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Oligopolistic Equilibrium and Financial Constraints*

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Abstract

We provide a model of dynamic duopoly in which firms face financial constraints and dissapear when they are unable to fulfill them. We show that, in some cases, Cournot outputs are no longer supported in equilibrium, because if these outputs were set, a firm may have incentives to ruin the other. In these cases, standard grim-trigger strategies in which collusion is sustained by infinite reversion to Cournot outputs cannot be used. We show that there is a stationary Markov equilibrium in mixed strategies where predation occurs with a positive probability. We also obtain a modified "folk theorem." We show that any bankruptcy-free outputs (outputs in which no firm can drive another firm to bankruptcy without becoming bankrupt itself) that attain individually rational profits (reflecting bankruptcy consideration) can be supported by a subgame perfect Nash equilibrium when firms are sufficiently long-sighted.

Key words: Financial Constraints, Bankruptcy, Firm Behavior, Dynamic Games.

Journal of Economic Literature Classification Number(s): D2, D4,L1,L2

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1. Introduction

There is ample evidence that financial constraints play an important role in the behavior of firms (Bernanke and Gertler, 1989; Kiyotaki and Moore, 1997). We begin with the observation that the punishment for violation of a financial constraint must be severe or otherwise firms would default all the time. Suppose that the punishment is so severe that firms violating financial constraints lose the capacity to compete and disappear (Bolton and Scharfstein, 1990).[1] Firms might then have incentives to take actions that would make it impossible for competitors to fulfill financial constraints in the hope of getting rid of them.

In this paper we provide a model of dynamic duopoly in which both firms take fully into account the financial constraints of the other firm as well as their own financial constraints. To simplify our task we make two assumptions: Profits cannot be transferred from one period to the next and the financial constraint requires that profits must be non-negative in each period. The second assumption entails just a normalization of profits. However, the first assumption is not innocuous and is discussed later on.

We introduce the concept of bankruptcy-free outputs (BF hereinafter). This is the set of outputs in which profits for all firms are non-negative (so no firm goes bankrupt) and no firm can make the other firm bankrupt without becoming bankrupt. The concept of BF captures the opportunities for ruining other firms that exist in our set up, while they are not captured by standard concepts such as Cournot equilibrium. Importantly, we show that Cournot equilibrium may not be BF. Consider a market with constant returns to scale but different marginal costs. If marginal costs are not significantly different, both firms produce positive outputs in Cournot equilibrium. These outputs are not BF because the most efficient firm can produce an output (larger than its Cournot output) for which the market price becomes strictly lower than the competitor's average cost but larger than its own average cost. As a result, the efficient firm earns a positive profit while the other firm incurs a negative profit and goes bankrupt. Why a firm would make such a move? Because in a dynamic game this move gets rid of a competitor so if the efficient firm is very patient this move will pay off in the future. This story suggests that the commonly used constant-marginal-cost Cournot model might be misleading if firms have different marginal costs and financial constraints

[1] Even though firms can be reorganized after bankruptcy and continue business, the survival rate of firms after bankruptcy is typically low, 18% US, 20% in UK and 6% in France, see Couwenberg (2001).

are relevant. In fact, the introduction of financial constraints implies that monopolization (by the efficient firm) occurs with positive probability in a competitive equilibrium as we explain shortly.

When financial constraints are considered, the grim-trigger strategies may not work because reversion to Cournot outputs does not guarantee that firms have incentives to stay there. More strikingly, tacit collusion cannot be supported in general by the grim-trigger strategies even if firms are sufficiently patient, which is in a sharp contrast with the standard collusion analysis. We consider a more general concept, namely stationary Markovian equilibrium, a dynamic equilibrium in which firms' outputs depend only on states, not on the detailed history. Let the state of each period be the set of active firms, i.e., the firms that have not been bankrupted until that period. The unique Markovian equilibrium in pure strategies, if it exists, is the Cournot equilibrium. However, as we argued above, Cournot equilibrium may fail to be an (Markovian) equilibrium. Therefore, in these cases, each stationary Markovian equilibrium must entail mixed strategies. We show that such equilibrium exists. Assuming constant average costs and concave profit functions we characterize the support of this equilibrium. Precisely, when discount factor is larger than certain cut-off value, the equilibrium in mixed strategies is the unique Markovian equilibrium and it has the following properties: i) the firm with larger average costs, called the inferior firm from now on, never takes an output larger than the one that maximizes per period profits, ii) the inferior firm becomes bankrupt with positive probability (so monopolization occurs with positive probability), and iii) the support of each firm's mixed strategy contains exactly one interval with a unique mass point. Moreover, for each firm the mass point coincides with the best reply of the static game (to the other firm's mixed strategy). For the firm with smaller average costs, called the superior firm, the mass point is isolated and lies strictly below the interval support. Since the superior firm would not produce a larger output than the best reply unless bankruptcy occurs, outputs in the interval support reflect the predatory activities of the superior firm. For the inferior firm, the mass point lies weakly above the interval. This reflects that the possibility of bankruptcy makes the inferior firm prudent when choosing outputs.

The failure of the standard grim-trigger strategies does not necessarily imply that collusion is not sustainable as an equilibrium. Firms may maintain collusion by employing other self-enforcing punishments rather than reverting to Cournot outputs. The celebrated "folk theorem" shows that any feasible and individually rational profits can be supported by a Subgame Perfect Nash

Equilibrium (SPNE) when firms are fully patient (i.e., their discount factors are close to 1). Our interest is whether a similar characterization can be obtained when bankruptcy considerations are introduced.

We show that the folk theorem remains to be valid in our environment, once the set of feasible and individually rational profits are appropriately modified. In the original folk theorem (without bankruptcy consideration), each firm's profit becomes individually rational if it exceeds a minimax value, i.e., the minimum profit that a firm can guarantee itself even though the opponent takes the severest output, and this becomes zero in our duopoly model. Therefore, any combination of feasible and positive profits can be sustained by an equilibrium. The modification of feasibility is straightforward. Instead of considering all possible profits, we should focus only on BF outputs, since at least one firm has an incentive to ruin the other if the output is not BF. The modification of the individual rationality condition comes from the fact that under financial constraints, firms that take severe punishments may become bankrupt as a consequence of their own action. To avoid such scenario, we define a new concept called the minimax BF value where minimization and maximization are taken over only BF outputs.[2] Then, we establish the modified folk theorem that reflects bankruptcy consideration. Namely, we show that any BF output profile that gives profits greater than the minimax BF value can be supported as an SPNE and that profits less than the minimax BF value cannot be sustained in any SPNE for a discount factor close to 1.

We end this introduction with a preliminary discussion of the literature (see more on this in the final section). Although a number of papers demonstrate that the financial structure does affect market outcomes in an oligopoly, most previous studies adopt either static or two-stage models. There are at least two exceptions, Spagnolo (2000) and Kawakami and Yoshida (1997). Both papers make use of repeated games like ours. The former examines the role of stock options in repeated Cournot games. In his model, unlike standard repeated games, firms do not necessarily maximize average discounted profits because stock options affect managers' incentives. Taking this effect into consideration, Spagnolo (2000) shows that collusion becomes easier to achieve. In our model, by contrast, collusion becomes more difficult to support, at least when firms adopt the grim-trigger strategies. The latter incorporates a simple exit constraint into the repeated prisoners' dilemma. In their model, each firm must exit from the market no matter how it plays if the rival

[2]For technical reasons we assume here that average costs are increasing.

deviates over certain number of periods, and hence no output profile can be bankruptcy free. They show that predations inevitably occur when bankruptcy constraints are asymmetric and firms are long-sighted.

Finally, our approach might provide support to the notion that firms may engage in predatory activities when pursuing profit maximization. Standard explanations of this behavior are based on incomplete information (Milgrom and Roberts, 1982), the learning curve (Cabral and Riordan, 1994) or firms playing an attrition game (Roth, 1996). In our model, firms have complete information, the technology is fixed and firms play standard quantity-setting games. Nevertheless, we obtain predation as a competitive equilibrium in mixed strategies. More importantly, both predation and tacit collusion can be derived (as different equilibria) in a single model, which is a completely new result to the best of our knowledge.

2. The model and preliminary results

Two firms compete in an infinite number of periods. In each period firms simultaneously choose quantities. Firms produce an homogeneous product. In order to focus in the strategic decisions regarding outputs we assume that firms cannot accumulate profits. Firms become bankrupt if they suffer losses in a period. A bankrupt firm exits the market (i.e., 'produces' zero every period thereafter). When making its quantity decision in a period, each firm knows what any firm has produced in all previous periods and which firms became bankrupt. The equilibrium concept that we use is Subgame Perfect Nash Equilibrium (SPNE). The formal definitions are given in Section 4. In the rest of this section we present the elements of the game that is played in each period. For simplicity, the time dimension is not considered yet.

We refer to one of the firms as the superior (S) and the other as the inferior (I). Let $j \in \{I, S\}$ denote a firm and $x_j \in \mathbb{R}_+$ the output of firm j. Let $C_i(x_i)$ denote the cost function. Assume that for all output, x, $AC_S(x) \leq AC_I(x)$, where $AC_j(.)$ is firm j's average-cost schedule. Assume that average cost is nondecreasing and twice differentiable. Let $\mathbf{x} = (x_S, x_I)$ denote an output profile, and let $X = x_S + x_I$ be the aggregate output. Let $p(X)$ be the inverse demand function assumed to be strictly decreasing in X for any positive price and twice differentiable. Derivatives are denoted by primes, i.e. $p'(X)$ is the slope of the inverse demand at X, etc. Profits for firm i are $\pi_i \equiv p(X)x_i - C_i(x_i)$, and written as $\pi_i(\mathbf{x})$ or as $\pi_i(x_i, x_j)$. We assume the classical conditions that guarantee existence and uniqueness of a Cournot equilibrium namely, for all $\mathbf{x} = (x_S, x_I)$,

$$p''(X)x_i + p'(X) < 0, \text{ for all } i \in \{S, I\}, \tag{2.1}$$

$$p'(X) - C_i''(x_i) < 0, \text{ for all } i \in \{S, I\}. \tag{2.2}$$

These conditions are satisfied if, for example, demand is linear and cost functions are quadratic.

