

MULTISECTOR DYNAMIC LINEAR SYSTEMS  
AND THE TURNPIKE THEOREM

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## ABSTRACT

The subject of this study is an analysis of two of the basic linear systems; Leontief's system and von Neumann's model. The properties of the dual prices in a turnpike example are examined. The chapters in this study are organized as follows:

Chapter I furnishes the mathematical background for the subsequent chapters and requires the basic knowledge in linear algebra. The mathematical content of Chapter I is

- (1) the theory of nonnegative matrices and the conditions for the existence of nonnegative solution to systems of linear equations or inequalities of economic significance (the system in Leontief's input-output analysis);
- (2) the theory of productive matrices and the examination of the mutual equivalence between the conditions for the productive matrices and the conditions of the solvability of a Leontief's system;
- (3) indecomposability of the matrices.

Chapter II deals with von Neumann's and Leontief's dynamic models with economic interpretation of the indecomposability of Leontief's system. It is proved there a sufficient condition for the indecomposability of this system. Then, it is investigated the dual Leontief's system and it is proved that the growth factor  $\alpha$  in the equilibrium conditions is equal to the interest factor  $\beta$  and it is equal to the reciprocal of the eigenvalue of the technical coefficients matrix. In the remaining part of this chapter it is shown that the rate of growth in the Leontief's dynamic model is greater than one if and only if the matrix of technical coefficients is productive.

Chapter III, dealing with efficient balanced growth and a turnpike example, starts with the formulation of our simple Leontief dynamic model of economy and the definition of the efficient and the balanced growth path. Then, we have investigated the properties of the efficient path in our dynamic model with the numerical data. It is shown that the growth factor of the value of objective function in the last period is exactly determined by a given price level and the technical coefficients matrix. The remaining part of the chapter presents the properties of the dual prices in this model and the dual turnpike theorem.

Several numerical examples are given in the appendix for an illustration at the presented results.

## CHAPTER I

### 1. Nonnegative Matrices

We define for any two real matrices  $A = \{a_{ij}\}$ ,  $B = \{b_{ij}\}$  of the same order, say, with  $m$  rows and  $n$  columns:

a)  $A \geq B$ , if  $a_{ij} \geq b_{ij}$  ( $i = 1, \dots, m, j = 1, \dots, n$ )

b)  $A \succ B$ , if  $A \geq B$  and  $A \neq B$

c)  $A > B$ , if  $a_{ij} > b_{ij}$  ( $i = 1, \dots, m, j = 1, \dots, n$ )

A matrix  $A$  is called nonnegative, semipositive or positive according to whether  $A \geq 0$ ,  $A \succ 0$ , or  $A > 0$ .

By the same way we can introduce a nonnegativity concept for vectors.

When  $m = n$ , a matrix  $A$  is said to be square.

Consider a system of equations

$$x_i - \sum_{j=1}^n a_{ij} x_j = c_i \quad (i = 1, \dots, n) \quad (*)$$

The system (\*) is known as basis in Leontief's input-output analysis. We may observe that (\*) is a special case (for  $\phi = 1$ ) of the system of equations:

$$\phi x_i - \sum_{j=1}^n a_{ij} x_j = c_i \quad (i = 1, \dots, n) \quad (1)$$

where  $\phi$  is a parameter.

In matrix notation:

$$(\phi E - A)x = c \quad (2)$$

where  $A$  is a nonnegative square matrix,  $E$  is the  $n$ -th order identity matrix and  $x = \{x_i\}$ ,  $c = \{c_i\}$   $i = 1, \dots, n$ .

We shall write  $D = (\zeta E - A)$ .

We now list four conditions for the existence of nonnegative solutions to (2):

(I) System (2) has a set of nonnegative solutions  $x_i \geq 0$   
( $i = 1, \dots, n$ ) for some set of positive  $c_i > 0$  ( $i = 1, \dots, n$ )

(II) System (2) is solvable in the nonnegative unknowns  
 $x_i \geq 0$  ( $i = 1, \dots, n$ ) for any set of nonnegative  $c_i \geq 0$   
( $i = 1, \dots, n$ )

(III) The square matrix  $D = \{d_{ij}\}$  of coefficients has the  $n$   
positive upper left-hand corner principal minors, i.e.

$$\begin{vmatrix} d_{11} & \cdots & d_{1k} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ d_{k1} & \cdots & d_{kk} \end{vmatrix} > 0 \quad (k = 1, \dots, n)$$

(IV) All the principal minors of  $D$  are positive.

Conditions (III) and (IV) are known as the Hawkins-Simon condition.

Theorem 1.1

Conditions (I), (II), (III), and (IV) are mutually equivalent.<sup>1)</sup>

Now let us consider the system dual to (\*), (1), and (2).

That is:

$$p_j - \sum_{i=1}^n a_{ij} p_i = v_j \quad (j = 1, \dots, n) \quad (**)$$

---

1) Proof see by Nikaido.

$$p_j - \sum_{i=1}^n a_{ij} p_i = v_j \quad (j = 1, \dots, n) \quad (1d)$$

$$p(E - A) = v \quad (2d)$$

where  $p = \{p_j\}$ , and  $v = \{v_j\}$  ( $j = 1, \dots, n$ ). We require that the unknowns  $p_j$ , as well as the constant terms  $v_j$  are non-negative. Theorem 1.1 ensures the equivalence of the following four conditions (Id) - (IVd), for the existence of nonnegative solutions to (2d):

(Id) System (2d) has a set of nonnegative solutions  $p_j \geq 0$  ( $j = 1, \dots, n$ ) for some set of positive  $v_j > 0$  ( $j = 1, \dots, n$ ).

(IIId) System (2d) is solvable in the nonnegative unknowns  $p_j \geq 0$  ( $j = 1, \dots, n$ ) for any set of nonnegative  $v_j \geq 0$  ( $j = 1, \dots, n$ ).

(IIIId) The square matrix  $D' = \{d_{ij}\}'$  of coefficients has the  $n$  positive upper left-hand corner principal minors, i.e.,

$$\begin{vmatrix} d_{11} & \dots & \dots & \dots & d_{k1} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ d_{1k} & \dots & \dots & \dots & d_{kk} \end{vmatrix} > 0 \quad (k=1, \dots, n)$$

(IVd) All the principal minors of  $D$  are positive.

Theorem 1.2

Conditions (I), (II), (III), (IV), (Id), (IIId), (IIIId), and (IVd) are mutually equivalent.<sup>1)</sup>

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1) Proof see by Nikaido.

We can now formulate the statements (I) - (IV) and (Id) - (IVd) in an alternative expression, which may be referred to as nonnegative invertibility:

Definition 1.1

A square matrix A is called nonnegatively invertible if A is non-singular and its inverse  $A^{-1}$  is nonnegative.

Theorem 1.3

The matrix D of the system (2) is nonnegatively invertible if and only if any one of the equivalent conditions (I) - (IV) and (Id) - (IVd) is satisfied.<sup>1)</sup>

As known we can express inverse  $(E - A)^{-1}$ , where A is a non-negative square matrix and E is the n-th order identity matrix as a matrix power series (C. Neumann series):

$$(E - A)^{-1} = E + A + A^2 + \dots + A^r + \dots$$

More generally, the expression of  $(\varphi E - A)^{-1}$ , where  $\varphi$  is a parameter in the matrix power series

$$\frac{1}{\varphi} \sum_{r=0}^{\infty} \frac{A^r}{\varphi^r} = \frac{1}{\varphi} \left( E + \frac{A}{\varphi} + \frac{A^2}{\varphi^2} + \dots + \frac{A^r}{\varphi^r} + \dots \right) \quad (3)$$

suggests itself.

Theorem 1.4

Let A be a nonnegative square matrix and  $\varphi$  be a numerical parameter. Then,

- (i) If  $\varphi E - A$  is a nonnegatively invertible matrix, then  $\varphi > 0$ , and (3) is convergent, its sum being  $(\varphi E - A)^{-1}$ .
- (ii) If  $\varphi > 0$  and (3) is convergent, then the matrix  $\varphi E - A$  is nonnegatively invertible, and the sum of (3) is equal to  $(\varphi E - A)^{-1}$ .<sup>1)</sup>

Lemma 1.1

For a nonnegative square matrix A define the set

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1) Proofs see by Nikaido.

$M(A) = \{ \lambda \mid \lambda E - A \text{ is nonnegatively invertible} \}$ .

Then,  $M(A)$  is a semiinterval  $(\hat{\lambda}, +\infty)$ , where  $\hat{\lambda}$  is a nonnegative number.<sup>1)</sup>

Theorem 1.5

For the above  $\hat{\lambda}$ , there is some  $x \geq 0$  satisfying  $Ax = \hat{\lambda}x$ .

P r o o f

We shall prove the lemma by appealing to a theorem by Gale.<sup>2)</sup>  
By the theorem as applied to  $\lambda E - A$ , one of the following alternatives holds:

$$(\lambda E - A)x = 0 \text{ for } x \geq 0$$

or

$$p(\lambda E - A) > 0$$

Suppose  $p(\lambda E - A) > 0$  for some  $p \geq 0$ . From condition (Id) and theorem 1.3 it follows that  $(\lambda E - A)^{-1} \geq 0$ , and this implies (by Lemma 1.1) that  $\lambda \in M(A)$ . However this contradicts the definition of  $\hat{\lambda}$  by Lemma 1.1.

Then the following holds:  $(\lambda E - A)x = 0$

$$\lambda Ex - Ax = 0$$

$$Ax = \lambda x \text{ for } x \geq 0$$

1) Proof see by Nikaido.

2) Theorem: (Gale [1] p. 77):

Exactly one of the following two alternatives holds.

Either the equation

$$xA = 0$$

has a semipositive solution, or the inequality

$$Ay > 0$$

has a solution.

Theorem 1.6

(Perron 1907, Frobenius 1908, 1909, 1912).

Let  $A$  be a nonnegative square matrix. Then, (i-iv) hold:

- (i)  $A$  has a nonnegative eigenvalue. A nonnegative eigenvector is associated with the largest  $\lambda$  of all the nonnegative eigenvalues.
- (ii)  $\rho E - A$  is nonnegatively invertible if and only if  $\rho > \lambda$ .
- (iii) If  $Ay \geq \rho y$  for a real number  $\rho$  and a semipositive vector  $y \geq 0$ , then  $\lambda \geq \rho$ .
- (iv)  $\lambda \geq |\omega|$  for any eigenvalue  $\omega$  of  $A$ .

The above  $\lambda$  of  $A$  is denoted by  $\lambda(A)$  and is called the Frobenius-Perron root or the dominant root of  $A$ .

2. Productive Matrices

We introduce now the concept of productive matrices.

Definition 2.1

The matrix  $A$  will be called productive if there exists a nonnegative vector  $\tilde{x}$  such that  $\tilde{x} > A\tilde{x}$ .

Lemma 2.1

If  $A$  is a productive matrix and  $x \geq Ax$ , then  $x \geq 0$ .

Theorem 2.1

If the matrix  $A$  is productive, then for any  $y \geq 0$  the equation

$$x(E-A) = y$$

has a unique nonnegative solution.

Corollary 1

If A is a productive matrix then  $(E - A)$  is regular (i.e., it has rank n).

Corollary 2

The matrix A is productive if and only if  $(E - A)^{-1}$  is non-negative.<sup>1)</sup>

From Corollary 2 and theorem 1.3 (for  $\phi = 1$ ) it follows: if the matrix A is productive, then the conditions (I) - (IV) and (Id) - (IVd) are satisfied and vice versa.

Theorem 2.2

If the matrix A is productive, then the sum of the entries in at least one column is less than one.

P r o o f

The matrix A is productive, i.e.,

$$x > Ax \quad \text{or} \quad (E - A)x > 0$$

From corollary 2 follows that  $(E - A)^{-1} \geq 0$ .

Then we can write

$$(E - A)^{-1}(E - A)x \geq 0$$

$$\underline{x \geq 0}$$

---

1) The proofs of lemma, theorem, and corollaries see by Gale, pp. 370-371 (1).

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & \cdot & a_{2n} \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & \cdot & \cdot & a_{nn} \end{pmatrix}$$

and

$$x = (x_1, x_2, \dots, x_n)$$

We know that  $x > Ax$ , i.e.,

$$\begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} > \begin{pmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & \cdot & a_{2n} \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & \cdot & \cdot & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$$

and

$$x_i > \sum_{j=1}^n a_{ij} x_j \quad (i = 1, \dots, n)$$

We have showed that  $x \geq 0$ , i.e.,

$$x_i \geq 0 \quad (i = 1, \dots, n)$$

Then

$$\begin{aligned} \sum_{i=1}^n x_i > \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_j &= \sum_{j=1}^n \sum_{i=1}^n a_{ij} x_j = \\ &= \sum_{j=1}^n x_j \sum_{i=1}^n a_{ij} \end{aligned}$$

and

$$\sum_{i=1}^n x_i > \sum_{j=1}^n x_j \sum_{i=1}^n a_{ij} \implies \sum_{i=1}^n a_{ij} < 1$$

for at least one  $j$ .

For example for  $n=3$  we can write:

$$x_1 > \sum_{j=1}^3 a_{1j} x_j = a_{11} x_1 + a_{12} x_2 + a_{13} x_3$$

$$x_2 > \sum_{j=1}^3 a_{2j} x_j = a_{21} x_1 + a_{22} x_2 + a_{23} x_3$$

$$x_3 > \sum_{j=1}^3 a_{3j} x_j = a_{31} x_1 + a_{32} x_2 + a_{33} x_3$$

$$\sum_{i=1}^3 x_i > x_1 \sum_{i=1}^3 a_{i1} + x_2 \sum_{i=1}^3 a_{i2} + x_3 \sum_{i=1}^3 a_{i3}$$

i.e., the sum in at least one column must be less than one.

From theorem 2.2 and its proof then follows a sufficient condition for productive matrix  $A$ . This condition is in fact an application of the Brauer - Solow Condition for solvability

of (1) in the nonnegative unknowns  $x_i \geq 0$ .<sup>1)</sup> From corollary (2) and theorem 1.3 (for  $\phi = 1$ ) we know that the solvability of (1) in above mentioned sense and the productivity of the matrix A are mutually equivalent.

Corollary

Either (i) or (ii) is sufficient for productivity of matrix A:

- (i) The sum of the entries in all columns is less than one.
- (ii) The sum of the entries in all rows is less than one.

P r o o f

It is very easy to find a nonnegative vector  $x$  such that  $x > Ax$ . For instance  $x_1 = x_2 = \dots = x_n = 1$ .

Theorem 2.3

The matrix A is productive if and only if

$$\lim_{n \rightarrow \infty} A^n = 0$$

P r o o f

A necessary condition: If the matrix A is productive, then

$$\lim_{n \rightarrow \infty} A^n = 0.$$

Since A is productive there exists  $x \geq 0$  such that  $Ax < x$ . Then from corollary 2 it follows that  $(E - A)^{-1} \geq 0$ . By theorem 1.4 we know (for  $\phi = 1$ ) that the matrix power series

$$\sum_{n=0}^{\infty} A^n = E + A + A^2 + \dots + A^n + \dots \quad (4)$$

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1) See Nikaido.

is convergent in that case. But this implies that  $\lim_{n \rightarrow \infty} A^n = 0$ .

