THREE-PERSON GAMES WITH IMPERFECT COALITIONS

A Sociologically Relevant Concept in Game Theory

by

Robert Reichardt

May 1966

Research Memorandum No.3

C O N T E N T S

A B S T R A C T

In chapter I, two main concepts are proposed for reconciling the diverging interests within a coalition, if side-payments are excluded. The first is to select the joint strategy from a top, defined by notions of dominance between correlated strategies. The second concept is called "ceremonial" and means a normative regulation of the process of consulting a random mechanism. Several examples of such "ceremonials" are given.

In the following chapter, the possibilities of relaxing the closedness condition for coalitions are discussed on a conceptual level.

The resulting notion of imperfect coalitions is applied in the third chapter to three-person games, where it means a chain in the form, that i is connected with j and j with k, but i and k are not connected directly. If player j has in both coalitions (i,j) and (j,k) the role of assigning strategies to signals of the random-device, he can establish a signal-linkage that is optimal for him. Given the probability with which the same random-device is used in the coalition (i,j) as well as in (j,k), given furthermore the influence that j has under certain ceremonials on this probability, we can compute tripels of expected values for the players. These three-element vectors are functions of the pair of ceremonials adopted by the coalitions (i,j) and (j,k).

The concluding chapter tries to relate the concepts and results obtained in the earlier chapters to the sociological literature. Some suggestions about experimental work on games with imperfect coalitions are made. Possible mathematical research on the relation of the vectors of expected values obtained in this paper and the solution theory is sketched.

Z U S A M M E N F A S S U N G

Im I. Kapitel werden zwei Konzepte vorgeschlagen, den Konflikt zwischen den divergierenden Interessen innerhalb einer Koalition zu lösen, wenn Seitenzahlungen ausgeschlossen sind. Das erste Konzept besteht darin, daß die gemeinsame Strategie der Koalition aus einer Menge von Maximalelementen ausgewählt wird, die durch die Anwendung von Dominanzrelationen auf die Menge der korrelierten Strategien entsteht. Das zweite Konzept wird "Zeremoniell" genannt und bedeutet, daß die Benützung eines Zufallsmechanismus nach einem streng fixierten System von Regeln erfolgt. Mehrere Beispiele für "Zeremonielle" werden gegeben.

Im II. Kapitel werden die Möglichkeiten, die Bedingung der Geschlossenheit für Koalitionen aufzuheben, auf einem begrifflichen Niveau diskutiert.

Das aus dieser Diskussion resultierende Konzept von unvollkommenen Koalitionen wird im dritten Kapitel auf Dreipersonen-Spiele angewandt. Dort bedeutet es, daß etwa Spieler i und j, sowie j und k miteinander verbunden sind, nicht aber i und k. Wenn der Spieler in den zentralen Positionen, hier also j, sowohl in Koalition (i,j) als auch in (j,k) die Aufgabe bekommt, die Signale des Zufallsmechanismus mit Strategien-Vektoren zu koppeln, kann er dies in einer für ihn optimalen Weise tun. Kennt man die Wahrscheinlichkeit, daß der gleiche Zufallsmechanismus sowohl in Koalition (i,j) als auch in (j,k) benützt wird, kennt man ferner den Einfluß, den Spieler j bei gewissen Zeremoniellen auf diese Wahrscheinlichkeit hat, so kann man Vektoren von Erwartungswerten für die drei Spieler berechnen. Ein solcher Vektor kann als Funktion des Paars von Zeremoniellen dargestellt werden, die durch die Koalitionen (i,j) und (j,k) akzeptiert worden sind.

Das letzte Kapitel versucht die Konzepte und Resultate der früheren Kapitel mit soziologischen Fragestellungen in Verbindung zu bringen. Es folgen einige Vorschläge, wie Spiele mit unvollkommenen Koalitionen für Experimente verwendet werden können. Schließlich wird auf einige interessante mathematische Fragen hingewiesen, die sich auf den Zusammenhang zwischen den Erwartungswert-Vektoren aus Kapitel III und der Lösungstheorie beziehen.

I. THE PROBLEM OF FAIR DIVISION IN PERFECTLY CLOSED COALITIONS WITHOUT SIDE-PAYMENT

A coalition can be closed in two different ways: (i) intentionally and (ii) communicando. In an intentionally closed coalition its members have decided to act jointly and to maximize hereby the expected utility of the outcome for each player simultaneously, regardless of what the outcome for players outside the coalition may be. We may think that at a certain stage of the bargaining process of forming the coalitions from all the players a certain subset gets intentionally closed and that each member of this subset signs a contract, by which he commits himself completely to a joint action. One may think that such a contract may include means by which the signing player can be forced to cooperation. However, intentional closedness does not imply communicando-closedness. The intentionally closed coalition may still communicate with the rest of the players, using threats or bluff in order to get an advantage. Of course, such negotiation strategies would constitute a supragame[1)] in relation to the original game. In a communicando-closed coalition S, no member $i \in S$ can communicate with any member $j \in (N - S)$ of the remaining set of players. The players in S may not even know what coalition structure exists among the players in N - S. Still communicando-closedness does not imply intentional closedness. There might be some players in S, who try to influence the action of S in such a way that the outcome of the play will be favorable for some players outside S, although these players cannot communicate with S. Situations of this

---

1) The term "supra-game" refers to any kind of formalization of a certain proceedure that is part of the actual carrying out of a game. Conversely, the term "super-game" refers to a sequence of plays of the same game. Instead of "supra-game", the term "pseudo-game" is sometimes used.

kind abound in certain types of adventure stories, where e.g. some prisoners are completely separated from the outside world but still try to influence their guards in such a way that they may abstain from attacking the prisoner's friends outside the camp. In other such situations, traitors, saboteurs etc. deliberately sneak into a group of people where they act against them but in favor of an enemy with whom they cannot communicate anymore. Goffman has described a whole variety of such people in a chapter "Discrepant Roles" of his book "The Presentation of Self in Everyday-Life." (Goffman, Chapter IV).

In the subsequent section we consider a coalition that is closed intentionally as well as communicando. We call such a coalition completely closed. By this assumption it will be possible to develop behavior rules related to the principle of fair division. These behavior-devices will later on help us to analyze situations, in which these assumptions are relaxed and we are confronted with coalitions that are no longer closed.

Let C^S be the set of all correlated strategies available for the coalition S. We define a function p on the elements $C_v^S \in C^S$.

(I.1) $$p(C_v^S) = \min_{C_\mu \in C^{N-S}} x_i \, (C_v^S \, , \, C_\mu^{N-S}) \, , \; i \in S$$

where x_i is the payoff to the i-th player associated with a given probability distribution over the possible strategy vectors. The pair of correlated strategies (C_v^S, C_μ^{N-S}) determine such a probability distribution. $p(C_v^S)$ thus collects the minima for all members of S over all possible correlated strategies of the players $N-S$.

We now define an ordering on the set of p's. Let Φ be a relation between any two payoff vectors. We abbreviate :
$p_r^S \equiv p(C_r^S)$

(I.2) $$p_r^S \; \Phi \; p_s^S \text{ if and only if } x_i^r > x_i^s$$

for all $i \in S$. Where the superscripts of the x's correspond to the subscripts of the p's and the subscripts of the x's refer to the individuals. We say

(I.3) $$p_r^S \; \theta \; p_s^S \text{, if neither } p_r^S \; \Phi \; p_s^S \text{ nor } p_s^S \; \Phi \; p_r^S \text{ holds.}$$

The binary relation Φ is transitive, whereas θ is not transitive.
We need furthermore another dominance relation between strategy vectors.

(I.4) $C_r^S \; \Lambda_i \; C_s^S$, if and only if

$x_i^r \,(C_r^S \, , \, C_k^{N-S}) \geq x_i^s \,(C_s^S \, , \, C_k^{N-S})$ for all

$C_k^{N-S} \in C^{N-S}$ and the strict inequality holds

at least for one C_k^{N-S}.

(I.5) $$C_r^S \; \Lambda \; C_s^S \text{ if and only if } C_r^S \; \Lambda_i \; C_s^S \text{ for all } i \in S.$$

The relations Φ and θ in (I.2) and (I.3) can also be established between correlated strategies. We may say $C_r^S \; \Phi_p \; C_s^S$, if $p_r^S \; \Phi \; p_s^S$ and $C_r^S \; \theta_p \; C_s^S$, if $p_r^S \; \theta \; p_s^S$, where Φ_p means dominance and θ_p equivalence both via the function p.

We now define a top of C^S. We call $\hat{C}^S$ a top of C^S, if the following conditions are fulfilled:

(i) $\hat{C}^S \subset C^S$

(ii) for any two elements of $\hat{C}^S$, $C_r^S \in \hat{C}^S$ and $C_s^S \in \hat{C}^S$, the relation $C_r^S \;\theta_p\; C_s^S$ holds.