We denote by $\mathbf{x}^C = (x_S^C, x_I^C)$ the Cournot output profile and by π_i^C firm i's payoff at the Cournot output profile.

Central to the analysis of our dynamic set up is the concept of bankruptcy-free (BF) output profiles. This is the set of output profiles in which no firm makes negative profit and no firm can drive another firm into bankruptcy without bankrupting itself. A motivation to focus on such outputs is that they describe a long run equilibrium in an industry in which all firms have incentives to stay in the market and not to engage in predatory activities. Of course these activities might be important but we look at the industry once the dust has settled and the predatory activities (if any) have been done in the past. Formally,

Definition 1. *An output profile $\hat{\mathbf{x}} = (\hat{x}_S, \hat{x}_I)$ is Bankruptcy-Free (BF) if:*
a) $\pi_i(\hat{\mathbf{x}}) \geq 0$, for all $i \in \{S, I\}$.
b) $\pi_i(\hat{x}_i, x) \geq 0$ for any x such that $\pi_j(x, \hat{x}_i) \geq 0$.

Note that if firm i is required to make some profit v_i (it could be either positive or negative) to avoid bankruptcy, we can define a new profit function as $\tilde{\pi}_i(\mathbf{x}) \equiv \pi_i(\mathbf{x}) - v_i$ and redefine BF with respect to this new profit function.

Next we characterize the BF output profiles. The characterizations will become important for the analysis of the dynamic game.

Lemma 1. *An output profile $\mathbf{x} = (x_I, x_S)$ is BF if and only if, for both j, $\pi_j(\mathbf{x}) \geq 0$ and*

$$AC_j(x_j) \leq AC_k(D(AC_j(x_j)) - x_j), \text{ for all } j \text{ such that } x_j \neq 0, \tag{2.3}$$

where $D(.)$ is aggregate demand and $k \neq j$.

Proof. The requirement $\pi_j(\mathbf{x}) \geq 0$ for both j follows from the definition of BF. If $\hat{x} > \hat{x}'$, then $\pi_j(x_j, \hat{x}') < 0$ entails $\pi_j(x_j, \hat{x}) < 0$. First note that, if a firm is not producing any output, this firm can not be driven to bankruptcy. Thus, consider a firm j such that $x_j \neq 0$, and let us see when the other firm k can drive firm j to bankruptcy. Define

$$x(x_j) = \inf\{x \in \mathbb{R}_+ / \pi_j(x_j, x) < 0\}. \tag{2.4}$$

By continuity $\pi_j(x_j, x(x_j)) = 0$. It follows that

$$p(x_j + x(x_j)) = AC_j(x_j) \Longrightarrow x(x_j) = D(AC_j(x_j)) - x_j. \tag{2.5}$$

It follows that firm k can drive firm j to bankruptcy without bankrupting itself if and only if

$$0 \leq p(x_j + \hat{x}_k) - AC_k(\hat{x}_k) < p(x_j + \underline{x}) - AC_k(\underline{x}) = AC_j(x_j) - AC_k(D(AC_j(x_j)) - x_j) \tag{2.6}$$

for some $\hat{x}_k > x(x_j)$. But (2.6) can hold if and only if (2.3) does not. ■

Corollary 1. *If the average cost is constant for both firms, $AC_j(x) = c_j$, $j \in \{I, S\}$, and $c_S < c_I$ no output profile with both firms active is BF. In a BF output profile only the superior firm is producing.*

Proposition 1. *Suppose $\mathbf{x} = (x_I, x_S)$ is BF. Then, any combinations of outputs smaller than $\mathbf{x}$, i.e., $\mathbf{x}' = (x'_I, x'_S)$ such that $x'_I \leq x_I$ and $x'_S \leq x_S$, must be BF.*

Proof. Because demand curves slope down and average cost is nondecreasing, $D(AC_j(x_j)) \leq D(AC_j(x_j\prime))$ for $x_j \geq x_j\prime$. This establishes the chain:

$$0 \leq AC_k(D(AC_j(x_j)) - x_j) - AC_j(x_j) \leq AC_k(D(AC_j(x'_j)) - x_j\prime) - AC_j(x_j\prime). \tag{2.7}$$

It follows, from Lemma 1 that $\mathbf{x}\prime$ is BF. ■

A useful characterization of the BF set can be provided under the following additional assumption.

Assumption 1. Both firms have increasing average cost, and there is a unique $\bar{\mathbf{x}} = (\bar{x}_S, \bar{x}_I)$ with $\bar{x}_i \neq 0$ for all $i \in \{S, I\}$ such that $\pi_i(\bar{\mathbf{x}}) = 0$ for all $i \in \{S, I\}$.

Assumption 1 always holds if for example demand is linear and $C_i(x_i) = \gamma_i x_i^2$ with $\gamma_i > 0$.

Lemma 2. *Suppose Assumption 1 holds. Then the set of BF output profiles is:*

$$BF = \{(x_S, x_I) \mid 0 \leq x_i \leq \bar{x}_i \text{ for } i \in \{S, I\}\}. \tag{2.8}$$

Proof. Note first that, trivially, $\bar{\mathbf{x}} = (\bar{x}_S, \bar{x}_I)$ is BF. By Proposition 1, all $\mathbf{x} < \bar{\mathbf{x}}$ are also BF. We can see that no other output profile can be BF. Let (x_S, x_I) be an output profile such that both firms have non-negative profits, and suppose that $x_I > \bar{x}_I$. Let us see that firm S, by increasing its output, can drive firm I into bankruptcy keeping positive profits for itself. Let $\hat{x}_S$ be such that $\hat{x}_S + x_I = \bar{x}_S + \bar{x}_I$, since $x_I > \bar{x}_I$, $\hat{x}_S < \bar{x}_S$, thus, at $(\hat{x}_S, x_I)$ firm S has positive profits. But since average cost is increasing and $x_I > \bar{x}_I$, firm I at $(\hat{x}_S, x_I)$ becomes bankrupt. ■

To close this section, we see under which conditions Cournot' output profile is BF in the particular case of linear demand, $p(x_S + x_I) = A - x_S - x_I$, and cost, $C_S(x) = \gamma_S x^2$, $C_I(x) = \gamma_I x^2$ with $0 < \gamma_S \leq \gamma_I$. By Lemma 2 the BF set is completely characterized by the output profile $\bar{\mathbf{x}} = (\bar{x}_S, \bar{x}_I)$ such that $AC_I(\bar{x}_I) = AC_S(\bar{x}_S) = p(\bar{x}_S + \bar{x}_I)$, thus, $\gamma_S \bar{x}_S = \gamma_I \bar{x}_I = A - \bar{x}_S - \bar{x}_I$.

$$\bar{x}_S = \frac{\gamma_I A}{(1+\gamma_S)(1+\gamma_I) - 1}; \ \bar{x}_I = \frac{\gamma_S A}{(1+\gamma_S)(1+\gamma_I) - 1}. \tag{2.9}$$

The Cournot equilibrium is given by

$$x_S^C = \frac{(1+2\gamma_I)A}{4(1+\gamma_I)(1+\gamma_S) - 1} : \tag{2.10}$$

$$x_I^C = \frac{(1+2\gamma_S)A}{4(1+\gamma_I)(1+\gamma_S) - 1} \tag{2.11}$$

For the superior firm is always the case that $x_S^C \leq \bar{x}_S$. For the inferior firm, $x_I^C \leq \bar{x}_I$ if and only if

$$(1+\gamma_I)(1-2\gamma_S) \leq 1. \tag{2.12}$$

Thus, the Cournot equilibrium is BF if and only if condition (2.12) holds.

Finally, note that the joint profit maximization output, $\mathbf{x}^J = (x_S^J, x_I^J)$, is always BF because marginal cost for both firms has to be equal, $\gamma_S x_S^J = \gamma_I x_I^J$, and profits are non-negative for both firms. Thus, trivially $x_j^J \leq \bar{x}_j$ for all $j \in \{S, I\}$.

3. Dynamic Competition with Bankruptcy

In this section we focus on the dynamic model.

In each period t each firm $i \in \{I, S\}$ chooses an output denoted by x_i^t. Let $\mathbf{x}^t = (x_I^t, x_S^t)$ be a profile of outputs in period t. The profits obtained by firm i in period t are $\pi_i(\mathbf{x}^t)$ $t = 0, 1, ..., \tau, ..$We define a state at t as the set of firms that did not fall into bankruptcy in previous periods called *active* firms. Let $\delta \in (0,1)$ be the common discount factor. Discounted profits for firm i are $\Pi_i =$

$\sum_{t=0}^{\infty} \delta^t \pi_i(\mathbf{x}^t)$. The continuation payoff in period t is given by $\Pi_i^t = \sum_{r=0}^{\infty} \delta^r \pi_i(\mathbf{x}^{t+r})$. At period 0 the game begins with the null history h^0. For $t \geq 1$, a history, denoted by $h^t = (\mathbf{x}^0, \mathbf{x}^1, .., \mathbf{x}^{t-1})$, is a list all outputs at all periods before t. A strategy for firm i, σ_i, (pure or mixed) is a sequence of maps, one for each period t, mapping all possible period t histories into a probability distribution in outputs. Let $\sigma = (\sigma_I, \sigma_S)$ denote a strategy profile (pure or mixed). A Markovian strategy for firm i is a mapping from the set of active firms into a probability distribution on outputs. A Nash Equilibrium (NE hereinafter) is a collection of strategies from which no firm finds it profitable to deviate. A Subgame Perfect Nash Equilibrium (SPNE hereinafter) is a collection of strategies which are a NE in every possible subgame. A stationary Markovian SPNE is a SPNE in which firms use stationary Markovian strategies only. To ease notation whenever no confusion can arise, we drop the time superindex.

