RULES OF FAIR DIVISION

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1. The subject of this note is the problem of equitable or fair division. We consider the simple case, where two individuals have to split a constant sum of money, say US-Dollars 100, which they will get by reaching an agreement. The division should satisfy both, in the sense that everyone considers his amount as a fair share. In welfare economics and game theory different rules or schemes were proposed to solve this problem. The most important of this fair division schemes are discussed in this paper. First we consider three rules, the Kemeny solution, the Nash solution, and the welfare solution, and compare this rules with the axioms of Nash's bargaining problem. In the second part we give a new interpretation and a rigorous proof of a remarkable theorem of Lerner. This rule is not consistent with the bargaining model of Nash, because it involves uncertainty, whereas the first three refer to decisions under certainty.

2. We assume that the preference system of both individuals may be represented by a von Neumann-Morgenstern utility index. This means that utility can be measured up to a linear transformation of the utility function. If $u_i(x)$ is the utility function representing the preferences of individual i, $a_i u_i(x) + b_i$ could be an equally well representation.

3. Assumptions on the bargaining set.
   We assume that the bargaining set S is
   3.1 closed
   3.2 bounded and
   3.3 convex.
   The intuitive meaning of 3.1 is that S contains the boundary points, and 3.3 denotes that a linear combination of two points of the set is also an element of the set.

4. The problem now is to find a point in the bargaining set, according to a rule of fair division. The following rules are known in game theory and welfare economics.

1 Equal utility proportions

2 Equal marginal rates of utility

3 Equal marginal utility.

1. Equal utility proportions(The Kemeny Solution)

4.1.1. Let $u_i(x)$ be the utility of individual i, which receives the share x of an amount of money z. $(0 \leq x \leq z)$. The function $u_i(x)$ is a strictly monotonic increasing function of x. The utility proportion p_i is the ratio of the utility, which player i receives from his share x, and the utility which he would get from the total amount z:

4.1 $$p_i = \frac{u_i(x) - u_i^*}{u_i(z) - u_i^*}$$

u_1^* represents the utility of the "nondivision point", the status quo point.

Kemeny rule: Divide z in such a way that the share h_i produces the same utility proportion p_i for every individual i. (i = 1, 2)

We first prove on the lines of Kemeny that this division is always possible for the convenient type of utility function (4.2).

4.2 $$u_i = x_i^{\varepsilon_i}$$ where ε_i is the elasticity of u_i

with respect to x_i and $0 \leq x_i \leq z$.

The utility proportion of i then is

4.3 $$p_i = \left(\frac{x_i}{z}\right)^{\varepsilon_i}$$

The inverse function of 4.3 h_i gives us the share of player i of z:

4.4 $$h_i = p_i^{\frac{1}{\varepsilon_i}} z$$

The function $s(p_1 \ldots p_n) = \sum_i h_i(p_i)$ is defined for the interval $[0, 1]$ and takes all values between $[0, n\, z]$ only once. (Strictly monotonic).

<u>Corollary</u>: There exists a unique α in $[0, 1]$ such that

4.5 $$S(\alpha) = \sum_i h_i(\alpha) = z$$

We have then:

4.6 $$\sum_i h_i = \sum_i p_i^{\frac{1}{\varepsilon_i}} z = z \qquad \text{or}$$

4.7 $$\sum_i p_i^{\frac{1}{\varepsilon_i}} - 1 = 0$$

The root $p_i = p_j = \alpha$ of (4.7) gives the equal utility proportion for every player.
Player i gets the share: $h_i = \alpha^{\frac{1}{\varepsilon_i}} z$; his utility proportion according to (4.3) is:

$$p_i(\alpha) = \left[\frac{z.\alpha^{\frac{1}{\varepsilon_i}}}{z}\right]^{\varepsilon_i} = \alpha$$

4.1.2 An example may illustrate this procedure: A pauper and a millionaire find together a bill of US-Dollars 100 and divide this according to Kemeny's rule. It is a well known stereotyp [Luce & Raiffa [1]] that a millionaire has for a small amount of money a constant, a pauper a decreasing marginal utility for money. The utility function of the pauper is $u_1 = \sqrt{x}$, of the millionaire $u_2 = 1/10(100 - x)$.