(iii) for any two elements $C_t^S \notin \hat{C}^S$ and $C_t^S \in C^S$, there exists at least one element $C_r^S \in \hat{C}^S$, such that $C_r^S \;\Phi_p\; C_t^S$ holds.

(iv) if between two elements of C^S, C_r^S and C_u^S, the relation $C_r^S \;\Lambda\; C_u^S$ holds, the latter is not a member of the top: $C_u^S \notin \hat{C}^S$.

(v) For any $C_r^S \in \hat{C}^S$, the function p yields a payoff x_i^r for player i, such that $x_i^r > v(\{i\})$, where $v(\{i\})$ is the minimal payoff which player i can guarantee himself without cooperation, and the inequality holds for all $i \in S$.

We assume the top as the set of strategies from which a coalition S picks one.

Let us consider a three-person game, in which the coalition $\{1,2\}$ is completely closed:

Example 1

strategies of player 3

strategies of $\{1,2\}$ — payoffs to players 1 and 2

	a	b
α	0,0	-1,-1
β	2,1	2,1
γ	2,1	1,1

Here strategy α is Λ-dominated by all other strategies of $\{1,2\}$, whereas any mixed strategy of β and γ is in the top. We have $p(\beta) = (2,1)$ and $p(\gamma) = 1,1$. One may ask why the coalition should not stick to β where player 1 is better off and player 2 gets the same as with γ . But as there are no side-payments, there is no possibility of convincing player 2 to choose β rather than γ , as he does not gain anything from the improvement player 1 attains.

It is easy to see, that any probability distribution over the members of a top $\hat{C}^S$ must also be in the top. Let C_r^S and C_s^S be two elements of the top.

If $x_i^r > x_i^s$ for a fixed i, we must find a $j \in S$ such that $x_j^r \leq x_j^s$. (Otherwise the condition of (I.2) would be fulfilled, $C_r^S \Phi_p C_s^S$ would hold and C_s^S could not be a member of $\hat{C}^S$.)

Let us now consider a mixture:

$$C_\pi^S = \pi C_r^S + (1-\pi)C_s^S, \qquad 0 < \pi < 1 .$$

Now we get: $x_i^r > x_i^\pi$ and $x_j^r \leq x_j^\pi$, as the function p takes already into account all correlated strategies of N-S. Thus C_π^S will always be an element of the top $\hat{C}^S$.

The top therefore can be described as a set of joint mixed strategies[1)] plus all probability distributions over these strategies. It should however be kept in mind that this does not include all possible mixtures of the joint pure strategies from which the original joint mixed strategies are derived. The following example may illustrate this:

<u>Example 2</u>

strategies of player 3

strategies of $\{1,2\}$	a	b
α	2,0	0,2
β	0,2	2,0
γ	2,1	2,1

payoffs to players 1 and 2

---

1) We use the term "joint mixed strategies" synonymously with "correlated strategies" as does <u>Aumann</u> in his paper "Acceptable Points in General Cooperative n-Person Games" (Aumann, 1959)

There the top $C^{\{1,2\}}$ is defined by: $(\delta \cdot \alpha + \delta \cdot \beta + (1-2\delta) \cdot \gamma)$ for $0 < \delta < \frac{1}{2}$.

The joint pure strategies α and β have to be played with the same probability, otherwise player 3 could bring down the payoff of one of players $\{1,2\}$ below the value 1 simply by chcosing a pure strategy.

The function p over all members of $\hat{C}^S$ thus describes a convex hyper-surface in a q-dimensional Euclidean space, where q is the number of players in S.

At this point clearly a problem of fair division arises. We first deal with the conceptual problems and discuss later on several techniques that may lead to an acceptable outcome for all participants. The problem of division stems from the fact, that the interests of the players within the coalition may be opposite. The following example (3) brings this out clearly:

<u>Example 3</u>

strategies of player 3

strategies of $\{1,2\}$		a	b	payoffs to players 1 and 2
	α	4,0	0,2	
	β	0,2	2,0	

There the top $\hat{C}^{\{1,2\}}$ consists of a mixture of the joint pure strategies α and β, namely $\pi\alpha + (1-\pi)\beta$, with $\frac{1}{3} \leq \pi \leq \frac{1}{2}$.

$p(\frac{1}{3}\alpha + \frac{2}{3}\beta) = (\frac{4}{3}, \frac{2}{3})$ and

$p(\frac{1}{2}\alpha + \frac{1}{2}\beta) = (1, 1)$. Thus player 1 will like to keep π at the lower boundary, whereas player 2 prefers $\pi = \frac{1}{2}$. In such a situation player 1 may propose to use a random device, which is supposed to emit a certain signal with a probability of $\frac{1}{2}$, which however has a bias and sends this signal only with probability $\frac{1}{3}$, this bias being known to player 1 only. Furthermore, the players may not be sure whether their coalition is really communicando closed, thus fearing that some cheaters among them may be able to influence players outside the coalition to act in such a way, that the cheaters benefit from it. (In the next sections, we will deal with cases, where this fear is justified.)

Any technique which aims to provide a fair-division-situation under the conditions discussed in this chapter, will have to answer the following two questions:

(i) which joint mixed strategy from the top should be chosen?
(ii) How can a fair use of a random device be guaranteed?

(i) Choice of a joint mixed strategy: It is possible to reduce the top further, by a notion of weak dominance. We say that $p_r^S \chi p_s^S$, if $x_i^r \geq x_i^s$ for all $i \in S$, and if the strict inequality holds at least for one $i \in S$. We then may delete all the C's from the top, which are χ-dominated via the function p. This procedure is perfectly convincing, if the set of p's that are χ-dominated is symmetric over all the members of S, i.e. if this set remains invariant for all permutations of the members of S. In all other cases this reduction of the top is only appealing to those who benefit from it. However, it then could be justified through the idea that the coalition will have to cooperate in a whole sequence of different games and that the situation concerning the χ-dominance in the top will eventually be reversed, so that the now indifferent players will profit later on from this deleting-procedure. Still, the strictly antagonistic interests within a coalition remain unreconciled after the dropping of the χ-dominated elements of the top. (as e.g., in example 3 above)

In order to solve this dilemma, the following method seems to us most suitable: Each member of the coalition is chosen with the same probability as an arbitrator who determines which joint mixed strategy from the top the coalition will play. I.e. the arbitrator determines an admissible probability distribution over the pure joint strategies of which the top makes use. This method is applicable whether the above described deletion-procedure was carried out or not.

If the top is symmetric over the members of S, this method will lead to the same result as would the adoption of the arithmetic means over the strategies that determine the corners of the hyper-surface (as is the case in example 3). However, in cases where the symmetry does not hold, the use of the arithmetic means is not convincing. Let us assume the top for a six-person coalition contains two joint pure strategies α and β and all mixtures of them. Let

$$p(\alpha) = (2, 2, 2, 2, 2, 1) \text{ and}$$
$$p(\beta) = (1, 1, 1, 1, 1, 2).$$

Here five out of six players will reject the use of the arithmetic mean $(\frac{1}{2}\alpha + \frac{1}{2}\beta)$. Conversely the above proposed method of choosing an arbitrator at random will lead (provided rationality) to an expected mixture of $(\frac{5}{6}\alpha + \frac{1}{6}\beta)$, which seems - at least intuitively - coming closer to a fair division.

(ii) <u>Use of the random device</u>. Let us assume the probability distribution over the joint pure strategies as given for a perfectly closed coalition. We then still can have mutual distrust and some members might try to cheat, using a random device that has a certain bias. Several procedures can be proposed in order to cope with this difficulty. We call such a procedure a <u>ceremonial</u>, which is a set of behavior rules describing exactly how the players have to act when they are using a random device. In the following, we introduce several ceremonials:

<u>Ceremonial a (Z_a)</u>. We assume that many different random devices are available, each of which emits the same signals. Furthermore we assume the number of the different signals being large enough, so that any probability distribution that comes up in the game situations under consideration can be approximated with sufficient accuracy by assigning joint pure strategies to certain signals.

Ceremonial a prescribes that the player who chooses the random-device cannot be the same player who assigns joint pure strategies to its signals. As we assume the different random devices emitting the same signals, the choice of the device and the assignment of the signals can be carried out independently and secretly. These choices may be written down and put into envelopes that are later opened before the whole coalition. In a two-person coalition, there are only two distributions of these two roles possible. The natural procedure therefore will be to adopt each of these distributions with the same probability, namely $\frac{1}{2}$. It is easily seen that even when the roles are not distributed at random and even if the players have some information about biases of the random devices, none of them will be able to take advantage from this situation, provided ceremonial a is carried out.

