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Ines Fortin
Jaroslava Hlouskova

Institut für Höhere Studien
Institute for Advanced Studies
Vienna

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Ines Fortin, Jaroslava Hlouskova

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Institute for Advanced Studies, Vienna

Contact:

Ines Fortin
Department of Economics and Finance
Institute for Advanced Studies
Stumpergasse 56
1060 Vienna, Austria
☎: +43/1/599 91-165
email: ines.fortin@ihs.ac.at

Jaroslava Hlouskova
Department of Economics and Finance
Institute for Advanced Studies
Stumpergasse 56
1060 Vienna, Austria
☎: +43/1/599 91-142
email: hlouskov@ihs.ac.at
and
Thompson Rivers University
School of Business and Economics
British Columbia, Canada

Abstract

Growing experimental evidence suggests that loss aversion plays an important role in asset allocation decisions. We study the asset allocation of a linear loss-averse (LA) investor and compare the optimal LA portfolio to the more traditional optimal mean-variance (MV) and conditional value-at-risk (CVaR) portfolios. First we derive conditions under which the LA problem is equivalent to the MV and CVaR problems. Then we analytically solve the two-asset problem, where one asset is risk-free, assuming binomial or normal asset returns. In addition we run simulation experiments to study LA investment under more realistic assumptions. In particular, we investigate the impact of different dependence structures, which can be of symmetric (Gaussian copula) or asymmetric (Clayton copula) type. Finally, using 13 EU and US assets, we implement the trading strategy of an LA investor assuming assets are reallocated on a monthly basis and find that LA portfolios clearly outperform MV and CVaR portfolios.

Keywords

Loss aversion, portfolio optimization, MV and CVaR portfolios, copula, investment strategy

JEL Classification

G11, G15, G24

Comments
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Contents

1 Introduction

Risk management and behavioral finance are core activities in the asset allocation conducted by banks, insurance and investment companies, or any financial institution that is concerned about risk and about the impact of psychology on individual choice behavior. Financial decision making under uncertainty has long been characterized by modeling investors to be risk averse. Risk aversion means that if several investment opportunities have the same expected return the one with the smallest variation in returns is preferred. In classical finance, investors characterized by risk aversion usually maximize a concave utility function of total wealth. For a certain class of such utility functions a number of results have been known long since, see, for example, Merton (1990) and Ingersoll (1987). However, this is not yet the case in the field of behavioral finance, where a theory of portfolio selection that takes investors' psychology into account is still being developed.

Although the idea of risk aversion is appealing and captures an important aspect of investment behavior, the choice of particular utility functions is probably more motivated from a rational-choice economics and tractability point of view than from realistic investors' preferences. Experiments in behavioral economics have shown that consumers' preferences cannot always be consistently explained using the traditional finance framework, see Kahneman and Tversky (1979). In particular, real investors seem to be characterized by a *loss-averse* behavior, which is a phenomenon describing asymmetric attitudes with respect to gains and losses, rather than by a purely risk averse behavior. Their aversion to losses seems to be considerably stronger than their liking of gains.

Lately, the issue of loss aversion has been receiving more and more interest. Odean (1998) finds empirical evidence of a behavior consistent with prospect theory known as the disposition effect (see also Shefrin and Statman, 1985), when investors tend to hold losing investments too long and sell winning investments too soon. De Bondt and Thaler (1985) investigate whether overreaction to unexpected and dramatic events matters at the market level and find positive evidence in the sense that prior losers are found to outperform prior winners. Benartzi and Thaler (1995), who examine the single-period portfolio choice for an investor with prospect-type utility, offer an explanation of the equity premium puzzle based on myopic loss aversion (interaction between loss aversion and frequent portfolio evaluation). This supports the idea that if investors review their portfolios annually, the resulting empirical premium is consistent with the loss aversion values estimated in the standard prospect theory framework.[1] Related to the work of Benartzi and Thaler (1995) is

[1] The term *narrow framing* is sometimes used to describe the underlying phenomenon.

also that of Barberis, Huang and Santos (2001) who consider loss aversion in a multiperiod context, where investors update the reference point through time and argue that these updating rules might explain the equity premium puzzle. The finding of Benartzi and Thaler is challenged by Durand, Lloyd and Tee (2004) who claim based on their empirical findings that the analysis of Benartzi and Thaler is not robust. On the other hand, Zeisberger, Langer and Trede (2007) implement a bootstrap approach and find results in line with the original results of Benartzi and Thaler. Gomes (2005) studies the optimal portfolio allocation of loss-averse investors and its implications for trading volume. An axiomatic characterization of the behavior depending on a reference point is provided in Apesteguia and Ballester (2009), who also give applications on modeling the status-quo bias and the addictive behavior. Recently, McGraw, Larsen, Kahneman and Schkade (2010) have pointed out the importance of the context of judgement for the presence of loss aversion. An extensive behavioral economic survey can be found in DellaVigna (2009).

While one strand of research, as indicated above, has recently been working with utility functions that are motivated by behavioral experiments and imply asymmetric or downside risk, another strand of research, mainly in the area of applied finance, has been occupied with the discussion and introduction of new downside risk measures, without (necessarily) building on utility maximizing agents.[2] The use of downside risk measures has been particularly promoted by banking supervisory regulations, which specify the risk of proprietary trading books and its use in setting risk capital requirements. The measure of risk used in this framework is value-at-risk (VaR), which explicitly targets downside risk, see the Basel Committee on Banking Supervision (2003, 2006). VaR has been developing into one of the industry standards for assessing the risk of financial losses in risk management and asset/liability management. Another risk measure, which is closely related to VaR but offers additional desirable properties like information on extreme events, coherence and computational ease, is conditional value-at-risk (CVaR). Computational optimization of CVaR has been made readily accessible through the results in Rockafellar and Uryasev (2000).

We contribute to the existing literature outlined above along different lines. First, we investigate how the maximization of (a certain form of) loss-averse utility relates to the optimization of CVaR. In doing so, we create a link between the two above-mentioned strands of literature, i.e., between maximizing loss-averse utility whose specific form is motivated by experiments, and between optimizing purely descriptive downside risk measures. More specifically, we extend the results of

[2] An overview of downside risk measures with an application to hedge funds can be found in Krokhmal, Uryasev and Zrazhevsky (2002).

Rockafellar and Uryasev (2000) by comparing CVaR as well as mean-variance optimization to the maximization of loss-averse utility. Second, we analytically solve the portfolio selection problem of a loss-averse investor in a way similar to Gomes (2005), where our set-up of the loss aversion problem differs from his. Third, we provide additional insight into the asset allocation decision by running simulation experiments for the case when analytical solutions cannot be obtained. In doing so, we explicitly account for asymmetric dependence by using (appropriate) copulas which have been found useful to model dependence beyond linear correlation. Fourth, we contribute to the empirical research involving loss-averse investors by investigating the portfolio performance under the optimal investment strategy, where the portfolio is re-allocated on a monthly basis using 13 European and U.S. assets. In addition to using fixed loss aversion parameters, we employ time-changing versions which depend on previous gains and losses and which have been suggested to better reflect the behavior of real investors. As opposed to a number of other authors, we do not consider a general equilibrium model but examine the portfolio selection problem from an investor's point of view.