In infinite repeated games without bankruptcy considerations there is only one state, and hence the stationary Markovian strategy (in the stationary Markovian SPNE) exactly coincides with the Cournot output. In those games, collusive outcomes can be supported as a SPNE using grim-trigger strategies in which any deviation from collusion triggers a switch to the Cournot outcome forever. Under bankruptcy considerations, when the Cournot output profile is BF, the unique stationary Markovian equilibrium is that both firms produce the Cournot outcome when both are active and, when only one firm is active, this firm produces the monopoly outcome. In equilibrium both firms are active in every period. It is not difficult to see that, in this case, the collusive outputs that can be supported without bankruptcy considerations by grim-trigger strategies, can be also supported with bankruptcy considerations. Formally, the outputs (x_i, x_j) that can be supported for a given δ without bankruptcy considerations are those that satisfy:

$$\pi_i(x_i, x_j) \geq (1-\delta)\pi_i(B_i(x_j), x_j) + \delta \pi_i^C, \tag{3.1}$$

where $B_i(x_j) = \arg\max \pi_i(x_i, x_j)$. Given that $\pi_i(x_i, x_j) \leq \pi_i(B_i(x_j), x_j)$,

$$\delta \pi_i(B_i(x_j), x_j) \geq \delta \pi_i^C. \tag{3.2}$$

Since $\pi_i(B_i(x_j), x_j)$ is decreasing in x_j and $\pi_i^C = \pi_i(B_i(x_j^C), x_j^C)$, the above inequality implies that $x_j \leq x_j^C$. Thus, the quantities produced at those collusive outputs are smaller than at Cournot outputs and by Proposition 1, if Cournot is BF, all these output profiles are BF.

A different analysis has to be made when the Cournot outcome profile is not BF because it could be the case that for some discounts factors one firm may have incentives to bankrupt the other firm.[3] In the following Lemma we provide the range of the discount factor that is needed in order to prevent such a deviation.

Lemma 3. *Suppose that* (x_S^C, x_I^C) *is not BF. Then, there exists* $\underline{\delta} < 1$ *such that the stationary Markov strategy* $x_i = x_i^C, i \in \{I, S\}$ *in the state with all firms active, and* $x_i = x_i^M$ *in states where only firm* i *is active constitutes a stationary Markovian SPNE if and only if* $\delta \leq \underline{\delta}$*.*

Proof. Deviations in states at which only one firm is active are not profitable because the active firm is producing the monopoly outcome and the other firm is out of the market. Thus, only deviations at states with both firms active are possible. Note that if both firms are active and produce Cournot outputs, profits for firm i are $\pi_i^C/(1-\delta)$. Given that the Cournot output profile is not BF, a potential profitable deviation is such that one firm drives the other to bankruptcy without bankrupting itself. The discounted profits for this move are $\pi_i^D + \delta\pi_i^M(1-\delta)$, where π_i^M are monopoly profits and π_i^D are profits in the deviation for firm i. Firm i drives firm j to bankruptcy by producing an outcome $\hat{x}_i > x(x_j^C)$, where $x(x_j^C)$ is such that $\pi_j(x_j^C, x(x_j^C)) = 0$. Given that $\pi_j(x_j^C, x_i^C) \geq 0$, $x(x_j^C) \geq x_i^C$, and therefore, for all $\hat{x}_i > x(x_j^C)$, $\pi_i(\hat{x}_i, x_j^C) < \pi_i(x(x_j^C), x_j^C)$. Thus, driving firm j to bankruptcy is not a profitable deviation for firm i if and only if

$$\pi_i^C \geq (1-\delta)\pi_i(x(x_j^C), x_j^C) + \delta\pi_i^M. \tag{3.3}$$

For $\delta \simeq 0$ the right hand side of 3.3 is approximately $\pi_i(x(x_j^C), x_j^C)$, and then the inequality holds because $\pi_i^C \geq \pi_i(x(x_j^C), x_j^C)$. For $\delta \simeq 1$ the right hand side of 3.3 is approximately π_i^M, and the inequality does not hold because $\pi_i^C < \pi_i^M$. Since the right hand side of 3.3 is decreasing in δ, by the intermediate value theorem there is $\underline{\delta}_i$ such that $\pi_i^C \geq (1-\delta)\pi_i(x(x_j^C), x_j^C) + \delta\pi_i^M$ if and only if $\delta \leq \underline{\delta}_i$. In conclusion, by taking $\underline{\delta} = \min\{\underline{\delta}_i\}_{i \in \{I,S\}}$, we get the result. ■

In order to grasp the implications of bankruptcy considerations, we start by analyzing the case of linear demand and constant marginal cost. We also assume that at Cournot equilibrium both firms are active.

As we have shown in Lemma 3, even though the Cournot output is not BF, for $\delta \leq \underline{\delta}$ firm S does not have incentives to bankrupt firm I. In the next proposition we show that if firm S does

[3] Recall that if firms have constant marginal costs, the Cournot outcome is BF if (i) costs are identical or (ii) they are sufficiently different so that only one firm produces in equilibrium. In cases where costs are different, but not quite, the Cournot outcome is not BF and therefore, for some discount factors, it is not an equilibrium.

not have incentives to predate at Cournot outputs, it does not have incentives to predate at any collusive output supported by triggering with the Cournot outcome forever without bankruptcy considerations. The intuition behind this is that in all those collusive outcomes, firms produce less than under Cournot. And since the cost of bankrupting a firm in the collusive output is larger than in the Cournot outputs, if bankruptcy was not profitable for the superior firm in the Cournot outputs, it is not profitable under collusion. Before formally introduce the proposition, we need the following auxiliary lemma.

Lemma 4. *Suppose that a collusive outcome $\mathbf{x} = (x_I, x_S)$ is supported by the grim-trigger strategies under bankruptcy consideration for some $\delta \leq \underline{\delta}$. Then, $\mathbf{x}$ can also be supported by the grim-trigger strategies without bankruptcy consideration for the same δ.*

Proof. Since the punishment phase (repeated production of the Cournot outcomes) constitutes a SPNE without bankruptcy consideration, it is enough to show that no deviation can occur on the cooperation phase, i.e., (3.1) holds for $\mathbf{x} = (x_I, x_S)$. By our assumption, $\mathbf{x}$ is supported by the grim-trigger strategies under bankruptcy consideration. Therefore, each firm i does not have incentive to switch its output from x_i to any other, in particular to $B_i(x_j)$, in the presence of bankruptcy constraints. If choosing $B_i(x_j)$ makes the rival firm to stay in the market, the incentive condition remains identical to (3.1). If choosing $B_i(x_j)$ makes the rival to go bankrupt, the condition becomes

$$\pi_i(x_i, x_j) \geq (1-\delta)\pi_i(B_i(x_j), x_j) + \delta\pi_i^M > (1-\delta)\pi_i(B_i(x_j), x_j) + \delta\pi_i^C. \tag{3.4}$$

The second inequality is satisfied since $\pi_i^M > \pi_i^C$. Thus, (3.1) must hold in both possible cases. ■

Proposition 2. *Suppose demand is linear, marginal cost is constant and both firms are active at the Cournot output. When $\delta \leq \underline{\delta}$, bankruptcy considerations do not change the collusive outcomes that can be supported by grim-trigger strategies.*

Proof. We have to show that, for $\delta \leq \underline{\delta}$, (i) any $\mathbf{x}$ that can be supported (by the grim-trigger strategies) under bankruptcy consideration can also be supported without the consideration, and (ii) any $\mathbf{x}$ that can be supported without bankruptcy consideration can also be supported under the consideration. Since (i) is already shown by Lemma (4) in general settings, we only need to show (ii).

The collusive outcomes that can be supported for a given δ without the bankruptcy considerations satisfy (3.1). For $i = S$, the following inequality holds.

$$\pi_S(x_S, x_I) \geq (1-\delta)\pi_S(B_S(x_I), x_I) + \delta\pi_S^C. \tag{3.5}$$

As we have shown, $x_I \leq x_I^C$. Under bankruptcy considerations, we have to add the condition that each firm does not have incentives to drive the other firm to bankruptcy. Since only firm S possibly has those incentives, the condition that will prevent that deviation is,

$$\pi_S(x_S, x_I) \geq (1-\delta)\pi_S(x(x_I), x_I) + \delta\pi_S^M. \tag{3.6}$$

Since $\delta \leq \underline{\delta}$, at the Cournot output profile firm S does not have incentives to bankrupt firm I,

$$\pi_S^C \geq (1-\delta)\pi_S(x(x_I^C), x_I^C) + \delta\pi_S^M. \tag{3.7}$$

Condition (3.5) can be rewritten as

$$\pi_S(x_S, x_I) \geq (1-\delta)\pi_S(B_S(x_I), x_I) - (1-\delta)\pi_S^C + \pi_S^C. \tag{3.8}$$

Substituting (3.7) into (3.8),

$$\pi_S(x_S, x_I) \geq (1-\delta)\pi_S(B_S(x_I), x_I) - (1-\delta)\pi_S^C + (1-\delta)\pi_S(x(x_I^C), x_I^C) + \delta\pi_S^M, \tag{3.9}$$

which is also rewritten as

$$\begin{aligned}
\pi_S(x_S, x_I) \geq\ & (1-\delta)\pi_S(B_S(x_I), x_I) - (1-\delta)\pi_S^C + (1-\delta)\pi_S(x(x_I^C), x_I^C) \\
& - (1-\delta)\pi_S(x(x_I), x_I) + (1-\delta)\pi(x_S(x_I), x_I) + \delta\pi_S^M \\
& = (1-\delta)[(\pi_S(B_S(x_I), x_I) - \pi_S(x(x_I), x_I)) - (\pi_S(B_S(x_I^C), x_I^C) - \pi_S(x(x_I^C), x_I^C))] \\
& + (1-\delta)\pi_S(x(x_I), x_I) + \delta\pi_S^M.
\end{aligned} \tag{3.10}$$

In order to derive (3.6), it is enough to show the third line in (3.10) is non-negative, that is,

$$\pi_S(B_S(x_I), x_I) - \pi_S(x(x_I), x_I) \geq \pi_S(B_S(x_I^C), x_I^C) - \pi_S(x(x_I^C), x_I^C). \tag{3.11}$$

Note that $\pi_S(B_S(x_I), x_I) - \pi_S(x(x_I), x_I)$ is decreasing in x_I for all $x_I < x_I^C$ in the wide range of situations, including the linear demand with constant marginal costs. Therefore, (3.11) holds under our assumption. ■

When $\delta > \underline{\delta}$, the stationary Markov strategy $x_i = x_i^C, i \in \{I, S\}$ in the state with all firms active, and $x_i = x_i^M$ in states where only firm i is active is not a stationary Markovian Perfect Equilibrium because firm S has incentives to bankrupt firm I. Thus, the Markovian Perfect equilibrium may involve mixed strategies. We formally discuss this point at the end of this section. Before that, we discuss first the possibility of collusion in this case. In particular, consider all collusive outcomes supported with grim-trigger strategies without bankruptcy considerations. In the next example we show that none of those collusive outcomes can be supported as an SPNE because in all of them, firm S have incentives to bankrupt firm I.