If S contains more than two players, a more sophisticated procedure is necessary. Let us assume that the mixed joint strategy chosen for the coalition is based on a set D of d different joint pure strategies $s_1, \ldots, s_d$. If we take any two members of D, s_κ and s_λ, and compare the values for $p(s_\kappa)$ and $p(s_\lambda)$, we get a trichotomy of the members of S, namely a subset $S_{\kappa\lambda}$ for which $x_i^\kappa > x_i^\lambda$, $i \in S_{\kappa\lambda}$, a subset $S_{\lambda\kappa}$ for which $x_j^\kappa < x_j^\lambda$, $j \in S_{\lambda\kappa}$, and a subset $S^{(\kappa\lambda)}$ for which $x_k^\kappa = x_k^\lambda$, $k \in S^{(\kappa\lambda)}$, with $S_{\kappa\lambda} \cup S_{\lambda\kappa} \cup S^{(\kappa\lambda)} = S$ and $S_{\kappa\lambda} \cap S_{\lambda\kappa} \cap S^{(\kappa\lambda)} = \emptyset$. (the x's being defined as in (I.2)). We now pick a pair $s_\kappa \in D$, $s_\lambda \in D$, $\kappa \neq \lambda$ and assign the role of the random-device-chooser to one member of $S_{\kappa\lambda}$ (each having the same chance) and the role of the signal-interpreter to one member of $S_{\lambda\kappa}$ (each having the same chance). The possible choices of s_κ, s_λ are performed with the same probability of $\frac{1}{d(d-1)}$. We thus guarantee that the two roles of ceremonial a are carried out as a rule by players with opposite interests. In case a $S_{\kappa\lambda}$ or $S_{\lambda\kappa}$ is empty, we replace it by the set $S^{(\kappa\lambda)}$, that in this case cannot be empty too.

<u>Ceremonial b (Z_b).</u> In this ceremonial, the random device is chosen first. Another player, having the role of signal-interpreter assigns joint pure strategies to the signals, only after he knows which random device has been chosen. This procedure will be necessary, if not all the available random devices emit the same set of signals. However the allocation of the two roles to players can follow the same method as described under ceremonial a.

<u>Ceremonial c (Z_c).</u> Here one member of the coalition chooses the random device as well as assigns its signals to joint pure strategies. This ceremonial requests a higher degree of trust among the coalition members, but can become necessary, if the players are physically separated and at the same time have to act fast, the former ceremonials Z_a and Z_b being too time-consuming. It seems convincing that each player of the coalition gets the same chance of getting this double-role.

<u>Ceremonial d (Z_d).</u> In this ceremonial one player is chosen as coxswain. The coxswain is supposed to consult a random device and to tell his fellow members, which joint pure strategy was signalized by this device. As there is no control, the coxswain may as well select the joint pure strategy he prefers for any reason. This ceremonial involves therefore the greatest degree of trust. Still, a coxswain does not necessarily have to cheat. He also may be in the best position if he really consults a random device as prescribed by the joint mixed strategy of its coalition. Here too, the most convincing way of assigning the coxswain-role will be to give each member the same chance of getting to be coxswain.

One may ask, why we don't also apply the cautious method of consulting a random-device as described for Z_a and Z_b to the chance mechanism by which the players are picked out for the roles of device-chooser and signal-interpreter. One could imagine a whole sequence of cautious steps preceding the final consultation of the random device. We do not deny that such more complicated ceremonials could be considered. One justification for omitting a more elaborate treatment of the role assignments is that in many real life situations the righteousness of these role-assignments can be controlled by the whole coalition, whereas the final consulting of the random device is cut off from immediate inspection by the coalition-members.

An example may clarify this point. Let the players in the coalition be several artillery units distant from each other. The two joint pure strategies of this coalition might be that all of them fire their guns at a certain target exactly at 5:30 p.m., or none of them fires. The probability distribution over these two strategies varies, according to the actions of the enemy until 5:28 p.m. (I.e., this coalition is in a different game according to the enemy's actions.) We now could imagine that the chiefs meet in the morning and distribute the roles of random-device-chooser and signal-interpreter. Here they can control their mutual actions. Once back at their command posts, such a mutual control is no longer possible and a random transmitter that is accessible by a communication network for all of them, as well as the interpretation of its signals, have to be determined at least shortly before 5:30 p.m. through long distance communication.

II. THE NOTIONS OF IMPERFECT COALITIONS AND OF A CONTRACT-NETWORK

In the foregoing chapter, the notion of perfect closedness of a coalition was defined as being based on intentional-closedness as well as communicando-closedness. An intentionally closed coalition is not thought of as a set of completely altruistic actors. Rather it is a coalition in which the cooperation of each member in a joint action is assured, but within which still antagonisms concerning the distribution of the commonly acquired wealth exist. But these antagonisms do not endanger the cooperation. (This may have been the case in a former stage of the coalition formation.) Here, the antagonistic wishes concerning the distribution of wealth are openly discussed and reconciled. There are no hidden intentions of some coalition-members. Chapter I was devoted to the means of reconciliation in the case where side-payments are excluded.

We now relax the condition of perfect closedness. It seems adviseable to concentrate first on the conditions of communication. Let us assume, that a certain communication network among the players is given, in which all communication channels are two-way-channels. Thus, if B can receive a message from A, he must also be able to send one to A. Such a network can be represented by a graph, in which the several players are symbolized by points and the possibility of communication by connecting lines. Not every player can necessarily communicate with any other player, not even indirectly. The network is imposed onto the players by an outside force. By this assumption we exclude the complications which would arise if the communication channels themselves were subject to inter-player negotiations. Furthermore, we do not consider temporal changes of the network. It will soon be seen that these more complicated cases which now are excluded can easily be derived from the simpler model.

Given a communication network, intentionally closed coalitions may form under the sole restriction, that within such a coalition every player can communicate with every other player at least indirectly, and that there is no overlapping of coalitions. Thus, there may exist communication channels also between players that are members of different coalitions. Of course, in order to make a joint strategy effective, it very often will be necessary for a coalition to hide its plans from the players outside. In such a case, these inter-coalition channels simply will not be used, or they may be interrupted definitely and deliberately.

Coalitions for which the conditions of intentional closedness do not hold are called henceforth imperfect coalitions. In an imperfect coalition, there exist one or more subsets of players that do not follow the over-all intention of the coalition. We may call them the deserting sets. In which way desertions are possible, will be determined by the general rules for the negotiations among players as well as by specific standards of behavior within the coalition under consideration. One condition for the possibility of desertion will be, that the deserters can hide some of their plans from the rest of the coalition-members. From this condition follows, that the rest of the coalition will base its actions on the assumption that the deserters are normal members of the coalition. The deserters themselves may form an intentionally closed coalition within the imperfect coalition. We then will speak of a deserting sub-coalition. Such a sub-coalition may eventually contain only one member. Within an imperfect coalition there can exist several deserting sub-coalitions. A deserting sub-coalition may itself be imperfect, by containing within it deserting members of the second order. And this principle of enveloping desertion can be carried further to higher orders.

Given the notion of imperfection of coalitions, one is lead immediately to the concept of overlapping coalitions. Let us consider two non-disjoint sets of players S and T. Then we can distinguish for the non-empty intersection $S \cap T$ the following possibilities: (i) The intersection is a deserting sub-coalition of S, but a fully integrated part of coalition T. Then S is imperfect, whereas T can be considered as perfect, i.e. an intentionally closed coalition that takes advantage from the fact that some of its members, namely $S \cap T$ are treated by the players S-T as co-members in the imperfect coalition S. (ii) The intersection is deserting toward both coalitions S and T. In this case $S \cap T$ is an intentionally closed coalition that exploits the fact that it is treated by S-T as well as by T-S as a part of the imperfect coalitions S and T.

More complicated cases can easily be constructed. $S \cap T$ may not belong to one intentionally closed coalition, but will be split in several subsets with different intentions. Or we may face an overlapping of more than two coalitions.

It should be noted here, that the concept of imperfect coalitions was developped from an investigation of 3-person games. There the notion of desertion and of overlapping gets a well-defined mathematical meaning, namely where the communication-network has the form of a chain and where ceremonial d - as explained in the foregoing chapter - is used. We may expect that further research in the direction of this paper, especially for games with a relatively large number of players, will lead to modifications of the concepts that were just exposed above.

Let us sketch another way to look at the problems being treated in this paper. We may imagine a network of contracts among the players. Any two players may sign a contract concerning their strategic behavior. Such an agreement may assign to each player a set of permitted probability distributions over his set of available strategies. Sub-cases of such assignments are on the one hand the complete exclusion of certain pure strategies and on the other hand the prescription of a specific pure strategy. The agreement between two players can also be based on the concept of a joint mixed strategy. In this case, the use of a random-device determining the strategy choice cannot be independent for the two players. Therefore, all the considerations of the foregoing chapter about the use of random-devices come in, and we will expect that one of the proposed ceremonials will be part of the contract. The concept of a contract-network, of course, includes the possibility that a player signs contracts with several other players. It is easily seen, that a necessary condition for defining rational behavior under these assumptions is a knowledge about the players' information of the existing contracts as well as about the order in which the agreements can be reached and are reached actually. It will be interesting to analyse cases, in which a legal framework is given, regulating the players' information of the negotiating and contracting process as well as the time-sequence of these processes.

