STABILITY IN NEO-CLASSICAL AND NEO-KEYNESIAN GROWTH MODELS

by

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I. Introduction

The aim of this paper is to investigate the properties of neo-classical and neo-Keynesian models of economic growth with respect to the stability of dynamic equilibria. The first question that immediately arrises is: What do we mean by stability?

To answer this question we must formulate our specific problem in such a way that we can first of all find out what concepts of stability we could theoretically apply and, secondly which ones make sense in this context.

We are considering dynamic systems, the structure of which we know via a set of equations. What we are interested in are the time-paths of the endogenous variables and whether there exists an equilibrium time path of these variables.

When these problems are solved, we have to ask ourselves how this equilibrium can be attained and what happens to it when disturbances occur.

Stability can now mean that:

1.) whatever the initial conditions of the system are it will tend to move towards a steady-state growth path, which

   a) is the same as if the relation of the relevant variables had been the("right" being the steady state solution ratios) from the outset on;

   b) is different from this steady-state path;

2.) whatever changes in the size or the sign of the structural parameters of the system may occur there will be a path back to equilibrium.

The present paper is mainly concerned with initial conditions as most of the other questions have been dealt with in great detail in other publications.

Another problem of great interest to us was to find out what values these parameters may take to yield a stable equilibrium and to what limit they may alter until the system does not return to its steady state.

The basis of our investigation were the growth models of Meade, Solow etc. on the neo-classical side and the Kaldor-model on the other, both of them in non-vintage form.

II. The Neo-Classical Model

A) Structure of the model

There are 3 relations to be specified (having to solve for 3 unknowns):

1.) A technical equation that relates output and inputs. (As the question of stability is sometimes best answered by working with a particular production function, we shall assume that the technology of our economy obeys a Cobb-Douglas type of function.)

2.) Behavioral equations that explain how people make use of their strategic variables income (in form of consumption and saving) and investment.

For the sake of simplicity a proportional saving-function will be used, investment is to depend on the rate of interest only.

The equality between saving and investment (ex-ante) is to be postulated to reach an equilibrium solution. Nothing is really said in the neo-classical model how this is to be achieved. Meade for example supposes that there exists a monetary authority in an economic system which fixes the interest rate (by various means of monetary policy) in such a way that the equality of saving and investment holds at any period of time.

It would be very interesting to find a function that shows how this is to be done. What sort of error adjustment mechanism is there to be assumed to ensure the stability of an equilibrium?

3.) Further an equation is required that shows the development of the labor-force (equilibrium-condition being that the total labor-force is employed).

B) Equilibrium and Stability

As it is the main task of this paper to examine the equilibrium solution for its stability properties, we can more or less skip the question of finding out the equilibrium solution as such. The main problem of this chapter is to find out whether an equilibrium position can be found starting from any set of initial conditions, or whether certain values for these have to be ruling to get to an equilibriumpath.

The following analysis follows the lines of argument of R.G.D. Allen's "Macro-economic Theory".

Assuming that all 3 equilibrium-conditions of the neo-classical model hold, it can be shown that starting from any set of initial conditions of the relevant variables, the growth path converges to the equilibrium path eventually.

The three conditions of equilibrium are:

1.) full capacity: $Y = F(K,L)$

2.) investment = saving: $DK = sY$

3.) full employment: $L = L_o e^{nt}$

where Y = national income (output)

$K = Le^{mt}$ Labor in efficiency units

L = Labor-force

s = propensity to save

m = rate of technical progress

n = rate of population growth

$D = \frac{d}{dt}$

Knowing the time path of the labor force, we are now interested in the time path of the capital stock K under these conditions. (K and L determining Y). Doing all the necessary transformations one obtains:

$DK = s\ F(K,L_o e^{ut})$ .... $u = m + n$

To make things more obvious, we shall now assume that productions follows a Cobb-Douglas function: $Y = K^a L^{1-a}$, where $0 < a < 1$.