Example 1. *Let* $p(X) = (a - x_S - x_I)$, $a > 0$, $a_I = a - c_I = 5, a_S = a - c_S = 7$, $\delta = 0.3$. *Note first that* $\delta > \underline{\delta} = 0.235\,29$. *Let* (x_S, x_I) *be an output profile satisfying (3.1). That is:*

$$(7 - x_I - x_S)x_S \geq 0.7\left(\frac{7 - x_I}{2}\right)^2 + 2.7, \tag{3.12}$$

$$(5 - x_I - x_S)x_I \geq 0.7\left(\frac{5 - x_S}{2}\right)^2 + 0.3 \tag{3.13}$$

For $\delta > \underline{\delta}$, *at the Cournot output firm* S *has incentives to bankrupt firm* I. *We can see that at all the output profiles that satisfied (3.12) and (3.13), firm* S *has incentives to bankrupt firm* I. *That is,*

$$(7 - x_I - x_S)x_S < 1.4(5 - x_I) + 0.3\left(\frac{7}{2}\right)^2. \tag{3.14}$$

In Figure 1 the area enclosed between the two solid lines corresponds to all the output profiles (x_I, x_S) *that satisfy (3.12) and (3.13). The area above the dash line corresponds to all the outputs that satisfy (3.14). Note that for all* (x_I, x_S) *that satisfy (3.12) and (3.13), firm* S *has incentives to bankrupt firm* I.[4]

[4] A similar example can be constructed with linear demand and cuadratic cost.

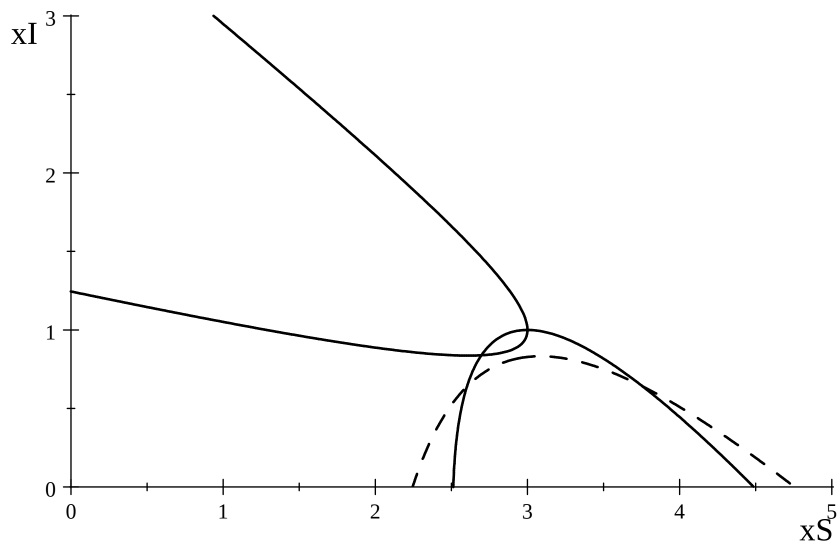

Figure 1

As we have mentioned above, in what follows we discuss the properties of equilibrium when $\delta > \underline{\delta}$. We start by a general observation that will be useful later on.

Lemma 5. *For any pure strategy SPNE, no firm goes bankrupt.*

Proof. Suppose that firm i goes bankrupt in some period t, which happens only if its profit in t is negative. Since the profits after bankruptcy are always zero, the i's continuation payoff at t is zero. However, producing nothing at t and at any following periods, assures zero profits, so firm i can profitably deviate by choosing $x_i^t = 0$ at t. Thus we derive contradiction. ■

Note that Lemma 5 holds even when strategies are not constrained to be Markovian. The following lemma shows that when the repeated Cournot outcome cannot be an equilibrium, no equilibrium in pure strategies exists when δ is large.

Lemma 6. *For any $\delta > \underline{\delta}$, there is no stationary Markov SPNE in pure strategies.*

Proof. Given no bankruptcy occurs in an equilibrium in pure strategies (Lemma 5), the repeated Cournot outcome when both firms are active is a unique mutual best reply in pure strategies that are Markovian. However, it cannot be an equilibrium for $\delta > \underline{\delta}$ by Lemma 3. ■

In light of Lemma 6, we study equilibria in mixed strategies when the infinite repetition of the Cournot output cannot be an equilibrium, i.e., $\delta > \underline{\delta}$. The existence of a stationary Markovian equilibrium is guaranteed by an extension of a theorem proved in Dasgupta and Maskin (1986) which we leave in an appendix.

Proposition 3. *For any δ, there exists at least one stationary Markovian equilibrium, possibly, in mixed strategies.*

Proof. See Appendix.

The characterization of the mixed strategy equilibrium is not an easy task. We limit our study to the characterization of the support of the mixed strategy. The details are developed in the Appendix but we highlight here some of the properties. i) The support of each firm's mixed strategy contains exactly one interval with a unique mass point. Moreover, for each firm the mass point coincides with the best reply of the static game (to the other firm's mixed strategy). ii) The inferior firm becomes bankrupt with positive probability (so monopolization occurs with positive probability). There is only one output in its support in which the probability of bankruptcy is zero, namely the inferior point of the support. As output increases, the probability of bankruptcy increases but is never one. The mass point lies weakly above the interval. This reflects that the possibility of bankruptcy makes the inferior firm prudent when choosing outputs. iii) For the superior firm, the mass point is isolated and lies strictly below the interval support. Since the superior firm would not produce a larger output than the best reply unless bankruptcy occurs, outputs in the interval support reflect the predatory activities of the superior firm.

4. Equilibrium with Increasing Average Cost and Patient Firms

The folk theorem of repeated games states that when firms are sufficiently patient, arbitrary feasible payoffs larger than the minimax can be obtained as the average payoff of an SPNE of the repeated game. Thus a natural question is to ask what kinds of payoffs can be supported as an SPNE in our model for sufficiently patient firms. This section is devoted to this task under Assumption 1 (see Section 3). We concentrate here on pure-strategy equilibria.

We first see that, for sufficiently patient firms (δ close to one), any SPNE of the dynamic game yields BF action profiles in each period. This result is independent on both demand and costs conditions. Denoting monopoly profits for firm i as π_i^M, we have the following:

Proposition 4. *Let* $((x_S^1, x_I^1), ..., (x_S^t, x_I^t), ...)$ *be a sequence of output profiles yielded by a SPNE for a sufficiently large* δ *and such that there is an* $\epsilon > 0$ *with* $\pi_i(\mathbf{x}^t) + \epsilon \leq \pi_i^M$ *for all* $t = 1, 2, ...,$ $i \in \{S, I\}$. *Then, when* $\delta \to 1$, (x_S^t, x_I^t) *is BF for all* t.

Proof. Suppose that in period t, (x_S^t, x_I^t) is not BF. Thus, one firm can bankrupt the other. Suppose, without loss of generality, that the superior firm can bankrupt the inferior one. Consider the following strategy for firm S. In period t, firm S produces an output $\tilde{x}_S$ that drives firm I into bankruptcy, and produces the monopoly output thereafter. The continuation payoff for firm S is

$$(1-\delta)(\pi_S(\tilde{x}_S, x_I^t) + \delta\pi_S^M + \delta^2\pi_S^M + ...). \tag{4.1}$$

The continuation payoff at t for the sequence $((x_S^1, x_I^1), ..., (x_S^t, x_I^t), ...)$ is:

$$(1-\delta)(\pi_S(\mathbf{x}^t) + \delta\pi_S(\mathbf{x}^{t+1}) + \delta^2\pi_S(\mathbf{x}^{t+2}) + ...). \tag{4.2}$$

By the definition of an SPNE,

$$\pi_S(\mathbf{x}^t) + \delta\pi_S(\mathbf{x}^{t+1}) + \delta^2\pi_S(\mathbf{x}^{t+2}) + ... \geq \pi_S(\tilde{x}_S, x_I^t)) + \delta\pi_S^M + \delta^2\pi_S^M + ... \tag{4.3}$$

or

$$\pi_S(\mathbf{x}^t) - \pi_S(\tilde{x}_S, x_I^t) \geq \delta(\pi_S^M - \pi_S(\mathbf{x}^{t+1})) + \delta^2(\pi_S^M - \pi_S(\mathbf{x}^{t+2})) + ... \geq \delta\epsilon + \delta^2\epsilon + ... = \delta\frac{\epsilon}{1-\delta}. \tag{4.4}$$

Clearly, when $\delta \to 1$, the above inequality is impossible, contradicting that we were in an SPNE. ■

The condition that $\pi_i(\mathbf{x}^t) + \epsilon \leq \pi_i^M$ is satisfied, for instance, for stationary sequences. This result shows that when firms are sufficiently patient, incentives for predation are high so firms only choose BF allocations in equilibrium.[5]

In what follows we characterize the payoff that can be supported as a SPNE of the dynamic game for sufficiently patient firms. For this purpose, we adapt the standard definition of a minimax payoff to the case in which outputs are constrained to be BF.

Assumption 1 guarantees that the set of BF output profiles is not empty and, as we have shown in Lemma 2, is characterized as $BF = \{(x_S, x_I)/0 \leq x_i \leq \bar{x}_i \text{ for } i \in \{S, I\}\}$, where $\bar{\mathbf{x}} = (\bar{x}_S, \bar{x}_I)$

[5] This result can not be extended to n firms. The difficulty is that, after a firm is bankrupted, the strategies of the surviving firms can be anything. An example of this is obtainable under request from the authors.

with $\bar{x}_i \neq 0$ for all $i \in \{S, I\}$ is such that $\pi_i(\bar{\mathbf{x}}) = 0$ for all $i \in \{S, I\}$. **The minimax** BF **payoff for firm** i is defined as:

$$\pi_{im} = \min_{x_j \in [0, \bar{x}_j]} \max_{x_i \in [0, \bar{x}_i]} \pi_i(x_i, x_j) = \max_{x_i \in [0, \bar{x}_i]} \pi_i(x_i, \bar{x}_j). \tag{4.5}$$

Note that, since $\pi_i(\bar{\mathbf{x}}) = 0$, $\pi_{im} > 0$ because at $(\bar{x}_i, \bar{x}_j)$ firm i by reducing his output gets positive profits. The standard minimax, when applied to our model, yields a minimax payoff of zero because firm $j \neq i$ could produce an output, call it x_j, such that the best reply of i is to produce zero. But x_j might not fulfill the definition of minimax BF payoffs because it might drive firm j to bankruptcy.