In the specific situations for three-person games that we are studying in the next chapter, both concepts - the imperfect coalition as well as the contracts-network concept - prove to be equally well applicable. They actually mean the same in the cases under consideration. We may however expect that in more complicated cases, the two concepts will be of different usefulness.

III. THREE-PERSON GAMES WITH IMPERFECT COALITIONS

The following analysis of three-person games with imperfect coalitions leaves out the case of imperfect three-person coalitions. Therefore, the phenomena we are going to describe can as well be represented by the concept of a network of two-person contracts, as outlined in the foregoing chapter. It follows furthermore, that the top has only to be considered for two-player sets. The case under study here can typically be represented by a chain of players i-j-k, in which the central player j is connected and has contracts with i as well as with k, but in which players i and k are not connected directly. The central player will henceforth be called the <u>pivot-player</u>.

The situations that have to be analysed depend on two main things: (a) the contents of the two contracts between the pivot-player and players i and k respectively, (b) the external framework of the players' behavior. The following two components of the external framework are of special interest here: (i) the legal framework regulating the possibilities of contracts and indicating, what mechanisms are available in order to enforce the fulfilling of the contracts. The legal framework may allow only certain types of contracts. If we consider the two tops $\hat{C}^{(ij)}$ and $\hat{C}^{(jk)}$, it is easily seen, that the joint pure strategy that comes up from the random-device for (i,j) may imply a different pure strategy for the pivot-player than the pure strategy pair for (j,k). In such a dilemma-situation, the pivot-player will be forced to break his contract with i or with k. The legal framework can settle this case in different manners: It may forbid completely the breaking of a contract. Then, the pivot-player will not be allowed to sign two contracts simultaneously, if it is possible that such dilemma-situations arise. Another regulation would be, that an umpire will solve such a dilemma according to certain prescribed rules. A third type of regulation may simply consist in a cost-function for breaking contracts. This cost-function may be constant for all cases of contract-breaking and for every player. A more sophisticated regulation would be, that the costs of breaking a contract vary according to the type of dilemma and to the player.

(ii) The second component of the external framework has to do with the random-devices that are available for the players. Here, one has to indicate how many random mechanisms can be used, how many and what kinds of signals they are emitting, whether or not and how strongly their random choices are interdependent on one another and finally, what is the players' information about the interdependence of the random-devices.

For this chapter's investigations we choose the following external framework: (i) The legal framework allows contracts, in which the two players settle to use one joint strategy from their top, and select a ceremonial for the use of the random-device. Furthermore, it is forbidden to break a contract.
(ii) A fixed number of random-devices is available, each having the same set of signals. Existing inter-dependencies among the probabilities of the random-choices are fully known to the players.

We use the following notations: Let $s^{(ij)*} \in \hat{C}^{(ij)}$ be the joint mixed strategy which the players (i,j) have chosen in their contract. With $d_1^{(ij)}, d_2^{(ij)}, \ldots, d_q^{(ij)}$ we denote the pure strategy-pairs contained in $s^{(ij)*}$. With $d_1^{(ij)j}$, $d_2^{(ij)j}, \ldots, d_q^{(ij)j}$ we indicate the j-components of these strategy-pairs. In replacing (ij) by (jk), we get the analogous symbols for the pair of players (j,k). The number of pure strategy-pairs for (j,k) may be r instead of q. According to our choice of the legal framework, we consider only situations, in which

$$d_1^{(ij)j} = d_2^{(ij)j} = \ldots = d_q^{(ij)j} = d_1^{(jk)j} = d_2^{(jk)j} = \ldots = d_r^{(jk)j} \tag{III.1}$$

Only under this condition, a chain of contracts in the form i-j-k can be formed.

We denote the set of i-components of the $d^{(ij)}$:
$D^{(ij)i} = d_1^{(ij)i}, d_2^{(ij)i}, \ldots\ldots, d_q^{(ij)i}$

Similarly we define
$D^{(jk)k} = d_1^{(jk)k}, d_2^{(jk)k}, \ldots\ldots, d_r^{(jk)k}$

The product of these two sets yields the set of all possible pure strategy-tupels for players i and k, that can arise in the chain of contracts i-j-k and given the choices $s^{(ij)*}$ and $s^{(jk)*}$ of the joint mixed strategies. Let us call the one pure strategy that can come up for player j, as indicated in (III.1), $d^{(j)}$. Then we get the set of all possible pure strategy-tripels, that might have to be played in the described situation:

(III.2) $$D^{(ij)(jk)} = D^{(ij)i} \times D^{(jk)k}, d^{(j)} .$$

We denote the several elements of $D^{(ij)(jk)}$ with:

(III.3) $$d_u^{(ij)(jk)} \qquad u = 1,2, \ldots\ldots, qr$$

Let us now introduce a payoff function. $M(d_u^{(ij)(jk)})$ denotes the payoff-tripel associated with the tripel of pure strategies $d_u^{(ij)(jk)}$.

$M_j(d_u^{(ij)(jk)})$ is the j-component of this payoff-tripel, i.e. the payoff that player j receives, if $d_u^{(ij)(jk)}$ is played.

In the following, we give a formal description of the pivot-player's optimal behavior for the several possibilities of the fixation of ceremonials within the two two-person contracts. The main advantage the pivot-player has is, that he can - under certain circumstances - assign pure strategies to signals of the same random-device for both other players. Thus he can manage to bring up the several strategy-tripels (III.3) with different probabilities than would be the case, if within the two imperfect two-person coalitions a random-device were used independently. We call the procedure, through which the pivot-player arrives at this effect, a <u>signal-linkage</u>.

In order to discuss signal-linkages, we assume for the moment, that the same random-device is used within coalition (i,j) as well as within (j,k), and that the pivot-player j assigns its signals to pure strategies for both pairs of players (i,j) and (j,k). The probabilities, with which the several elements of $D^{(ij)i}$ have to be played - fixed by the choice of $s^{(ij)*}$ - may be written as fractions. Assuming we have found the common denominator of these fractions, we can write the probabilities assigned to the elements of $D^{(ij)i}$ as:

(III.4) $$\frac{p_1}{P}, \frac{p_2}{P}, \ldots\ldots \frac{p_q}{P} \qquad \sum_{v=1}^{q} p_v = P$$

Similarly, we write for the probabilities referring to the elements of $D^{(jk)k}$:

(III.5) $$\frac{q_1}{Q}, \frac{q_2}{Q}, \ldots\ldots \frac{q_r}{Q} \qquad \sum_{w=1}^{r} q_w = Q$$

As we assume that the random-device has a sufficient large number of signals, so that any probability distribution under consideration can be approximated with the necessary accuracy. (If we wish absolute accuracy, we can of course assume the set of signals being of a countable infinity.) In order to do his job for coalition (i,j), the pivot-player has only to divide the set of signals in P classes, each of which coming up with the same probability, and then assign p_1 of these classes to $d_1^{(ij)i}$, p_2 classes to $d_2^{(ij)i}$ and so on. Within the coalition (j,k), the pivot-player j will divide the set of signals into Q classes, each having the same probability of being emitted, and associate q_1 of these classes to $d_1^{(jk)k}$, q_2 classes to $d_2^{(jk)k}$ and so on.

However, in order to find an optimal signal-linkage, player j has to divide the set of signals in PQ classes of the same emission-probability. He then will assign Qp_v of these classes to $d_v^{(ij)i}$, v = 1,2, ..., q. This is merely the pivot-player's private computation. For his message to player i, he will treat Q of the Qp_v classes as units. Analogously, j will assign Pq_w of the PQ classes to each $d_w^{(jk)k}$, w = 1,2, ..., r. For his message to player k, the pivot-player will treat P-tupels of classes as units.

From the point of view of combinatorics, the assignment of the signals to the pure strategies can be considered as a permutation of elements with classes of identical elements. Therefore, the number of possible assignments of the PQ elements to the strategies in $D^{(ij)i}$ is:

$$A^{(ij)i} = \frac{(PQ)!}{(Qp_1)! \cdot (Qp_2)! \cdot \cdots\cdots (Qp_q)!} \qquad \text{(III.6)}$$

Similarly, we get:

$$A^{(jk)k} = \frac{(PQ)!}{(Pq_1)! \cdot (Pq_2)! \cdot \cdots\cdots (Pq_r)!} \qquad \text{(III.7)}$$

Calling the PQ classes of signals <u>signal-elements</u>, we denote a distribution of these elements on the pure strategies in $D^{(ij)i}$ and $D^{(jk)k}$, such that each signal-element is assigned to one of the $d_v^{(ij)i}$ as well as to one of the $d_w^{(jk)k}$, a signal-linkage L_x. The set of all signal-linkages is denoted with L. This set contains A elements, namely:

$$A = A^{(ij)i} \cdot A^{(jk)k} \qquad \text{(III.8)}$$

Now, each signal-linkage L_x, x = 1,2, ..., A, determines a specific probability-distribution on the strategy-tripels - described in (III.2) and (III.3) - $d_u^{(ij)(jk)}$. Let us denote the set of signal-elements that are allocated to $d_v^{(ij)i}$ through the linkage L_x with $T_{(vx)}^{(ij)i}$. Similarly we define $a = T_{(wx)}{}^{(jk)k}$. The number of elements in $T_{(vx)}{}^{(ij)i}$ is Qp_v and in $T_{(wx)}{}^{(jk)k}$ it is Pq_w.