A sufficient condition: if  $\lim_{n \rightarrow \infty} A^n = 0$ , then the matrix  $A$  is productive.

We denote  $T_n = \sum_{r=0}^n A^r$ .

Then

$$T_n(E - A) = (E - A)T_n = E - A^{n+1} = E,$$

because  $\lim_{n \rightarrow \infty} A^{n+1} = 0$  and  $A$  is a nonnegative matrix. Then

$$T_n = (E - A)^{-1}.$$

Since  $A$  is nonnegative,  $T_n$  is also nonnegative. Hence,  $(E - A)^{-1}$  is nonnegative too and by corollary 2 the matrix  $A$  is productive.

### 3. Indecomposability

#### Definition 3.1

A nonnegative  $n \times n$  matrix  $A = \{a_{ij}\}$  is called decomposable if there is a nonempty proper subset  $J$  of  $\{1, 2, \dots, n\}$  such that

$$a_{ij} = 0 \quad (i \notin J, j \in J)$$

#### Definition 3.2

A matrix  $A = \{a_{ij}\}$  is called indecomposable, if it is not decomposable and is not a zero matrix of order 1.

When  $A$  is an  $n \times n$  matrix, then  $A_T = TAT^{-1}$  denotes the transform of  $A$  by the nonsingular  $n \times n$  matrix  $T$ .

Now we can introduce an alternative definition of an indecomposable matrix.

Definition 3.3

An  $n \times n$  matrix  $A$  ( $n \geq 2$ ) is said to be indecomposable if for no permutation matrix <sup>1)</sup>  $\pi$  holds

$$A_{\pi} = \pi A \pi^{-1} = \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix}$$

where  $B_{11}$ ,  $B_{22}$  are square submatrices.

If the matrix  $A$  is decomposable with respect to a subset  $J$  consisting of  $k$  integers, then it can be transformed into a matrix

$$\pi A \pi^{-1} = \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix}$$

where  $B_{11}$  is a  $k \times k$  submatrix,  $B_{22}$  is an  $(n - k) \times (n - k)$  submatrix and  $B_{12}$  is a  $k \times (n - k)$  submatrix.

Lemma 3.1

A matrix  $A = \{a_{ij}\}$  is decomposable if and only if its transpose  $A'$  is decomposable.

Theorem 3.1

Let a nonnegative square matrix  $A$  be indecomposable. Then:

---

1) A permutation matrix is a matrix such that each row and each column has a single nonzero entry of value 1.  $\pi A \pi^{-1}$  is obtained by performing the same permutation on the rows and on the columns of  $A$ .

- (i) Any nonnegative eigenvector associated with  $\lambda(A)$  is positive. Moreover  $\lambda(A) > 0$ .
- (ii) The eigenvector of  $A$  associated with  $\lambda(A)$  is unique up to the multiplication by scalars.

P r o o f

(i) Suppose a contradiction, let  $x = (0, x_2, \dots, x_n)$ . Then the following must hold:

$$Ax = \lambda x$$

$$Ax = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} 0 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$$

$$\text{thus } a_{11} \cdot 0 + a_{12}x_2 + \dots + a_{1n}x_n = 0$$

so  $a_{12} = a_{13} = \dots = a_{1n} = 0$ , violating the indecomposability of  $A$ .

Next, if  $\lambda(A) = 0$ , then  $Ax = 0$ . Hence,  $\sum_j a_{ij}x_j = 0$ , for all  $i$ , which implies  $a_{ij} = 0$  for all  $i, j$  because  $a_{ij} \geq 0$  and  $x_j > 0$  for all  $i, j$ . Thus  $A = 0$ , contradicting the indecomposability of  $A$ . This proves that  $\lambda(A) > 0$ .

(ii) Let  $y$  be an eigenvector associated with  $\lambda(A)$ . Take another positive eigenvector  $y' = (y'_1, \dots, y'_n)$  associated with  $\lambda(A)$ .

Let  $\alpha = \min_i y_i/y'_i$  ( $i = 1, \dots, n$ ), for example  $\alpha = y_1/y'_1$

Then  $\alpha > 0$ , because from (i) of this theorem it follows that

$$y > 0. \text{ Let } y_j'' = y_j - \alpha y_j' \geq 0$$

$$\text{or } y'' = y - \alpha y' \geq 0.$$

Then  $Ay'' = Ay - \alpha Ay' \geq 0$  where  $A$  is a square nonnegative matrix.

However  $Ay = \omega(A)y$ .

Then we can write:

$$Ay'' = \omega(A)y - \alpha\omega(A)y' = \omega(A)\underbrace{(y - \alpha y')}_{y''}$$

$$Ay'' = \omega(A)y''$$

however  $y'' \not\geq 0$ , because  $y_1'' = 0$ .

If  $Ay'' = \omega(A)y''$ , where  $y'' = (0, y_2'', \dots, y_n'')$ , then necessarily

$$A = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

But this is a violation of indecomposability of the matrix  $A$  and therefore  $y'' = 0$  must hold.

$$\text{Then } y - \alpha y' = 0$$

$$y = \alpha y',$$

what completes the proof.

## CHAPTER II

### 1. Von Neumann's Model

We consider as a model of an economy a finite set of  $n$  processes which operate at discrete time intervals and which produce a finite number  $m$  of different goods. From an economic point of view one usually expects that  $n > m$ , since for most of goods, there are alternative ways of production so that there is a choice among production processes. From a mathematical point of view we need not assume any relationship between  $m$  and  $n$ . We assume constant returns to scale and the unlimited availability of the natural production factors such as labor and land. We denote by  $a_{ij}$  an amount of good  $i$  consumed in process  $j$ , (per unit of intensity) and by  $b_{ij}$  an amount of good  $i$  produced in process  $j$  (per unit intensity). It is assumed that  $a_{ij}$  and  $b_{ij}$  are nonnegative real numbers for all  $i$  and  $j$ . Hence, the  $j$ -th process is represented by two nonnegative vectors, an input vector  $a_j = (a_{1j}, a_{2j}, \dots, a_{mj})$  and an output vector  $b_j = (b_{1j}, b_{2j}, \dots, b_{mj})$ . The corresponding matrices  $A = \{a_{ij}\}$  and  $B = \{b_{ij}\}$  are called the input and output matrices, respectively.

Each process operates at an intensity  $x$  (a column vector) where  $x$  is a real semipositive  $n$ -vector and the corresponding input and output vectors are given by  $Ax$  and  $Bx$ . The inputs required for the processes at any time  $t$  are the goods produced during the time period  $t-1$ , plus the natural production factors.

Symbolically we can represent physical production change during one time period as follows:

$$(\text{time } t-1)Ax \longrightarrow Bx (\text{time } t)$$

To each good there is assigned a price  $p_i$  and  $p$  is a semipositive price  $m$ -vector (a row vector) and the vector  $pA$  gives the value of inputs entering into the processes, and the vector

$pB$  gives the value of goods produced by the processes.  
Symbolically we can represent value changes during one time period as follows:

$$(\text{time } t-1) \ pA \longrightarrow pB \ (\text{time } t)$$

It is assumed here that there is an interest rate  $b$  (per cent) from which we derive interest factor  $B = 1 + b/100$ .

It is also assumed that there is an expansion rate  $a$  (per cent) - common for all goods - from which we derive in the expansion factor  $\alpha = 1 + a/100$ .

For the matrices  $A$  and  $B$ :

$$\sum_{i=1}^m a_{ij} > 0 \quad (j = 1, \dots, n) \quad (1)$$

$$\sum_{j=1}^n b_{ij} > 0 \quad (i = 1, \dots, m) \quad (2)$$

Condition (1) implies that each process  $j$  needs at least one kind of input. On the other hand, (2) means that every good can be produced by some process. Now we can write the well known five conditions which the vectors  $p = (p_i)$ ,  $x = (x_j)$  and two scalars  $\alpha$ ,  $B$  must satisfy:

$$(B - \alpha A)x \geq 0 \quad (3)$$

$$p(B - BA) \leq 0 \quad (4)$$

$$p(B - \alpha A)x = 0 \quad (5)$$

$$p(B - BA)x = 0 \quad (6)$$

$$pBx > 0 \quad (7)^{1)}$$

$$p \geq 0, \quad x \geq 0 \quad (8)$$

Condition (3) means that no more goods can be used during any time period than were produced during a preceding time period.

Condition (4) makes the economy a profitless one, i.e., one in which a process cannot yield a return greater than that, yielded by the corresponding interest factor.

Condition (5) requires that a zero price must be charged for goods that are overproduced.

Condition (6) says that inefficient processes must be used with zero intensity.

Condition (7) means that the total value of all goods produced must be positive.

#### Definition 1.1

A price vector  $p$  having the property (4) is termed a von Neumann price vector.

Kemeny (1956), Thompson (1956), and Howe (1960) proved the following theorem:

#### Theorem 1.1

Assume (1), (2) and  $A \geq 0$ . Then (i) and (ii) hold:

---

1) Von Neumann proved the existence of solution  $(p, x, \alpha, B)$  only to (3) - (6) and (8). His proof was without assumptions (1) and (2), but under other - very strict - assumption

$$a_{ij} + b_{ij} > 0 \quad (i = 1, \dots, m; j = 1, \dots, n)$$

that is hardly acceptable by economists. The assumptions (1) and (2) - economically very plausible - and condition (7) were introduced by Kemeny, Morgenstern, Thompson.

(i) There is a solution  $(p, x, \alpha, B)$  of (3) - (8) such that

$$p_i = 0 \text{ if and only if } \alpha \sum_{j=1}^n a_{ij} x_j < \sum_{j=1}^n b_{ij} x_j \quad (9)$$

and

$$x_j = 0 \text{ if and only if } \beta \sum_{i=1}^m a_{ij} p_i > \sum_{i=1}^m b_{ij} p_i \quad (10)$$

(ii)  $\alpha = B$  for any  $(p, x, \alpha, B)$  satisfying (5), (6), (7).

## 2. Leontief's Model

The Leontief's model is a special case of the von Neumann's model in which there are exactly  $n$  activities,  $n$  goods and  $B$  is the identity matrix  $E$ . In other words, there is only one process with fixed proportions for each good and the output matrix of the Leontief's model is a diagonal matrix. Therefore we can speak about sector or industry, which produces exactly one good, and the matrix  $A$  can be considered as matrix known from classical input-output analysis. The matrix  $A$  may or may not have zero elements. In some cases, the layout of these elements allows the system to split into subgroups of sectors where production activities take place more or less independently of each other. A typical concept of splitability is decomposability defined in mathematical sense in Chapter I. An economic interpretation of decomposability for our model is given by following definition and theorem.

### Definition 2.1

Let  $A = \{a_{ij}\} \geq 0$ . Then, the  $i$ -th sector is said to be connected to the  $j$ -th sector if there exists a chain of sectors  $\{k_0, k_1, k_2, \dots, k_r\}$  joining  $k_0 = i$  to  $k_r = j$  such that any two consecutive sectors  $k_s$  and  $k_{s+1}$  are directly connected in the sense that  $a_{k_s k_{s+1}} > 0$ .

Theorem 2.1

$A = \{a_{ij}\} \geq 0$  is indecomposable if and only if the  $i$ -th sector can be connected to the  $j$ -th sector for any  $i, j = 1, \dots, n$ .<sup>1)</sup>

We know from the input-output analysis that  $a_{ij}$  represents the direct requirement of input  $i$  in sector  $j$ . But if sector  $j$  is considered as a part of a complete system, then unit output of this sector requires the use of some of the inputs  $k = 1, 2, \dots, i-1, i+1, \dots, n$  as well as input  $i$ . To produce these other inputs, other sectors have to operate and these will, in general, also require the use of input  $i$ . Furthermore, these other sectors will also require  $j$  as an input, so that sector  $j$  will have to produce more than a unit amount in order to supply the sectors that are themselves supplying inputs to sector  $j$ . The requirement of input  $i$  which arises from these sources constitutes an indirect requirement of input  $i$  in sector  $j$ . The total requirement of input  $i$  for unit output of sector  $j$  is the sum of the direct and indirect requirements.

We can compute then the total mutual requirements for any  $i, j = 1, \dots, n$  as the inverse of the matrix  $D = E - A$ . The coefficients of the matrix  $(E - A)^{-1}$  denoted by  $d_{ij}^*$ , represent the total requirements between any  $i, j = 1, \dots, n$ . A coefficient  $d_{ij}^* > 0$  then means that the  $i$ -th sector is connected to the  $j$ -th sector in sense of our definition. If all  $d_{ij}^* > 0$ , i.e., the  $i$ -th sector can be connected to the  $j$ -th sector for any  $i, j = 1, \dots, n$ , then from theorem 2.1 follows that the matrix  $A$  is indecomposable. More exactly the following theorem holds:

Theorem 2.2

If  $(E - A)^{-1} > 0$ , the matrix  $A$  is indecomposable.

-----  
1) Proof see by Nikaido.

P r o o f

Suppose that A is decomposable. Then it can be transformed into a matrix

$$\overline{\Pi A \overline{\Pi}^{-1}} = \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix}$$

where  $B_{11}$  is  $k \times k$  submatrix,  $B_{22}$  is  $(n - k) \times (n - k)$  submatrix and  $B_{12}$  is  $k \times (n - k)$  submatrix.

Then

$$E - A = \begin{pmatrix} E_1 - B_{11} & -B_{12} \\ 0 & E_2 - B_{22} \end{pmatrix}$$

where  $E_1$  is the  $k$ -th order and  $E_2$  is the  $(n - k)$ -th order identity matrix. We compute the inverse of  $(E - A)$ . To obtain the inverse of a matrix in a partitioned form we can use the following result.<sup>1)</sup>

Let D be an  $n \times n$  nonsingular matrix partitioned as

$$D = \begin{pmatrix} K & F \\ G & H \end{pmatrix} \quad \text{where } K \text{ is } k \times k, F \text{ is } k \times (n - k), \\ G \text{ is } (n - k) \times k \text{ and } H \text{ is } (n - k) \times (n - k).$$

Suppose that the matrices  $K$  and  $C = H - GK^{-1}F$  are nonsingular.

---

1) Proof see by A. S. Goldberger, p. 27.

Then

$$D^{-1} = \begin{pmatrix} K^{-1}(E + FG^{-1}GK^{-1}) & -K^{-1}FC^{-1} \\ -C^{-1}GK^{-1} & C^{-1} \end{pmatrix}$$

We denote  $K = E_1 - B_{11}$ ;  $F = -B_{12}$ ,  $G = 0$  and  $H = E_2 - B_{22}$ .

Then we can see:

$$(E - A)^{-1} = \begin{pmatrix} K^{-1}(E + FC^{-1}GK^{-1}) & -K^{-1}FC^{-1} \\ 0 & C^{-1} \end{pmatrix}$$

However this is a contradiction to the assumption that  $(E - A)^{-1} > 0$ . Hence, the matrix  $A$  is indecomposable and this proves the theorem.

