Since $s_i^k \notin supp(\sigma_i) \iff \sigma_i^k = 0$, all off-diagonal elements in the row of $D_\sigma b(\sigma)$ corresponding to $s_i^k \in S_i$ are zero and the diagonal element is given by

$$\frac{\partial}{\partial \sigma_i^k} b_i^k(\sigma) = U_i(\sigma_{-i}, s_i^k) - U_i(\sigma)$$

and, therefore, is an eigenvalue of $D_\sigma b(\sigma)$. Next suppose $s_i^k \notin supp(\sigma_i)$ is such that $k = K_i$. Then subtract $[U_i(\sigma_{-i}, s_i^{K_i}) - U_i(\sigma)]$ from all diagonal elements of $D_\sigma b(\sigma)$ and sum the rows corresponding to s_i^h, $h = 1, \ldots, K_i - 1$. This yields for the columns corresponding to s_i^l, $l = 1, \ldots, K_i - 1$,

$$\begin{aligned}
&\sum_{h=1}^{K_i-1} \sigma_i^h [U_i(\sigma_{-i}, s_i^{K_i}) - U_i(\sigma_{-i}, s_i^l)] + \\
&+ U_i(\sigma_{-i}, s_i^l) - U_i(\sigma) - [U_i(\sigma_{-i}, s_i^{K_i}) - U_i(\sigma)] = \\
&= (1 - \sum_{h=1}^{K_i-1} \sigma_i^h)[U_i(\sigma_{-i}, s_i^l) - U_i(\sigma_{-i}, s_i^{K_i})] = 0,
\end{aligned}$$

because $s_i^{K_i} \notin supp(\sigma_i) \iff (1 - \sum_{h=1}^{K_i-1} \sigma_i^h) = 0$. For the columns corresponding to s_j^l, $j \in \mathcal{N}\backslash\{i\}$, $l = 1, \ldots, K_j - 1$, this operation yields

$$\sum_{h=1}^{K_i-1} \sigma_i^h [U_i(\sigma_{-ij}, s_i^h, s_j^l) - U_i(\sigma_{-ij}, s_i^h, s_j^{K_j}) - U_i(\sigma_{-j}, s_j^l) +$$

$$+ U_i(\sigma_{-j}, s_j^{K_j})] = (1 - \sum_{h=1}^{K_i-1} \sigma_i^h)[U_i(\sigma_{-j}, s_j^l) - U_i(\sigma_{-j}, s_j^{K_j}) -$$

$$- U_i(\sigma_{-ij}, s_i^{K_i}, s_j^l) + U_i(\sigma_{-ij}, s_i^{K_i}, s_j^{K_j})] = 0,$$

$(\sigma_{-ij}, \hat{\sigma}_i, \hat{\sigma}_j) = (\sigma_1, \ldots, \sigma_{i-1}, \hat{\sigma}_i, \sigma_{i+1}, \ldots, \sigma_{j-1}, \hat{\sigma}_j, \sigma_{j+1}, \ldots, \sigma_n)$. It follows that these $K_i - 1$ rows are linearly dependent and, therefore, $[U_i(\sigma_{-i}, s_i^{K_i}) - U_i(\sigma)] \in \Re$ is an eigenvalue. ∎

THEOREM 1. *Let $\bar{\Gamma} \in G(S_1, \ldots, S_n)$ and assume that $\bar{\sigma} \in E(\bar{\Gamma})$ is a regular equilibrium. Let $\bar{u} \in \Re^{nK}$ denote the payoff vector of $\bar{\Gamma}$. Then there exists a neighbourhood $\mathcal{U}$ of $\bar{u}$ in $\Re^{nK}$ and a neighbourhood $\mathcal{V}$ of $\bar{\sigma}$ in $\Re^M$, such that*

(i) $|E(\Gamma(u)) \cap \mathcal{V}| = 1$, $\forall u \in \mathcal{U}$, *and*

(ii) the mapping $\sigma: \mathcal{U} \to \mathcal{V}$, $u \mapsto \sigma(u)$, where $\sigma(u)$ is the unique equilibrium of $\Gamma = \Gamma(u)$ in $\mathcal{V}$, is continuous.