We now form the intersection

$$T_{(vw,x)}{}^{(ij)(jk)} = T_{(vx)}{}^{(ij)i} \cap T_{(wx)}{}^{(jk)k} \tag{III.9}$$

The number of elements in $T_{(vw,x)}{}^{(ij)(jk)}$ is denoted with $N(T_{(vw,x)}{}^{(ij)(jk)})$, or shortly $N(vw,x)$. Clearly:

$$0 \leq N(vw,x) \leq \min\ (Qp_v, Pq_w) \tag{III.10}$$

The probability with which a given strategy tripel $d_u{}^{(ij)(jk)}$ comes up as a consequence of L_x may be written as $P_{(ux)}$. The u is determined by the choice of a v and a w. Then:

$$P_{(ux)} = \frac{N(vw,x)}{PQ}, \qquad u = u(v,w), \qquad \sum_{u=1}^{qr} P_{(ux)} = 1 \tag{III.11}$$

Given a specific linkage L_x, the pivot-player's expected payoff will be:

$$E_j(x) = \sum_{u=1}^{qr} P_{(ux)} \cdot M_j(d_u{}^{(ij)(jk)}) \tag{III.12}$$

It is now possible to define the optimal behavior of j as finding a signal-linkage L_x that maximizes his $E_j(x)$. We define L_{x^*} as follows:

$$E_j(x^*) = \max_x E_j(x) \tag{III.13}$$

Associated with the strategy tripels $d_u{}^{(ij)(jk)}$ are payoff-tripels. As a L_x determines a probability-distribution over the set of strategy tripels, i.e. over $D^{(ij)(jk)}$, we can compute the expected value for all three components of the payoff-function $M(d_u^{(ij)(jk)})$. Let us write for the tripel of expected values produced by the choice of the linkage L_x by the pivot-player $E(x)$. From (III.12) we get:

$$E(x) = \sum_{u=1}^{qr} P_{(ux)} \cdot M_y(d_u^{(ij)(jk)}), \qquad y = i,j,k \tag{III.14}$$

It remains to discuss the case, where two different and independent random-devices are used within the coalitions (i,j) and (j,k) respectively. Under this condition, the choice of the signal-linkage by the pivot-player has no influence on the probabilities with which the different strategy-vectors will come up. It is easily seen, that the strategy vector $(d_v^{(ij)i}, d^{(j)}, d_w^{(jk)k})$ will be chosen with the probability

$$\frac{p_v}{P} \cdot \frac{q_w}{Q}$$

We denote the vector of the three expected values for players i, j and k with E, leaving out the variable x, as E is now independent from the choice of an L_x. Of course E depends on $s^{(ij)*}$, $s^{(jk)*}$. We have:

$$E = \sum_{v=1}^{q} \sum_{w=1}^{r} \frac{p_v}{P} \cdot \frac{q_w}{Q} \cdot M(d_v^{(ij)i}, d^{(j)}, d_w^{(jk)k}) \tag{III.15}$$

Similarly, one can define E_i, E_j and E_k as the i-, j- and k-component respectively of the vector E.

Still another case is possible. Two different random mechanisms can be chosen within the coalitions (i,j) and (j,k). But now we can assume, that the probability with which a given signal is emitted by one random mechanism depends on the signal that comes up from the other device. In this situation, a specific signal-linkage will influence the expected value for the players, however, not necessarily in the same manner as in the case, where the same random-device is used by (i,j) as well as by (j,k). We do not discuss this case, but it should be obvious from the foregoing part how this situation can be treated formally. It does not present further conceptual problems.

In the following section, we analyze the influence of the several ceremonials Z_a, Z_b, Z_c and Z_d on the expected values for the players. Let us indicate the choice of a specific ceremonial by a pair of players as follows: $ij{:}Z_c$ means that the players (i,j) have included into their contract the ceremonial Z_c. In the same manner, we write the ceremonial-choice of (j,k). A contract-situation concerning the ceremonials has to specify the ceremonial within (i,j) as well as within (j,k). Such a contract-situation may e.g. be written like that: $(ij{:}Z_b;jk{:}Z_a)$. As we are treating here the general case for all choices of joint mixed strategies within the pairs of players, we can consider (i,j) and (j,k) symmetrically. Therefore, we have to distinguish only ten contract-situations, namely six with different ceremonials within the coalitions and four with identical ceremonials for both coalitions. Of course, if we discuss numerical examples, we have to distinguish more cases, namely sixteen.

$(ij{:}Z_a;jk{:}Z_a)$

As will be remembered from chapter I, ceremonial a prescribes that from a contracting pair of players one chooses the random-device and the other one assigns pure strategies to the signals. These assignments are made independently, as each of the available random mechanisms has the same set of signals. We are discussing here the general case, where the two roles are not necessarily allocated to the two players with the probability of $\frac{1}{2}$ each. We denote the probability, that player j will get the role of signal-interpreter within the pair (i,j) with $a^{(ij)j}$. Similarly $a^{(jk)j}$ is the probability that j is chosen as signal-interpreter in (j,k). Clearly, the pivot-player can only establish an optimal signal-linkage L_x* if he is chosen as signal-interpreter in both pairs (i,j) and (j,k), a case which will arise with a probability of $a^{(j)} = a^{(ij)j} \cdot a^{(jk)j}$. But even if j is signal-interpreter in both pairs, his linkage L_x* will only work, if players i and k select the same random-device. We call the probability that this is the case $m^{(i)(k)}$. If R is the number of random mechanisms and if each of them has the same chance of being chosen by i as well as by k, the choices being independent, then $m = \frac{1}{R}$. In the following, we denote the vector of the three expected values according to a given contract-situation by E, followed by the two ceremonials for (i,j) and (j,k). For the discussed situation, we write $E(Z_a, Z_a)$.

From the foregoing considerations, it whould be obvious that

$$(III.16) \qquad E(Z_a,Z_a) = a^{(j)} \cdot m^{(i)(k)} \cdot E(x^*) + (1-a^{(j)} \cdot m^{(i)(k)}) \cdot E$$

$$\underline{(ij:Z_a;jk:Z_b)}$$

Differently from ceremonial a, Z_b prescribes that the signal-interpreter makes his signal-assignments only after he knows what random-device was chosen by his companion. In this situation, the choice of the signal-linkage will only influence the expected payoffs, if j is interpreter in both pairs and if i and k select the same device, exactly as in the previous case. However, if we admit, that the probability of the emission of certain signals in one device depends on the signal-emission of the other one, and if the pivot-player knows this dependence, then he can exploit his situation of being signal-interpreter in both pairs, even if i and k do not select the same random mechanism. Of course, the notion of an optimal signal-linkage depends in such a case on the pair of selected random-devices. Thus, under the assumption of dependence of the signal-emissions and of j's information on this fact, the pivot-player will have a greater expected value in this contract-situation, than in the first one. Otherwise, the two E's are the same.

$(ij{:}Z_a; jk{:}Z_c)$

In ceremonial c in a given pair of players, one player chooses a random-device as well as assigns its signals to strategies. In the contract-situation discussed here, a further complication arises, as it plays a role, within which of the two pairs (i,j) and (j,k) the choice of the random-device and the assignment of the signals to strategies is made first. The order in which these choices are made determines, how far the pivot-player can exploit his central position. An inspection of the eight possible cases that can come up in this contract-situation will reveal this. For simplicity, we assume that the pair of players that makes its choices first will make both choices before the other pair. So we exclude cases like this: (i,j) select first their random mechanism, then (j,k) choose a random mechanism, thirdly (i,j) assign signals to strategies and finally the same is done by (j,k). It should, however, be kept in mind that "making choices" does not imply, that a pure strategy within a pair of players is already fixed. It only means that a random mechanism is chosen and its signals are allocated to pure strategies. But the actual signal-emission of this device will happen immediately before the game is played. Thus, the pivot-player will never be informed about the actual decision of the random mechanism, as long as he himself is in the state of making device-selections and signal-assignments. We get the following eight cases:

1) j is signal-interpreter in (i,j) and has the double-function in (j,k). The pair (i,j) makes his choices first. Having the double-function in (j,k), and knowing what was

done in (i,j), the pivot-player can establish here with certainty the optimal signal-linkage L_{x^*}. The expected value for this case is $E(x^*)$.