The utility proportions are: (1) $p_1 = (\frac{x}{100})^{1/2}$

(2) $p_2 = \frac{x}{100}$

The inverse functions of (1) and (2) are:

$$h_1 = 100\ p_1^2$$

$$h_2 = 100\ p_2$$

The sum of the shares equals US-Dollars 100:

$$\sum_i h_i = 100(p_1^2 + p_2) = 100\ , \quad \text{where } p_1 = p_2 = x,$$

this gives: $x^2 + x - 1 = 0$

the root is: $\alpha = 0{,}62$

the pauper's share is: $h_1 = 100(0{,}62)^2 = 38$

the millionaire's share is: $h_2 = 100\ .\ 0{,}62 = 62$

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[1] R.D. Luce and H. Raiffa, Games and Decisions, New York 1966, p. 129.

The utility proportions for both ar equal:

$$p_1 = \left(\frac{100.(0,62)^2}{100}\right)^{1/2} = 0,62$$

$$p_2 = \frac{100.0,62}{100} = 0,62$$

<u>2. Equal marginal rates of utilities (The Nash solution)</u>

4.2.1 A fair distribution of a sum z between player i and j requires that the rates of marginal utilities are equal for both individuals:

$$2.1 \qquad \frac{du_i(x)}{dx} \cdot \frac{1}{u_i(x)} = - \frac{du_j(z-x)}{dx} \cdot \frac{1}{u_j(z-x)}$$

If these two (percentual) rates were unequal (say that the marginal rate of utility of i were greater than that of j), then the transfer of some x from individuum j to i, would make the society better off, in the sense that the product of their utilities $u_i(x).u_j(z-x)$ is larger than before. The Nash solution 2.1 implies that the two individuals maximize the product of their utilities. This can be seen by the following manipulation:

The utility of the status quo point is: (u_i^*, u_j^*)

We maximize the function:

$$g(u_i, u_j) = (u_i(x) - u_i^*)(u_j(x) - u_j^*)$$

$$\frac{du_i}{dx}(u_j - u_j^*) + \frac{du_j}{dx}(u_i - u_i^*) = 0$$

Rearranging and the substitution $\bar{u}_i = u_i - u_i^*$

$$\bar{u}_j = u_j - u_j^*$$

give the rule 2.1.

2.2 The Nash solution has a straightforward economic interpretation. If we consider a continuous concave utility possibility curve $u_2 = f(u_1)$, then the elasticity of this function will take on all values from zero to infinity. There exists one point, where the elasticity of this curve is $\varepsilon = -1$, this point represents the Nash solution:

$$\varepsilon = \frac{du_i(x)}{du_j(z-x)} \cdot \frac{u_j(z-x)}{u_i(x)} = -1$$

2.3 The beggar-millionaire bargaining will give the following division of the US-Dollars 100 according to the Nash solution:

$$\max \left[x^{1/2} \cdot \frac{1}{10}(100 - x) \right]$$

$$\frac{1}{20} x^{-1/2} (100 - x) - \frac{1}{10} x^{1/2} = 0$$

$$\underline{x_1 = 33.3}$$

$$\underline{x_2 = 66.7}$$

The utility ratios are: $\frac{du_1}{dx} \cdot \frac{1}{u_1} = \frac{1}{2x} = \frac{1}{66.6} = 0.015$

$$\frac{du_2}{dx} \cdot \frac{1}{u_2} = \frac{1}{66.7} = 0.015$$

3. Equal marginal utilities (the welfare solution)

According to this principle a fair division of z between two (or more) players involves that the marginal utility of everyone is equal.

$$3.1 \qquad \frac{du_i(x)}{dx} = -\frac{du_j(z-x)}{dx}$$

If these two numbers are unequal, the situation can not be optimal. A redistribution of z between i and j would increase the sum of utilities $h(u_i, u_j) = (u_i(x) - u_i^*) + (u_j(x) - u_j^*)$ until 3.1 is satisfied. This will give us a unique point on the utility possibility curve $u_2 = f(u_1)$, where the tangent to it has the slope -1.