PROOF: Define the mapping $\bar{b}: \Re^{nk} \times \Re^M \to \Re^M$ by

$$\bar{b}_i^k(u, \sigma) = \sigma_i^k [U_i(\sigma_{-i}, s_i^k) - U_i(\sigma)] = b_{u,i}^k(\sigma),$$

$\forall k = 1, \ldots, K_i - 1$, $\forall i \in \mathcal{N}$. Since $\bar{\sigma} \in E(\bar{\Gamma}) = E(\Gamma(\bar{u}))$, one has $\bar{b}(\bar{u}, \bar{\sigma}) = 0$, and since $\bar{\sigma}$ is regular, $|D_\sigma \bar{b}(\bar{u}, \bar{\sigma})| \neq 0$. Then by the implicit function theorem there exists a neighbourhood $\mathcal{U}_0$ of $\bar{u}$ in $\Re^{nK}$ and a unique function $\sigma: \mathcal{U}_0 \to \Re^M$ such that σ is differentiable on $\mathcal{U}_0$, $\sigma(\bar{u}) = \bar{\sigma}$, and $\bar{b}(u, \sigma(u)) = 0$, $\forall u \in \mathcal{U}_0$.

Choose an open neighbourhood $\mathcal{V}$ of $\bar{\sigma}$ such that

$$\sigma^{-1}(\mathcal{V}) \subseteq \mathcal{U}_0$$

$$\bar{\sigma}_i^k > 0 \implies \sigma_i^k > 0, \ \forall \sigma \in \mathcal{V}, \ \forall k = 1, \ldots, K_i - 1,$$

$$\sum_{h=1}^{K_i-1} \bar{\sigma}_i^h < 1 \implies \sum_{h=1}^{K_i-1} \sigma_i^h < 1, \ \forall \sigma \in \mathcal{V},$$

for all $i \in \mathcal{N}$. By continuity of σ, $\sigma^{-1}(\mathcal{V})$ is an open set in $\mathcal{U}_0$, with $\bar{u} \in \sigma^{-1}(\mathcal{V})$. Define the continuous mapping

$$\lambda_{i,k}: \sigma^{-1}(\mathcal{V}) \to \Re, \ u \mapsto (u, \sigma(u)) \mapsto U_i(\sigma_{-i}(u), s_i^k) - U_i(\sigma(u)).$$

It follows from continuity of $\lambda_{i,k}$ that the set

$$W_{i,k} = \lambda_{i,k}^{-1}(-\infty,\, 0) \cap \sigma^{-1}(\mathcal{V})$$

is open. Consider now the following intersection

$$\mathcal{U} = \bigcap_{i \in \mathcal{N}} \bigcap_{s_i^k \notin supp(\bar{\sigma}_i)} W_{i,k}\,.$$

As a finite intersection of open sets, $\mathcal{U}$ is open and, because $\bar{u} \in W_{i,k}$ for all $(i,\, k)$ such that $\bar{\sigma}_i^k = 0$ ($\bar{\sigma}$ is a regular and, therefore, quasi-strict equilibrium of $\bar{\Gamma} = \Gamma(\bar{u})$), it follows that $\mathcal{U}$ is an open neighbourhood of $\bar{u}$.

Next it will be shown that for all $u \in \mathcal{U}$, $\sigma(u)$ is an equilibrium of $\Gamma(u)$:

$$\forall u \in \mathcal{U}\colon\ \sigma_i^k(u) = 0 \Rightarrow \bar{\sigma}_i^k = 0 \Rightarrow U_i(\sigma_{-i}(u),\, s_i^k) - U_i(\sigma(u)) < 0.$$

Moreover, $\bar{b}(u,\, \sigma(u)) = 0$ implies that $\sigma_i^k(u) > 0 \Rightarrow U_i(\sigma_{-i}(u),\, s_i^k) - U_i(\sigma(u)) = 0$, such that $\sigma(u)$ is an equilibrium of $\Gamma(u)$. ∎