2) The distribution of the roles is the same as in case (1), however (j,k) come first for making their choices. Although the pivot-player has the possibility of making the optimal signal-linkage here, this linkage will be effektive only if player i chooses the same random-device as has already been chosen by j for (j,k). Let us denote the probability that this is the case with $m^{(i)}$. Then the expected value will be for (2):

$$m^{(i)}E(x^*) + (1 - m^{(i)})E .$$

3) Player j is signal-interpreter in (i,j), but has no function in (j,k). The choices for (i,j) are made before those of (j,k). When making the signal-assignments for (i,j), the pivot-player does not yet know what assignments will be made within (j,k). He therefore cannot establish any signal-linkage. The expected value in this case is E.

4) Player j is signal-interpreter in (i,j), but has no function in (j,k). The pair (j,k) comes first in making its choices. Here j is informed of the chosen random-device as well as of the signal assignments for (j,k) and furthermore about the choice of a random-device by i for (i,j). He therefore can establish an optimal signal-linkage, which however is only worthwhile, if i happened to select the same device as did k. This fortunate event for j comes up with the probability $m^{(i)}$. The expected value for case (4) is:

$$m^{(i)}E(x^*) + (1 - m^{(i)})E.$$

5) Player j chooses the random mechanism for (i,j) and has the double-role in (j,k). The choices in (i,j) are made first. Being informed about the device-selection as well as the signal-assignment in (i,j), the pivot-player can in his

double-role for (j,k) choose the same device as for (i,j) and establish an optimal signal-linkage. The expected value is therefore $E(x^*)$.

6) The roles are distributed as in case (5), but the choices for (j,k) are made first. In contrast to case (5), player j does not yet know the signal-assignments for (i,j), when making his choices for (j,k). The signal-linkage therefore cannot be optimalized. It follows that it is not worthwhile for j, to choose the same random-device for both pairs. The expected value is E.

7) j is device-chooser in (i,j) and has no function in (j,k). The choices of (i,j) are made first. As j has no control over the signal-linkage here, the expected value is E.

8) The roles are distributed as in (7), but the order of choices is inversed between (i,j) and (j,k). The same argument as above holds here too. The expected value is E.

It remains to determine the probabilities with which the cases (1) to (8) occur. Let $a^{(ij)j}$ again be the probability that j is the signal-interpreter for (i,j), and let us denote with $c^{(jk)j}$ the probability, that j gets the double-role in (j,k). For the probability that the choices for (i,j) are made before those of (j,k) we write $h^{(i/k)}$. Leaving out the superscripts in order to abbreviate, we get for the foregoing eight cases the following probabilities:

1) ach
2) $ac(1-h)$
3) $a(1-c)h$
4) $a(1-c)(1-h)$
5) $(1-a)ch$
6) $(1-a)c(1-h)$
7) $(1-a)(1-c)h$
8) $(1-a)(1-c)(1-h)$

It is easily seen that these eight probabilities sum up to 1. Now we multiply the expected values for these eight cases, as computed above, with the corresponding probabilities.

The result is:

(III.17) $$E(Z_a,Z_c) = [\, ch + ma(1-h)] \cdot E(x^*) + \\ + [\,(1-a)(1-c) + a(1-c)h + (1-a)c(1-h) + (1-m)a(1-h)] \cdot E$$

$(ij:Z_a;jk:Z_d)$

The coxswain, the role that is foreseen by ceremonial d, is supposed to consult a random-device for a pair of players and to communicate to his fellow-player what pure strategy came up from this consultation. The role of coxswain can serve as a device to exploit the situation. In fact, the coxswain simply can prescribe to his fellow-player, which pure strategy he must use. If player k gets coxswain, he cannot take advantage of this fact, as there is left only one pure strategy to j, namely $d^{(j)}$. But j being the coxswain can find that element from $D^{(jk)k}$ which maximizes his expected value, given the probability distribution on the elements of $D^{(ij)i}$, resulting from $s^{(ij)*}$. We denote the expected value that is thus produced $E(o)^{(jk)}$. We can say:

(III.18) $$E_j(o)^{(jk)} = \max_w \sum_{v=1}^{q} \frac{p_v}{p} \cdot M_j(d_v^{(ij)i}, d^{(j)}, d_w^{(jk)k})$$

It is easily seen, that in this contract-situation, it does not make any difference, whether the choices of (i,j) or those of (j,k) are made first. Also the role of j in the pair (i,j) has no influence on the outcome, as the best j can do in (i,j) is to help to bring upon the correct probabilities $\frac{p_v}{p}$. If we denote the probability that j is chosen coxswain in (j,k) with $o^{(jk)j}$, we get:

(III.19) $$E(Z_a,Z_d) = o^{(jk)j} \cdot E(o)^{(jk)} + (1 - o^{(jk)j}) \cdot E$$

$(ij:Z_b;jk:Z_b)$

Here in both pairs, the signal-interpreter knows, which random-device was chosen by his fellow-player. However, only if being signal-interpreter in both pairs, the pivot-player can establish a signal-linkage, but in this case, he cannot exploit his knowledge about the selection of random mechanisms. Therefore this contract-situation is identical with $(ij:Z_a;jk:Z_a)$ and $(ij:Z_a;jk:Z_b)$. This holds only, if the signal-emissions of the several devices are independent of each other. If this is not the case, we have
$E_j(Z_b,Z_b) \geq E_j(Z_a,Z_a)$.

$(ij:Z_b;jk:Z_c)$ and $(ij:Z_b;jk:Z_d)$

These contract-situations are identical with $(ij:Z_a;jk:Z_c)$ and $(ij:Z_a;jk:Z_d)$ respectively under the assumption of independent random-devices.

$(ij:Z_c;jk:Z_c)$

Here, the pivot-player can establish an optimal signal-linkage, if he gets the double-role in both pairs (i,j) and (j,k) simultaneously. If this is the case, he can also manage that the same random-device is used for (i,j) as well as for (j,k). Calling the probability, that j is getting the double function in both pairs simultaneously, $c^{(j)}$, we get:

$$E(Z_c,Z_c) = c^{(j)}E(x^*) + (1 - c^{(j)})E \qquad \text{(III.20)}$$

$(ij:Z_c;jk:Z_d)$

If the pivot-player is the coxswain in (j,k), the best he can do in all cases is to reach $E_j(o)$. If he is not coxswain, it does not matter whether he gets the double-role in (i,j) or not. $o^{(jk)j}$ being the probability that j is coxswain in (j,k), we have:

$$\text{(III.21)} \quad E(Z_c,Z_d) = o^{(jk)j}E(o)^{(jk)} + (1-o^{(jk)j})E = E(Z_a,Z_d) = E(Z_b,Z_d)$$

<u>$(ij:Z_d;jk:Z_d)$</u>

This contract-situation was already treated in an earlier paper (Reichardt). If j is coxswain in (i,j) as well as in (j,k), he can order players i and k to play such pure strategies that his payoff is maximized. In other words, he picks out a strategy-tripel $d_u{*}^{(ij)(jk)}$, such that

$$\text{(III.22)} \quad M_j(d_u{*}^{(ij)(jk)}) = \max_u M_j(d_u^{(ij)(jk)}).$$

We write for the payoff-tripel:

$$\text{(III.23)} \quad M(d_u{*}^{(ij)(jk)}) = M(u^*)$$

If j is coxswain in only one of the two pairs (i,j) and (j,k), we have to make an assumption about the behavior of the other coxswain. If e.g. player i is coxswain, he does not know whether coalition (j,k) exists or not and which mixed strategies players j and k are choosing. The best i can do therefore is to stick to the optimal joint mixed strategy for (i,j) and to consult really a random-device. We assume in the following, that players i and k behave in this manner, if they get the role of coxswain. It now depends, whether j is coxswain in (i,j) or in (j,k). Analogously to (III.18) we define:

$$\text{(III.24)} \quad E_j(o)^{(ij)} = \max_v \sum_{w=1}^{r} \frac{q_w}{Q} \cdot M_j(d_v^{(ij)i}, d^{(j)}, d_w^{(jk)k})$$

The corresponding tripel of expected values is denoted with $E(o)^{(ij)}$. The probability, that j is coxswain in (i,j) is written $o^{(ij)j}$. It follows:

(III.25) $$E(Z_d,Z_d) = o^{(ij)j} \cdot o^{(jk)j} \cdot M(u^*) + o^{(ij)j} \cdot (1-o^{(jk)j}) \cdot E(o)^{(ij)} + o^{(jk)j} \cdot (1-o^{(ij)j}) \cdot E(o)^{(jk)} + (1-o^{(ij)j}) \cdot (1-o^{(jk)j}) \cdot E$$

Generalization to the case of contract-breaking.