In the following example we show the isoprofits corresponding to the minimax BF payoff.

Example 2. *Let firm I be such that $\pi_I(x_I, x_S) = (10 - x_I - x_S)x_I - x_I^2$, and firm S be such that $\pi_S(x_S, x_I) = (10 - x_S - x_I)x_S - \frac{1}{5}x_S^2$. In Figure 2, the intersection of the linear solid lines provides $(\bar{x}_I, \bar{x}_S)$. The dash lines correspond to the best replies of the firms and the curve lines corresponds to the minimax BF isoprofits. Minimax outputs are those between the two isoprofit corresponding to minimax BF profits.*

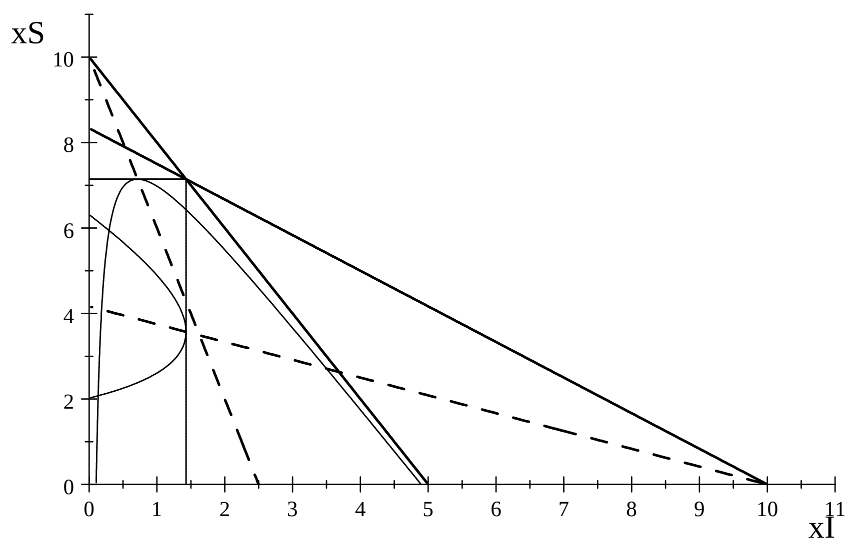

Figure 2

The next proposition shows that, for a sufficiently large δ, no SPNE of the dynamic game can give any firm a payoff lower than its minimax BF payoff.

Proposition 5. *Under Assumption 1, $\delta' \in (0,1)$ exists such that for all $\delta \in (\delta', 1)$, $\pi_i < \pi_{im}$ cannot be supported in any SPNE.*

Proof. For each $i \in \{S, I\}$, let $\delta^i \in (0,1)$ be such that $\delta^i \pi_i^M = \pi_{im}$ where π_i^M is the monopoly profits for firm i and π_{im} is the minimax BF payoff. Since $\pi_i^M > \pi_{im}$, δ^i exists. Let $\delta' = \max_{i \in N} \delta^i$ and let $\delta \in (\delta', 1)$. Suppose that $\pi_i < \pi_{im}$ is supported as an SPNE for $\delta \in (\delta', 1)$. If $x_j^t \in [0, \bar{x}_j]$ for all t on and off the equilibrium path, firm i could have achieved at least π_{im} irrespective of δ by choosing an output $x_i^t \in [0, \bar{x}_i]$ (the standard argument in repeated games can be applied here because in this case the output profile at each t is in the BF set). Therefore, if $\pi_i < \pi_{im}$ happens in equilibrium, $x_j^t > \bar{x}_j$ must hold for some t either on or off the equilibrium path. We show that if this is the case, the continuation payoff for i at t in equilibrium, Π_i^t, must be such that $\Pi_i^t \geq \delta \pi_i^M$. Suppose that $\Pi_i^t < \delta \pi_i^M$; since $x_j^t > \bar{x}_j$, firm i can make firm j bankrupt retaining non-negative profits, and can achieve a monopoly profit in every period from $t+1$. Although the bankruptcy of firm j has a cost at period t, the continuation payoff for firm i if it deviates from equilibrium will be at least $\delta \pi_i^M$. However, if $\delta \pi_i^M > \Pi_i^t$ such a deviation would be profitable for firm i and would contradict the notion that we are in equilibrium. Therefore, $\Pi_i^t \geq \delta \pi_i^M$. Since $\delta \in (\delta', 1)$, $\Pi_i^t > \pi_{im}$. Thus, π_i must exceed π_{im} which concludes the proof. ■

Note that when δ is very small, firms may have little incentives to engage in predatory activities and allocations which are not BF might be supported as an SPNE. For instance, if $\delta = 0$ only the payoffs corresponding to the Cournot equilibrium can be supported as an SPNE, but Cournot equilibrium outputs may be not BF (see Figure 2).

We are now ready to prove a folk theorem regarding BF allocations. We say that π_i is an **individually rational** BF **payoff** if $\pi_i > \pi_{im}$. An individually rational BF vector payoff $(\pi_i)_{i \in \{S,I\}}$ is **feasible** if a BF output profile (x_i, x_j) exists such that $\pi_i = \pi_i(x_i, x_j)$ for all $i, j \in \{S, I\}$, $i \neq j$. In Figure 2 the BF output profiles that give an individually rational BF payoff are the ones in the area limited by the minimax BF isoprofits.

Proposition 6. *Suppose Assumption 1 holds. Let $\pi = (\pi_i)_{i \in \{S,I\}}$ be a feasible and individually rational BF payoff vector. Then, $\delta' \in (0,1)$ exists such that for all $\delta \in (\delta', 1)$, π is the average payoffs in some SPNE.*

Proof. The proof is given by constructing an equilibrium which is originally proposed by Fudenberg and Maskin (1986). Let $(\pi_i)_{i \in \{S,I\}}$ be feasible and individually rational BF payoff vector. By the definition of feasibility, there is a BF output profile (x_i, x_j) such that $\pi_i = \pi_i(x_i, x_j)$ for $i, j \in \{S, I\}$, $i \neq j$.

Suppose each firm $i \in \{S, I\}$ produces output x_i in each period if no deviation has occurred, but both $i \in \{S, I\}$ produce $\bar{x}_i$, for T periods once one of them unilaterally deviates from the equilibrium path. If no one deviates during these T periods, then firms go back to the original path. Otherwise, if one of them deviates, then firms restart this phase for T more periods. We prove that this strategy constitutes an SPNE.

First consider a deviation from the equilibrium path. Suppose firm i produces $x'_i \neq x_i$ in some period, say period t. By the one-stage-deviation principle (e.g. Fudenberg and Tirole, 1991, p.110), a deviation is profitable if and only if firm i could profit by deviating from the original strategy in period t only and conforming thereafter. Therefore, firm i can benefit by deviation if and only if x'_i exists such that

$$(1-\delta)\pi_i(x'_i, x_j) + (1-\delta)(\delta + .. + \delta^T)\pi_i(\bar{x}_i, \bar{x}_j) + \delta^{T+1}\pi_i > \pi_i, \tag{4.6}$$

or equivalently,

$$(1-\delta)\pi_i(x'_i, x_j) + \delta^{T+1}\pi_i > (1-\delta)(1 + \delta + ... + \delta^T)\pi_i + \delta^{T+1}\pi_i, \tag{4.7}$$

which it holds whenever:

$$(1-\delta)\{(\pi_i(x'_i, x_j) - \pi_i) - (\delta + ... + \delta^T)\pi_i\} > 0. \tag{4.8}$$

Let $\Delta_i = \max_{x'_i} \pi_i(x'_i, x_j) - \pi_i$ and choose T such that $\Delta_i < T\pi_i$. Note that the left hand side of (4.8) is weakly less than $(1-\delta)\{\Delta_i - (\delta + ... + \delta^T)\pi_i\}$. This term is non-positive when δ is close to 1. Therefore, (4.8) cannot be satisfied for such T.

By the same argument as above, firm i can benefit by deviating from the mutual minmax phase if and only if x''_i exists such that

$$\begin{aligned}&(1-\delta)\pi_i(x''_i, \bar{x}_j) + (1-\delta)(\delta + ... + \delta^T)\pi_i(\bar{x}_i, \bar{x}_j) + \delta^{T+1}\pi_i \\ &> (1-\delta)(1 + \delta + ... + \delta^{T-1})\pi_i(\bar{x}_i, \bar{x}_j) + \delta^T\pi_i,\end{aligned} \tag{4.9}$$

which can be written as:

$$\pi_i(x''_i, \bar{x}_j) > \delta^T\pi_i. \tag{4.10}$$

Note that $\pi_i(x_i'', \bar{x}_j) \leq \max_{x_i \in [0, \bar{x}_i]} \pi_i(x_i, \bar{x}_j) = \pi_{im}$. Since $\pi_i > \pi_{im}$ by assumption, (4.10) never holds when δ is close to 1.

Thus there is no profitable deviation when δ is sufficiently close to 1. Since π is an arbitrary feasible and individually rational BF payoff vector, the proof is complete. ■

5. Final Remarks

In this paper we have developed a theory of dynamic competition in which firms may bankrupt other firms. We have shown that this theory provides new insights into the theory of dynamic games. Cournot may not constitute a Markovian equilibrium. When this is the case, collusive outcomes supported as an SPNE by grim-trigger strategies without bankruptcy considerations may not be supported now, because in those outcomes, the superior firm have incentives to predate. For sufficiently high δ the Markov equilibrium involves mixed strategies and predation occurs with positive probability. Finally, we have shown limit results, a folk theorem kind of result. Collusion is more difficult to sustain than in standard supergames and, in particular, not every individually rational payoff can be supported by a SPNE.