Now we consider a simple indecomposable Leontief's model as a dynamic version of the linear programming model. Let

$p_i$  = the price of the  $i$ -th good

$y_i(t)$  =  $p_i$  + the rental rate of the  $i$ -th good at time  $t$ .

$x_j(t)$  = an intensity at which the  $j$ -th process is operated at time  $t$ .

For the time being the prices  $p_i$  will be thought as given nonnegative constants. Next, suppose that the nonnegative levels of  $y_i(t)$ ,  $x_j(t)$  at  $t$  are already determined. Then analogously to condition (3) of von Neumann's model we cannot use more goods during any time period than we produced during a preceding time period. Then the primal linear programming problem is given by

$$\sum_{j=1}^n a_{ij} x_j(t+1) \leq x_i(t) \quad (i = 1, \dots, n) \quad (11)$$

$$x_j(t+1) \geq 0 \quad (j = 1, \dots, n) \quad (12)$$

and

$$\sum_{j=1}^n p_j x_j(t+1) \implies \max \quad (13)$$

i.e., the value of output is to be maximized.

On the valuation side, services rendered by goods are to be priced subject to

$$\sum_{i=1}^n a_{ij} y_i(t+1) \geq p_j \quad (j = 1, \dots, n) \quad (14)$$

$$y_i(t+1) \geq 0 \quad (i = 1, \dots, n) \quad (15)$$

and

$$\sum_{i=1}^n x_i(t) y_i(t+1) \implies \min \quad (16)$$

so that the value in each process is exhausted by remunerating, their owners at the rental rates  $y_i(t+1) - p_i$ .

From the linear programming theory we know that an equilibrium solution  $\{x_j^0(t+1), y_i^0(t+1)\}$ ,  $i, j = 1, \dots, n$  satisfies the condition.

$$\sum_{j=1}^n p_j x_j^0(t+1) = \sum_{i=1}^n x_i(t) y_i^0(t+1) \quad (17)$$

Starting at given initial values  $x_j(0) \geq 0$  and constant prices  $p_i$ , we can obtain sequences  $\{y_i(t)\}$ ,  $\{x_j(t)\}$  satisfying (11) - (16)

and

$$y_i(t) = \beta p_i \quad i = 1, \dots, n \quad (18)$$

$$x_j(t) = \alpha^t x_j \quad j = 1, \dots, n \quad (19)$$

for  $\alpha$ ,  $\beta$ ,  $p_i$ ,  $x_j$ . We know from von Neumann's model that the price vector  $p$  is unique up to the multiplication by scalars. We have defined  $y_i(t)$  as  $p_i$  plus the rental rate of the  $i$ -th good. This means that the investment of  $p_i$  dollars for the purchase of one unit of the  $i$ -th good yields the return of  $y_i(t) - p_i$  dollars as the rental for the productive service rendered by this good at period  $t$ . Therefore  $y_i(t)/p_i - 1$ , represents the rate of interest for the investment in the  $i$ -th good which may generally differ from good to good as well as from period to period. In (18) we consider the very special situation in which the rate of interest is a constant  $\beta - 1$ , common to all the goods over time. On the other hand, (19) means the balanced growth of intensities at the rate  $\alpha - 1$ . We can see that this expansion factor  $\alpha$  and interest factor  $\beta$  are equivalent to  $\alpha$  and  $\beta$  in the von Neumann model.

Now we can write our model in a matrix form. The primal problem is given by

$$\alpha Ax \leq x \tag{20}$$

$$x \geq 0 \tag{21}$$

$$\alpha px \implies \max. \tag{22}$$

where  $A = \{a_{ij}\}$  is an  $n \times n$  nonnegative matrix and  $p = (p_1 \dots p_n)$  is a row vector,  $x = (x_1, \dots, x_n)$  is a column vector,  $\alpha$  is a scalar.

The dual problem:

$$yA \geq p \tag{23}$$

$$y \geq 0 \tag{24}$$

$$xy \implies \min. \tag{25}$$

where  $y = (y_1, \dots, y_n)$  is a row vector or by (18):

$$BpA \geq p \quad (23')$$

$$p \geq 0 \quad (24')$$

$$Bpx \implies \min. \quad (25')$$

If  $p$  is a given constant von Neumann's price vector then our model is equivalent to simple models of the expansion problem:

$$\alpha Ax \leq x \quad (26)$$

$$x \geq 0 \quad (27)$$

$$\alpha \implies \max. \quad (28)$$

regarded by Gale [1]. We now show that for the Leontief's models the above inequality becomes an equation.

Theorem 2.3

If  $x_0$  and  $\alpha_0$  solve (26) - (28) then

$$x_0 = \alpha_0 Ax_0^{1)} \quad (29)$$

The dual problem takes a form:

$$BpA \geq p \quad (30)$$

$$p \geq 0 \quad (31)$$

$$B \implies \min. \quad (32)$$

---

1) Proof see by Gale [1].

Theorem 2.4

If  $p_0$  and  $\beta_0$  solve (30) - (32) then

$$p_0 = \beta_0 p_0 A \quad (33)$$

P r o o f

The idea of proof is from Gale [1] and it is similar to the proof of Theorem 3.

Suppose that there is an optimal vector  $p_0$  such that  $p_0 \leq \beta_0 p_0 A$ . We cannot have  $p_0 < \beta_0 p_0 A$ , since  $\beta_0$  could be replaced by a smaller number. Now choose an optimal vector  $p = (p_1, p_2, \dots, p_n)$  such that the strict inequality

$$p_j < \beta_0 p a_j = \beta_0 \sum_{i=1}^n p_i a_{ij} \quad (34)$$

where  $a_j$  is  $j$ -th column of matrix  $A$ , holds for as many indices as possible, say all indices in the set  $S$ , while for the remaining indices of  $S'$  we have

$$p_j = \beta_0 p a_j = \beta_0 \sum_{i=1}^n p_i a_{ij} \quad \text{for } j \in S' \quad (35)$$

We now assert

$$p_i a_{ij} = 0 \quad \text{for } i \in S, \quad j \in S' \quad (36)$$

for if this were not the case, then say,

$$p_{i_0} a_{i_0 j_0} > 0, \quad i_0 \in S, \quad j_0 \in S'. \quad \text{Then we could replace } p_{i_0} \text{ by } p_{i_0} + \epsilon,$$

where  $\epsilon$  is a positive but sufficiently small number so that inequalities (34) remain valid. But from (35):

$$p_{j_0} = \beta_0 \sum_{i=1}^n p_i a_{ij_0} < \beta_0 [p_1 a_{1j_0} + \dots + (p_{i_0} + \varepsilon) a_{i_0 j_0} + \dots + p_n a_{nj_0}]$$

so we would have a vector which increases the number of strict inequalities (34) contrary to the choice of  $p$ , and this proves (36)

Now let  $p' = (p'_1 \dots p'_n)$ , where  $p'_i = p_i$  for  $i \in S$ ,  $p'_i = 0$  for  $i \in S'$ . Replacing  $p$  by  $p'$  in (34) we see that the inequalities are not strict and in (35) we get  $0 = 0$  because of (36). But this means that we could again decrease  $\beta_0$ , at this contradicts its definition. Hence, there are no strict inequalities (34) and the theorem is proved.

Theorem 2.5

If the matrix  $A$  is indecomposable then for the Leontief's expanding model hold:

- (i)  $\alpha_0 = \frac{1}{\lambda(A)} > 0$ , where  $\lambda(A)$  is eigenvalue of  $A$ .
- (ii) the optimal intensity vector and optimal price vector are positive and unique up to multiplication by a positive number.
- (iii)  $\alpha_0 = \beta_0$

P r o o f

The equation (29) we may rewrite in the form

$$Ax^0 = \frac{1}{\alpha_0} \cdot x^0.$$

If the matrix  $A$  is indecomposable then from Theorem 3.1 of Chapter I, it follows that  $1/\alpha_0$  is a positive eigenvalue of  $A$  and  $x_0$  is a corresponding positive eigenvector and it is unique

up to multiplication by a positive number. Or we can say that in a simple expanding model is the maximal expansion factor - which is called the technological expansion factor - equal to the reciprocal of the eigenvalue of  $A$  and the level of production in all sectors will be positive over time. <sup>1)</sup>

Analogously the equation (33) we can rewrite again in the form

$$A^T p_0 = \frac{1}{\beta_0} p_0 ,$$

where  $A^T$  is transpose matrix of  $A$  and  $p_0$  is an optimal price vector (a column vector). By lemma 3.1 from Chapter I we know that if the matrix  $A$  is indecomposable, then its transpose  $A^T$  is also indecomposable. Then we can apply theorem 3.1 from Chapter I and see that  $\frac{1}{\beta_0}$  is a positive eigenvalue of  $A^T$  and  $p_0$  is a corresponding positive eigenvector and it is unique up to multiplication by a positive number. In other words, the minimal interest factor - which is called an economic expansion factor - is equal to the reciprocal of the eigenvalue of  $A^T$  and the prices in all sectors of economy are positive. <sup>2)</sup>

Now from linear algebra we know that

$$\lambda(A) = \lambda(A^T)$$

---

1) If the matrix  $A$  is decomposable then the expansion factor is nonnegative and vector  $x$  is semipositive (by Theorem 1.5 from Chapter I).

2) If the matrix  $A$  is decomposable then the interest factor is nonnegative and price vector  $p$  is semipositive (by Theorem 1.5 from Chapter I).

and this implies  $\alpha_0 = \beta_0$ , i.e., in equilibrium situation the expansion factor is equal to the interest factor.

Another alternative proof of this statement is the following one. We know that the optimal intensity vector  $x^0$  and the optimal price vector  $p^0$  are positive. By using (29) and (33) we can write:

$$\begin{aligned} \alpha_0 A x^0 &= x^0 & | \cdot p^0 & & \beta_0 p^0 A &= p^0 & | \cdot x^0 \\ \alpha_0 p^0 A x^0 &= p^0 x^0 & & & \beta_0 p^0 A x^0 &= p^0 x^0 \end{aligned}$$

From this it follows that

$$\alpha_0 p^0 A x^0 = \beta_0 p^0 A x^0$$

By using (1) and  $x^0 > 0$ ,  $p^0 > 0$  hold  $p^0 A x^0 > 0$ , thereby  $\alpha_0 = \beta_0$ .

### Theorem 2.6

The expansion factor of an indecomposable simple model is greater than 1 if and only if the matrix A is productive.

### P r o o f

Necessarity: It holds

$$\alpha A x \leq x$$

and the expansion factor in sector i is

$$\alpha_i = \frac{x_i}{a_i x}$$

$$\text{Then } \alpha = \min_i \{\alpha_i\} = \min_i \frac{x_i}{a_i x} \quad (i = 1, \dots, n)$$

where  $x = (x_1, x_2, \dots, x_i, \dots, x_n)$  is a column-vector and  $a_i$  is the i-th row of the matrix A. The matrix A is productive,

i.e.,

$$x > Ax \quad \text{and} \quad \frac{x_i}{a_i x} > 1 \quad \text{for } i = 1, \dots, n$$

This implies  $\alpha_i > 1$  for  $i = 1, \dots, n$ , it means that

$\alpha = \min_i \{\alpha_i\}$  is also greater than 1.

Sufficiency:

$$\alpha > 1 \implies \alpha_i > 1 \quad \text{for } i = 1, \dots, n$$

however  $\alpha_i = \frac{x_i}{a_i x}$  and thereby  $\frac{x_i}{a_i x} > 1$

for  $i = 1, \dots, n$

or  $x_i > a_i x$  for  $i = 1, \dots, n$ ,

i.e.,  $x > Ax$  and the matrix  $A$  is productive.

CHAPTER III

Efficient Balanced Growth and a Turnpike Example

Consider a self-sustained economy producing and consuming  $n$  categories of goods. It is assumed that all these goods are reproducible and emerge as both, inputs and outputs. Thus, an input vector  $x$  and an output vector  $y$ , which altogether compose a process  $(x, y)$ , are of the same dimension.

The technology set  $M$  is a set of technically possible processes of the above type. We shall consider our model of economy as a model of linear programming as its dynamic version, analogue to the model regarded in Chapter II.

Consider our Leontief model of an economy with  $n$  goods, with an arbitrary initial stock of goods  $x(0)$  (a column vector), with an  $n \times n$  input-output matrix  $A$  and with a given positive von Neumann price vector  $p$  (a row vector).

The dynamic specifications of the model are that the output of period  $t$  forms the capacity constraints of the period  $t + 1$ . We only assume a finite time period  $t = 1, \dots, T$ , or a finite horizon  $T$ . The objective function  $\Psi[x_1(T), x_2(T), \dots, x_n(T)]$  is a function of the terminal stock of goods only. Then our dynamic model takes the following form:

$$\begin{array}{rcl}
 Ax(1) & & \leq x(0) \\
 -Ex(1) + Ax(2) & & \leq 0 \\
 \quad - Ex(2) + Ax(3) & & \leq 0 \\
 \quad \quad \quad \vdots & & \vdots \\
 \quad \quad \quad - Ex(T-1) + Ax(T) & \leq & 0
 \end{array} \tag{1}$$

$$x(1) \geq 0, x(2) \geq 0, \dots, x(T) \geq 0 \tag{2}$$

$$px(T) \implies \max \tag{3}$$

where  $x(t)$   $t = 1, \dots, T$  is a production vector of period  $t$ . Condition (1) implies that no more goods can be used during any time period than were produced during preceding time period. Our criterion of terminal optimality will be the maximization of the value of terminal outputs  $x(T)$  for given von Neumann prices  $p$ .

A feasible path will be defined as a finite sequence of vectors  $\{x(t)\}_{t=1}^T$  satisfying (1) and (2).

An efficient path is a feasible path satisfying (3). In other words an efficient path will be defined as a path which is terminally optimal for a given von Neumann price vector  $p$ . The technology set  $M$  in our model is a set of solutions of inequalities (1) and (2) or it is a set of all feasible paths. From the theory of linear programming we know that  $M$  is a closed convex set.

A (nontrivial) feasible path  $\{x(t)\}_{t=1}^T$  of finite horizon is called a balanced growth path if

$$x(t+1) = \alpha x(t) \quad (t=0, 1, \dots, T)$$

where  $\alpha$  is a nonnegative constant.  $\alpha$  is termed the growth factor while  $\alpha-1$  is the rate of growth.

A balanced growth path can be put in the form:

$$x(t) = \alpha^t x, \quad x = x(0) \geq 0$$

A balanced growth path having a maximum growth factor over all balanced growth paths is called as the von Neumann path or the von Neumann ray, generalizing from the original von Neumann model.