The remaining paper is organized as follows. In Section 2 we first derive conditions under which the linear loss-averse utility maximization (LA) problem is equivalent to the traditional mean-variance (MV) and conditional value-at-risk (CVaR) problems, under the assumption of normally distributed asset returns. Then we look at the two-asset case, where one asset is risk-free, and derive the optimal weight of the risky asset as well as the (slightly modified form of the) Sharpe ratio under the assumption of binomially and normally distributed returns. Section 3 reports simulation results for the two-asset case on the sensitivity of the (modified) Sharpe ratio and the asset allocation of the optimal LA portfolio with respect to the the loss aversion parameter and the reference point, and with respect to the degree and structure of dependence. We implement the trading strategy of a linear loss-averse investor, who re-allocates this portfolio on a monthly basis, and study the performance of the resulting optimal portfolio in Section 4. We also compare the optimal LA portfolio to the more traditional optimal MV and CVaR portfolios. Section 5 concludes.

2 Portfolio optimization under linear loss aversion

Loss aversion, which is a central finding of Kahneman and Tversky's (1979) prospect theory, describes the fact that people are more sensitive to losses than to gains, relative to a given reference point. More specifically (i) returns are measured relative to a given reference value and (ii) the decrease in utility implied by a marginal loss (relative to the reference point) is always greater than the increase in utility implied by a marginal gain (relative to the reference point).[3] We consider a *linear* form of loss-averse utility, which is a special case of the originally introduced loss-averse utility. Another feature of the original (S-shaped) loss aversion is an explicit risk seeking behavior in the domain of losses, something which is not captured by our linear loss-averse utility. Under linear loss aversion investors are characterized by the following utility of (portfolio/asset) return y

$$g(y) = \left\{ \begin{array}{rl} y, & y > \hat{y} \\ (1+\lambda)y - \lambda\hat{y}, & y \leq \hat{y} \end{array} \right\} = y - \lambda[\hat{y} - y]^+$$

where $\lambda \geq 0$ is the loss-averse, or penalty, parameter, $\hat{y} \in \mathbb{R}$ is the given reference point, and $[t]^+$ denotes the maximum of 0 and t, see Figure 1. Under the given utility, investors face a trade-off between return on the one hand and shortfall below the reference point on the other hand. Interpreted differently, the utility function contains an asymmetric or downside risk measure, where losses are weighted differently from gains.

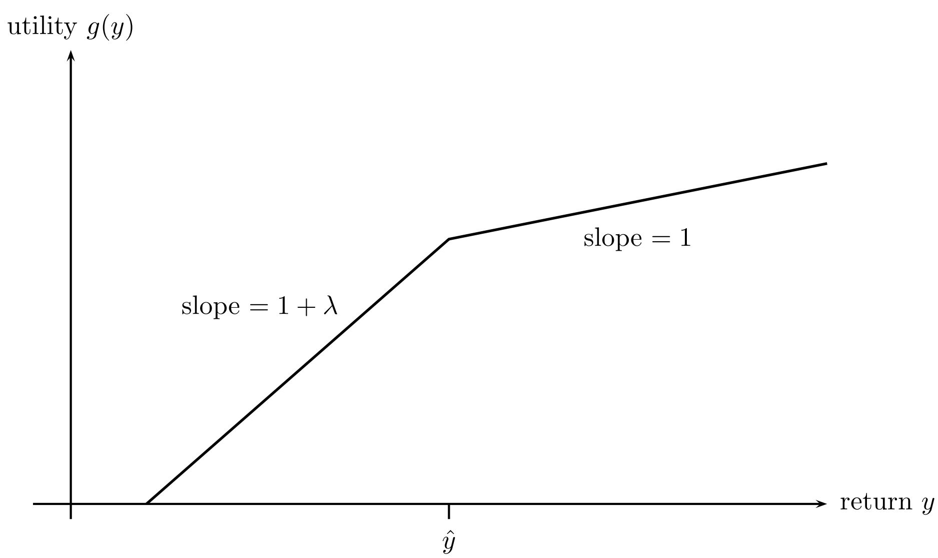

Figure 1: Linear loss-averse utility function

We start by studying the optimal asset allocation behavior of a linear loss-averse investor. This

[3]This is also referred to as the first-order risk aversion (see Epstein and Zin, 1990).

behavior depends crucially on the reference return $\hat{y}$ and, in particular, on whether this reference return is below, equal to, or above the risk-free interest rate or the (requested lower bound on the) expected portfolio return. Investors maximize their expected utility of returns as

$$\max \left\{ \mathbb{E}\left(r'x - \lambda\left[\hat{y} - r'x\right]^{+}\right) \,\middle|\, Ax \leq b \right\} \tag{2.1}$$

where $x = (x_1, \ldots, x_n)'$, with $x_i \in \mathbb{R}$ denoting the proportion of wealth invested in asset i,[4] $i = 1, \ldots, n$, and r is the $n-$dimensional random vector of returns, subject to the usual asset constraints $Ax \leq b$, where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$. Note that in general the proportion invested in a given asset may be negative or larger than one due to short-selling.

2.1 Linear loss-averse utility versus mean-variance and conditional value-at-risk

In this section we show the relationship between the linear loss-averse utility maximization problem (2.1) and both the MV and the CVaR problem, under the assumption of normally distributed asset returns. The MV problem seeks to minimize the variance of an asset portfolio, the CVaR problem optimizes the asset portfolio with respect to its conditional value-at-risk. Both problems may include restrictions on the expected portfolio return and/or the assets' weights.[5]

Let Z be a (continuous) random variable describing the stochastic portfolio return and $f_Z(\cdot)$ and $F_Z(\cdot)$ be its probability density and cumulative distribution functions. Then we define the expected linear loss-averse utility of return Z, given the penalty parameter $\lambda \geq 0$ and the reference point $\hat{y} \in \mathbb{R}$, as[6]

$$\begin{aligned}
\mathrm{LA}_{\lambda,\hat{y}}(Z) &= \mathbb{E}(Z - \lambda[\hat{y} - Z]^{+}) \\
&= \mathbb{E}(Z) - \lambda(\hat{y} - \mathbb{E}(Z|Z \leq \hat{y}))P(Z \leq \hat{y}) \\
&= \mathbb{E}(Z) - \lambda F_z(\hat{y})\left(\hat{y} - \mathrm{CVaR}_{\mathrm{F_z(\hat{y})}}(\mathrm{Z})\right) && (2.2) \\
&= \mathbb{E}(Z) - \lambda \int_{-\infty}^{\hat{y}} (\hat{y} - z) f_Z(z)dz && (2.3) \\
&< \mathbb{E}(Z)
\end{aligned}$$

[4]Throughout this paper, prime ($'$) is used to denote matrix transposition and any unprimed vector is a column vector.