Our results are obtained at the cost of making several simplifying assumptions to make the model tractable. Here we discuss some of the issues arising from these simplifications.

No accumulation

In this paper we focused on outputs that make other firms bankrupt, but we did not consider the funds that might support or deter aggressive strategies (the "deep pocket" argument). Our research strategy is to analyze the incentives to prey in the simplest possible case where no funds can be accumulated. A full fledged model of accumulation and predation is, no doubt about it, preferable but it is beyond the scope of our paper. In other cases, accumulation of profits might play an important role in shaping the SPNE set as in the model of Rosenthal and Rubinstein (1984).[6]

Credit

If credit is given on the basis of past performance, the redefinition of the BF set can be applied here and credits can be incorporated into the model. However, if credit is given on the basis of

[6] They characterize a subset of the Nash equilibria in the repeated game with no discounting (i.e., $\delta = 1$) where each player regards ruin of the other player as the best possible outcome and his own ruin as the worst possible outcome.

future performance, future performance also depends on credit (via the BF constraints), which makes this problem extremely complex. This points to a deep conceptual problem about credit in oligopolistic markets where firms might be made bankrupt. This topic should be the subject of future research.

Entry

In this paper we assumed that the disappearance of a firm does not bring a new one into the market. Of course this should not be taken literally. What we mean is that if entry does not quickly follow, it makes sense, as a first approximation, to analyze the model with a given number of firms. For instance it can be shown that when firms are very patient and costs and demand are linear, ruining a firm is a good investment even if monopoly lasts for one period. In other cases, though, the nature of equilibria will be altered if, for instance, entry immediately follows the ruin of a competitor as in the model of Rosenthal and Spady (1989).[7]

Buying Competitors

In our model, there is no option to buy a firm. Sometimes it is argued that buying an opponent may be a cheaper and safer strategy than ruining it. We do not deny that buying competitors plays an important role in business practices. However, we do not agree that under the option of buying, ruining a competitor is irrational. First, buying competitors may be forbidden by a regulatory body because of anticompetitive effects. Second, when the owner of a firm sells it to competitors, this does not stop her from creating a new firm and financing it with the money received from selling the old one. In other words, selling a firm is not equivalent to a contract in which the owner commits not to enter into a market again. Thus, bankruptcy may be the only credible way of getting rid of a competitor. Finally, buying and ruining competitors may complement each other because the acquisition value may depend on the aggressiveness of the buyer in the past; see Burns (1986) for some evidence in the American tobacco industry. Thus, it seems that a better understanding of the mechanism of ruin might help to further enhancement of our understanding of how the buying mechanism works in this case.

Summing up, the model presented in this paper sheds some light on certain aspects of the equilibrium in oligopolistic markets in which firms may make each other bankrupt . We hope that

[7] They consider a prisoner's dilemma in continuous time in a market with room for two firms only. When a firm goes bankrupt, this firm is immediately replaced by a new entrant. They show that some kind of predatory behavior can arise in equilibrium.

the insights obtained here can be used in further research in this area.

6. APPENDIX

Proof of Proposition 3. There are 4 possible states in our dynamic game, i.e., the set of active firms is (i) S and I, (ii) S (iii) I, and (iv) empty. Note that continuation payoffs in (ii) to (iv) can be derived straightforwardly. Then, a stationary Markov SPNE of our dynamic game is identical to a Nash equilibrium of a static game with the following payoff functions for each firm:

For all $\mathbf{x}$ such that $\pi_i(\mathbf{x}) \geq 0$ for all $i \in \{S, I\}$,

$$V_i(\mathbf{x}) = \frac{\pi_i(\mathbf{x})}{1-\delta} \; i \in \{S, I\}; \tag{6.1}$$

for all $\mathbf{x}$ such that $\pi_i(\mathbf{x}) \geq 0$ and $\pi_j(\mathbf{x}) < 0$, $i \neq j$, $i, j \in \{S, I\}$,

$$V_j(\mathbf{x}) = \pi_j(\mathbf{x}); \tag{6.2}$$

$$V_i(\mathbf{x}) = \pi_i(\mathbf{x}) + \frac{\delta}{1-\delta}\pi_i^M; \tag{6.3}$$

for all $\mathbf{x}$ such that $\pi_i(\mathbf{x}) < 0$ for all $i \in \{S, I\}$,

$$V_i(\mathbf{x}) = \pi_i(\mathbf{x}), \; i \in \{S, I\}. \tag{6.4}$$

This game has discontinuous payoff functions so the usual existence theorems can not be applied here. The discontinuities in our game arises because of the possibility of bankruptcy. When only one firm goes bankrupt, the bankrupted firm disappears from the market and the other one gets monopoly profits. Fortunately, Theorem 5b in Dasgupta and Maskin (1986) (D&M hereinafter) can be invoked to show the existence of a Nash equilibrium in mixed strategies. Roughly speaking, the existence of equilibrium is guaranteed when utility functions are bounded and continuous except in a set of measure zero in the (joint) strategy space. More precisely, the theorem requires that (a) discontinuities occur in a set whose dimension is strictly lower than the dimension of the strategy space, (b) strategy sets are intervals, and (c) when we approach a discontinuity if a firm payoff falls, another rises. [8]

[8] The original proof of D&M presumes that the set of points in which discontinuities occur is the main diagonal, i.e., $x_S = x_I$. However, as the authors discuss in subsection 4.1, it is easy to verify that the essentially same proof also holds as long as the dimensionality assumption is satisfied.

To apply this theorem we need to construct an auxiliary game in which we disregard the possibility that both firms go bankrupt. We will now show that this auxiliary game fulfills the conditions of Theorem 5b. The bankruptcy-free constraint of a firm i is $p(x_i + x_j) \geq AC_i(x_i)$ and, since demand is strictly decreasing (and average costs are non-decreasing), the discontinuity set for a firm i is

$$A(i) = \{(x_i, x_j) \in \mathbb{R}^2_+ \mid \text{either } x_j = D(AC_i(x_i)) - x_i \text{ and } \pi_j(x_i, x_j) > 0,$$
$$\text{or } x_i = D(AC_j(x_j)) - x_j \text{ and } \pi_i(x_i, x_j) > 0\}, \tag{6.5}$$

which is of lower dimension than the set of possible outputs which is a subset of $\mathbb{R}^2_+$ with dimensionality 2. For instance, under linear demand ($p = a - x_1 - x_2$) and quadratic costs ($\alpha_i x_i^2$) the set of outputs for which a discontinuity occurs for firm i is the intersection of the segment given by $a - x_i - x_j - \alpha_i x_i = 0$ and $a - x_i - x_j - \alpha_j x_j > 0$, $j \neq i$.

Profits are bounded and continuous except in the aforementioned set, and at the discontinuity point when the profit of a firm falls (because this firm is bankrupt) the profit of the other firm rises (because it becomes a monopolist). Thus this auxiliary game fulfills the conditions of Theorem 5b in D&M and, therefore, it has an equilibrium in mixed strategies. The final step is to show that the Nash equilibrium of this auxiliary game is a Nash equilibrium of the original game. This is done by realizing that no profitable deviation exists in the original game from the Nash equilibrium of the auxiliary game: for the superior firm because it will never bankrupt itself and for the inferior firm because in order to bankrupt the superior firm it will become bankrupt itself. ■

Characterization of the support of the Markovian mixed strategy equilibrium.

In the rest of the appendix, we study the structure of the Markovian mixed strategy equilibrium which we simply call "equilibrium." To characterize the equilibrium support, we impose two assumptions.

Assumption AC The average cost for each firm is constant.

Let c_S denote the average constant cost of the superior firm and c_I the average constant cost of the inferior firm, $c_S > c_I$.

Assumption EC Each firm i's expected profit function (given other firm's mixed strategy) is strictly concave in x_i for any $x_i > 0$.

Let $E[\pi_i(x_i, x_j) \mid \sigma_j]$ be expected profits of firm i, given that firm j is using the mixed strategy σ_j.

Assumption EC is often imposed in the IO literature, which holds under the wide range of environments, such as linear cost functions with linear demand.

In order to characterize the support of the Markovian mixed strategy we need the following auxiliary lemmas.

Lemma 7. *For any mixed strategy of the other firm, the optimal output that maximizes a firm's expected profit in a period (i.e. $E[\pi_i(x_i, x_j) \mid \sigma_j]$) is always unique.*

Proof. From Assumption EC, $\arg\max_{x_i>0} E[\pi_i(x_i, x_j) \mid \sigma_j]$ is unique (if it exists), and i's optimal output is either $\arg\max_{x_i>0} E[\pi_i(x_i, x_j) \mid \sigma_j]$ or 0. Assumption AC implies that the expected profit of the former is always positive, so it cannot be the case that both become optimal. ■

For constant marginal costs with linear demand, we can easily derive the optimal output as follows. Firm i's expected profit function (given j's mixed strategy σ_j) is

$$\begin{aligned} E[\pi_i(x_i, x_j) \mid \sigma_j] &= E[(A - (x_i + x_j) - c_i)x_i \mid \sigma_j] \\ &= (A - (x_i + E[x_j]) - c_i)x_i, \end{aligned} \tag{6.6}$$

which is a quadratic function of x_i. Then, the optimal solution is derived by

$$x_i = \max\left\{0, \frac{A - c_i}{2} - \frac{E[x_j]}{2}\right\}, \tag{6.7}$$

which is clearly unique.

Lemma 8. *Given $\delta > \underline{\delta}$, for any equilibrium in mixed strategies,*

(i) at least one firm goes bankrupt with strictly positive probability,

(ii) no firm uses pure strategy.