Dorfman, Samuelson, Solow showed that for an efficient path the marginal rate of substitution between any two goods as inputs in period  $t+1$  is equal to their marginal rate of substitution as outputs in period  $t$  (intertemporal efficiency conditions)<sup>1)</sup>

In terms of linear programming this means that the ratio of dual prices for any two goods is constant over time (for an efficient path). Our dual model is given by:

$$\begin{aligned}
 y(1)A - y(2)E & \geq 0 \\
 y(2)A - y(3)E & \geq 0 \\
 y(3)A - y(4)E & \geq 0 \\
 & \dots \\
 y(T)A & \geq p
 \end{aligned}
 \tag{1d}$$

$$y(1) \geq 0, y(2) \geq 0 \dots \dots y(T) \geq 0
 \tag{2d}$$

$$y(1).x(0) \xrightarrow{\quad\quad\quad} \min
 \tag{3d}$$

We have defined  $y_i(t) = p_i$  + the rental of the  $i$ -th good at  $t$ . Then the condition (1d) means that the revenue per unit cannot exceed costs per unit or that profit per unit of a good cannot be positive. In equilibrium situation or for efficient paths must hold:

$$p.x(T) = y(1).x(0)
 \tag{4}$$

i.e., the costs for initial input are equal to the revenue at the end of planning horizon. In other words, the value of initial input is equal to the value of terminal output.

Dorfman, Samuelson, Solow's intertemporal efficiency conditions

1) See Lancaster or Dosso chapter 12 although the notation is different.

in our notation are given by:

$$\frac{y_i(t)}{y_j(t)} = \frac{y_i(t+1)}{y_j(t+1)} \quad \text{for } i, j = 1, \dots, n \quad (5)$$

and  $t = 1, 2, \dots, T-1$

The principle of optimality

A very useful and interesting concept in dynamic models is Bellman's principle of optimality which is a basis for dynamic programming:

An optimal policy has the property that whatever the initial state and initial decision are, the remaining decision must constitute an optimal policy with regard to the state resulting from the first decision.

This means, if  $\{q_1, \dots, q_n\}$  is an optimal strategy of n-stage process with initial state  $r_1$ , then the sequence of decisions  $\{q_2, \dots, q_n\}$  is an optimal strategy of n-1 stage process with initial state  $r_2$ .

In our dynamic model the principle implies that if a path is terminally optimal, commences from a point  $x(0)$  and passes through  $x(t)$  on its way to the terminal point  $x(T)$ , then the sequence from  $x(t)$  to  $x(T)$  is a terminally optimal with respect to initial point  $x(t)$ .

Lancaster showed that the terminally optimal paths satisfy the principle of optimality. We shall illustrate this statement later on several numerical examples.

Starting at any nonzero point  $x(0)$  the efficient path needs not conform to the pattern of balanced growth. We can prove that there will always exist one and only one relative configuration of initial stocks that will permit all capacities to grow at the same percentage rate with no excess capacity.<sup>1)</sup>

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1) See Dosso, chapter 12.

However - in general - in dynamic models with an arbitrary nonzero initial point  $x(0)$  the efficient path has an inclination for leaning to the von Neumann path and then to move off to the terminal point. The longer the time horizon is the more persistently their proximity to the von Neumann path will occur. This class of theorems that are concerned with the relationship of optimal growth paths to the von Neumann path, are generally referred to as turnpike theorems. The name was originally given by Dorfman, Samuelson and Solow. Samuelson compares a maximal balanced growth ray to a turnpike in an automobile highway system. If a person goes from a town to a nearby town, he proceeds directly to the final goal, not using a turnpike. However, when he makes a very long journey to a remote town, he first reaches a turnpike, then goes on it for most of the time, and finally gets off it to use a local road leading to the goal town.

Several turnpike theorems of rather kinds have been proved, the best known being those of Radner-Nikaido<sup>1)</sup>, McKenzie<sup>2)</sup> and Morishima<sup>3)</sup>. Radner's turnpike theorem with terminal optimality states<sup>4)</sup>:

For an economy having a neoclassical transformation relationship  $F(y,x)=0$  which is strictly convex around the von Neumann way, then, for any  $\epsilon > 0$ , a terminally optimal path over a finite number of periods  $T$  will be at a distance<sup>5)</sup> greater than, or equal to,  $\epsilon$

1) See Radner and Nikaido

2) See McKenzie [1], [2], and [3]

3) See Morishima [1], [2]

4) See Lancaster p. 185, but with other notation

5) The distance between two vectors  $x, y$  is defined:  
where  $|v|$  denotes the Euclidean

$$d(x,y) = \left| \frac{x}{|x|} - \frac{y}{|y|} \right| \text{ norm of } v.$$

from the von Neumann path for not more than  $S$  periods, where  $S$  is a finite number independent of  $T$  but, in general, dependent on  $\xi$ .

McKenzie<sup>1)</sup> extended the Radner's theorem for a Leontief model and proved - under assumption of the indecomposability of the input-output matrix - the convergence of the efficient path to the Neumann facet.

We have attempted to illustrate this theorem and to investigate additional properties of efficient path on our Leontief dynamic model with numerical data. Our examples are very simple with only two sectors, with arbitrary initial point  $x(0)$  and with the terminal von Neumann prices. We have investigated two different planning horizons, 5 years and 10 years such that we can better compare the convergence to the von Neumann path. We have constructed our input-output matrix such that it is indecomposable and productive. We see from our numerical examples and from our pictures that the efficient path in fact for most of the planning horizon goes in the neighbourhood of the von Neumann path (approximately until 7-th year) and then gets off to the terminal point. The balanced growth factor is exactly to the reciprocal of the eigen-value of matrix  $A$ . (See theorem 5 from chapter II). In our examples the maximal eigenvalue of matrix  $A$  is  $\lambda = 0,7$  and the balanced growth factor - as we can see -  $\alpha = 1,42$ .

Very interesting is the "jump" in the value of objective function in the last period resp. the growth factor between the period  $T-1$  and  $T$ . This growth factor or the movement to the terminal point is determined by given price level. We can prove the following theorem:

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1) See McKenzie [2]

Theorem 1

The growth factor of the value of objective function in the last period  $\alpha_T$  is determined by

$$\alpha_T = \beta = \max_j \frac{p_j}{pa_j} \quad j = 1, \dots, n \quad (6)$$

where  $p$  is a given price vector (a row vector) and  $a_j$  is the  $j$ -th column of the matrix  $A$ .

P r o o f

From duality theorem of linear programming follows:

$$[y(T) \cdot A - p] \cdot x(T) = 0$$

$$y(T) \cdot A \cdot x(T) = p \cdot x(T)$$

However  $Ax(T) = x(T-1)$  from theorem 5 from chapter II under assumption of indecomposability of matrix  $A$ .

Then we can write:

$$y(T) \cdot x(T-1) = p \cdot x(T) \quad (7)$$

We have defined  $y_t = \beta \cdot p$  in (18) in chapter II.

Then our dual problem takes a form as in chapter II:

$$\beta p A \geq p$$

$$\beta \geq \frac{p_j}{pa_j} \quad j = 1, \dots, n$$

$$\beta = \max_j \frac{p_j}{pa_j} \quad (8)$$

where  $a_j$  is the  $j$ -th column of matrix  $A$  and  $p$  is the given price vector.

Then (7) we can rewrite:

$$\beta \cdot p \cdot x(T-1) = p \cdot x(T)$$

$$\frac{p \cdot x(T)}{p \cdot x(T-1)} = \beta = \alpha_T$$

where  $\beta$  is defined by (8). This completes the proof.

Gale in [2] p. 290 says:  $\beta(p)$  represents the maximum rate at which the value of goods in the model can expand at prices  $p$ . In all our examples we see that this rate of growth is actually maximal at which the value of goods - in other words our objective function - expands during the planning horizon  $T$  and is determined by (8).

We have said that the terminally optimal path satisfies the principle of optimality. In figures 1 and 4 we can compare the efficient paths for two different planning horizons, 5 years and 10 years. We see that the efficient path for 5 years oscillates more or the distance from the von Neumann path is greater than for 10 years. The planning period of 5 years is short for the present turnpike theorem. We know that the longer the time horizon is, the more persistently the proximity to the von Neumann path will occur. This corresponds Samuelson's a maximal balanced-growth ray to a turnpike in an automobile highway system.

However very interesting is the comparison between the efficient path for 5 years and the segment from  $x(6)$  to  $x(10)$  of the efficient path for 10 years. We see that these two segments are parallel or in other words the production in both sectors and the value of objective function grow at the same rate, analogue the dual solutions. This corresponds to the principle optimality by Bellman, which we have already spoken about.

Very useful tools of economic analysis in linear programming dynamic models are the dual variables or the shadow prices. We have said that for an efficient path the ratio of dual prices for any two goods is constant over time. But in our examples with arbitrary initial stock  $x(0)$  and with the present turnpike theorem we see that this ratio is not constant over time however with growing  $t$  converges to the constant ratio (see our appendix). This means that the marginal rate of substitution converges to the constant.

Let us investigate the ratio of dual prices  $y_{i,t}$  and  $y_{i,t-1}$  for the  $i$ -th sector and for  $t = 1, 2, \dots, T$ . From this ratio we can analyse the use of capacities between periods  $t$  and  $t-1$ . Laščiak defined this ratio as the rate of normal efficiency. We see from all our examples that this ratio for both sectors with growing  $t$  converges to the constant - denoted by  $\beta_0$  - and it is equal exactly to the balanced growth rate. In our examples we can write:

$$\frac{y_{t-1}}{y_t} \longrightarrow \beta_0 = \alpha_0 = 1,42$$

with growing  $t$ . This means that the efficient path for growing  $t$  leads to a better use of capacities or in other words, the losses due to insufficiency of capacity decrease exactly with the rate equal to the balanced growth rate of production and value of objective function from the primal problem.

This property of dual prices is known as dual turnpike theorem.<sup>1)</sup>

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1) see Tsukui.

APPENDIX

The examples with solutions.

Example 1

$$A = \begin{pmatrix} 0,3 & 0,4 \\ 0,5 & 0,2 \end{pmatrix}$$

$$x(0) = \begin{pmatrix} 60 \\ 80 \end{pmatrix} \quad p = (10, 12) \quad \alpha_0 = \frac{1}{\lambda(A)} = \frac{1}{0,7} = 1,42$$

a) T = 10 years

The optimal solution to the primal problem:

$x_1 = 85,71$	}	1,42	$x_2 = 85,71$	}	1,42
$x_3 = 122,44$			$x_4 = 122,45$		
$x_5 = 174,95$	}	1,42	$x_6 = 174,89$	}	1,42
$x_7 = 249,76$			$x_8 = 250,06$		
$x_9 = 357,67$	}	1,43	$x_{10} = 356,14$	}	1,42
$x_{11} = 506,59$			$x_{12} = 514,24$		
$x_{13} = 745,55$	}	1,41	$x_{14} = 707,31$	}	1,37
$x_{15} = 955,83$			$x_{16} = 1.147,01$		
$x_{17} = 1.911,63$	}	1,28	$x_{18} = 955,84$	}	1,62
$x_{19} = 0$			$x_{20} = 4.778,94$		

The optimal solution to the dual problem:

$y_1 = 955,78$		$y_2 = 0$	
$y_3 = 286,71$		$y_4 = 382,31$	$y_3/y_4 \doteq 0,74$
$y_5 = 277,18$		$y_6 = 191,16$	$y_5/y_6 \doteq 1,44$
$y_7 = 178,74$		$y_8 = 149,11$	$y_7/y_8 \doteq 1,19$
$y_9 = 128,17$	$\left. \begin{array}{l} 1,43 \\ 1,42 \\ 1,42 \\ 1,42 \\ 1,42 \\ 1,42 \end{array} \right\}$	$y_{10} = 101,31$	$y_9/y_{10} \doteq 1,26$
$y_{11} = 89,11$		$y_{12} = 71,53$	$y_{11}/y_{12} \doteq 1,24$
$y_{13} = 62,50$		$y_{14} = 49,95$	$y_{13}/y_{14} \doteq 1,25$
$y_{15} = 43,72$		$y_{16} = 34,99$	$y_{15}/y_{16} \doteq 1,24$
$y_{17} = 30,61$		$y_{18} = 24,48$	$y_{17}/y_{18} \doteq 1,25$
$y_{19} = 21,42$		$y_{20} = 17,14$	$y_{19}/y_{20} \doteq 1,24$

The values of objective function:

$AZ(1) = 1,885,62$	$\left. \begin{array}{l} 1,42 \\ 1,42 \\ 1,42 \\ 1,42 \\ 1,43 \\ 1,41 \\ 1,46 \\ 1,31 \\ 1,87 \end{array} \right\}$
$AZ(2) = 2.693,80$	
$AZ(3) = 3.848,18$	
$AZ(4) = 5.498,32$	
$AZ(5) = 7.850,38$	
$AZ(6) = 11.236,78$	
$AZ(7) = 15.943,22$	
$AZ(8) = 23.322,42$	
$AZ(9) = 30.586,38$	
$AZ(10) = 57.347,304$	

b) T = 5 years

The optimal solution to the primal problem:

$$\begin{array}{rcl}
 x_1 = 84,98 & \left. \vphantom{x_1} \right\} & 1,47 \\
 x_3 = 125,06 & \left. \vphantom{x_3} \right\} & 1,28 \\
 x_5 = 160,34 & \left. \vphantom{x_5} \right\} & 2 \\
 x_7 = 320,68 & & \\
 x_9 = 0 & & \\
 x_2 = 86,26 & \left. \vphantom{x_2} \right\} & 1,37 \\
 x_4 = 118,65 & \left. \vphantom{x_4} \right\} & 1,62 \\
 x_6 = 192,41 & \left. \vphantom{x_6} \right\} & 0,83 \\
 x_8 = 160,34 & \left. \vphantom{x_8} \right\} & 5 \\
 x_{10} = 801,71 & & 
 \end{array}$$

The optimal solution to the dual problem:

$$\begin{array}{rcl}
 y_1 = 160,34 & & y_2 = 0 \\
 y_3 = 48,10 & & y_4 = 64,13 \quad y_3/y_4 \doteq 0,74 \\
 y_5 = 46,49 & & y_6 = 32,06 \quad y_5/y_6 \doteq 1,45 \\
 y_7 = 29,98 & & y_8 = 25,01 \quad y_7/y_8 \doteq 1,2 \\
 y_9 = 21,50 & & y_{10} = 16,99 \quad y_9/y_{10} \doteq 1,26
 \end{array}$$

The values of objective function:

$$\begin{array}{rcl}
 AZ(1) = 1.884,92 & \left. \vphantom{AZ(1)} \right\} & 1,41 \\
 AZ(2) = 2.674,40 & \left. \vphantom{AZ(2)} \right\} & 1,46 \\
 AZ(3) = 3.912,32 & \left. \vphantom{AZ(3)} \right\} & 1,31 \\
 AZ(4) = 5.130,88 & \left. \vphantom{AZ(4)} \right\} & 1,87 \\
 AZ(5) = 9.620,534 & & 
 \end{array}$$

$$B = \max_j \frac{p_j}{p_{a_j}} = 1,87$$

Example 2

$$A = \begin{pmatrix} 0,3 & 0,4 \\ 0,5 & 0,2 \end{pmatrix}$$

$$x(0) = \begin{pmatrix} 60 \\ 80 \end{pmatrix} \quad p = (2,3) \quad \alpha_0 = \frac{1}{\lambda(A)} = \frac{1}{0,7} = 1,42$$

a) T = 10 years

The optimal solution to the primal problem:

$x_1 = 85,71$	} 1,42	$x_2 = 85,71$	} 1,42
$x_3 = 122,44$		$x_4 = 122,45$	
$x_5 = 174,95$	} 1,42	$x_6 = 174,89$	} 1,42
$x_7 = 249,76$		$x_8 = 250,06$	
$x_9 = 357,67$	} 1,41	$x_{10} = 356,14$	} 1,44
$x_{11} = 506,59$		$x_{12} = 514,24$	
$x_{13} = 745,55$	} 1,28	$x_{14} = 707,31$	} 1,62
$x_{15} = 955,83$		$x_{16} = 1.147,01$	
$x_{17} = 1.911,63$	} 1,99	$x_{18} = 955,84$	} 0,83
$x_{19} = 0$		$x_{20} = 4.778,94$	

The optimal solution to the dual problem:

$y_1 = 238,94$		$y_2 = 0$		
$y_3 = 71,67$		$y_4 = 95,57$	$y_3/y_4 \doteq 0,74$	
$y_5 = 69,29$		$y_6 = 47,79$	$y_5/y_6 \doteq 1,44$	
$y_7 = 44,68$		$y_8 = 37,27$	$y_7/y_8 \doteq 1,19$	
$y_9 = 32,04$		$y_{10} = 25,32$	$y_9/y_{10} \doteq 1,26$	
$y_{11} = 22,27$		$y_{12} = 17,88$	$y_{11}/y_{12} \doteq 1,24$	
$y_{13} = 15,62$	} 1,42	} 1,42	$y_{14} = 12,48$	$y_{13}/y_{14} \doteq 1,25$
$y_{15} = 10,93$			$y_{16} = 8,74$	$y_{15}/y_{16} \doteq 1,25$
$y_{17} = 7,65$			$y_{18} = 6,12$	$y_{17}/y_{18} \doteq 1,25$
$y_{19} = 5,35$			$y_{20} = 4,28$	$y_{19}/y_{20} \doteq 1,25$

The values of objective function:

$AZ(1) = 428,55$	} 1,42	
$AZ(2) = 612,23$		
$AZ(3) = 874,57$		
$AZ(4) = 1.249,70$		
$AZ(5) = 1.783,76$		
$AZ(6) = 2.555,90$		1,43
$AZ(7) = 3.613,03$		1,41
$AZ(8) = 5.352,69$		1,48
$AZ(9) = 6.690,78$		1,24
$AZ(10) = 14.336,82$		2,14

b) T = 5 years

The optimal solution to the primal problem:

$$\begin{array}{l} x_1 = 84,98 \\ x_3 = 125,06 \\ x_5 = 160,34 \\ x_7 = 320,68 \\ x_9 = 0 \end{array} \left. \begin{array}{l} \\ \\ \\ \\ \end{array} \right\} \begin{array}{l} 1,47 \\ 1,28 \\ 2 \\ \end{array} \quad \begin{array}{l} x_2 = 86,26 \\ x_4 = 118,65 \\ x_6 = 192,41 \\ x_8 = 160,34 \\ x_{10} = 801,51 \end{array} \left. \begin{array}{l} \\ \\ \\ \\ \end{array} \right\} \begin{array}{l} 1,37 \\ 1,62 \\ 0,83 \\ 5 \end{array}$$

The optimal solution to the dual problem:

$$\begin{array}{l} y_1 = 40,08 \\ y_3 = 12,02 \\ y_5 = 11,62 \\ y_7 = 7,49 \\ y_9 = 5,37 \end{array} \quad \begin{array}{l} y_2 = 0 \\ y_4 = 16,03 \\ y_6 = 8,01 \\ y_8 = 6,25 \\ y_{10} = 4,24 \end{array} \quad \begin{array}{l} \\ y_3/y_4 \doteq 0,74 \\ y_5/y_6 \doteq 1,45 \\ y_7/y_8 \doteq 1,19 \\ y_9/y_{10} \doteq 1,26 \end{array}$$

The values of objective function:

$$\begin{array}{l} AZ(1) = 428,74 \\ AZ(2) = 606,07 \\ AZ(3) = 897,91 \\ AZ(4) = 1.122,38 \\ AZ(5) = 2.405,133 \end{array} \left. \begin{array}{l} \\ \\ \\ \\ \end{array} \right\} \begin{array}{l} 1,41 \\ 1,48 \\ 1,24 \\ 2,14 \end{array}$$

$$B = \max_j \frac{P_j}{p_{aj}} = 2,14$$

Example 3

$$A = \begin{pmatrix} 0,3 & 0,4 \\ 0,5 & 0,2 \end{pmatrix}$$

$$\alpha_0 = \frac{1}{\lambda(A)} = \frac{1}{0,7} = 1,42 \quad x(0) = \begin{pmatrix} 75 \\ 75 \end{pmatrix} \quad p = (2,3)$$

T = 10 years

The optimal solution to the primal problem:

$x_1 = 107,14$	}	1,42	$x_2 = 107,14$	}	1,42
$x_3 = 153,05$			$x_4 = 153,06$		
$x_5 = 218,69$	}	1,42	$x_6 = 218,61$	}	1,42
$x_7 = 312,19$			$x_8 = 312,58$		
$x_9 = 447,08$	}	1,43	$x_{10} = 445,17$	}	1,42
$x_{11} = 633,23$			$x_{12} = 642,79$		
$x_{13} = 931,93$	}	1,47	$x_{14} = 884,13$	}	1,37
$x_{15} = 1.194,78$			$x_{16} = 1.433,75$		
$x_{17} = 2.389,51$	}	2	$x_{18} = 1.194,79$	}	0,83
$x_{19} = 0$			$x_{20} = 5.973,62$		

The values of objective function:

AZ(1)	=	535,70	
AZ(2)	=	765,28	1,42
AZ(3)	=	1.093,18	1,42
AZ(4)	=	1.562,12	1,42
AZ(5)	=	2.229,67	1,42
AZ(6)	=	3.194,83	1,43
AZ(7)	=	4.516,25	1,41
AZ(8)	=	6.690,81	1,48
AZ(9)	=	8.363,39	1,24
AZ(10)	=	17.920,886	2,14

$$B = \max_j \frac{P_j}{pa_j} = 2,14$$

$y_1$	=	0		$y_2$	=	238,94	
$y_3$	=	119,46		$y_4$	=	47,78	$y_3/y_4 \doteq 2,5$
$y_5$	=	59,73		$y_6$	=	57,34	$y_5/y_6 \doteq 1,04$
$y_7$	=	46,59		$y_8$	=	35,36	$y_7/y_8 \doteq 1,31$
$y_9$	=	31,66		$y_{10}$	=	25,71	$y_9/y_{10} \doteq 1,23$
$y_{11}$	=	22,35		$y_{12}$	=	17,80	$y_{11}/y_{12} \doteq 1,25$
$y_{13}$	=	15,60	$\left. \begin{array}{l} 1,42 \\ 1,42 \\ 1,42 \end{array} \right\}$	$y_{14}$	=	12,50	$y_{13}/y_{14} \doteq 1,24$
$y_{15}$	=	10,93		$y_{16}$	=	8,74	$y_{15}/y_{16} \doteq 1,25$
$y_{17}$	=	7,65		$y_{18}$	=	6,12	$y_{17}/y_{18} \doteq 1,25$
$y_{19}$	=	5,35		$y_{20}$	=	4,28	$y_{19}/y_{20} \doteq 1,25$

Example 4

$$A = \begin{pmatrix} 0,3 & 0,3 \\ 0,5 & 0,2 \end{pmatrix}$$

$$\alpha_0 = \frac{1}{\lambda(A)} = \frac{1}{0,7} = 1,42$$

$$x(0) = \begin{pmatrix} 80 \\ 60 \end{pmatrix} \quad p = (10,12)$$

T = 10 years

$$\begin{array}{l} x_1 = 85,71 \\ x_3 = 122,44 \\ x_5 = 174,95 \\ x_7 = 249,75 \\ x_9 = 357,67 \\ x_{11} = 506,58 \\ x_{13} = 745,55 \\ x_{15} = 955,82 \\ x_{17} = 1911,61 \\ x_{19} = 0 \end{array} \left. \begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \right\} \begin{array}{l} 1,42 \\ 1,42 \\ 1,42 \\ 1,43 \\ 1,41 \\ 1,47 \\ 1,28 \\ 1,99 \end{array}$$

$$\begin{array}{l} x_2 = 85,71 \\ x_4 = 122,45 \\ x_6 = 174,89 \\ x_8 = 250,06 \\ x_{10} = 356,14 \\ x_{12} = 514,23 \\ x_{14} = 707,31 \\ x_{16} = 1.147,00 \\ x_{18} = 955,83 \\ x_{20} = 4.778,90 \end{array} \left. \begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \right\} \begin{array}{l} 1,42 \\ 1,42 \\ 1,42 \\ 1,42 \\ 1,44 \\ 1,37 \\ 1,62 \\ 0,83 \\ 4,99 \end{array}$$

The values of objective function:

AZ(1)	=	1.885,62	}	1,42
AZ(2)	=	2.693,80		
AZ(3)	=	3.848,18	}	1,42
AZ(4)	=	5.498,32		
AZ(5)	=	7.850,38	}	1,42
AZ(6)	=	11.236,78		
AZ(7)	=	15.943,22	}	1,43
AZ(8)	=	23.322,42		
AZ(9)	=	30.586,38	}	1,41
AZ(10)	=	57.346,834		

$$B = \max_j \frac{p_j}{p_{aj}} = 1,87$$

The optimal solution to the dual problem:

$y_1 = 0$	$y_2 = 955,78$	
$y_3 = 477,87$	$y_4 = 191,15$	$y_3/y_4 \doteq 2,49$
$y_5 = 238,95$	$y_6 = 229,39$	$y_5/y_6 \doteq 1,04$
$y_7 = 186,38$	$y_8 = 141,46$	$y_7/y_8 \doteq 1,31$
$y_9 = 126,64$	$y_{10} = 102,84$	$y_9/y_{10} \doteq 1,23$
$y_{11} = 89,41$	$y_{12} = 71,22$	$y_{11}/y_{12} \doteq 1,25$
$y_{13} = 62,43$	} 1,42	} 1,42
$y_{15} = 43,73$		
$y_{17} = 30,61$		
$y_{19} = 21,42$		
	$y_{14} = 50,01$	$y_{13}/y_{14} \doteq 1,24$
	$y_{16} = 34,97$	$y_{15}/y_{16} \doteq 1,25$
	$y_{18} = 24,49$	$y_{17}/y_{18} \doteq 1,24$
	$y_{20} = 17,14$	$y_{19}/y_{20} \doteq 1,24$

Example 5

$$A = \begin{pmatrix} 0,3 & 0,4 \\ 0,5 & 0,2 \end{pmatrix}$$

$$\alpha_0 = \frac{1}{\lambda(A)} = \frac{1}{0,7} = 1,42$$

$$x(0) = \begin{pmatrix} 80 \\ 60 \end{pmatrix} \quad p = (2,3)$$

T = 10 years

The optimal solution to the primal problem:

$x_1 = 85,71$	}	1,42	$x_2 = 85,71$	}	1,42
$x_3 = 122,44$			$x_4 = 122,45$		
$x_5 = 174,95$	}	1,42	$x_6 = 174,89$	}	1,42
$x_7 = 249,75$			$x_8 = 250,06$		
$x_9 = 357,67$	}	1,41	$x_{10} = 356,14$	}	1,44
$x_{11} = 506,58$			$x_{12} = 514,23$		
$x_{13} = 745,55$	}	1,28	$x_{14} = 707,31$	}	1,62
$x_{15} = 955,82$			$x_{16} = 1.147,00$		
$x_{17} = 1.911,61$	}	1,99	$x_{18} = 955,83$	}	0,83
$x_{19} = 0$			$x_{20} = 4.778,90$		

The values of objective function:

AZ(1)	=	428,55	}	1,42
AZ(2)	=	612,23		
AZ(3)	=	874,57	}	1,42
AZ(4)	=	1.249,70		
AZ(5)	=	1.783,76	}	1,42
AZ(6)	=	2.555,90		
AZ(7)	=	3.613,03	}	1,41
AZ(8)	=	5.352,69		
AZ(9)	=	6.690,78	}	1,48
AZ(10)	=	14.336,708		

$$\beta = \max_j \frac{p_j}{pa_j} = 2,14$$

The optimal dual solution:

$y_1$	=	0	$y_2$	=	238,94						
$y_3$	=	119,46	$y_4$	=	47,78	$y_3/y_4$	$\hat{=}$	2,5			
$y_5$	=	59,73	$y_6$	=	57,34	$y_5/y_6$	$\hat{=}$	1,04			
$y_7$	=	46,59	$y_8$	=	35,36	$y_7/y_8$	$\hat{=}$	1,31			
$y_9$	=	31,66	$y_{10}$	=	25,71	$y_9/y_{10}$	$\hat{=}$	1,23			
$y_{11}$	=	22,35	$y_{12}$	=	17,80	$y_{11}/y_{12}$	$\hat{=}$	1,25			
$y_{13}$	=	15,60	}	1,42	}	$y_{14}$	=	12,50	$y_{13}/y_{14}$	$\hat{=}$	1,24
$y_{15}$	=	10,93									
$y_{17}$	=	7,65	}	1,42	}	$y_{16}$	=	8,74	$y_{15}/y_{16}$	$\hat{=}$	1,25
$y_{19}$	=	5,35									
			}	1,42	}	$y_{18}$	=	6,12	$y_{17}/y_{18}$	$\hat{=}$	1,25
			}	1,42	}	$y_{20}$	=	4,28	$y_{19}/y_{20}$	$\hat{=}$	1,25

Example 6

$$A = \begin{pmatrix} 0,3 & 0,4 \\ 0,5 & 0,2 \end{pmatrix}$$

$$\alpha_0 = \frac{1}{\lambda(A)} = \frac{1}{0,7} = 1,42$$

$$x(0) = \begin{pmatrix} 75 \\ 75 \end{pmatrix} \quad p = (10,12)$$

T = 10 years

The values of objective function:

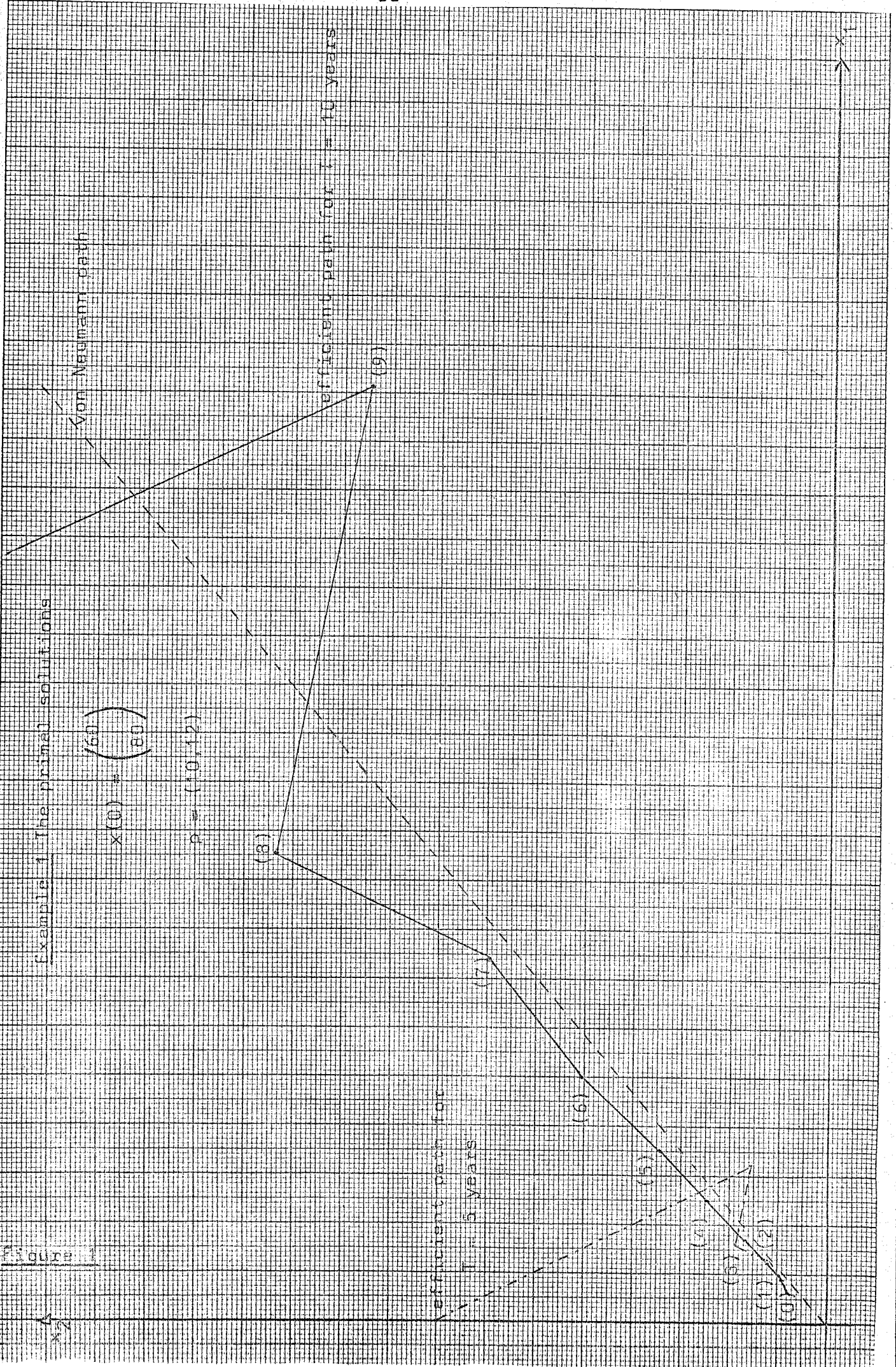
AZ(1) = 2.357,08	}	1,42
AZ(2) = 3.367,22		
AZ(3) = 4.810,22	}	1,42
AZ(4) = 6.812,84		
AZ(5) = 9.812,84	}	1,42
AZ(6) = 14.045,78		
AZ(7) = 19.928,86	}	1,41
AZ(8) = 29.152,80		
AZ(9) = 38.232,58	}	1,46
AZ(10) = 71.683,545		

$$\beta = \max_j \frac{p_j}{pa_j} = 1,87$$

The optimal dual solution:

$y_1 = 0$			$y_2 = 955,78$		
$y_3 = 477,87$			$y_4 = 191,15$	$y_3/y_4 \doteq 2,49$	
$y_5 = 238,95$			$y_6 = 229,39$	$y_5/y_6 \doteq 1,04$	
$y_7 = 186,38$			$y_8 = 141,46$	$y_7/y_8 \doteq 1,31$	
$y_9 = 126,64$			$y_{10} = 102,84$	$y_9/y_{10} \doteq 1,23$	
$y_{11} = 89,41$			$y_{12} = 71,22$	$y_{11}/y_{12} \doteq 1,25$	
$y_{13} = 62,43$	} 1,42	1,42 {	$y_{14} = 50,01$	$y_{13}/y_{14} \doteq 1,24$	
$y_{15} = 43,73$			$y_{16} = 34,97$	$y_{15}/y_{16} \doteq 1,25$	
$y_{17} = 30,61$			$y_{18} = 24,49$	$y_{17}/y_{18} \doteq 1,24$	
$y_{19} = 21,42$			$y_{20} = 17,14$	$y_{19}/y_{20} \doteq 1,24$	

Figure



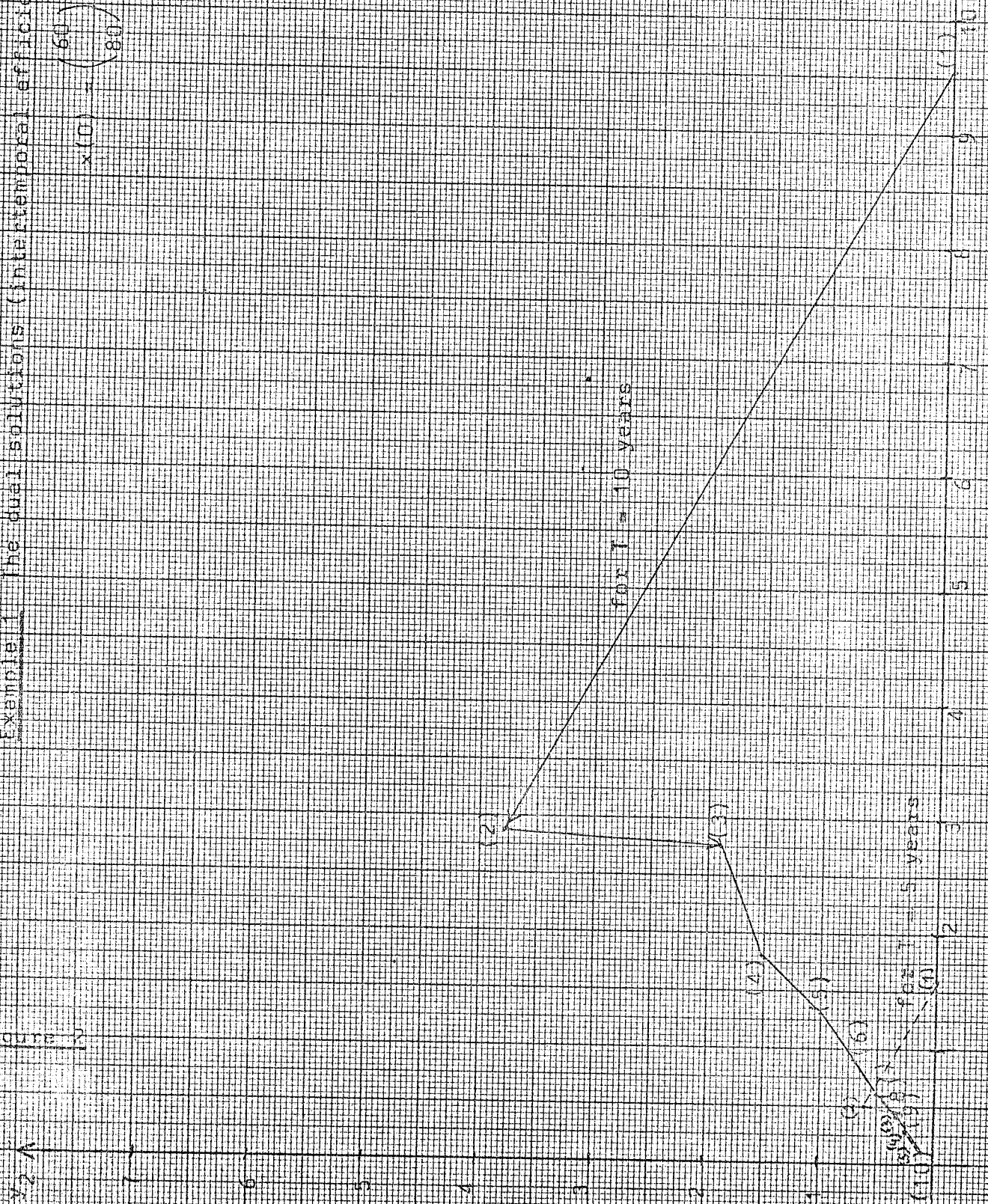
Example 1 (the dual solutions (intertemporal efficiency conditions))

$$x(0) = \begin{pmatrix} 60 \\ 80 \end{pmatrix}$$

$$p = (10, 12)$$

Figure 2

$y_2$



Example 1 The dual solutions:

$$x(0) = \begin{pmatrix} 60 \\ 80 \end{pmatrix} \quad p = (10, 12)$$

for  $T = 10$  years

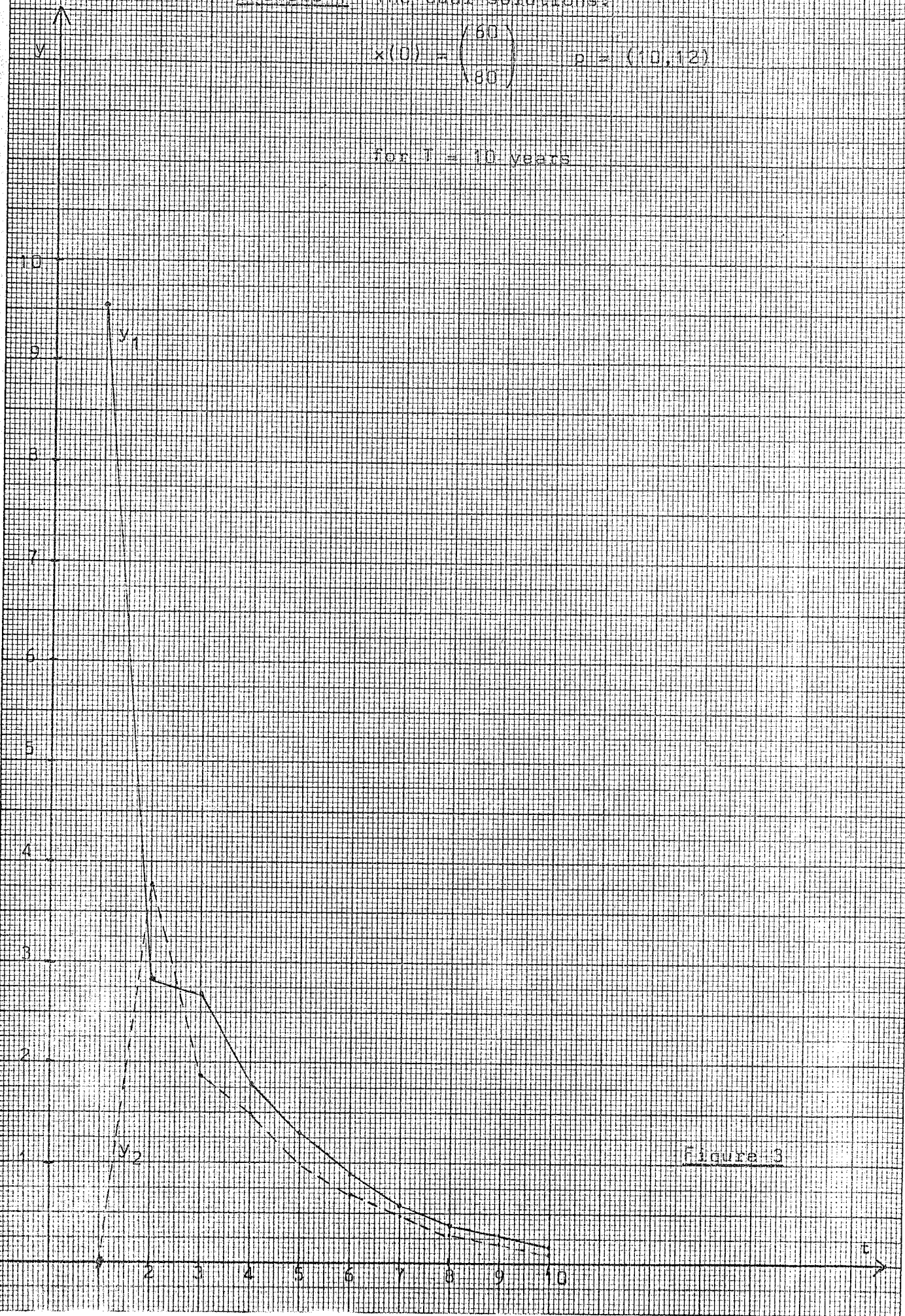
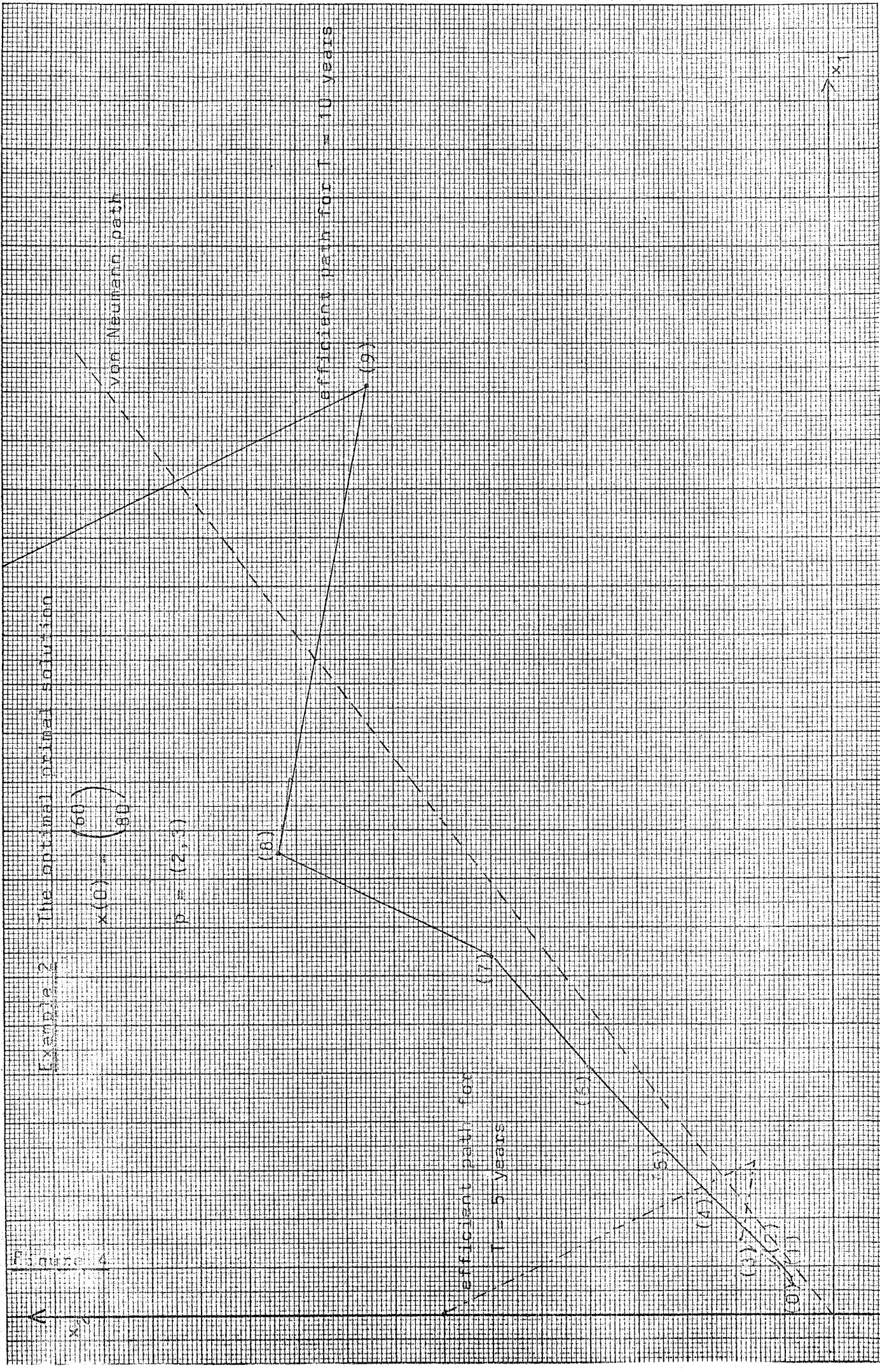