It may be helpful to mention the difference between the Nash solution and the solution 3.1. Both concepts of a "fair" solution pick out one point on the utility possibility curve. The Nash solution is characterized by the elasticity of $u_2 = f(u_1)$ is -1; the principle 3.1 gives us a unique point, where

$$\frac{df(u_1)}{du_1} = -1.$$

5. Comparison of the three fair division schemes with Nash axioms.

How can we compare these rules? A reasonable approach for a comparison may be to formulate some axioms (or "desiderata") where most (or at least some) people will agree that a fair division scheme should be satisfying. According to Nash's famous paper, there are 4 axioms, which a "fair" solution of a two person bargaining game should satisfy:

<u>5.1. Pareto optimality</u>: If S is a bounded, convex, and closed bargaining set, the solution $(u_1^{\cdot}, u_2^{\cdot})$ must have the following property:

1.1 $u_1^{\cdot} \geq u_1^{*}$, $u_2^{\cdot} \geq u_2^{*}$

1.2 $(u_1^{\cdot}, u_2^{\cdot}) \in S$

1.3 If $(u_1, u_2) \geq (u_1^{\cdot}, u_2^{\cdot})$ then $(u_1, u_2) = (u_1^{\cdot}, u_2^{\cdot})$

u_1^{*}, u_2^{*} refer to the status quo point (non division point).

The axiom is trivial. It only means that the solution picks out the point on the utility possibility curve, which all three concepts satisfy. There is no other point in the bargaining set that dominates the solution.

<u>5.2. Independence of linear transformations of the utility functions</u>

5.2.1. If T is obtained from S by a linear transformation,

$$u_1' = a_1 u_1 + b_1$$

$$u_2' = a_2 u_2 + b_2$$

and if the solution of S is $g(S, u_1^*, u_2^*) = (u_1^{\bullet}, u_2^{\bullet})$ there must be

$$g(T, a_1u_1^* + b_1, a_2u_2^* + b_2) = (a_1u_1^{\bullet} + b_1, a_2u_2^{\bullet} + b_2)$$

If the bargaining set T is obtained from S by a linear transformation, i.e., by changing the origin and the units of the utility function, then the solutions of these two games should be related by the same linear transformation.

<u>5.2.2. Proposition</u>: The Nash rule and the Kemeny rule are independent of linear transformation of the utility function. The welfare solution is not invariant with respect to a change in the unit of measurement.

<u>Proof</u>: The transformed variables are (u_1', u_2')

$$u_1' = a_1u_1 + b_1$$

$$u_2' = a_2u_2 + b_2$$

<u>Nash</u>: $g'(u_1', u_2') = [u_1' - (a_1u_1^* + b_1)][u_2' - (a_2u_2^* + b_2)]$

$$= a_1a_2(u_1 - u_1^*)(u_2 - u_2^*)$$

If $(u_1^{\bullet}, u_2^{\bullet})$ maximizes $g(u_1, u_2) = (u_1 - u_1^*)(u_2 - u_2^*)$, it follows that $(u_1^{\bullet\prime}, u_2^{\bullet\prime})$ maximizes $g'(u_1', u_2')$.

<u>Kemeny</u>: $$p_i' = \frac{u_1'(x) - (a_1u_1^* + b_1)}{u_1'(z) - (a_1u_1^* + b_1)} = \frac{u_1(x) - u_1^*}{u_1(z) - u_1^*}$$

If $p_i = \dfrac{u_i(x) - u_i^*}{u_i(z) - u_i^*}$ is a Kemeny solution, p_i' is also a Kemeny solution.

Welfare solution: $h'(u_1', u_2') = [u_1' - (a_1 u_1^* + b_1)] + [u_2' - (a_2 u_2^* + b_2)]$

$$= a_1(u_1 - u_1^*) + a_2(u_2 - u_2^*)$$

The maximum of this linear function depends on the scale values (a_1, a_2) which follows from the first order condition

$$\frac{du_1(x)}{du_2(x)} = - \frac{a_2}{a_1} .$$

In the case of equal marginal utilities, we cannot change the unit of measurement, without changing the bargaining solution.

5.3. Independence of irrelevant alternatives

5.3.1. If $(u_1^\bullet, u_2^\bullet) \in T$ and $T \subset S$ and if $(u_1^\bullet, u_2^\bullet) = g(S, u_1^*, u_2^*)$ where $g(S, u_1^*, u_2^*)$ is the "bargaining solution" of the game (S, u_1^*, u_2^*), then $(u_1^\bullet, u_2^\bullet) = g(T, u_1^*, u_2^*)$.