Our differential equations is now of the form

$DK = s\ F(K,L_o e^{ut}) = sL_o^{\ 1-a} K^a e^{u(1-a)t}$.

The equilibrium time path of K is obtained by integration over all t:

$$\int \frac{dK}{K^a} = s\ L_o^{\ 1-a} \int e^{u(1-a)T}\ dt + C \Rightarrow$$

$$\frac{1}{1-a} K^{1-a} = \frac{s}{u\ (1-a)}\ L_o^{\ 1-a}\ e^{u(1-a)t} + C$$

i.e. $K^{1-a} = A + \frac{s}{u} L_o^{\ 1-a} e^{u(1-a)t}$

The constant A can be found by setting $t = 0$: $A = K_o^{\ 1-a} - \frac{s}{u} L_o^{\ 1-a}$

Hence the equilibrium path of K is given by:

$$K^{1-a} = K_o^{1-a} + \frac{s}{u} L_o^{1-a} \left[e^{u(1-a)t} - 1\right].$$

As this time path does not correspond to the steady state path we have to check whether it converges to it when $t \to \infty$. The term involving time in the whole expression is:

$$K^{1-a} = \frac{s}{u} L_o^{1-a} e^{u(1-a)t} .$$

When t becomes very large, the terms not involving t become negligible and our path converges to: $K = (\frac{s}{u})^{1/(1-a)} L_o e^{ut}$ which is the steady state path of K.

III. Neo-Keynesian Models (Kaldor).

A) Structure of the model

While the neo-classical model is based on the assumption of perfect competition in all markets, the Kaldor growth model rests on the theory of "monopolistic competition". Income distribution is no longer determined by the marginal products of the production factors. Instead the share of profits (and wages as a residual) out of the national income becomes an explicit variable of the model to be solved for.

Another problem arising in a system of non perfect competition is the determination of investment. As profit-maximisation is not applicable in its classical form (via the marginal productivity of capital) one has to specify an explicit investment function.

A saving function and a technical relation between inputs and outputs is needed again to close the model.

The Kaldor model as presented here does not by far give the full "flavor" of Kaldor's considerations. It only uses the mathematical relations as stated in the 4 following equations with 4 unknowns.

1.) Entrepreneurial behaviour concerning investment is characterised in the following way:

$$\frac{K^*}{Y} = v + \beta \frac{P}{K} \qquad (P \text{ .... Profits})$$

Entrepreneurs have a very precise idea about what their capital-stock should look like (desired capital-stock K^*). Income Y and the profitability of capital are the variables their investment decision is based on.

2.) The second behavioral equation deals with the saving-structure of a society. Generally speaking the following relation must hold: $S = sY$. Kaldor, however, prefers to break this global relation into 2 parts. He supposes that wage-earners save a different proportion of their income (s_w) than profit income receivers (s_p). The total savings of the society consequently are: $S = s_w(Y-P) + s_p P$. In the context of this paper we will make the (not too unrealistic) assumption that all saving comes out of profits (as Kaldor does in his work to facilitate matters).

3.) Instead of a production-function Kaldor thinks that the technology of the economy, which is constantly changing, can best be described by means of a "Technical Progress-Function". Technology becomes relevant for the economy only when it is "embodied" in the form of investment. Only investment (i.e. the change of the capital stock per time unit) can thus lead to a rise in output.

Using per capita variables ($k = K/L$, $y = Y/L$) we can write:

$$\frac{Dy}{y} = f\left(\frac{Dk}{k}\right)$$

4.) The expansion of the labor-force is again characterised by the function $L = L_o e^{nt}$.

B) Equilibrium

The equilibrium conditions are the same as in the neo-classical model.

1.) full capacity: $K^{*} = K$

2.) investment = saving: $DK = S = sY = s_w (Y-P) + s_p P =$
$= s_p P (\text{for } s_w = 0)$

3.) full employment: $L = L_o e^{nt}$

Nothing is said in Kaldor's work what entrepeneurs do when they are unable to reach their desired capital-stock right away. (This question will be considered later in this paper). Kaldor did, however, study the problem whether or whether not ex-ante savings will equal ex-ante investment very carefully. His lines of argument are only reviewed very briefly in this context.