THEOREM 2. *Almost all games $\Gamma \in G(S_1, \ldots, S_n)$ have all equilibria regular.*

PROOF: First "slice" the polyhedron Σ in the following way: Set $\Sigma_M = int\ \Sigma$. Then for each $0 < m < M$ let Σ_m be the set of all interiors of all boundary faces of Σ with dimension m and denote by Σ^m a typical element of Σ_m, $\Sigma^m \in \Sigma_m$. Finally, let Σ_0 be the set of all "corners" of Σ (pure strategy combinations) and again denote by $\Sigma^0 \in \Sigma_0$ a typical point representing a pure strategy combination. For each $0 \leq m \leq M$ every $\Sigma^m \in \Sigma_m$ is a differentiable manifold without boundary of dimension m. Each of the sets Σ_m is finite.

Next identify $G(S_1, \ldots, S_n)$ with the space of payoff vectors $u \in \Re^{nK}$ and let the dependence of the mapping b, defined in (3), on the payoff vector be expressed by writing b_u for it. Let $\mathcal{C}^\infty(\Sigma,\, \Re^M)$ be the set of all mappings taking Σ to $\Re^M$ which are infinitely often continuously differentiable. Define the mapping

$$b\colon \Re^{nK} \to \mathcal{C}^\infty(\Sigma,\, \Re^M) \quad \text{by} \quad b(u) = b_u\,.$$

(The same symbol b is used here as in (3) to avoid extra notation, because no confusion can arise.) Analogously denote by $b_u|\Sigma^m\colon \Sigma^m \to \Re^m$ the mapping b_u restricted to the boundary face Σ^m and define $b|\Sigma^m\colon \Re^{nK} \to \mathcal{C}^\infty(\Sigma^m,\, \Re^m)$ by setting $b|\Sigma^m(u) = b_u|\Sigma^m$. Also let the evaluation map $b^{ev}|\Sigma^m\colon \Re^{nK} \times \Sigma^m \to \Re^m$ be defined by $(u,\, \sigma) \mapsto b_u|\Sigma^m(\sigma)$. To ensure that these definitions yield something well defined, the following two Lemmas are needed:

LEMMA A.1. *For each* $0 \leq m \leq M$ *and any* $u \in \Re^{nK}$

$$Image(b^{ev}|\Sigma^m) \subseteq \Re^m\,,\ \forall \Sigma^m \in \Sigma_m\,.$$

PROOF: Lemma A.1 is a weak version of Lemma 2 and follows from the implications $\sigma_i^k = 0 \implies b_i^k(\sigma) = 0$ and $\sigma_i^k = 1 \implies b_i^k(\sigma) = 0$. ∎

LEMMA A.2. *For any* $0 \leq m \leq M$ *and each* $\Sigma^m \in \Sigma_m$ *the derivative* $D_{(u,\,\sigma)}b^{ev}|\Sigma^m(\bar{u},\,\bar{\sigma})$ *is surjective, i.e. has rank* m*, at each point* $(\bar{u},\,\bar{\sigma}) \in \Re^{nK} \times \Sigma^m$ *such that* $b^{ev}|\Sigma^m(\bar{u},\,\bar{\sigma}) = 0$.

PROOF: Since $D_{(u,\,\sigma)}b^{ev}|\Sigma^m = [D_u b^{ev}|\Sigma^m,\ D_\sigma b^{ev}|\Sigma^m]$ it suffices now to show that $D_u b^{ev}|\Sigma^m(\bar{u},\,\bar{\sigma})$ has rank m. Calculating partial derivatives yields

$$\frac{\partial b_i^k|\Sigma^m}{\partial u_i(s_{-i},\,s_i^l)}(\bar{u},\,\bar{\sigma}) = (\delta_{kl} - \bar{\sigma}_i^l)\bar{\sigma}_i^k \prod_{j \in \mathcal{N}\setminus\{i\}} \bar{\sigma}_j(s_{-i})\,,$$