In the beginning of this chapter, we made the assumption, that the legal framework forbids contract-breaking. Let us discuss here one form of relaxation of this condition. We assume, that contract-breaking is only allowed, if it resolves a dilemma of a pivot-player, to whom were assigned different pure strategies from the pairs of contracting players, of which he is a member. Two strategy pairs $d_v^{(ij)}$ and $d_w^{(jk)}$ come up from the two pairs. As (III.1) no longer holds, we have $d_v^{(ij)j} \neq d_w^{(jk)j}$. We introduce a cost-function for the breaking of contracts. Let us denote the loss of utility that j faces, if he resolves the dilemma using $d_v^{(ij)j}$ with $T_j(\underline{v},w)$. Similarly, we write $T_j(v,\underline{w})$ for j's cost if he uses $d_w^{(jk)j}$. As j maximizes his payoff, we assume that his utility associated with $(d_v^{(ij)}, d_w^{(jk)})$ will be

(III.26) $$M^*(v,w) = \max \left[(M_j(d_v^{(ij)i}, d_v^{(ij)j}, d_w^{(jk)k}) - T_j(\underline{v},w)) ; (M_j(d_v^{(ij)i}, d_w^{(jk)j}, d_w^{(jk)k}) - T_j(v,\underline{w})) \right]$$

IV. RELATION OF THE GAMES WITH IMPERFECT COALITIONS TO SOCIOLOGICAL CONCEPTS

In this final chapter we try to outline the connection between the concept of games with imperfect coalitions and some parts of the sociological theory. The best way to understand what the games with imperfect coalitions mean is to consider them as a procedure of simulation of social processes. Let us recall how we came to our conclusions. In order to make visible the isomorphisms between the logico-mathematical structure and the social phenomena we have in mind, we translate the description of our steps from the game-theoretical language into the sociological terminology. We use for that purpose two columns:

	Game-theoretical language	Sociological language
1)	A set of players, a so-called coalition, decides to correlate the strategy-choices of its members, in order to bring about a result that increases the members' utilities.	Within a larger social context, a group of people wants to co-operate in order to reach certain goals that are desirable for them.
2)	Side-payments are excluded. Therefore not every coalition-member has the same preference-order on the set of outcomes to which the coalition aspires through the set of correlated strategies taken into consideration.	The joint actions taken into consideration by the group may not all lead to the same result. There is no unanimity among the members about which of these possible results are most desirable and these differences in aspirations cannot simply be balanced by a redistribution of gratifications within the group.

Game-theoretical language (cont.)	Sociological language (cont.)
3) The coalitions are not necessarily closed, i.e. there may exist overlapping coalitions or a network of two-players contracts concerning the correlation of strategies.	An individual may be a member of several groups each of which is striving after different goals. The same phenomenon can be described by the concept of a sociometric network concerning the common goal-striving of connected individuals.
4) Correlated mixed strategies require the consultation of a random-device by the connected players.	The joint cooperative actions are based on the interpretation of events coming up within the environment of the group.
5) The procedure of consulting a random-device by a coalition can be regulated by one of several possible "ceremonials". The adoption of a ceremonial can be motivated partly by considerations of fairness, partly simply by functional necessities. Some ceremonials are based on a distribution of functions among players, as Z_a and Z_b. Other ceremonials, as Z_c and Z_d are more monopolistic in character in concentrating all decision-functions on one person. However, the allocation of these functions to persons is still subject to random-choices through which the principle of fairness is maintained at least	The decision process by which a group finds its common action is subject to a norm-system, called "ceremonial" in the game-theoretical part. Some of the proposed norm-systems are based on the principle of "dividing the functions", others concentrate all decision power to one individual. The power of the deciding individual(s) is, however, restricted by two facts. (i) The deciding persons are only selecting possible events from the environment and declaring them as "essential". Furthermore, they give interpretations of these events in the sense, that certain reactions of the group are linked with the happening of a given

Game-theoretical language (cont.)

partly. In a sense, the ceremonials together with the allocation mechanisms are substitutes for side-payments.

Sociological language (cont.)

event. But they cannot influence the probability with which these events come up actually. (ii) The distribution of the decision-functions is governed itself by principles of fairness that can either be represented by a system of circulation of power or by a supervision of the process of function-allocation by an outside institution, considered as neutral. If a person gets coxswain under ceremonial Z_d, he is in a position to give orders to others. But still then, in some cases it is best for the coxswain to strive strictly for the common good. In other cases, the coxswain is able to exploit egoistically his power.

The main result of this paper can be stated as follows:

6) Given two overlapping pairs of players in a three-person game, given a state of nature, namely a set of random-devices, the probabilities for the allocation of functions according to the ceremonials, the probabilities for the order in which choices are made, probabilities

We consider a sociometric structure of three persons in the form of a chain, where A is connected with B and B with C, but A is not connected with C. Given the decisions of the two cooperating pairs about their common action and given the influence of the environ-

Game-theoretical language(cont.)	Sociological language (cont.)
for the fact, that the same random-device is selected independently by two players, given furthermore the joint mixed strategy for each of the two pairs of players, then the vector of expected payoffs for the three players is a function of the pair of two ceremonials adopted by the two pairs of players.	ment also on the process of dividing the decision-functions within the connected pairs, then the relative advantages of the resulting actions and their consequences depend solely on the norm-systems concerning the decision-making that were adopted by the two connected pairs.
A further result was not derived explicitely in this paper, but could be proved without much difficulty by the study of symmetrical three-person games.	
7) Ceteris paribus, the expected payoff of a player is equal or larger, when he is in the central position of a chain (when he is the pivot-player), than the expected value obtained in another position within the coalition structure.	An individual in a central position of a sociometric network has, ceteris paribus, a greater chance to realize his personal goals through a cooperative process, than an individual in a more peripherical place.

Let us clarify what was called the main result and was described under (6). Of course, one could consider the vector of expected values also as a function of the several probabilities like $a^{(ij)j}$, $c^{(ij)j}$, $o^{(ij)j}$, $h^{(i/k)}$, $m^{(i)}$ etc., that were ascribed in the explanation of (6) to the state of nature. E.g. the value of $o^{(ij)j}$, indicating the probability with which player j gets coxswain in the coalition (i,j) can be thought of as reflecting the relative negotiation ability of players i and j. A further interesting result would be, to express the vector of expected values as a function of these abilities. In a sense, such a result would reflect, how worthwhile it is to strive after the role of coxswain in a given situation. But the logical tool developed in this paper, cannot go further into the process of these negotiations.

In what sense is the theory set forth in this paper normative? We have to distinguish two phases, where normative elements come in: (i) The notion of a top over the set of the possible joint strategies for a coalition - as developped in chapter I - is based on the maximin-concept, i.e. the concept of optimalization under relatively pessimistic assumptions about the actions of the players outside the coalition. The rule, that a coalition adopts a joint strategy from its top, was introduced in order to give the probabilities on the set of strategy-vectors a certain amount of plausibility. It should, however, not be overlooked, that the concept of games with imperfect coalitions does by no means stand or fall with the concept of the top. One could apply everything stated in chapter III. to cases, where the $s^{(ij)*}$ and $s^{(jk)*}$ do not belong to the top and might appear as less plausible or even irrational. (ii) The actions of the pivot-player of finding an optimal signal-linkage L_x*, of finding optimal elements $d_w^{(jk)k}$ and $d_v^{(ij)i}$, as explained in (III.18) and (III.24), and of finding $d_u*^{(ij)(jk)}$ are also normative.

An important difference between this concept of rationality and the one inherent in the notion of optimal behavior as expressed in the minimax-theorem should be stressed here. The pivot-player's action is rational rather in the sense of a decision under risk - taking the probabilities for the occurence of certain events as given. The degree of freedom, with which the pivot-player can take these actions depends on the ceremonials as norm-systems adopted in the players' contracts.