[5]For details on the MV and CVaR optimizations, see Markowitz (1952) and Rockafellar and Uryasev (2000).

[6]Note that $\mathrm{LA}_{\lambda,\hat{y}}$ already accounts for the expectation of utility.

where the conditional value-at-risk $\mathrm{CVaR}_{F_z(\hat{y})}(Z)$ is the conditional expectation of Z below $\hat{y}$; i.e. $\mathrm{CVaR}_{F_z(\hat{y})}(Z) = \mathbb{E}(Z|Z \leq \hat{y})$. As $\int_{-\infty}^{\hat{y}}(\hat{y}-z)f_Z(z)dz \geq 0$, the loss-averse utility of the random variable Z is its mean reduced by some positive quantity, where the size of the reduction depends positively on the values of the penalty parameter λ and the reference point $\hat{y}$. The expected linear loss-averse utility $\mathrm{LA}_{\lambda,\hat{y}}(\cdot)$ is thus a decreasing function in both the penalty parameter and the reference point. Using the fact that Z is normally distributed with $Z \sim N(\mu, \sigma^2)$ we have

$$\mathrm{LA}_{\lambda,\hat{y}}(Z) = \mu - \lambda\sigma\left(\frac{\hat{y}-\mu}{\sigma}F\left(\frac{\hat{y}-\mu}{\sigma}\right) + f\left(\frac{\hat{y}-\mu}{\sigma}\right)\right) \tag{2.4}$$

where $f(\cdot)$ and $F(\cdot)$ are the probability density and the cumulative probability functions of the standard normal distribution. Since $\frac{\hat{y}-\mu}{\sigma}F\left(\frac{\hat{y}-\mu}{\sigma}\right) + f\left(\frac{\hat{y}-\mu}{\sigma}\right)$ is an increasing function of $\frac{\hat{y}-\mu}{\sigma}$, the linear loss-averse utility depends negatively on $\frac{\hat{y}-\mu}{\sigma}$.[7] Figure 2 presents different positionings of the expected linear loss-averse utility of a given asset return, provided it is normally distributed, with respect to the reference point $\hat{y}$ and the CVaR_β, under the assumption that $\lambda = 1/\beta$ and $\beta = F_z(\hat{y})$. The return's loss-averse utility is larger than its conditional value-at-risk when $\mu > \hat{y}$, it is smaller when $\mu < \hat{y}$, and it is equal when $\mu = \hat{y}$.

If asset returns are normally distributed, i.e., $r \sim N(\mu, \Sigma)$, where $\mu, r \in \mathbb{R}^n$ and $\Sigma \in \mathbb{R}^{n\times n}$, then the portfolio return is also normally distributed, i.e., $r'x \sim N(\mu'x, x'\Sigma x)$, where $x \in \mathbb{R}^n$, and using our formulation of linear loss aversion given normal returns, see (2.4), we introduce the following linear loss-averse utility maximization problem

$$\max\left\{\mathrm{LA}_{\lambda,\hat{y}}(r'x) = \mu'x - \lambda\sqrt{x'\Sigma x}\left(\frac{\hat{y}-\mu'x}{\sqrt{x'\Sigma x}}F\left(\frac{\hat{y}-\mu'x}{\sqrt{x'\Sigma x}}\right) + f\left(\frac{\hat{y}-\mu'x}{\sqrt{x'\Sigma x}}\right)\right)\,\middle|\, Ax \leq b, \mu'x = \bar{R}\right\} \tag{2.5}$$

Under the same assumptions, the MV problem can be stated as

$$\min\left\{\mathrm{var}(r'x) = x'\Sigma x \,\middle|\, Ax \leq b, \mu'x = \bar{R}\right\} \tag{2.6}$$

and the CVaR optimization problem can be written as

$$\max\left\{\mathrm{CVaR}_{F(\frac{\hat{y}-\mu'x}{\sqrt{x'\Sigma x}})}(r'x) = \mu'x - \sqrt{x'\Sigma x}\frac{f(\frac{\hat{y}-\mu'x}{\sqrt{x'\Sigma x}})}{F(\frac{\hat{y}-\mu'x}{\sqrt{x'\Sigma x}})}\,\middle|\, Ax \leq b, \mu'x = \bar{R}\right\} \tag{2.7}$$

[7]Thus, the positioning of the mean with respect to the reference point is important. Based on this observation we will later introduce a performance measure labeled the *relative* Sharpe ratio $\frac{\mu-\hat{y}}{\sigma}$.

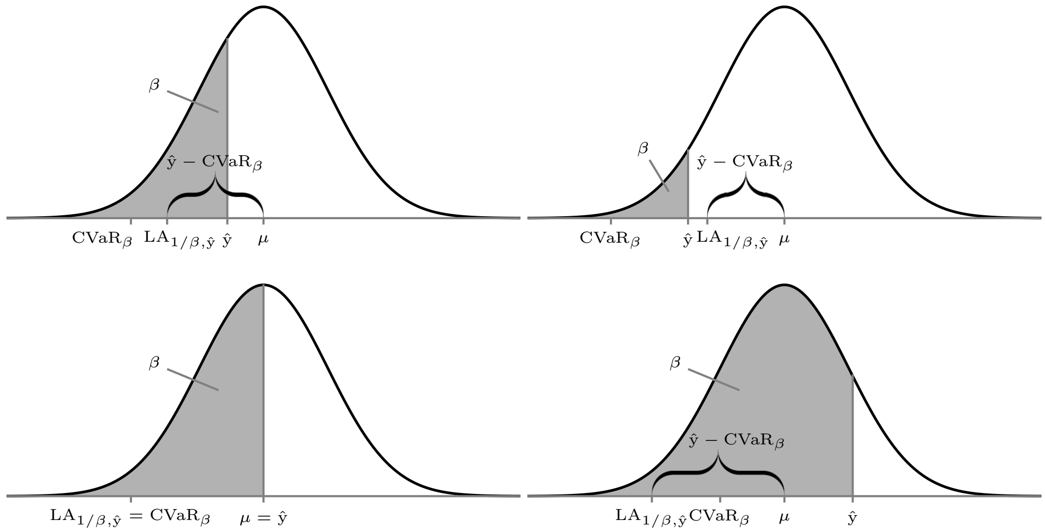

Figure 2: Linear loss-averse utility and conditional value-at-risk
The expected linear loss-averse utility $\mathrm{LA}_{1/\beta,\hat{y}}(Z)$ and the conditional value-at-risk $\mathrm{CVaR}_\beta(Z)$ of the stochastic return Z, with $\beta = F_Z(\hat{y})$ and $\lambda = 1/\beta$, are shown for $\hat{y} < \mu$, and $\mu - \hat{y} \le \hat{y} - \mathrm{CVaR}_\beta(Z)$ (top left), $\hat{y} < \mu$, $\mu - \hat{y} > \hat{y} - \mathrm{CVaR}_\beta(Z)$ (top right), $\hat{y} = \mu$ (bottom left), and $\hat{y} > \mu$ (bottom right).