where $\delta_{kl} = 0$, if $k \neq l$, and $\delta_{kl} = 1$, if $k = l$. The partial derivatives of $b_i^k|\Sigma^m$ with respect to the payoffs of any other player $j \neq i$ are zero. Therefore, it suffices to consider $D_{u_i}b_i^{ev}|\Sigma^m(\bar{u},\,\bar{\sigma})$, where $u_i = [(u_i(s_{-i},\,s_i))_{s_{-i}\in S_{-i}}]_{s_i \in S_i}$ is player i's payoff vector. Consider linear combinations of rows of $D_{u_i}b_i^{ev}|\Sigma^m(\bar{u},\,\bar{\sigma})$: Along any column corresponding to $(s_{-i},\,s_i^k) \in S$ the linear combination with weights α_l, $l = 1,\ldots,K_i - 1$, over an arbitrary subset of rows yields

$$\begin{aligned}
&\bar{\sigma}_i^k \prod_{j \in \mathcal{N}\setminus\{i\}} \bar{\sigma}_j(s_{-i})[\alpha_k(1 - \bar{\sigma}_i^k) - \sum_{l \neq k} \alpha_l \bar{\sigma}_i^l] = \\
&= \bar{\sigma}_i^k \prod_{j \in \mathcal{N}\setminus\{i\}} \bar{\sigma}_j(s_{-i})[\alpha_k - \sum_l \alpha_l \bar{\sigma}_i^l]\,.
\end{aligned}$$

If $s_i^k \notin supp(\bar{\sigma}_i)$, then this trivially equals zero. If $(s_{-i},\,s_i^k) \in supp(\bar{\sigma})$, an assumption that the rows are linear dependent would imply $\alpha_k = \alpha = \sum_l \alpha_l \bar{\sigma}_i^l$ with $\alpha \neq 0$, such that

$$\alpha_k - \sum_l \alpha_l \bar{\sigma}_i^l = \alpha(1 - \sum_l \bar{\sigma}_i^l)\,.$$

But the RHS of this equation can only equal zero, if the summation is over the entire support of $\bar{\sigma}_i$. This implies that

$$\mathrm{rank}(D_{u_i}b_i^{ev}|\Sigma^m(\bar{u},\,\bar{\sigma})) = |supp(\bar{\sigma}_i)| - 1\,.$$

But by construction $\sum_{i \in \mathcal{N}} |supp(\bar{\sigma}_i)| - n = \dim\Sigma^m = m$, such that the Lemma follows. ∎

(Proof of Theorem 2 continued): Pick a $\Sigma^m \in \Sigma_m$, $0 \leq m \leq M$. The map $b^{ev}|\Sigma^m$ is infinitely often differentiable and by Lemma A.2 the $0 \in \Re^m$ is a regular value of $b^{ev}|\Sigma^m$. Then the parametric transversality theorem [**Hirsch**, 1976,p.79] states that the set

$$V_{\Sigma^m} = \{u \in \Re^{nK} \,|\, 0 \in \Re^m \text{ is a regular value of } b|\Sigma^m\}$$

is dense in $\Re^{nK}$.

Next let W be defined as the set of all $u \in \Re^{nK}$, such that the corresponding game $\Gamma = \Gamma(u)$ has only quasi- strict equilibria. From Theorem 2 in **Harsanyi** [1973] it follows that the complement of W in $\Re^{nK}$ is a closed set with Lebesgue measure zero. Hence W is dense in $\Re^{nK}$ (Suppose not: Then there exists an open set $\mathcal{O}$ contained in the complement of the closure of W and a compact set $\mathcal{Q} \subset \mathcal{O}$. But then, since the measure of $\mathcal{Q}$ is non-zero, this must also be true for the measure of $\mathcal{O}$. Because $\mathcal{O}$ is a subset of the complement of the closure of W it must be contained in the complement of W - a contradiction to Harsanyi's Theorem 2.). Now consider the set of $u \in \Re^{nK}$ which have only quasi-strict equilibria and all regular equilibria in every Σ^m, $\forall 0 \leq m \leq M$. Each u in this set satifies

$$u \in \bigcap_{0 \leq m \leq M} \bigcap_{\Sigma^m \in \Sigma_m} V_{\Sigma^m} \cap W = \bigcap V_{\Sigma^m} \cap W \,.$$

The Baire-Theorem [**Hirsch**, 1976, p.213] implies that $\bigcap V_{\Sigma^m} \cap W$ is dense in $\Re^{nK}$.