Proof. (i) Since $\delta > \underline{\delta}$, at least one firm is not using a pure strategy. Suppose without loss of generality that is firm j and suppose that no firm goes bankrupt. Pick any two outputs x, x' from the support of the mixed strategy of firm j. Then, the firm must be indifferent between choosing x and x'. However, given that no bankruptcy occurs, the firm's optimal output (to the other firm's equilibrium strategy) is always unique by Lemma 7. Thus we get a contradiction.
(ii) Suppose on the contrary that firm i uses pure strategy x_i (and j uses mixed strategy σ_j). Then,

firm j does not go bankrupt in equilibrium, since choosing such an output in the support of j is clearly suboptimal. Let $x < x'$ be two different outputs in the support of σ_j. Lemma 7 implies that, in order for j to be indifferent between x and x', bankruptcy must occur in either output. Since π_i is decreasing in x_j, i must go bankrupt under x' but not under x. Moreover, the support of σ_j cannot contain a point other than x and x' (since choosing such an output cannot yield the same profit as x and x' do). However, given that j randomizes only over the two points x and x', x must be best reply to x_i, since bankruptcy does not occur in such case. Then, we can conclude that either i or j has a profitable deviation; j has incentive to set x' as small as possible to make i going bankrupt, but then i can avoid bankruptcy by slightly reducing x_i. In this way, (x_i, σ_j) cannot be a mutual best reply. ■

Let us denote smallest and largest outputs in the support of the equilibrium (mixed) strategy for firm i by $\underline{x_i}$ and $\overline{x_i}$, respectively. By Lemma 8 (ii), we have $\underline{x_i} < \overline{x_i}$ for each i.

Lemma 9. *In equilibrium, the following condition must hold for every firm i:*

$$\pi_i(\overline{x_i}, \underline{x_j}) > 0.$$

Proof. Suppose $\pi_i(\overline{x_i}, \underline{x_j}) \leq 0$. By choosing $x_i = \overline{x_i}$, firm i always receives non-positive profit, and strictly negative profit when $x_j > \underline{x_j}$ (note $\pi_i(x_i, x_j)$ is decreasing in x_j). Since the latter case occurs with positive probability, i would always become strictly better off by choosing $x_i = 0$. ■

Lemma 10. *In equilibrium, the following condition must hold for at least one firm:*

$$\pi_i(\underline{x_i}, \overline{x_j}) > 0.$$

Proof. Suppose on contrary that $\pi_i(\underline{x_i}, \overline{x_j}) \leq 0$ and $\pi_j(\underline{x_j}, \overline{x_i}) \leq 0$ hold simultaneously. Combining with Lemma 9, the following inequalities must hold.

$$\pi_i(\overline{x_i}, \underline{x_j}) > 0 \text{ and } \pi_j(\underline{x_j}, \overline{x_i}) \leq 0 \Rightarrow AC_i(\overline{x_i}) < p(\overline{x_i} + \underline{x_j}) \leq AC_j(\underline{x_j}), \tag{6.8}$$

$$\pi_j(\overline{x_j}, \underline{x_i}) > 0 \text{ and } \pi_i(\underline{x_i}, \overline{x_j}) \leq 0 \Rightarrow AC_j(\overline{x_j}) < p(\overline{x_j} + \underline{x_i}) \leq AC_i(\underline{x_i}). \tag{6.9}$$

Since average costs are non-decreasing, the above conditions imply

$$p(\overline{x_j} + \underline{x_i}) \leq AC_i(\underline{x_i}) \leq AC_i(\overline{x_i}) < p(\overline{x_i} + \underline{x_j}), \tag{6.10}$$

$$p(\overline{x_i} + \underline{x_j}) \leq AC_j(\underline{x_j}) \leq AC_j(\overline{x_j}) < p(\overline{x_j} + \underline{x_i}), \tag{6.11}$$

which further implies $p(\overline{x_i}+\underline{x_j}) < p(\overline{x_j}+\underline{x_i})$ and $p(\overline{x_j}+\underline{x_i}) < p(\overline{x_i}+\underline{x_j})$, which is a contradiction. ■

Lemma 11. *If $\pi_i(\underline{x_i}, \overline{x_j}) > 0$ holds, then*
(i) $\underline{x_i}$ maximizes the expected per period profit given j's mixed strategy, $E[\pi_i(x_i, x_j) \mid \sigma_j]$,
(ii) $\underline{x_i}$ must be isolated from other part of the support of i's equilibrium strategy.

Proof. Since $\pi_j(\overline{x_j}, \underline{x_i}) > 0$ by Lemma 9, j's profit $\pi_j(x_j, \underline{x_i})$ is non-negative for any $x_j \in [0, \overline{x_j}]$. This implies that probability such that j goes bankrupt is 0 when i chooses output sufficiently close to $\underline{x_i}$. Therefore, (i) $\underline{x_i}$ must be optimal given that no bankruptcy occurs, and (ii) no output close to $\underline{x_i}$ can be contained in i's equilibrium support. ■

Lemma 12. *In equilibrium, firm I never produces strictly higher output than the one which maximizes its per period profit given firm S's equilibrium mixed strategy. That is,*

$$\overline{x_I} \leq \arg\max_{x_I} E[\pi_I(x_I, x_S) \mid \sigma_S].$$

Proof. Note first that I can make S bankrupt only if I itself goes bankrupt. Hence, firm I can never be better off by bankrupting firm S. Choosing $x_I > \overline{x_I}$ weakly increases the risk of bankruptcy and strictly reduces π_I in that period. Therefore, it must be suboptimal. (Note that I makes S bankrupt only when I itself goes bankrupt.) ■

Lemma 13. *The following conditions must hold:*

(i) $\pi_I(\underline{x_I}, \overline{x_S}) \leq 0$.

(ii) $\pi_S(\underline{x_S}, \overline{x_I}) > 0$.

Proof. We first verify (i). Suppose on the contrary that $\pi_I(\underline{x_I}, \overline{x_S}) > 0$ holds. Then, by Lemma 11, I must choose a strictly larger output than $\arg\max_{x_I} E[\pi_I(x_I, x_S) \mid \sigma_S]$, which contradicts Lemma 12. Given that (i) holds, (ii) must be satisfied by Lemma 10. ■

Lemma 14. *Let $x_S^0 = \arg\max_{x_S} E[\pi_S(x_S, x_I) \mid \sigma_I]$. The equilibrium support of firm S's mixed strategy is such that $\underline{x_S} = x_S^0$.*

Proof. By Lemma 13, $\pi_S(\underline{x_S}, \overline{x_I}) > 0$. Thus, by Lemma 11, $\underline{x_S}$ maximizes S's expected profit (per period) given I's mixed strategy and it is an isolated point. Therefore, $\underline{x_S} = x_S^0$.

Lemma 15. *In equilibrium, firm I goes bankrupt with positive probability.*

Proof. By Lemma 8 at least one firm goes bankrupt. If I does not go bankrupt then firm S is bankrupt, but this is impossible because whenever firm S is bankrupt firm I is also bankrupt. ■

Lemma 16. *For all x_I in I's mixed strategy support, $\pi_I(x_I, \overline{x_S}) \leq 0$.*

Proof. This follows immediately from Lemma 13. ■

Lemma 17. *In equilibrium:*
(i) There is at most an $\tilde{x}_I$ in the support of I's mixed strategy such that the probability of bankruptcy for firm I is zero.
(ii) If such $\tilde{x}_I$ exists, then $\tilde{x}_I = \underline{x_I}$.

Proof. By Lemma 12, $\overline{x_I} \leq \arg\max_{x_I} E[\pi_I(x_I, x_S) \mid \sigma_S]$. If there are two points with zero probability of bankruptcy, $\tilde{x}_I^1$ and $\tilde{x}_I^2$, it should be the case that $\tilde{x}_I^1 < \tilde{x}_I^2 < \overline{x_I}$. But then, since each firm i's expected profit function (given other firm's mixed strategy) is strictly concave in x_i, it could not be that the payoff of firm I is the same at $\tilde{x}_I^1$ and at $\tilde{x}_I^2$. The proof of (ii) follows immediately. ■

Lemma 18. *The probability of bankruptcy for firm I at $\underline{x_I}$ is zero.*

Proof. Suppose on the contrary that the probability that I goes bankrupt at $\underline{x_I}$ is positive. This probability is the probability that the superior firm produces $x_S \in (\tilde{x}_S, \overline{x_S}]$, for $\tilde{x}_S$ such that $p(\underline{x_I} + \tilde{x}_S) = c_I$. By Lemma 9, $\pi_I(\underline{x_I}, \underline{x_S}) > 0$, thus, $\tilde{x}_S \in (\underline{x_S}, \overline{x_S}]$. If $\tilde{x}_S$ is in the support of S's mixed strategy, firm S, by concentrating all the mass placed at $[\tilde{x}_S + \varepsilon, \overline{x_S}]$ in $\tilde{x}_S + \varepsilon$, will not change the probability of bankruptcy for firm I and will increase its per period payoff. If $\tilde{x}_S$ is not in the support of S's mixed strategy, let $\hat{x}_S > \tilde{x}_S$ be the closest point to $\tilde{x}_S$ in the support of S's mixed strategy. Again, firm S, by placing all the mass placed at $[\hat{x}_S, \overline{x_S}]$ in $\hat{x}_S$, will not change the probability of bankruptcy for firm I and will increase its per period payoff. ■

Lemma 19. *If the support of firm I's mixed strategy contains an interval $[\underline{x_I^*}, \overline{x_I}^*]$, then $\underline{x_I^*} = \underline{x_I}$, and there exists no other interval.*

Proof. Suppose that $[\underline{x_I^*}, \overline{x_I}^*]$ is in the support and $\underline{x_I^*} > \underline{x_I}$. Then, by Lemma 17 the probability of bankruptcy for firm I is positive at all $x_I \in [\underline{x_I^*}, \overline{x_I}^*]$. Suppose without loss of generality that $[\underline{x_I^*}, \overline{x_I}^*]$ is the first interval in the support. The argument in the proof of Lemma 18 can be replicated here applied to $\underline{x_I^*}$. Thus, the probability of bankruptcy for firm I at $\underline{x_I^*}$ should be zero. By Lemma 17 and 18, $\underline{x_I^*} = \underline{x_I}$. Therefore, it can not be another interval in the support. ■

Lemma 20. *The support of I's mixed strategy contains exactly one interval.*

Proof. By Lemma 19, if the support contains an interval, it must be unique. Suppose that the support does not contains an interval. Since all firms are playing mixed strategies, the support should contain at least two isolated mass points. Let these two points be x_I^1 and x_I^2, and assume that $x_I^1 < x_I^2$. At x_I^2 firm I must go bankrupt with positive probability. Thus, $\pi_I(x_I^2, \overline{x_S}) < 0$. But this implies that probability that I goes bankrupt is unchanged for any $x_S \in [\overline{x_S} - \varepsilon, \overline{x_S}]$ for sufficiently small ϵ. However, if $\overline{x_S} - \varepsilon$ is in the support of S's mixed strategy, S cannot be indifferent over this interval since there is a unique optimal output by Lemma 7. If $\overline{x_S} - \varepsilon$ is not in the support, it is a profitable deviation for firm S, because choosing $\overline{x_S} - \varepsilon$ does not change the probability of bankruptcy and gives a higher per period payoff (note $\underline{x_S} = x_S^0 < \overline{x_S}$ by Lemma 14). ■

Lemma 21. *The equilibrium support of firm S's mixed strategy contains at least one interval* $[\underline{x_S^*}, \overline{x_S}^*]$.