We can now state the two main theorems of equivalence, which describe how the LA problem is related to the more traditional MV and CVaR problems.

Theorem 2.1 *Let* $\left\{ x \,|\, Ax \le b,\ \mu' x = \bar{R} \ \right\} \ne \emptyset$, $r \sim N(\mu, \Sigma)$ *and* $\lambda > 0$. *Then the LA problem (2.5) and the MV problem (2.6) are equivalent, i.e., they have the same optimal solution, if either (i)* $\hat{y} = \bar{R}$ *or (ii)* $\hat{y} > \bar{R}$ *and* $\lambda = 1/F\left(\frac{\hat{y}-\bar{R}}{\sqrt{x'\Sigma x}}\right)$.

Proof: If $\hat{y} = \bar{R}$ and $\lambda > 0$ then $\mathrm{LA}_{\lambda,\hat{y}}(r'x) = \hat{y} - \lambda\sqrt{x'\Sigma x} f(0)$ and the equivalence between (2.5) and (2.6) follows from $\lambda f(0) > 0$.

If $\hat{y} > \bar{R}$ and $\lambda = 1/F\left(\frac{\hat{y}-\bar{R}}{\sqrt{x'\Sigma x}}\right)$ then the objective functions of (2.5) can be stated as

$$\mathrm{LA}_{1/F\left(\frac{\hat{y}-\bar{R}}{\sqrt{x'\Sigma x}}\right),\hat{y}}(r'x) = 2\bar{R} - \hat{y} - \sqrt{x'\Sigma x}\frac{f\left(\frac{\hat{y}-\bar{R}}{\sqrt{x'\Sigma x}}\right)}{F\left(\frac{\hat{y}-\bar{R}}{\sqrt{x'\Sigma x}}\right)}$$

Maximizing this is equivalent to minimizing the variance $x'\Sigma x$ over the same set of feasible solutions, which follows from the fact that $F(\cdot)$ is an increasing function and that $f(z)$ is decreasing for $z \ge 0$. $\square$

Theorem 2.2 *Let* $\{\, x \mid Ax \leq b,\ \mu' x = \bar{R}\ \} \neq \emptyset$, $r \sim N(\mu, \Sigma)$ *and* $\lambda > 0$. *Then the LA problem (2.5) and the CVaR problem (2.7) are equivalent, i.e., they have the same optimal solution, if* $\hat{y} \geq \bar{R}$ *and* $\lambda = 1/F\left(\frac{\hat{y}-\bar{R}}{\sqrt{x'\Sigma x}}\right)$.

Proof: If $\hat{y} \geq \bar{R}$ and $\lambda = 1/F\left(\frac{\hat{y}-\bar{R}}{\sqrt{x'\Sigma x}}\right)$ then problems (2.5) and (2.7) can be written as

$$\mathrm{LA}_{\lambda,\hat{y}}(r'x) = 2\bar{R} - \hat{y} - \sqrt{x'\Sigma x}\frac{f\left(\frac{\hat{y}-\bar{R}}{\sqrt{x'\Sigma x}}\right)}{F\left(\frac{\hat{y}-\bar{R}}{\sqrt{x'\Sigma x}}\right)}$$

and

$$\mathrm{CVaR}_{F(0)}(r'x) = \bar{R} - \sqrt{x'\Sigma x}\frac{f\left(\frac{\hat{y}-\bar{R}}{\sqrt{x'\Sigma x}}\right)}{F\left(\frac{\hat{y}-\bar{R}}{\sqrt{x'\Sigma x}}\right)}$$

and the statement of the theorem follows. $\square$

Theorem 2.1 states the conditions under which the LA and MV problems are equivalent provided returns are normally distributed: they are equivalent (i) when the reference point is equal to the mean of the portfolio return at the optimum, or (ii) when the reference point is strictly larger than the mean of the portfolio return at the optimum and the loss aversion parameter is equal to some specific value (depending on the reference point). In the latter case, the loss aversion parameter yielding equivalence is smaller for larger reference points. Under the conditions stated in (ii), the LA (MV) problem is also equivalent to the CVaR problem; and finally the LA and CVaR problems are equivalent also when $\hat{y} = \bar{R}$ and $\lambda = 2$ (see Theorem 2.2). The condition $\mu' x = \bar{R}$, which is required in both theorems, can be interpreted as setting a lower bound on the portfolio return, $\bar{R} \leq \mu' x$, which is binding at the optimum.[8]

Similar relationships between the maximization of loss-averse utility and the MV and CVaR problems are true under the assumption of t-distributed portfolio returns and additional assumptions.[9]

2.2 Analytical solution for one risk-free and one risky asset

To better understand the attitude with respect to risk of linear loss-averse investors, we consider a simple two-asset world, where one asset is risk-free and the other is risky, and analyze what

[8] A similar constraint is required in Rockafellar and Uryasev (2000), when the equivalence of the CVaR, VaR and MV problems is shown.

[9] Detailed derivations can be obtained from the authors upon request.

proportion of wealth is invested in the risky asset under linear loss aversion. Another motivation for looking at this problem is that, when Tobin's separation theorem holds, the investor's investment decision problem can be simplified to deciding which proportion to invest in the safe asset and which to invest in some risky portfolio. As Levy, De Giorgy and Hens (2004) have shown, Tobin's separation principle does hold under the assumption of the Tversky and Kahneman's prospect theory utility, of which our linear loss-averse utility is a special case.[10]

Let r^0 be the certain (deterministic) return of the risk-free asset and let r be the (stochastic) return of the risky asset. Then the portfolio return is $R(x) = xr + (1-x)r^0 = r^0 + (r - r^0)x$, where x is the proportion of wealth invested in the risky asset, and the maximization problem under consideration of the linear loss-averse investor is

$$\begin{aligned}\max\{\,\mathrm{LA}_{\lambda,\hat{y}}(R(x)) &= \mathbb{E}(R(x) - \lambda[\hat{y} - R(x)]^+) \\ &= \mathbb{E}(r^0 + (r - r^0)x) - \lambda\mathbb{E}([\hat{y} - r^0 - (r - r^0)x]^+)\,|\,x \in \mathbb{R}\} \qquad (2.8)\end{aligned}$$

where $\lambda \geq 0$, $\hat{y} \in \mathbb{R}$ and $[t]^+ = \max\{0, t\}$.