Figure 3

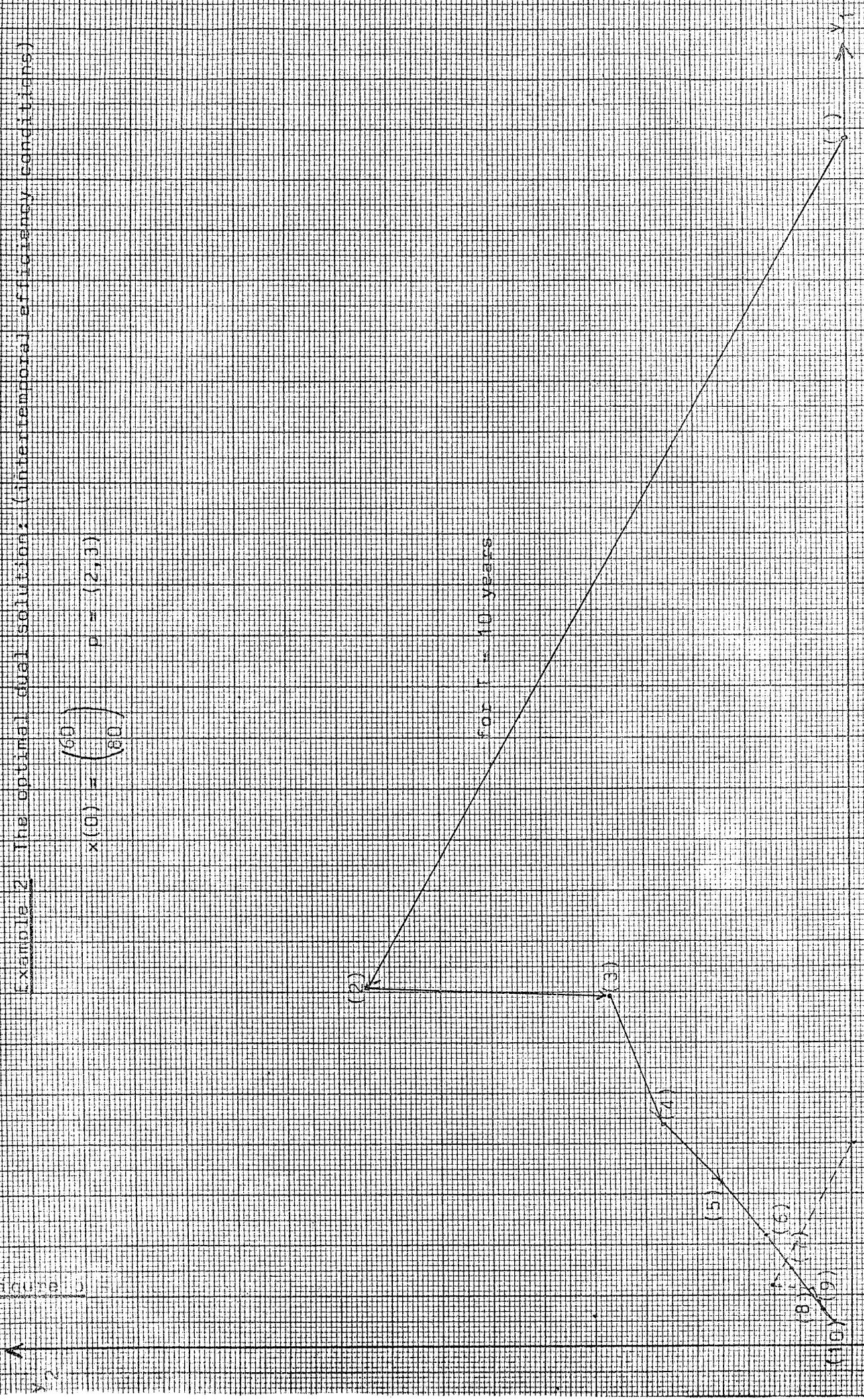


Example 2 The optimal primal solution

Example 2 The optimal dual solution: (intertemporal efficiency conditions)