The intersection point between the saving and investment function is only stable in the course of time if the slope of the saving-function exceeds that of the investment-function. By simple transformations one can bring both functions into a form where P/Y is the independent and I/Y and S/Y are the dependent variables. Kaldor then describes the process (by means of a diagramm) as follows:
"the point of intersection Q indicates the short period equilibrium level of profits and of investment as a proportion of income. If profits are a lower proportion of income the investment plans (although lower than the equilibrium level) will tend to exceed the available savings; prices will rise in relation to costs, until the discrepancy is eliminated through the consequential rise in profits."
Kaldor also argues that a system of the above described structure will tend to achieve full employment. The gist of his argument is that as long as full employment does not exist there are profitable investment-opportunities which will be made use of under the assumption of a "Kaldor invest-

ment-function". It is actually the assumption of full capacity that presents problems. How realistic is it really to believe that entrepreneurs always manage to adjust their actual capital-stock perfectly and instantly to their desired stock? This problem is dealt with in the next section of this paper.

Postulating that all those equilibrium conditions are actually ruling we want to find the equilibrium-solutions of this given system.

The structure of our economy is described by the following equations:

Model I

(1) $\frac{K}{Y} = v + \beta \frac{P}{K}$

(1a) $K = K^*$

(2) $S = I = s_p \; P$

(3) $\frac{Dy}{y} = f\left(\frac{Dk}{k}\right)$

(4) $L = L_o e^{nt}$

To solve this system of equations for the steady state paths of K and Y we have to find a differential equation in these unknowns.

First K^* and P are eliminated: out of (2) we get (5): $P = \frac{DK}{s_p}$

Changing to per capta variables now we obtain:

$\left(\frac{Y}{L} = y;\right) \frac{d}{dt}\left(\frac{Y}{L}\right) = D\left(\frac{Y}{L}\right) = Dy = \frac{DY}{L} - \frac{YDL}{L^2}$; $\frac{K}{L} = k$, $K = L\,k$;

$DK = DkL + kDL = L_o e^{nt} \cdot (Dk + nk)$ [ex (4)]

Substituting (5) into (1) and using (1a) we obtain:

$$\frac{K}{Y} = v + \frac{\beta}{s_p} \frac{DK}{K}$$

Changing to per capita terms again yields:

$$\frac{k}{y} = v + \frac{\beta}{s_p} \frac{L_o e^{nt}}{L_o e^{nt}} \frac{(Dk+nk)}{k} \qquad \left(\frac{K}{Y} = \frac{K/L}{Y/L} = \frac{k}{y} \right)$$

$$\frac{k}{y} = v + \frac{\beta}{s_p} \frac{Dk}{k} \frac{\beta}{s_p} n$$ .... substituting a_1 for $v + \frac{\beta}{s_p} n$ and a_2 for $\frac{\beta}{s_p}$ gives:

$$\frac{k}{y} = a_1 + a_2 \frac{Dk}{k}$$ .... using logarithms differentiating with respect to time gives the following result:

$$\frac{Dk}{k} - \frac{Dy}{y} = \frac{a_2 D \left(\frac{Dk}{k}\right)}{a_1 + a_2 \left(\frac{Dk}{k}\right)}$$ .... setting $\frac{Dk}{k} = x$ yields:

$$(x - F(x))\,(a_1 + a_2\, x) = a_2\, Dx \quad \left[\text{using (3) } \frac{Dy}{y} = f\left(\frac{D\,k}{k}\right) = F\,(x)\right]$$

solving for Dx we obtain a differential equation in x and t:

$$(6) \quad Dx = \left[x - F(x)\right] \left(\frac{a_1}{a_2} + x \right) = g(x) = \frac{dx}{dt}$$