The risky asset is binomially distributed

First we assume for the sake of simplicity and because in this case a number of results can be shown analytically, that the return of the risky asset follows a binomial distribution. We assume two states of nature: a good state of nature which yields return r_g such that $r_g > r^0$ and which occurs with probability p; and a bad state of nature which yields return r_b such that $r_b < r^0$ and which occurs with probability $1 - p$. In the good state of nature the portfolio thus yields return $R_g(x) = r^0 + (r_g - r^0)x$ with probability p, in the bad state of nature it yields return $R_b(x) = r^0 + (r_b - r^0)x$ with probability $1 - p$. Note that

$$\mathbb{E}(r) = pr_g + (1-p)r_b = p(r_g - r_b) + r_b, \qquad (2.9)$$

$$\mathrm{var}(r) = p(1-p)(r_g - r_b)^2, \qquad (2.10)$$

$$\begin{aligned}\mathbb{E}(R(x)) &= \mathbb{E}(r^0 + (r - r^0)x) = p(r^0 + (r_g - r^0)x) + (1-p)(r^0 + (r_b - r^0)x) \\ &= r^0 + (p(r_g - r_b) - r^0 + r_b)x, \qquad (2.11)\end{aligned}$$

[10] In the same place, it has been shown, however, that no financial market equilibria exist under the assumption of Tversky and Kahneman's prospect theory utility.

$$\mathrm{var}(R(x)) \quad = \quad p(1-p)(r_g - r_b)^2 x^2, \tag{2.12}$$

$$[\hat{y} - R_g(x)]^+ = \begin{cases} \hat{y} - r^0 - (r_g - r^0)x, & \text{for } x \leq \frac{\hat{y}-r^0}{r_g - r^0} \\ 0, & \text{for } x > \frac{\hat{y}-r^0}{r_g - r^0} \end{cases} \tag{2.13}$$

$$[\hat{y} - R_b(x)]^+ = \begin{cases} 0, & \text{for } x \leq \frac{r^0-\hat{y}}{r^0 - r_b} \\ \hat{y} - r^0 - (r_b - r^0)x, & \text{for } x > \frac{r^0-\hat{y}}{r^0 - r_b} \end{cases} \tag{2.14}$$

Thus, the loss-averse utility of the two-asset portfolio including the risk-free asset and the binomially distributed risky asset, can be written as

$$\mathrm{LA}_{\lambda,\hat{y}}(R(x)) = r^0 + (p(r_g - r_b) - r^0 + r_b)x - \lambda \left(p[\hat{y} - R_g(x)]^+ + (1-p)[\hat{y} - R_b(x)]^+\right) \tag{2.15}$$

The next theorem presents the analytical solution of the loss-averse utility maximization problem (2.8) for the binomially distributed risky asset with respect to a certain threshold value of the loss aversion parameter λ.

Theorem 2.3 *Let $r_b < r^0 < r_g$, $p > \frac{r^0 - r_b}{r_g - r_b}$, x^* be the optimal solution of (2.8) and*

$$\hat{\lambda} \equiv \frac{p(r_g - r_b) - r^0 + r_b}{(1-p)(r^0 - r_b)} \tag{2.16}$$

where the risky asset's return r is assumed to be binomially distributed with r_g (r_b) being the return in the good (bad) state of nature, which occurs with probability p $(1-p)$. Then the following holds:

(i) *If $0 \leq \lambda < \hat{\lambda}$ then $x^* = +\infty$*

(ii) *If $\lambda = \hat{\lambda}$ then $x^* \in [\max\left\{\frac{\hat{y}-r^0}{r_g-r^0}, \frac{r^0-\hat{y}}{r^0-r_b}\right\}, +\infty)$*

(iii) *If $\lambda > \hat{\lambda}$ then $x^* = \max\left\{\frac{\hat{y}-r^0}{r_g-r^0}, \frac{r^0-\hat{y}}{r^0-r_b}\right\}$*

Proof: in the appendix.

Note that the threshold value of λ can also be written as $\hat{\lambda} = (\mathbb{E}(r) - r^0)/((1-p)(r^0 - r_b))$ and can thus be interpreted as a scaled equity premium, where the scaling factor is greater than 1 (and thus $\hat{\lambda}$ greater than the equity premium) when $r^0 - r_b < 1/(1-p)$. Note, in addition, that if an investor is not sufficiently loss-averse then the solution of the loss-averse linear utility coincides with the solution of linear utility.

As a performance measure we will consider in the addition to the Sharpe ratio, $SR(Z) = \frac{\mu_z - r^0}{\sigma_z}$, also the *relative* Sharpe ratio of a random variable Z, which we define as $RSR(Z) = \frac{\mu_z - \hat{y}}{\sigma_z}$, where μ_z is the mean and σ_z is the standard deviation of Z. We consider the relative Sharpe ratio because we think that the loss-averse investor with a given reference point $\hat{y}$ is more concerned about excess returns above his/her individual reference return than about excess returns above the risk-free return. Our modified version of the Sharpe ratio is also supported by equation (2.4), where the expected return always enters with the reference point and the standard deviation as $-(\mu - \hat{y})/\sigma$.

The next remark characterizes the optimal risky asset's weight x^* and, in particular, states its sensitivity with respect to the loss aversion parameters λ and $\hat{y}$. In addition, the relative Sharpe ratio is given for the optimal weight.

Remark 2.1 *Let $r_b < r^0 < r_g$, $\mathbb{E}(r) > r^0$, x^* be the optimal solution of (2.8) and $\lambda > \hat{\lambda}$, where $\hat{\lambda}$ is defined by (2.16). Then*

(i) $x^* \in [0, 1)$ *if* $r_b \leq \hat{y} \leq r_g$*;*

(ii) *the mean-variance problem*

$$\min \ \mathrm{var}(\mathrm{R(x)}) = \min\{\, p(1-p)(r_g - r_b)^2 x^2 \,\}$$

has the same solution, namely $x^* = 0$, as the loss-averse utility maximization problem (2.8) if $\hat{y} = r^0$;

(iii)

$$\frac{dx^*}{d\lambda} = 0, \quad \frac{dx^*}{d\hat{y}} = \begin{cases} -\frac{1}{r^0 - r_b} < 0, & \text{for} \ \ \hat{y} < r^0 \\ 0, & if \ \ \hat{y} = r^0 \\ \frac{1}{r_g - r^0} > 0, & \text{for} \ \ \hat{y} > r^0 \end{cases} \tag{2.17}$$