If different bargaining games (S, u_1^*, u_2^*) and (T, u_1^*, u_2^*) have the same status quo point and the trading possibilities of one are enclosed in the other, and if the solution of the game with the larger set of alternatives is an element of the smaller set, then it should also be the solution of the game with the smaller bargaining alternatives. The difference between the larger set S and the smaller set T are points, that were rejected alternatives as compared with the solution point $(u_1^\bullet, u_2^\bullet)$ in the larger set. As R.L. Bishop pointed out[1], "this axoim might better be referred to as implying the independence (or irrelevance) of rejected alternatives."

<u>5.3.2. Proposition</u>: The Nash solution and the welfare solution are independent of irrelevant alternatives, but not the Kemeny solution.

<u>Proof</u>: In the case of the Nash solution and the equal marginal utility rule we have a simple argument. If the $(u_1^\bullet, u_2^\bullet)$ is the solution of the bargaining set (S, u_1^*, u_2^*) and if $(u_1^\bullet, u_2^\bullet) \in T$ and $T \subset S$, then if $g(u_1, u_2) = u_1(x).u_2(x)$, (or $h(u_1, u_2) = u_1(x) + u_2(x)$) is maximized over S, it will a fortiori be maximized over the smaller set T. This argument holds for every subset $T \subset S$, with the same origin and which contains the solution point of the larger set. The solution does not depend on the "corners" of the subset T.

$(o, u_2), (u_1, o)$

The Kemeny solution is not independent, with respect to irrelevant alternatives. Altering the bargaining set by denying the bargainers some rejected alternatives, changes the "corners" of the bargaining set and changes the solution.

If h_i is the final share of player i, then according to the rule the ratio

$$\frac{u_i(h_i)}{u_i(z)},$$

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1) R.L. BISHOP : "Game-Theoretic Analysis of Bargaining", Quarterly Journal of Economics, 1961, p.566.

(where z is the total amount) must be equal for every player i. For two players we rewrite the rule:

$$\frac{u_i(h_i)}{u_j(z-h_i)} = \frac{u_i(z)}{u_j(z)}$$

The player divides in a ratio equal to the ratio of the "corners"

$\frac{u_i(z)}{u_j(z)}$ of the bargaining set. Altering the corners, by removing some alternatives changes the solution.

5.4. Symmetry: If S has the properties:

4.1 $(u_1, u_2) \in S \longleftrightarrow (u_2, u_1) \in S$

4.2 $u_1^* = u_2^*$

4.3 $(u_1^\bullet, u_2^\bullet) = g(S, u_1^*, u_2^*)$ then

4.4 $u_1^\bullet = u_2^\bullet$

If the bargaining set is symmetric with respect to the line $u_1 = u_2$ the solution point lies on that line.

All three rules satisfy the definition of symmetry. If the bargaining set is symmetric, the Nash solution, the marginal utility principle, and the Kemeny rule produce the same solution. The solution must be on the $u_1 = u_2$ line. The Nash solution becomes:

$\frac{du_i}{du_j} = \frac{u_i}{u_j} = -1$: This means that not only the elasticity of the utility possibility curve on the solution point is -1; but also the slope of the tangent at this point is -1. (Rule 3). In the Kemeny case $u_i(z) = u_j(z)$, so we can multiply $p_i = p_j$, by $u_i(z)$, which gives $u_i(x_i) = u_j(z-x_i)$.

Comparison of the three concepts with the Nash-Axioms

Concept of fair division	Pareto optimal	Independence of linear utility transformation	Independence of irrelevant alternatives	Symmetry
Kemeny rule	Yes	Yes	No	Yes
Nash solution $\frac{du_i}{u_i} = -\frac{du_j}{u_j}$	Yes	Yes	Yes	Yes
Welfare solution $\frac{du_i}{dx} = -\frac{du_j}{dx}$	Yes	No	Yes	Yes

This table shows the axioms which are satisfied by these three concepts. The Nash solutions satisfies (by construction) all 4 axioms, the Kemeny solution is not independent with respect to irrelevant alternatives, the sum of utilities not with respect to linear transformations of the utility functions.

6. Lerner's rule of fair division

6.1. In his "Economics of Control" Lerner is concerned with the problem of an optimal division of income. The constant money income z should be divided between two individuals. He uses the following model:

1. The utilities of the two individuals are comparable and additive.[1)]
2. Both have a decreasing marginal utility function for money.
3. There is complete ignorance whether individual A or B has the greater marginal utility for money.[2)]
4. There is no interdependence of the utilities of A and B.