Finally, let $u \in \bigcap V_{\Sigma^m} \cap W$ and $\bar{\sigma}$ a zero of $b_u|\Sigma^m$. By elementary operations on determinants the following decomposition is obtained

$$|D_\sigma b_u(\bar{\sigma})| = \prod_{i \in \mathcal{N}} \prod_{s_i^k \notin supp(\bar{\sigma}_i)} (U_i(\bar{\sigma}_{-i}, s_i^k) - U_i(\bar{\sigma}))\, |D_\sigma b_u|\Sigma^m(\bar{\sigma})|\,,$$

if $\bar{\sigma} \in \Sigma^m$. Since all equilibria of $\Gamma(u)$ are quasi-strict and the determinant in the above decomposition is non-zero, all equilibria are regular. This holds for all u in the dense set $\bigcap V_{\Sigma^m} \cap W$. ∎

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FIGURE 1

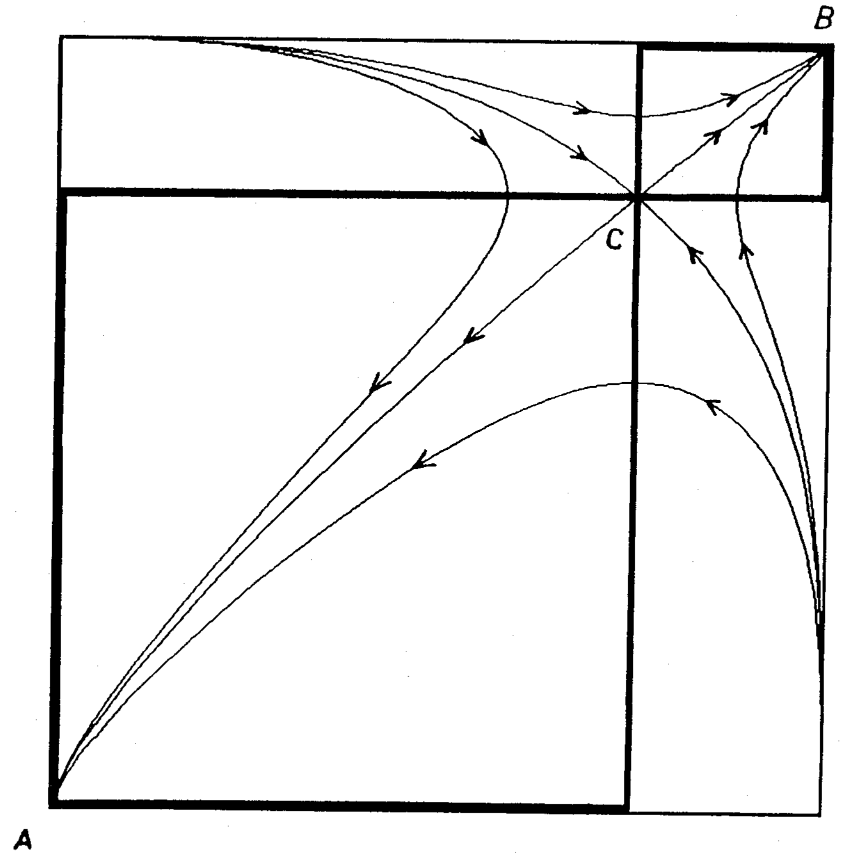

Figure 2

	L	M	R
T	3, 3	0, 2	3, 0
B	0, 0	3, 2	0, 3