(iv) $\mathrm{SR}(R(x)) = \mathrm{SR}(r) > 0$;

(v)

$$\mathrm{RSR}(R(x^*)) = \begin{cases} \sqrt{\frac{p}{1-p}} > 0, & \text{for } \hat{y} < r^0 \\ -\sqrt{\frac{1-p}{p}} < 0, & \text{for } \hat{y} > r^0 \end{cases} \tag{2.18}$$

(vi)

$$\mathrm{RSR}(r) = \sqrt{\frac{p}{1-p}} + \frac{r_b - \hat{y}}{\sqrt{\mathrm{var}(r)}}. \tag{2.19}$$

Theorem 2.3(iii) implies that the sufficiently loss-averse investor ($\lambda > \hat{\lambda}$) with the reference point being equal to the risk-free rate ($\hat{y} = r^0$), will not invest invest in the risky asset ($x^* = 0$) when its return has the binomial distribution and the expected equity premium is positive. In case when the reference point is below (above) the risk-free rate, $\hat{y} < r^0$ ($\hat{y} > r^0$), the fraction invested in the risky asset decreases (increases) with an increasing reference point. However, in all cases is the fraction invested in the risky asset insensitive to the change of the penalty parameter λ, provided the investor is sufficiently loss-averse, i.e., $\lambda > \hat{\lambda}$. In addition, it follows from Remark 2.1(iv), (v) that both the Sharpe ratio of the portfolio return $R(x)$ and the relative Sharpe ratio of the optimal portfolio return $R(x^*)$ are insensitive with respect to the penalty parameter and the reference point.

The risky asset is normally distributed

Let us now assume that the risky asset's return is normally distributed, i.e., $r \sim N(\mu, \sigma^2)$ where $\sigma > 0$. Then also the portfolio return $R(x) = r^0 + (r - r^0)x$ is normally distributed with $R(x) \sim N(r^0 + (\mu - r^0)x, x^2\sigma^2)$ and the loss-averse utility function can be formulated as

$$\mathrm{LA}_{\lambda,\hat{y}}(R(x)) = \begin{cases} r^0 + (\mu - r^0)x - \lambda\sigma x \left[\left(\frac{\hat{y}-r^0}{\sigma x} - \frac{\mu - r^0}{\sigma} \right) F\left(\frac{\hat{y}-r^0}{\sigma x} - \frac{\mu - r^0}{\sigma} \right) + f\left(\frac{\hat{y}-r^0}{\sigma x} - \frac{\mu - r^0}{\sigma} \right) \right], & x \neq 0 \\ r^0 - \lambda[\hat{y} - r^0]^+, & x = 0 \end{cases} \tag{2.20}$$

The following theorem proves properties of the optimal solution of the loss-averse utility maximization problem (2.8) under the assumption that the risky asset's return is normally distributed.

Theorem 2.4 *Let x^* be the optimal solution of problem (2.8), $\lambda > 0$, $\hat{\lambda}_N = \frac{\mu - r^0}{\sigma h}$, where $h =$*

$-\frac{\mu-r^0}{\sigma}F\left(-\frac{\mu-r^0}{\sigma}\right)+f\left(-\frac{\mu-r^0}{\sigma}\right)$ *and the risky asset is assumed to be normally distributed such that* $r\sim N(\mu,\sigma^2)$ *and* $\sigma>0$*. Then the following holds*

(i) *If* $\mu\le r^0$ *then the maximum of (2.20) is reached at* $x^*=-\infty$

(ii) *If* $\mu>r^0$ *and* $\lambda<\hat\lambda_N$ *then the maximum of (2.20) is reached at* $x^*=+\infty$

(iii) *If* $\mu>r^0$*,* $\lambda=\hat\lambda_N$ *and* $\hat y\ne r^0$ *then the maximum of (2.20) is reached at* $x^*=+\infty$

(iv) *If* $\mu>r^0$*,* $\lambda=\hat\lambda_N$ *and* $\hat y=r^0$ *then the maximum of (2.20) is reached for any* $x^*\in\mathbb{R}$

(v) *If* $\mu>r^0$ *and* $\lambda>\hat\lambda_N$ *then the maximum of (2.20) is reached at* $x^*=-\infty$

Proof: in the appendix.

Assuming the more reasonable case that $\mu>r^0$, the two main results are stated in parts (ii) and (v) of the theorem.[11] As before, the investor's optimal investment behavior depends crucially on the value of his/her penalty parameter. If it is below some threshold value ($\lambda<\hat\lambda_N$) the risky asset's optimal weight goes to plus infinity, if it is above that same threshold ($\lambda>\hat\lambda_N$), the risky asset's optimal weight goes to minus infinity. So, if investors are sufficiently loss-averse, they want to go infinitely short in the risky asset. In reality, however, investors usually face a short-sales constraint on risky assets. Therefore, the next theorem presents properties of the solution of the constrained loss-averse utility maximization problem

$$\max\{\,\mathrm{LA}_{\lambda,\hat y}(R(x))\,|\,x\ge 0\} \tag{2.21}$$

where $\mathrm{LA}_{\lambda,\hat y}(R(x))$ is given by (2.20) and the no-short-sales restriction is applied to the risky asset.

Theorem 2.5 *Let* x^* *be the optimal solution of problem (2.21),* $\lambda>\hat\lambda_N$*, where* $\hat\lambda_N>0$ *is defined as in Theorem 2.4,* $\mu>r^0$*,* $\hat y\ne r^0$ *and the risky asset is assumed to be normally distributed such that* $r\sim N(\mu,\sigma^2)$ *and* $\sigma>0$*. Then* $x^*>0$*,*

$$\frac{dx^*}{d\lambda}=\frac{\sigma^2(x^*)^3}{\lambda(\hat y-r^0)^2}\left[\frac{\mu-r^0}{\sigma}\frac{F\left(\frac{\hat y-r^0}{\sigma x^*}-\frac{\mu-r^0}{\sigma}\right)}{f\left(\frac{\hat y-r^0}{\sigma x^*}-\frac{\mu-r^0}{\sigma}\right)}-1\right]<0 \tag{2.22}$$

[11] Parts (iii) and (iv) cover the special case when the penalty parameter is exactly equal to some threshold value, i.e., $\lambda=\hat\lambda_N$.

and

$$\frac{dx^*}{d\hat{y}} = \frac{x^*}{\hat{y} - r^0} \begin{cases} > 0, & \text{for} \quad \hat{y} > r^0 \\ < 0, & \text{for} \quad \hat{y} < r^0 \end{cases} \tag{2.23}$$

Proof: in the appendix.