This differential equation can be solved by separation of variables:

$$dt = \frac{dx}{g(x)} \; ; \quad t - t_o = \int_c^x \frac{dx}{g(x)}$$

Resubstituting $\left(\frac{Dk}{k}\right)$ for x and using the logarithmic transformation

$$\frac{d\,(\ln\,k)}{dt} = x \quad \text{leads to} \quad d\,\ln\,k = xdt = \frac{xdx}{g(x)}$$

thus: $$k = k_o \exp \int \frac{xdx}{g(x)}$$

and: $$y = \frac{k_o \exp \int \frac{xdx}{g(x)}}{a_1 + a_2 x}$$ ..... using $\frac{k}{a_1 + a_2 x} = y$

We can now proceed to discuss the stability properties of this system, described by the differential equation (6):

$$Dx = g(x) = [x - F(x)] \left(\frac{a_1}{a_2} + x\right)$$

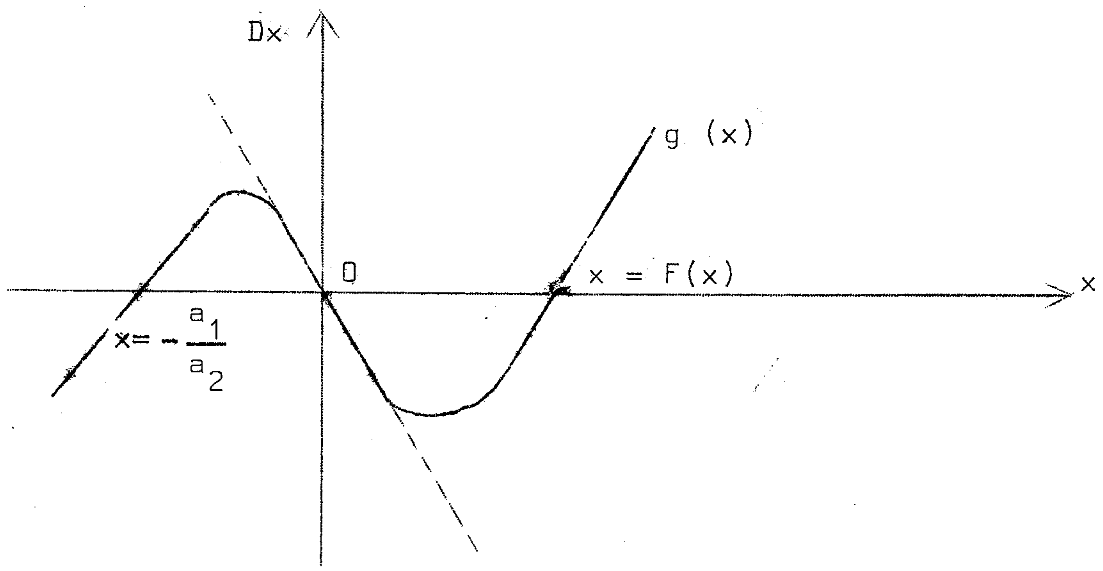
As $Dx = o$ in the steady state, any disturbance resulting in $Dx \lessgtr o$ must be corrected by the system eventually, leading back to $Dx = o$ to achieve a stable solution.

If $Dx = o$ (steady state solution) two cases are possible:

(1) $\left(\frac{a_1}{a_2} + x\right) = o$; $x = -\frac{a_1}{a_2}$

(2) $[x - F(x)] = o$; $x = F(x)$

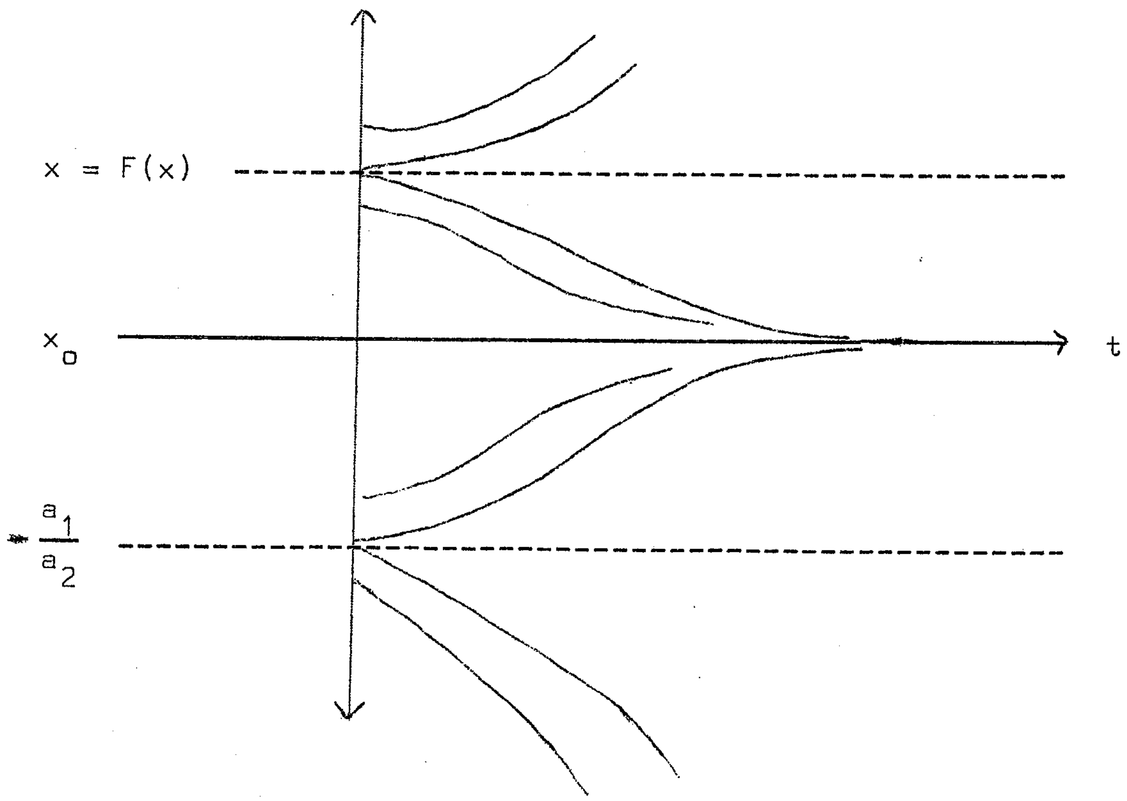
(3) If $x = 0$ & $F(0) = 0$, then $Dx = 0$.



The only stable solution we get for this situation is around 0 . Whenever disturbances occur that make $x > F(x)$, the system explodes.

When saving equals investment (ex - ante), profits are given in our model by: $P = \frac{DK}{s_p}$; As s_p is a constant, profits are a rising function of investment and it is thus always "profitable" for entrepreneurs to invest more, even beyond the point $F(x) = x$. Profits would then not fall to stop further investment per head. When a growth rate of the capital stock per head of less than $x = -\frac{a_1}{a_2}$ is reached, the system decays.

As we have three possible stationary solutions, it is easy to see that the initial conditions of this system will play a vital role in determining what solution will be attained. The following graph is to illustrate the dependance of the solutions on initial conditions.



Model II

We want to drop the assumption that $K^* = K$ now and see what happens to the system, when the adjustment of the capital to its desired level cannot be achieved immediately. A simple distributed lag, simply saying that it takes entrepreneures T years to carry out their adjustment plans, is to serve as "explanation" of how K^* is to be attained.