Using a diagrammatic exposition, Lerner proves the following theorem: "If it is impossible, on any division of income, to discover which of any individuals has a higher marginal utility of income, the probable value of total satisfaction is maximized by dividing income evenly."

Using a new interpretation of Lerners model, I prove two theorems:

Theorem I (Lerner's Rule of Fair Division):
The expected value of the total utility (sum of utilities) is maximized by an equal distribution of the income among the individuals.

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1) "... it is not meaningless to say that a satisfaction one individual gets is greater or less than a satisfaction enjoyed by somebody else". A. Lerner, "The Economics of Control", New York, 1961, p. 25.

2) "There is no way of discovering with certainty whether any individual's marginal utility of income is greater than, equal to or less than that of any other individual." A. Lerner, op.cit. p.28.

Theorem II

This solution is independent of linear transformations of the utility functions, so we can apply the von Neumann-Morgenstern utility theory.

6.2. Assume we have a dictator, who has to divide the income z among two individuals A and B.[1] He knows that both have different, but concave utility functions. If he would like to maximize total welfare (the sum of utilities) he has to devide the amount z unequal, in such a way, that the marginal utilities for A and B are equal. But there is no way for him to discover, who has the higher marginal utility function, because every individual could declare exceptionally high utilities by using arbitrarily (large) scale values. He can try to escape the problem of interpersonal comparison of utilities by using the following procedure: He makes an unequal split to z into

$$x \text{ and } z-x \text{ where } x \neq \frac{z}{2}$$

He then offers A (and B) a lottery-ticket L(x, z-x), where the players A and B can get a prize x or z-x on a flip of a coin.

He argues as follows: I know that A and B are different, but I do not know, who has the higher m.u.f. for money (the larger capacity for satisfaction at all income levels). I also know that in this situation an equal division would with certainty not be optimal. Using this lottery I produce

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1) I owe Prof. M. Maschler the following interpretation.

a chance, to approach the optimum without making interpersonal comparisons of utilities.

The utility for A, for the events x, and z-x is $u_a(x)$ and $u_a(z-x)$.

The expected utility for A (if the lottery is "fair") is:

$1/2\ u_a(x) + 1/2\ u_a(z-x)$, the actuarial value of the lottery is $1/2\ x + 1/2\ (z-x) = \frac{z}{2}$.

Proposition: If A has a concave utility function, (i.e., a decreasing marginal utility for money) the expected utility is smaller than the utility of the expected value of the lottery:

$$(1) \quad u_a(\tfrac{z}{2}) \geq \tfrac{1}{2}\, u_a(x) + \tfrac{1}{2}\, u_a(z-x)$$

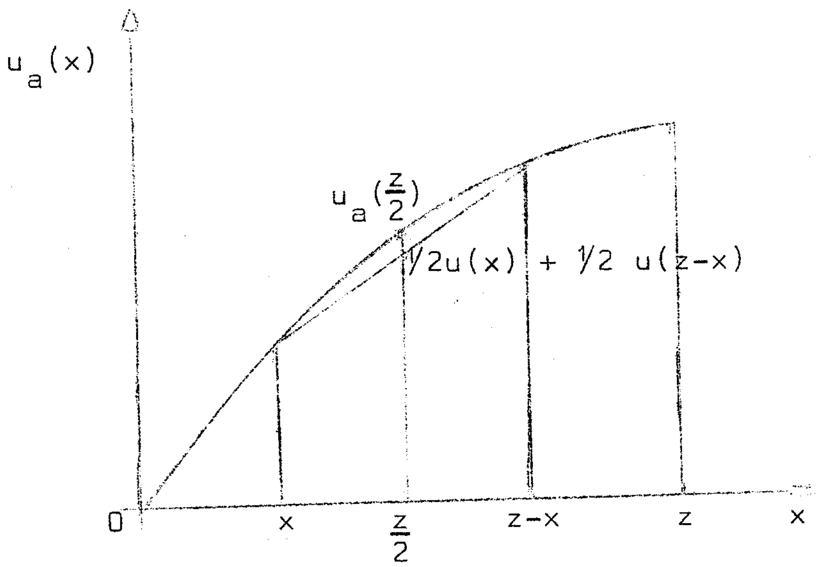

Figure 1

The same argument holds for B:

(2) $u_b(\frac{z}{2}) \geq \frac{1}{2} u_b(x) + \frac{1}{2} u_b(z-x)$

Adding (1) and (2) gives:

$$(3)\quad u_a(\tfrac{z}{2}) + u_b(\tfrac{z}{2}) \geq \tfrac{1}{2} u_a(x) + \tfrac{1}{2} u_a(z-x) + \tfrac{1}{2} u_b(x) + \tfrac{1}{2} u_b(z-x)$$

The expression on the right side of the inequality is the social welfare, which is produced if the dictator gives one of both x and the other z-x via the lottery.

6.3. We will prove that the welfare of the society is maximized, if A and B get $\frac{z}{2}$; in this case both sides of (3) are equal.

<u>Proposition</u>: The distribution $(\frac{z}{2}, \frac{z}{2})$ is optimal, because the welfare function $W(u_a, u_b)$ has a maximum at this division of z.

<u>Proof</u>: Let the social welfare be defined by the function:

$$W = 1/2\, u_a(x) + 1/2\, u_a(z-x) + 1/2\, u_b(x) + 1/2\, u_b(z-x)$$

we have to verify that $x = \frac{z}{2}$ is a maximum of $W(x)$.

(1) It was assumed that every $u_i(x)$ is a strictly concave function of x.

Now we use two lemmas on concave functions:

<u>Lemma 1</u>: If $f_k(x)$ $(k = 1 \ldots v)$ are concave functions and $\lambda_k \geq 0$ $(k = 1 \ldots v)$, then $\sum_k \lambda_k f_k(x)$ is also a concave function.

<u>Lemma 2</u>: A necessary and sufficient condition for concavity of a function $f(x)$ is $f''(x) \leq 0$, a sufficient condition for concavity is $f''(x) < 0$.

(2) The welfare function $W(x)$ is a linear combination of concave functions, by lemma 1 $W(x)$ is a concave function.

(3) The first order conditions of a maximum of $W(x)$ is:

$$\frac{dW}{dx} = u_a'(x) - u_a'(z-x) + u_b'(x) - u_b'(z-x) = 0$$

We verify that this is satisfied at the point:
$x = \frac{z}{2}$

(4) The second order condition is satisfied, if:

$$\frac{d^2W}{dx^2} = u_a''(x) + u_a''(z-x) + u_b''(x) + u_b''(z-x) < 0$$

By definition of concavity (lemma 2) the second derivative of every u_i in (3) is non positive, using the assumption of strict concavity lemma 1, the sum must be negative.

Therefore, if $W(x)$ has an absolute maximum, this maximum is unique and occurs where $W'(x) = 0$; i.e., where $x = \frac{z}{2}$.

<u>Theorem II</u>: The solution $(\frac{z}{2}, \frac{z}{2})$ is independent of linear transformations of the utility function $u_a(x)$ and $u_b(x)$.

<u>Proof</u>: Let the transformed utilities be:

$v_1 = a_1 u_a + b_1$

$v_b = a_2 u_b + b_2$ with $a_i > 0$

$b_i \geq 0$

and the status quo point be (u_a^*, u_b^*)

The transformed welfare function is:

$$v(x) = 1/2\left[v_a(x) - (a_1 u_a^* + b_1)\right] + 1/2\left[v_a(z-x) - (a_1 u_a^* + b_1)\right] + $$
$$+ 1/2\left[v_b(x) - (a_2 u_b^* + b_2)\right] + 1/2\left[v_b(z-x) - (a_2 u_b^* + b_2)\right]$$

$$v(x) = a_1\left[u_a(x) - u_a^*\right] + a_1\left[u_a(z-x) - u_a^*\right] + a_2\left[u_b(x) - u_b^*\right] + $$
$$+ a_2\left[u_b(z-x) - u_b^*\right]$$

$$\frac{dv}{dx} = a_1\left[u_a'(x) - u_a'(z-x)\right] + a_2\left[u_b'(x) - u_b'(z-x)\right] = 0$$

If $(\frac{z}{2}, \frac{z}{2})$ is a maximum of $W(x)$, it is also a maximum of the transformed welfare function $v(x)$.