Like before, when the risky asset's return was binomially distributed, the optimal fraction invested in the risky asset decreases (increases) with an increasing reference point, provided the reference point is below (above) the risk-free interest rate, i.e., $\hat{y} < r^0$ ($\hat{y} > r^0$). So, when the investor's reference return is below the risk-free interest rate, he/she will invest less into the risky asset with an increasing $\hat{y}$, as the risk-free asset yields a high enough return anyway. On the other hand, when the investor's reference return is above the risk-free rate, he/she will want to invest more into the risky (and more profitable) asset with an increasing $\hat{y}$ in order to meet the return target. The optimal weight of the risky asset also decreases with an increasing penalty parameter. So, if the investor's reference point is below the risk-free interest rate, his/her optimal weight of the risky asset is a decreasing function of both the penalty parameter and the reference point. The following remark presents our findings regarding the relative Sharpe ratio.

Remark 2.2 *Let the assumptions of Theorem 2.5 be satisfied. Then the following holds*

(i) $\mathrm{SR}(R(x)) = \mathrm{SR}(r) = \frac{\mu - r^0}{\sigma}$*;*

(ii) *Let* $x \neq 0$*. Then* $\mathrm{RSR}(R(x)) = \frac{r^0 - \hat{y}}{x\sigma} + \mathrm{SR}(r)$*;*

(iii) *Let* x^* *be the solution of problem (2.21). Then*

$$\frac{d\mathrm{RSR}(R(x^*))}{d\lambda} = -\frac{x^*\sigma}{\lambda(r^0 - \hat{y})} \left[\frac{\mu - r^0}{\sigma} \frac{F\left(\frac{\hat{y} - r^0}{\sigma x^*} - \frac{\mu - r^0}{\sigma}\right)}{f\left(\frac{\hat{y} - r^0}{\sigma x^*} - \frac{\mu - r^0}{\sigma}\right)} - 1 \right] = \begin{cases} > 0, & \text{for } \hat{y} < r^0 \\ < 0, & \text{for } \hat{y} > r^0 \end{cases}$$

(iv) *Let* x^* *be the solution of problem (2.21). Then* $\frac{d\mathrm{RSR}(R(x^*))}{d\hat{y}} = 0$.

Note that the portfolio performance (in terms of RSR) of the loss-averse investor facing a short-sales constraint increases with an increasing degree of the loss aversion if the reference point is below the risk-free rate; and it decreases if the reference point exceeds the risk-free rate. Note in addition that a positive expected equity premium and the reference point being below the risk-free rate are sufficient conditions such that the relative Sharpe ratio of the optimal portfolio of problem (2.21) is positive.

2.3 Numerical solution

In empirical applications or simulation experiments, the linear loss-averse utility maximization problem (2.1) has to be solved numerically. We thus reformulate the original problem as the bilinear parametric problem of $n-$variables

$$\max \left\{ \frac{1}{S} \sum_{s=1}^{S} \left(r_s' x - \lambda \left[\hat{y} - r_s' x \right]^+ \right) \,\middle|\, Ax \leq b \right\} \tag{2.24}$$

where λ, x, $\hat{y}$, A and b are defined as above, and r_s is the $n-$vector of observed returns, $s = 1, \ldots, S$.

It can be shown that (2.24) is equivalent to the following $(n+S)-$dimensional linear programming (LP) problem

$$\max_{x,y^-} \left\{ \hat{\mu}' x - \frac{\lambda}{S} e' y^- \,\middle|\, Ax \leq b,\ Bx + y^- \geq \hat{y} e,\ y^- \geq 0 \right\} \tag{2.25}$$

where $\hat{\mu} = (\hat{\mu}_1, \ldots, \hat{\mu}_n)'$ is the vector of estimated expected returns; i.e., $\hat{\mu}_i = \frac{1}{S} \sum_{s=1}^{S} r_{si}$, e is an $S-$vector of ones, $B' = [r_1, r_2, \ldots, r_S]$ and $y^- \in \mathbb{R}^S$ is an auxiliary variable.[12] The equivalence should be understood in the sense that if x^* is the x portion of an optimal solution for (2.25), then x^* is optimal for (2.24). On the other hand, if x^* is optimal for (2.24) then $((x^*)', (y^-)')'$ is optimal for (2.25) where $y_s^- = [\hat{y} - r_s' x^*]^+$, $s = 1, \ldots, S$. Thus, the utility function of problem (2.25) maximizes the expected return of the portfolio penalized for cases when its return drops below the reference value $\hat{y}$. However, Best, Grauer, Hlouskova and Zhang (2010) introduced a method that efficiently (and directly) solves the problem (2.24).

3 Simulation study assuming different structures and degrees of dependence

There is growing evidence that dependence in financial markets is not symmetric but of an asymmetric nature. Stock returns, for example, appear to display stronger dependence in bear than

[12] De Giorgi, Hens and Mayer (2007) suggested to solve the original problem by a different equivalent LP problem with $(n + 2S)$ dimensions.

in bull markets. The presence of such asymmetry violates the assumption of normally distributed asset returns, which underlies traditional mean-variance analysis and also our previous analytical analysis. The consequences of falsely assuming symmetric dependence are diverse: value-at-risk levels, for example, will generally be too optimistic when the potential for extreme co-movements is underestimated. A growing number of studies on portfolio management consider the asymmetric dependence of stock returns in an explicit way. These studies vary in how portfolios are optimized but agree to a large part in the use of copula theory. A copula links together two or more marginal distributions to form a joint distribution, where the marginal distributions can be of any form. The multivariate distribution can thus be split in two parts which may be treated completely separately, the univariate marginal distributions and the copula.[13]

The Gaussian (normal) copula is the one that is implicitly used all the time. It is implied by the joint normal distribution and is completely determined by the linear correlation ρ. One property of the Gaussian copula is that it displays symmetric dependence, i.e., dependence in the lower and upper tail of the distribution is the same. On the other hand, the Clayton copula is often used to model asymmetric dependence: under the Clayton copula the probability of joint negative extreme co-movements is greater than the probability of joint positive extreme co-movements, i.e., dependence in the lower tail of the distribution is larger than in its upper tail. We use the term *structure* of dependence to describe different copulas and the term *degree* of dependence to describe the level or amount of dependence, given a specific copula.

We study properties of the optimal linear loss-averse portfolio and its differences with respect to the optimal mean-variance portfolio by running simulation experiments when the distribution of returns is implied by the Gaussian and the Clayton copulas for different degrees of dependence. The rank correlation Kendall's tau[14] is used to find those Gaussian and Clayton copula parameters which reflect the same degree of dependence. We consider a simple two-asset world, where the investor can invest into two risky assets, such that one is considerably safer than the other. We thus simulate two asset returns which, first, are distributed as $N(\mu, \Sigma)$ involving the Gaussian copula and, second, display the same marginal (i.e., normal) distributions as above but instead of the Gaussian use the Clayton copula. The expected return, $\mu \in \mathbb{R}^2$, and the covariance matrix,

[13] Formally, the copula C of two random variables X and Y with marginal distribution functions $F_x(\cdot)$ and $F_y(\cdot)$ is implicitly defined by $F(x, y) = C(F_x(x), F_y(y))$, where $F(\cdot)$ is the two-dimensional distribution function. For a thorough introduction to copula theory see Joe (1997) and Nelsen (2006).