Our system of simultaneous equations then looks as follows:

(1) $K^* = vY + \beta \frac{P}{K} Y$

(2) $(K^* - K)\frac{1}{T} = I = DK$

(3) $S = s_p P$

(4) $\frac{Dy}{y} = f(\frac{Dk}{k})$

(5) $S = I = DK$

from (2) $K^* = T \,.\, DK + K$

(1) and (2) $K^* = T \,.\, DK + K = vY + \beta\,({}^P/K)\,Y$

from (3) $P = {}^S/s_p = {}^I/s_p = {}^{DK}/s_p$ ... by (5)

(1), (2) & (3) ... $T \,.\, DK + K = vY + ({}^\beta/s_p)\,({}^{DK}/K) \,.\, Y$

solving for DK yields:

$*$: $DK = \dfrac{vY - K}{T - {}^\beta/s_p \; {}^Y/K}$;

changing to per capita variables now, gives:

$y = {}^Y/L$, $k = {}^K/L$; $\frac{y}{k} = {}^Y/K$

thus: $K = L \,.\, k$ & $Y = L \,.\, y$

hence: $\overset{*}{*}$: $DK = DL \,.\, k + LDk = nLk + LDk$ ... (by $L = L_o e^{nt}$, $DL = n$)

$* = \overset{*}{*}$: $nLk + LDk = \dfrac{vY - K}{T - {}^\beta/s_p \; {}^Y/K}$;

solving for ${}^{Dk}/k$ ... ${}^{Dk}/k$ ... ${}^{Dk}/k = \dfrac{v \cdot {}^y/k - 1}{T - {}^\beta/s_p \,.\, {}^y/k} - n$;

The last equation to be used is (4), so we solve the last expression for y; bringing n over to the left side and cross-multiplying results in:

$$(kT - \alpha y)(nk + Dk) = vyk - k^2 \quad \ldots \text{ where } \alpha = \frac{\beta}{s_p}$$

After the necessary transformations and solving for y one obtains:

$$y = \frac{k(1+nT)+Dk}{(v+\alpha n)+ \alpha DK/k}$$

Taking the logarithms of both sides and differentiating with respect to time gives the desired expression in ${}^{Dy}/_{y}$ to be plugged into (4).

$$\frac{Dy}{y} = \frac{{}^{Dk}/_{k}\,(1+n\ T) + \frac{D(Dk)T}{k}}{(1+nT) + {}^{DK}/_{k}\ T} - \frac{\frac{D(Dk)k-(Dk)^2}{k^2}}{v+ \alpha n+ \alpha\, {}^{Dk}/_{k}},$$

substituting $x = {}^{Dk}/_{k}$ (hence: $Dx = \frac{D(Dk)k-(Dk)^2}{k^2}$ ...

$$\ldots \frac{D(Dk)}{k} = Dx + (\frac{Dk}{k})^2 = Dx + x^2)$$

yields (using (4)):

$$F(x) = \frac{x(1+nT) + (Dx+x^2)\ T}{1 + nT + xT} - \frac{\alpha\ Dx}{v+ \alpha n+ \alpha\ x};$$

solving for Dx:

$$Dx = \frac{\left[x(1+nT+xT)-F(x)\right]\ (v+\alpha n+\alpha x)}{\alpha(1+nT+xT)-T(v+\alpha n+\alpha x)};$$

dividing by $(1+nT+xT)$ and
then by $(\frac{v+\alpha n+\alpha x}{1+nT+xT})$

yields: $Dx = (x-F(x))\ (\frac{v+\alpha n\ +\alpha x}{\alpha(1+nT+\alpha T)}) = f(x)$

Again we are first interested in the stationary points of our differential equation. Letting $D_x = 0$ yields 3 points.