Proof. Note that σ_I contains an interval by Lemma 20 and $\overline{x_I} \leq \arg\max_{x_I} E[\pi_I(x_I, x_S) \mid \sigma_S]$ by Lemma 12. Since I's per period profit is strictly increasing and also continuous in x_I in the interval, the probability of bankruptcy must be continuously increasing (in order for I to be indifferent in the interval). This becomes possible only when σ_S also contains an interval where the distribution of x_S does not jump. ■

Lemma 22. *For any interval in the support of the firms' equilibrium mixed strategy, the following must hold:*

(i) $\pi_I(\underline{x_I^*}, \overline{x_S}^*) = 0$.

(ii) $\pi_I(\overline{x_I}^*, \underline{x_S^*}) = 0$

Proof. (i) Lemma 18 implies $\pi_I(\underline{x_I^*}, \overline{x_S}^*) \geq 0$. Since $\pi_I(\underline{x_I^*}, \overline{x_S}^*) \leq 0$ by Lemma 13, equality must hold.

(ii) If $\pi_I(\overline{x_I}^*, \underline{x_S^*}) < 0$, then $\pi_I(\overline{x_I}^*, x_S) < 0$ must hold for any $x_S \geq \underline{x_S^*}$. This implies that probability such that I goes bankrupt is unchanged if I chooses x_I from $[\overline{x_I}^* - \epsilon, \overline{x_I}^*]$ for sufficiently small ϵ. However, I cannot be indifferent over this interval. Therefore, $\pi_I(\overline{x_I}^*, \underline{x_S^*}) \geq 0$.

Now assume $\pi_I(\overline{x_I}^*, \underline{x_S^*}) > 0$. Suppose first that there is no isolated (mass) points in the support of I's mixed strategy. By Lemma 20 the support of I's mixed strategy contains exactly one interval, then, $\pi_I(\overline{x_I}^*, \underline{x_S^*}) > 0$ implies that $\pi_I(x_I, \underline{x_S^*}) > 0$ for any $x_I \in [\underline{x_I^*}, \overline{x_I}^*]$. Thus, the probability that I goes bankrupt is unchanged if S chooses x_S from $[\underline{x_S^*}, \underline{x_S^*} + \varepsilon]$ for sufficiently small ϵ. But if this is the case, S cannot be indifferent over this interval.

If there is an isolated point in the support of I's mixed strategy, by Lemmas 19 and 20, the largest output in I's support, $\overline{x_I}$, has to be the one. Suppose that $\pi_I(\overline{x_I}, \underline{x_S^*}) \geq 0$, then $\pi_I(x_I, \underline{x_S^*}) > 0$ for all $x_I < \overline{x_I}$ in I's support. Then for ε sufficiently small, for any $x_S \in [\underline{x_S^*}, \underline{x_S^*} + \varepsilon]$ the probability of bankruptcy for firm I is zero. But if this is the case, firm S can not be indifferent in the interval $[\underline{x_S^*}, \underline{x_S^*} + \varepsilon]$. Consequently, $\pi_I(\overline{x_I}, \underline{x_S^*}) < 0$. Since we assume $\pi_I(\overline{x_I}^*, \underline{x_S^*}) > 0$, when firm S produces $\underline{x_S^*}$ the probability of bankruptcy for firm I is the probability that I produces $\overline{x_I}$. By producing $\underline{x_S^*} + \varepsilon$ the probability of bankruptcy for I does not change. But in this case, S can not be indifferent among the outputs in $[\underline{x_S^*}, \underline{x_S^*} + \varepsilon]$. Thus $\pi_I(\overline{x_I}^*, \underline{x_S^*}) = 0$. ■

Lemma 23. *In equilibrium, the support of each firm strategy contains exactly one interval.*

Proof. The property in Lemma 22 cannot hold if a firm has more than one (disjoint) interval. By Lemma 21 and 20, we obtain the result. ■

Lemma 24. *The interval in the equilibrium support of firm S cannot contain $\underline{x_S^*}$.*

Proof. First, note that our argument so far has not concluded whether the interval is open or closed, i.e., endpoints are contained within the support or not. The lemma claims that the smaller endpoint of S's interval is not contained.

Suppose that $\underline{x_S^*}$ is included in the support. If I's support does not contain an isolated mass point, when firm S produces $\underline{x_S^*}$ the probability of bankruptcy for firm I is zero because by Lemma 22, $\pi_I(\overline{x_I}^*, \underline{x_S^*}) = 0$. Then, at x_S^0 (the isolated point in S's support which is the smallest output in the support), $\pi_I(x_I, x_S^0) \geq 0$ for all x_I in I's support. Thus, by producing x_S^0 firm S cannot bankrupt firm I. But in this case, firm S cannot be indifferent between x_S^0 and $\underline{x_S^*}$. If I's support contains an isolated point, $\overline{x_I}$, by producing $\underline{x_S^*}$ the probability that I goes bankrupt is the probability that I produces $\overline{x_I}$ because by Lemma 22, $\pi_I(\overline{x_I}^*, \underline{x_S^*}) = 0$. But then at $\underline{x_S^*} - \varepsilon$ for ε sufficiently small, the probability of bankruptcy for firm I will not change and firm S will be better off because $x_S^0 < \underline{x_S^*} - \varepsilon < \underline{x_S^*}$. ■

Lemma 25. *Let* $x_I^0 = \arg\max_{x_I} E[\pi_I(x_I, x_S) \mid \sigma_S]$, *Then, the following must hold:*
(i) $\overline{x_I} = x_I^0$.
(ii) $\pi_I(x_I^0, x_S^0) > 0$.

Proof. (i) Suppose that $\overline{x_I} < \arg\max_{x_I} E[\pi_I(x_I, x_S) \mid \sigma_S]$. By Lemma 22, $\pi_I(\overline{x_I}, \underline{x_S^*}) \leq 0$ (either because there is no isolated points in I's support and therefore $\overline{x_I} = \overline{x_I}^*$ and $\pi_I(\overline{x_I}, \underline{x_S^*}) = 0$, or there are isolated points and then $\overline{x_I}$ is an isolated point and $\pi_I(\overline{x_I}, \underline{x_S^*}) < 0$). Then $\pi_I(\overline{x_I}, x_S) < 0$ for all $x_S \in (\underline{x_S^*}, \overline{x_S}^*]$, and by Lemmas 9 and 14 $\pi_I(\overline{x_I}, x_S^0) > 0$. Thus, the probability of I being bankrupt when producing $\overline{x_I}$ is the probability that firm S produces $x_S \in (\underline{x_S^*}, \overline{x_S}^*]$. Take $x_I = \overline{x_I} + \varepsilon$ with ε sufficiently small, then $\pi_I(x_I, x_S^0) > 0$, and $\pi_I(x_I, \underline{x_S^*}) < 0$. Therefore $\pi_I(x_I, x_S) < 0$ for all $x_S \in (\underline{x_S^*}, \overline{x_S}^*]$. Thus, the probability of I being bankrupt at $\overline{x_I}$ is the same that at x_I. Given that $\overline{x_I} < x_I \leq \arg\max_{x_I} E[\pi_I(x_I, x_S) \mid \sigma_S]$, the one period payoff at x_I is higher and firm I would be better off at x_I than at $\overline{x_I}$. Thus, $\overline{x_I} = x_I^0$.

(ii) By (i) and Lemmas 9 and 14, $\pi_I(\overline{x_I}, \underline{x_S}) = \pi_I(x_I^0, x_S^0) > 0$ ■

Lemma 26. *If firm I has an isolated point in its support it has to be x_I^0, and in this case, the interval in the equilibrium support of firm I has to exclude $\overline{x_I}^*$.*

Proof. It follows directly from Lemma 25 that if there is an isolated point, x_I^0 must be isolated. Suppose that in this case, $\overline{x_I}^*$ is in the support. By Lemma 22 $\pi_I(\overline{x_I}^*, \underline{x_S^*}) = 0$, thus, the probability of bankruptcy for I when producing $\overline{x_I}^*$ is the probability that S produces $x_S \in (\underline{x_S^*}, \overline{x_S}^*]$. But given that $\pi_I(x_I^0, x_S^0) > 0$ by Lemma 25 (ii), the probability of bankruptcy for I when producing x_I^0 is also the probability that S produces $x_S \in (\underline{x_S^*}, \overline{x_S}^*]$. But if this is the case, I can not be indifferent between x_I^0 and $\overline{x_I}^*$. Therefore, $\overline{x_I}^*$ can not be in the support. ■

Summarizing we have the following characteristics of the equilibrium strategy support for each firm.

Proposition 7. *Every equilibrium must satisfy the following conditions:*

(i) *Firm S randomizes over $x_S^0 \cup (\underline{x_S^*}, \overline{x_S}^*]$ where $x_S^0 < \underline{x_S^*}$.*

(ii) *Firm I randomizes over $[\underline{x_I^*}, \overline{x_I}^*) \cup x_I^0$ where $\overline{x_I}^* \leq x_I^0$.*

(iii) $x_I^0 = \arg\max_{x_I} E[\pi_I(x_I, x_S) \mid \sigma_S]$.

(iv) *The probability of bankruptcy for firm I is positive at every x_I in the support except at $\underline{x_I^*}$.*

(v) $x_S^0 = \arg\max_{x_S} E[\pi_S(x_I, q_S) \mid \sigma_I]$.

(vi) $\pi_I(\underline{x_I^*}, \overline{x_S}^*) = 0$.

(vii) $\pi_I(\overline{x_I}^*, \underline{x_S^*}) = 0$.

(viii) $\pi_I(x_I^0, x_S^0) > 0$.

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