$$x(0) = \begin{pmatrix} 60 \\ 80 \end{pmatrix} \quad p = (2, 3)$$

Figure 1



Example 2 The optimal dual solution

$$x(0) = \begin{pmatrix} 60 \\ 80 \end{pmatrix}$$

$$p = (2, 3)$$

for  $l = 10$  years

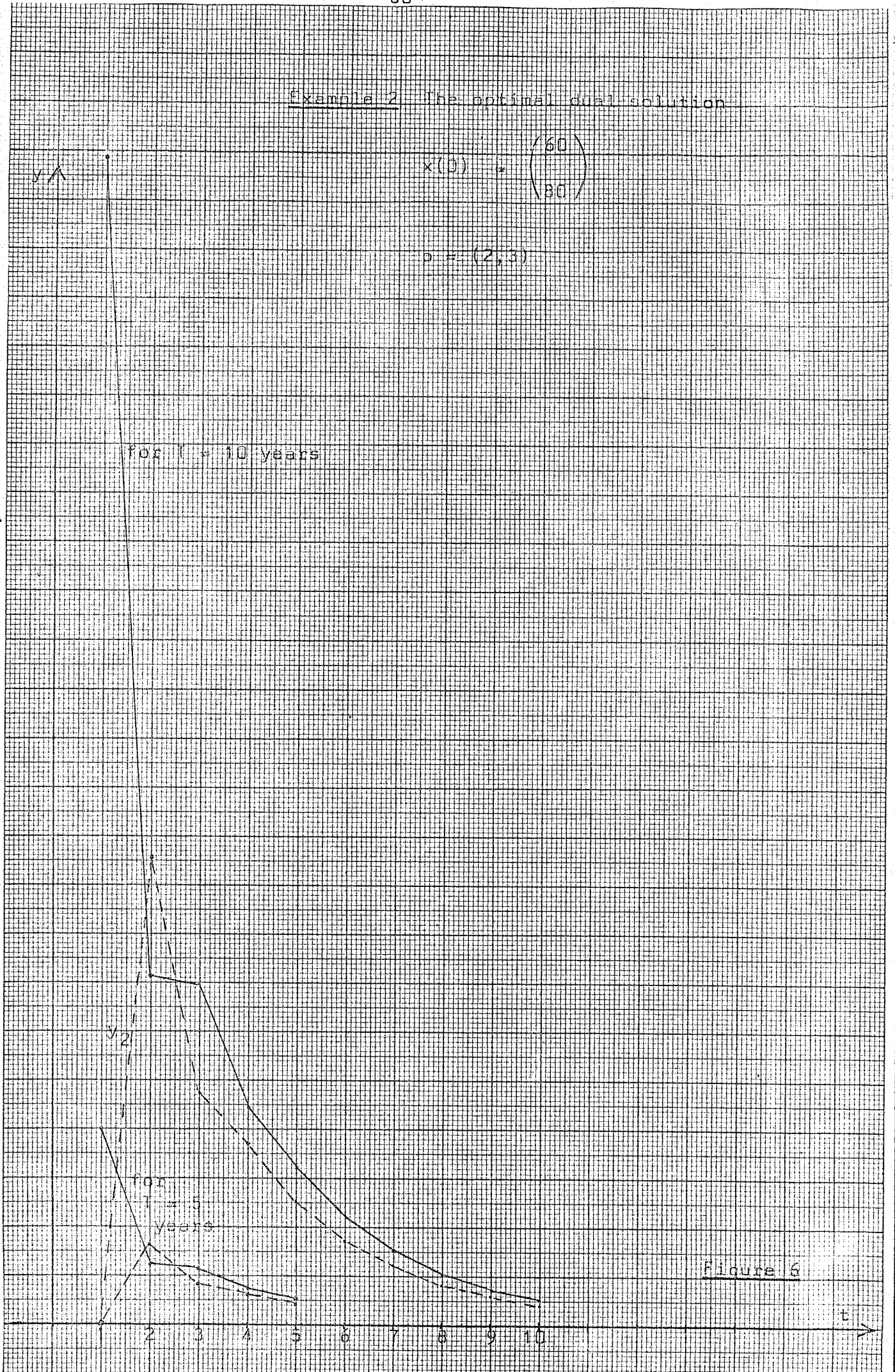
$y \wedge$

$y/2$

for  
 $l = 5$   
years

Figure 6

$t$

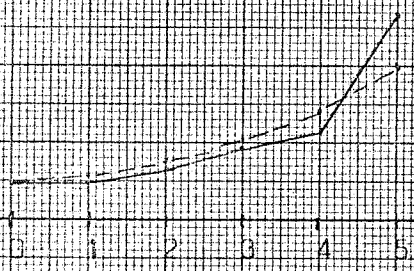


Example 2

$$x(0) = \begin{pmatrix} 60 \\ 80 \end{pmatrix}$$

$$p = (2, 3)$$

$T = 5$  years



$T = 10$  years

the value of objective function corresponding to the efficient path

the value of objective function corresponding to the von Neumann path

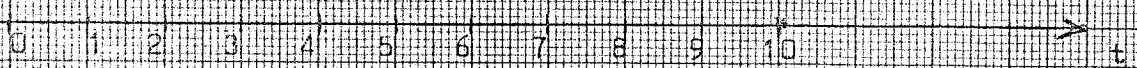


Figure 7

Example 3 The optimal primal solution

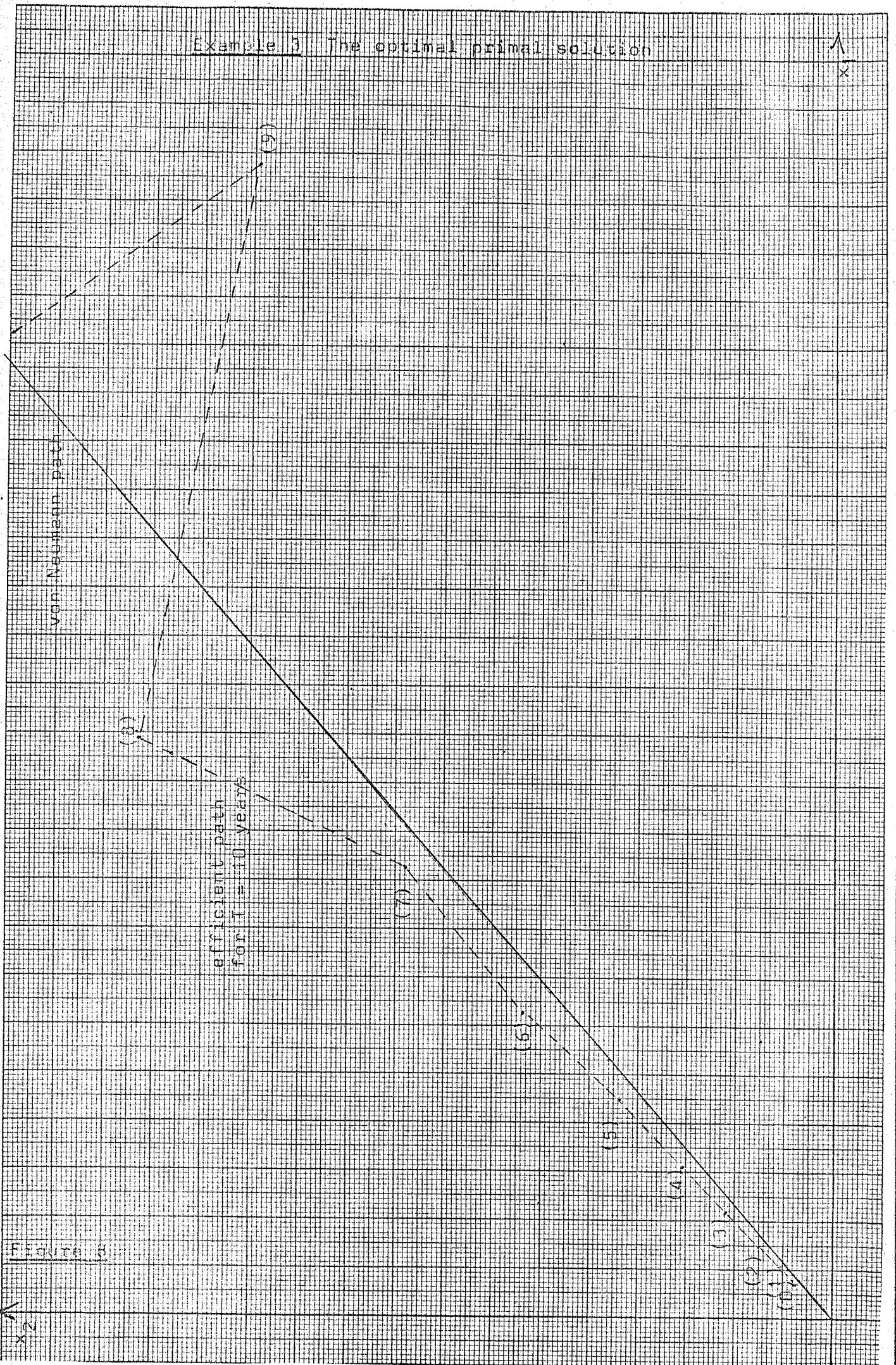


Figure 8

Example 3 The optimal dual solution

$v_1$  ↑  
 $v_2$  ↓

$$p = (2, 3)$$

$$x(0) = \begin{pmatrix} 75 \\ 75 \end{pmatrix}$$

$T = 10$  years

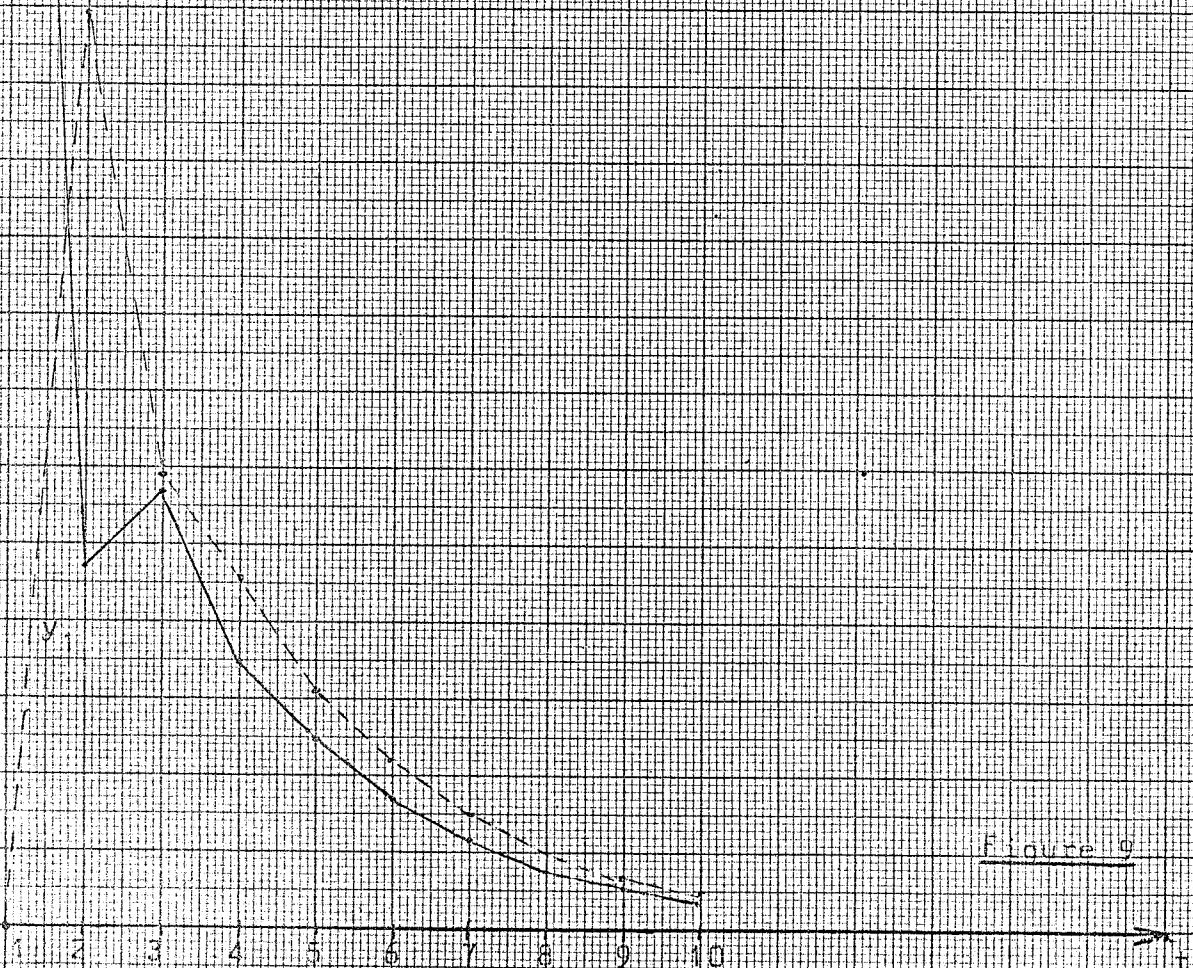


Figure 9

Example 3 The dual prices  
 (intertemporal efficiency conditions)

$$p = (2, 3)$$

$$x(0) = \begin{pmatrix} 10 \\ 75 \end{pmatrix}$$

$T = 10$  years

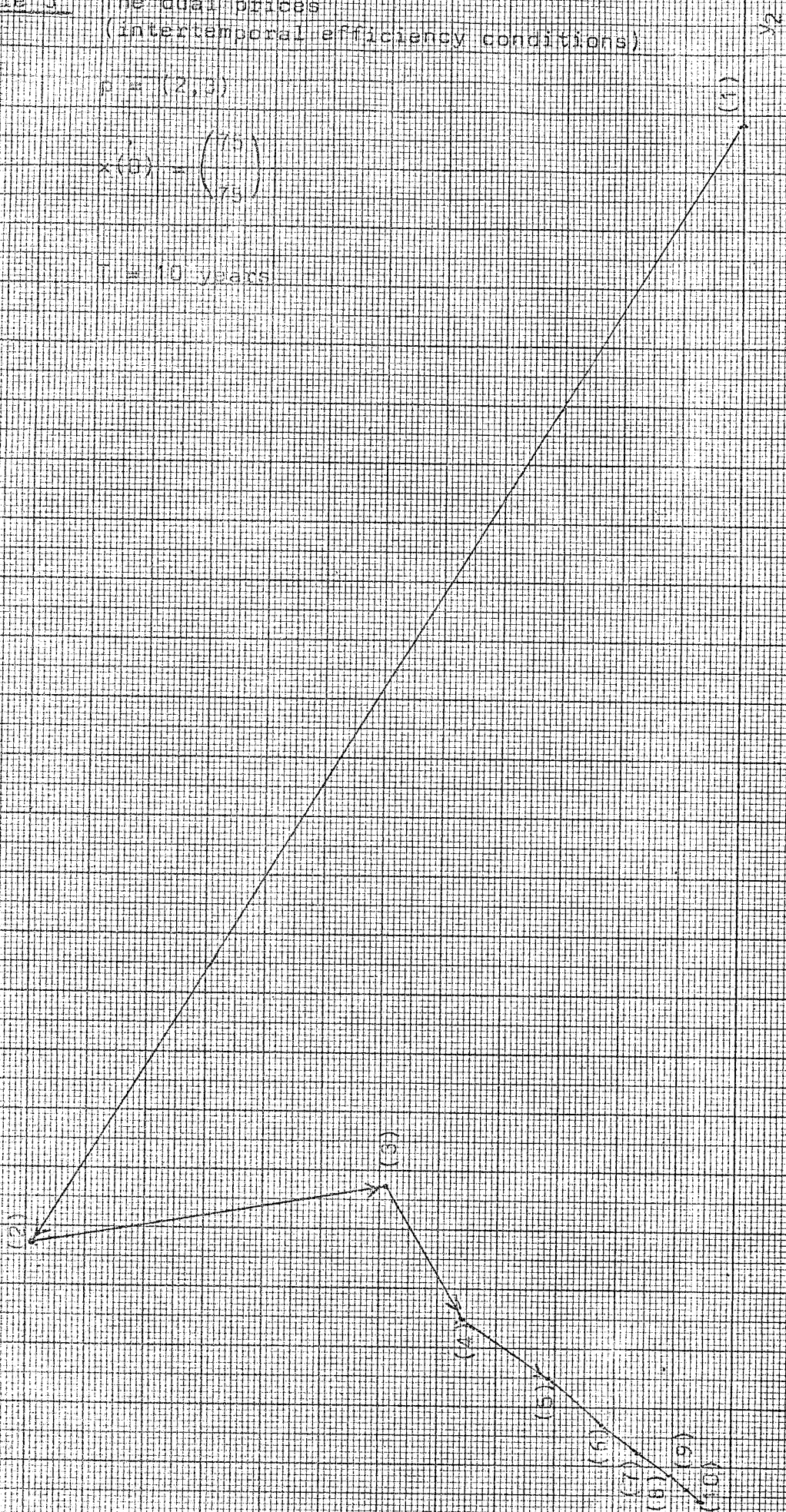


Figure 10

Example 3 for  $p = (2, 3)$

$$x(0) = \begin{pmatrix} 75 \\ 75 \end{pmatrix}$$

$T = 10$  years

$AZ$

The value of objective function corresponding to the efficient path

the value of objective function corresponding to the von Neumann path

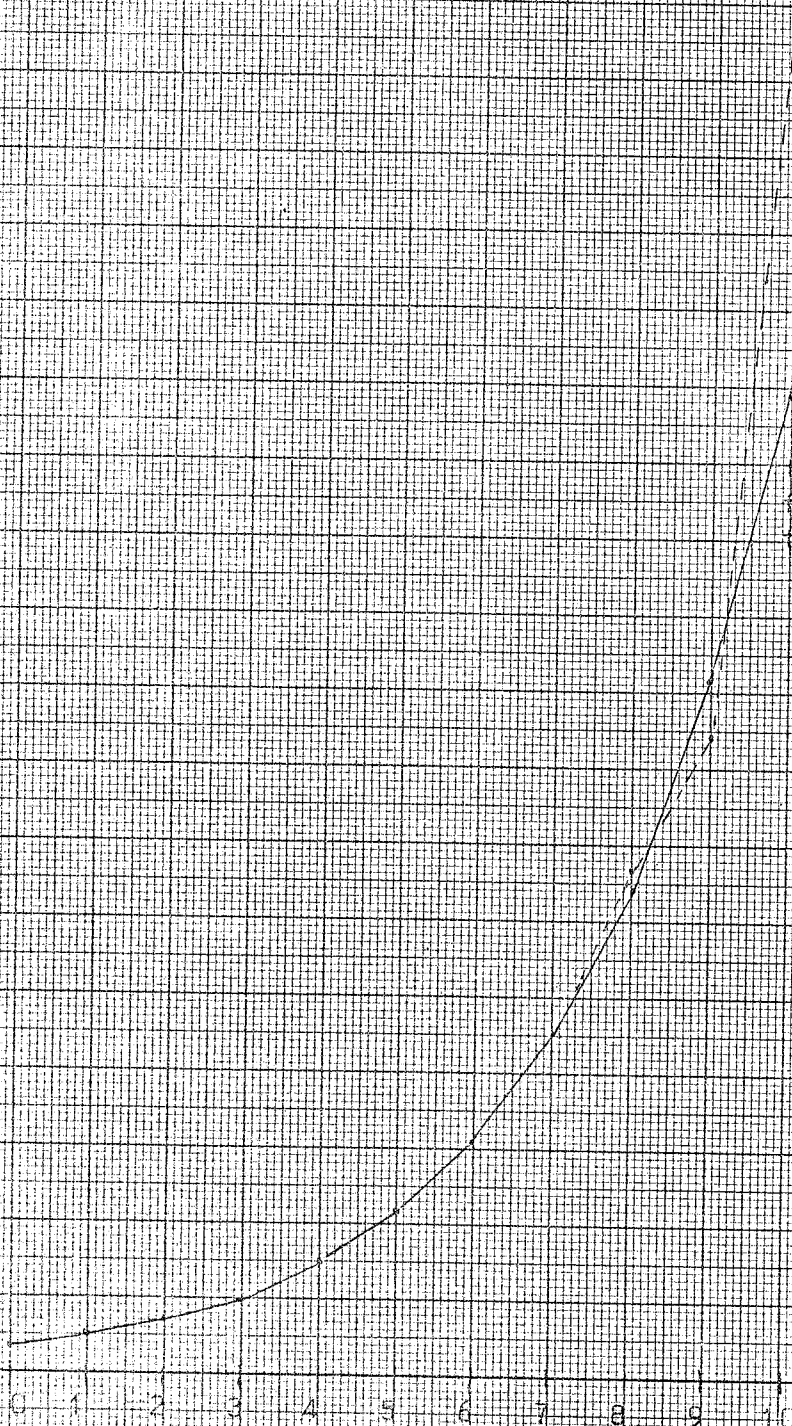


Figure 11

Example 4 The optimal primal solution

$$x(0) = \begin{pmatrix} 80 \\ 60 \end{pmatrix}$$

$$p = (10, 12)$$

$T = 10$  years

von Neumann path

efficient path

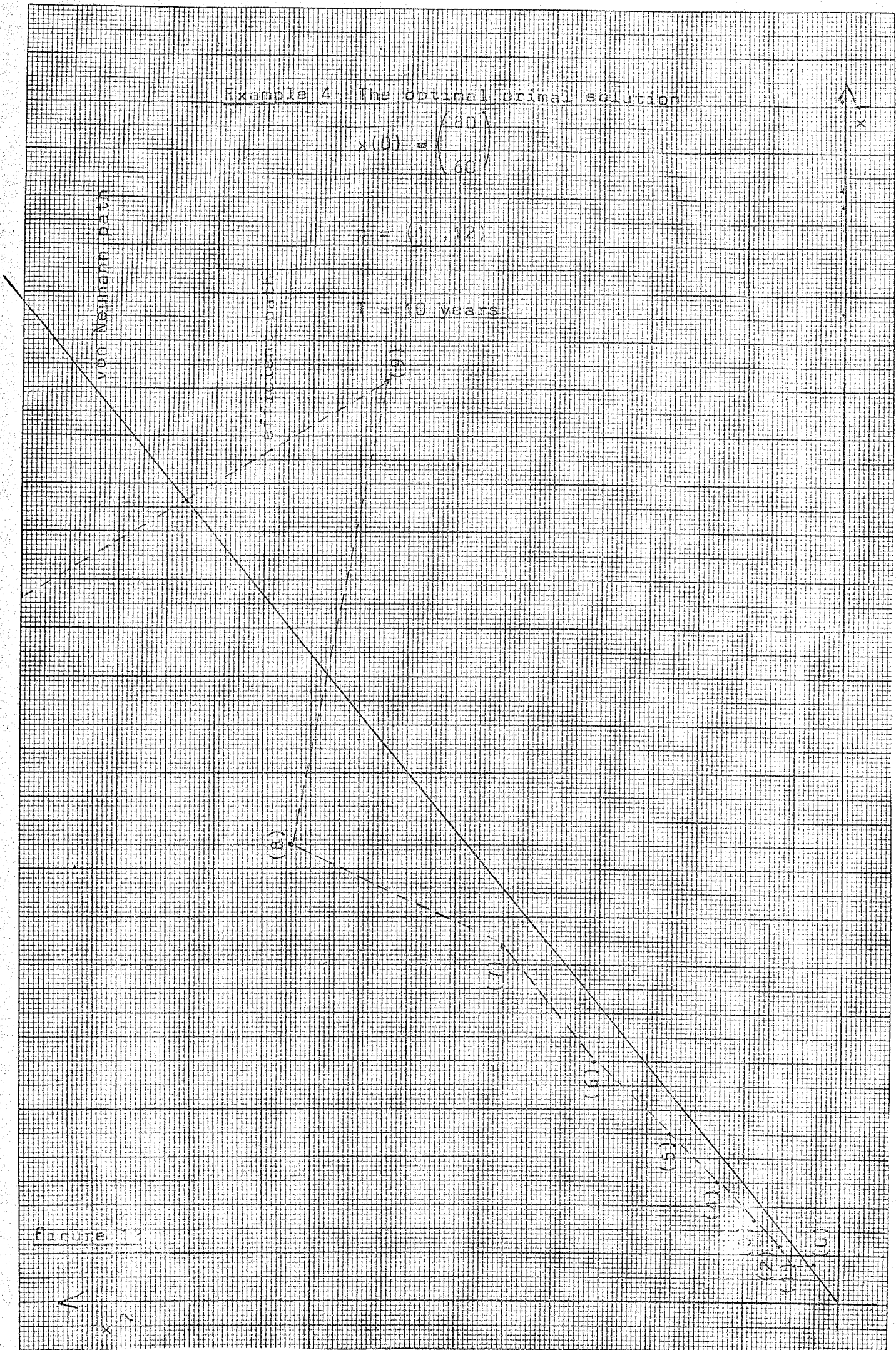


figure 12

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