(1) $$\frac{v + \alpha n + \alpha x}{\alpha(1 + nT + \alpha T)} = 0$$

hence: $x = - \frac{v}{\alpha} - n$

(2) $[x - F(x)] = 0$

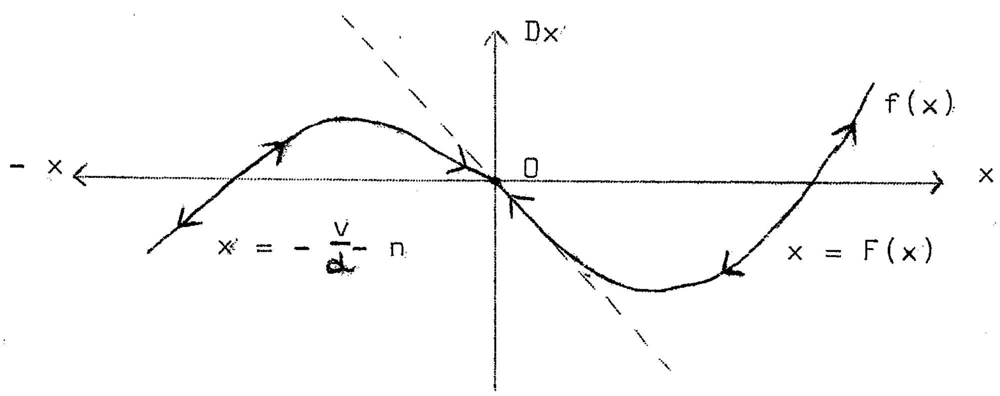
thus $x = F(x)$

(3) $x = 0$

making the assumption that $F(0) = 0$ will result in

$D_x = 0$ at $x = 0$.

Turning to the question of stability again, let us first see at what intersection-point (if at any) of f(x) with the x-axis the tangent to the function f(x) is negative.



Making the above mentioned assumptions about $F(x)$ yields a stable equilibrium point at $x = 0$ (graph just like model I). The model thus does not yield a globally stable equilibrium at all. Whenever either $x = F(x)$, or $x = -\frac{v}{\alpha} - n$ are reached the system will either explode or decay. We can also see that whatever point is going to be reached will depend on the initial conditions of the variables.

It is worth noting also, that the stable point $x = 0$ exists only if $F(0) = 0$ (or if $n < 0$ and $v = \alpha n$).

Assuming that there is growth in output per head without capital accumulation per head (as Kaldor does with his linear "Technical Progress - Function" for solving his model) yields an entirely different result. The point $x = 0$ is no longer a stationary point. $X = -\frac{v}{\alpha} - n$ (if $n \geq 0$, v & $\alpha > 0$) is the only stable solution, having a negative tangent in this point.

The point where $x = F(x)$ is Kaldor's "solution" of his model. The logic of this solution being, that even without any growth of the capital stock per head output per head would increase, thus inducing investment (accelerator). $F(x) = x$ as a "solution" however, implies that capital per head goes to infinity.

So the question is really, can there be an increase in output without investment? (all per capita). That in turn depends on category human capital assumed to be part of. If it is labelled "capital", then $F(0) = 0$ means that a society has to "save" income for the formation of human capital ("investments in education").

Assuming $F(0) = 0$ would also make the use of a linear function highly unwarranted, as there would be no intersection between x and $F(x)$, except at 0, removing the second intersection with the axis at $F(x)$, thus making the point $F(x) = 0$ at $x = 0$ an unstable point.

If T approaches 0, that is if the capital stock is adjusted to its desired level right away, model II reduces to model I.

IV. Final Remarks

The neo-classical and the neo-Keynesian model come to the conclusion (if one make the usual assumptions), that in the "very long run" the growth rate of the per capita capital stock will tend toward zero. When so much capital per labor unit is "piled up" that a further increase would not result in increased output, investment is limited to replacement of obsolete capital and the new investment opportunities arising from population growth. The "standard of living" (growth of output per head) will then only depend on "technical progress" embodied into the replaced capital goods. While the neo-classical model predicts this stationary point to be achieved from all initial conditions and to be globally stable, the Kaldor model (and its extensions) exhibit a stability range thus making the "steady state solution" (and hence the time paths of the variables) dependent on the initial conditions of the system.

Changes of parameters in both models yield the same directions of change in the endogenous variables of the system. While stability in Kaldor's model imposes certain restrictions on the size of those changes, the neo-classical model being globally stable, does not.

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