We think, that the main advantage of the theory developed in this paper is the fact, that it enlarges the domain of social phenomena for which a logico-mathematical analogy is possible. In the concept of the characteristic function, as introduced by von Neumann and Morgenstern (von Neumann & Morgenstern, chapter VI), a social fact, the relative advantages of several possible coalitions, found a quantitative expression. The theory set forth in this paper quantifies the influence of several norm-systems on the relative chances of goal-attainment among players, given a network of cooperation. Several other typically social concepts are interwoven with this theory and will be discussed in the sequel.

a) G.C. Homans gives the following definition of a norm: "A norm is a statement made by some members of a group that a particular kind or quantity of behavior is one they find valuable for the actual behavior of themselves, and others whom they specify, to conform to." (Homans, p.116). Homans makes clear that with the term "valuable" rewarding or punishing effects of certain events are meant. The above quotation implies that the conformity to a given norm-system can be rationalized (or motivated) by a principle which comes close to the concept of maximizing the member's utilities within a group. The rationalization for the adoption of a specific ceremonial in a coalition would be, that it serves the principle of fairness. Now, we have found that under some circumstances, a given ceremonial may as well turn out as a tool for the exploitation of one coalition-member by another. This gives us an analogy to the often stated fact, that norms can be preserved in situations, where their effect contradicts their original purpose or sense, in other words, where they are dysfunctional. R.K. Merton's theory of manifest and latent functions applies to such phenomena. (Merton, chapter I). The present model gives a mathematical description - however microscopically small in range and far from reality - of the survival of norms into situations where they are dysfunctional.

b) The ceremonials, being a substitute for side-payments, are models for systems of norms of fairness or distributive justice combined with other functional requirements. E.g. in chapter I, the adoption of ceremonial Z_d was motivated by the necessity of quick action. The principle of fairness is reached by dividing the functions, a concept which abounds in the sociological literature. Let us recall here only K. Davis' catalogue of societal necessities (Davis, chapter II). Fairness is on the other hand also brought upon either by a mechanism of rotation of power (in Z_d, each coalition member should have the same chance of becoming coxswain), or by the actions of a neutral arbiter outside the coalition.

c) In the process of contracting among coalition members, the adoption of a relatively "monopolistic" ceremonial may be balanced by the choice of a specific joint mixed strategy. Thus, in a coalition (i,j) player i may be willing to agree to the ceremonial Z_c or Z_d, if in turn joint strategy $s^{(ij)*}$ is selected from the top that favors him slightly in comparison with other possible strategy choices. This deal may reflect the expectation that player j will not exploit the role of coxswain, or the double-role in Z_c, if he gets it. Thus, the phenomenon of trust and distrust can come into the picture.

d) The consultation of a random-device reflects the relation of the group to nature. Power, in our model, means the function of interpreting the events of the outside world and to link the actions of the group to stimuli from the outside world.

e) The position of an individual within the network of social relations influences the range of possible actions of that individual. The principles stated under d) and e) remind us of the observations made by W.F. Whyte on the functioning of leadership in small informal groups. (Whyte).

The author is aware that the present model is still a relatively remote reflection of the phenomena just mentioned. He thought it worthwhile, however, at least to start to simulate them by mathematical structures. The foregoing section's intention is to point to the hooks, where by the attachment of more mathematical refinement and conceptual reflection we can hope to approximate important social structures and processes by a rigid formal treatment, rather than to suggest having reached these goals already.

In the following two final sections, we want to outline the bearing of the present model on experimental work and to give some suggestions about further mathematical investigations. The reader familiar with experimentation in game theory will probably soon have recognized that the games with imperfect coalitions can be used for the set-up of experiments. Most of the above mentioned social phenomena could be the subject of experimental investigation. E.g. we could record the players' verbal motivations for the adoption of a specific ceremonial and of a joint mixed strategy and compare them with their statements after a series of plays was completed. We thus would get some information about trust and distrust processes, as were mentioned under c). We concentrate in the sequel on three main approaches to the study of ceremonials or norm-systems. One would be to test rationality, i.e. to find out how far a pivot-player is really able and willing to make the necessary arrangements in order to exploit his situation. A second approach would be, to put three players in the "chain"-situation simply by establishing the corresponding communication network. We then would let them play the game several times and compare the resulting vector of the sums of payoffs with the several vectors of expected values derived from the mathematical theory. We thus may find out, what contract-situation reflects best the processes of reality. This approach would suffer especially from the difficulty which hampers most game-theoretical experimentation, namely that it is hard to make the number and series of experiments so large, that the results can be generalized with a sufficient degree of significance. We therefore would like to stress a third approach that seems to be more promising at the present moment and can, furthermore, be combined with the second approach. We can use the same set-up as in the experiments described above, but will concentrate on the ceremonial-like arrangements the players will reach. The social fantasy and imagination of the players may then reveal a great variety of forms of obligations that can fulfill the functions of our concept of ceremonials and show some bearings of these forms on behavioral phenomena.

The theory set forth in this paper is semi-normative in the sense used by Coleman.(Coleman, p. 525). What was just said about experimentation seems to us to come close to Coleman's comment about the possible merits of such theories; namely that they can be used as starting points for empirical investigations and that the contradiction between them and the results found in reality can lead to a series of refinements and improvements of the original theory, as was often the case in physics.

We conclude this paper with some remarks about further mathematical developments in game theory. If one looks at the recent game-theoretical works, one gets the impression that most emphasis is put on the characteristic function form. We would like to underline, that we fully recognize the great merits of the concept of the characteristic function. But it should not be overlooked that if we pass from the normal form to the characteristic function form, we drop a certain amount of information, and that the concept of the characteristic function form is a good deal further from reality than is the normal form. The latter says that each player has a set of possible actions and that the outcome depends on the actions actually chosen by the participants and gives different gratifications to the several actors. Therefore, we can have opposing interests. We can have furthermore cooperation through the correlation of strategies, a procedure which comes very close to the process of synchronization, as it was so convincingly described by Thibaut and Kelley for interactions in dyadic relations. (Thibaut & Kelley, p. 60-62). On the other hand, the concept of the characteristic function depicts a situation, where it is sufficient for a set of players to form a coalition in order to get a certain amount of utility. In real life situations, we never find such arrangements, because there, a coalition always will have to find a common action, to carry it out actually and to make sure that no member will deviate from the common decision. If we concentrate solely on the characteristic function, we bypass certain phenomena, which are of great importance from the point of view of social science, namely the process of finding a common strategy within a coalition, the problem how conformity of the members can be assured and the problem of an inner structure and of leadership within the coalition. This paper is a modest trial to bring these phenomena into the realm of game theory. We think, that the most fruitful development will be an interplay between the characteristic function theory and theories based on the normal form.

Some connections between the solution theory and the games with imperfect coalitions should be considered. The vectors E of expected values can be paralleled with payoff vectors. We may introduce the following assumption. A player entering into a coalition is aware, that his partner may have further connections with other players, in other words, that the coalition he is planning to enter, might be imperfect. He must then know all the possible expected values that can result for him from these further connections. If a player is only willing to enter a coalition, if his expected utility is greater than or equal to his security level, then the set of the possible E's of a game with imperfect coalition is a subset of the set $\bar{I}$ of individually rational payoff-vectors as defined by Shapley and Gillies. (Luce and Raiffa, p. 216). Some interesting cases may, however, lie outside this set, so that this assumption should not be made throughout.

A large, and in our opinion important field, would be the investigation of connections between the several solution-concepts and the games with imperfect coalitions. One may ask e.g. under what assumptions about the several behavioral probabilities of chapter III and about the ceremonials, the E's will lie inside the core, a von Neumann-Morgenstern or an Aumann-Peleg solution. (von Neumann & Morgenstern, chapter IV), (Aumann, 1961) and (Aumann & Peleg). In an unpublished paper, the author showed the connection between a specific concept of games with imperfect coalitions and the Aumann-Peleg solution for a special relatively small range of concepts. (Reichardt).

We conclude with the hope that the hitherto rather imperfect tool of games with imperfect coalitions can be made more and more efficient.

B I B L I O G R A P H Y

1) Aumann, Robert J. "The Core of a Cooperative Game Without Side Payments", Transact. of the Amer. Math. Soc. Vol. 98, No. 3 (1961), pp. 539-552.

2) Aumann, Robert J. "Acceptable Points in General Cooperative n-Person Games", in Contributions to the Theory of Games, Vol. IV, ed. by A.W. Tucker & R.D. Luce, pp. 287-324. Princeton 1959.

3) Aumann, Robert J. & Peleg, B. "Von Neumann-Morgenstern Solutions to Cooperative Games Without Side Payments", Bulletin of the Amer. Math. Soc., Vol. 66, No. 3 (1960), pp. 173-179.

4) Coleman, James S. "Introduction to Mathematical Sociology" Glencoe, Ill. 1964.

5) Davis, Kingsley "Human Society", New York, 1948.

6) Goffman, Erving "The Presentation of Self in Everyday Life", Garden City, N.Y. 1962.

7) Homans, George C. "Social Behavior: Its Elementary Forms", New York 1961.

8) Luce, R. Duncan & Raiffa, Howard "Games and Decisions", New York 1957.

9) Merton, Robert K. "Social Theory and Social Structure", rev. and enlarged ed. Glencoe, Ill. 1957.

10) Reichardt, Robert "Games With Imperfect Coalitions", unpublished manuscript, 1961.

11) Riordan, John "An Introduction to Combinatorial Analysis", New York 1958.

12) Thibaut, John W. & Kelley, Harold H. "The Social Psychology of Groups", New York 1959.

13) Vickrey, William "Self-Policing Properties of Certain Imputation Sets" in Contributions to the Theory of Games, Vol. IV, ed. by A.W. Tucker & R.D. Luce, Princeton 1959, pp. 213-246

14) von Neumann, John & Morgenstern, Oskar "Theory of Games and Economic Behavior", 3rd, rev. ed., Princeton 1953.

15) Whyte, William F. "Street Corner Society", Chicago 1943.