[14] Kendall's tau is a measure of dependence which does not depend on the marginal distributions (while linear correlation does) but is completely specified by the copula.

$\Sigma \in \mathbb{R}^{2\times 2}$, are the sample estimates of the EU dataset employed in the empirical section. We use the bond as a proxy of the safer (and less profitable) asset and the overall stock index as a proxy of the riskier (and more profitable) asset. For the EU market the corresponding quantities are $\mu_1 = 7.98\%$, $\mu_2 = 13.27\%$, $\sigma_1^2 = 5.66\%$, $\sigma_2^2 = 16.92\%$ and $\rho = 0.07$, where σ_1^2 and σ_2^2 are the estimated variances of asset 1 (bond) and asset 2 (stock) and ρ is their estimated correlation. As markets usually move together, we only consider a positive degree of dependence in the simulation exercise. The sample size of the simulation is 1,000.

Our experiments are conducted for changing the loss aversion parameter λ while the reference point $\hat{y}$ remains fixed, $\hat{y} \in \{0\%, 3.98\%, 7.98\%, 13.27\%\}$,[15] and while the dependence parameter for both copulas remain fixed, $\rho \in \{0.07, 0.5, 0.9\}$. We thus examine the performance and the asset allocation of optimal linear loss-averse and mean-variance portfolios when returns are generated by the Gaussian and Clayton dependence models.

Figure 3 presents optimal weights of the riskier asset for different reference points and dependence parameters, assuming Gaussian and Clayton dependence models. As the assumptions in our simulation experiment deviate from those in our theoretical analysis, i.e., we now consider a “safer” (but not totally risk-free) asset and we assume different structures and degrees of dependence, it will be interesting to see which of our analytical findings still hold. The very first observation is that the weight of the riskier asset is a decreasing function of the penalty parameter λ. So investors allocate less wealth to the riskier asset when their loss aversion parameter is higher. This seems to be true for both the Gaussian and Clayton copulas and is in line with our findings for the case when a no-short-sales restriction is imposed on the risky asset and the return of the risky asset is normally distributed, see Theorem 2.5.

On the other hand, what seems to be different with respect to the dependence structure is the weight of the riskier asset: for smaller λ, the weight seems to be larger when dependence is modeled by the Clayton copula, for larger λ (above some threshold level) the weight seems to be larger when dependence is modeled by the Gaussian copula. What can also be observed is that the threshold penalty parameter λ^* beyond which the weight of the riskier asset falls below one is increasing with an increasing dependence parameter ρ; i.e., $x(\lambda, \hat{y}, \rho) \leq 1$ for $\lambda \geq \lambda^*(\hat{y}, \rho)$ where $\lambda^*(\hat{y}, \rho)$ is increasing in ρ for a fixed reference point $\hat{y}$. So, in order to display a given investment behavior, i.e., a given optimal investment in the riskier asset, investors need to be more loss-averse (in terms of

[15] Note that $\hat{y} = 3.98\%$ corresponds to the mean of the risk-free interest rate from our EU dataset.

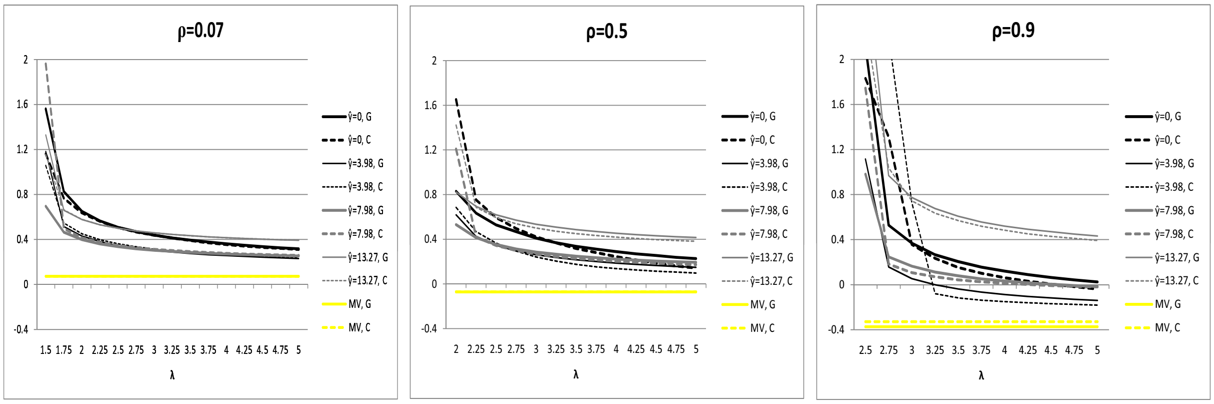

Figure 3: Sensitivity analysis of riskier asset's optimal weight with respect to λ for different ρ Returns are simulated using the Gaussian (solid line) and Clayton (dashed line) dependence models with $\mu_1 = 7.98\%$, $\sigma_1^2 = 5.66\%$ (safer asset) and $\mu_2 = 13.27\%$, $\sigma_2^2 = 16.92\%$ (riskier asset). These parameters are estimates for the bond and overall stock index returns in our empirical EU dataset ($1982 - 2008$). We use values of zero, the risk-free interest rate and the reported means of the safer and riskier assets for the reference point. The sample size of the simulation is 1,000. G (C) indicates that the returns were simulated with the Gaussian (Clayton) copula, MV indicates that the optimization procedure used was mean-variance.

the penalty parameter) with an increasing degree of dependence. Finally, for all three dependence parameters ρ is the weight of the riskier asset of the mean-variance investor significantly smaller than the weight of the loss-averse investor, however, they are nearly identical for both dependence models.

Certain monotonicity properties of the riskier asset's weight with respect to the dependence parameter ρ can be seen in Figure 4. The optimal weight of the riskier asset under both dependence models is an increasing function of the dependence parameter ρ for smaller penalty parameters λ; and is a decreasing function of ρ for higher penalty parameters; i.e., higher positive dependence seems to induce "riskier" behavior of less loss-averse investors who seem to invest more into the riskier asset when the dependence between assets increases. However, investors with a higher degree of loss aversion seem to become more conservative when dependence increases.

Figure 5 suggests that the dependence of the riskier asset's weights with respect to the reference point is in line with our findings for the case when a no-short-sales restriction is imposed on the risky asset and the risky asset's return is normally distributed, see Theorem 2.5. We can observe a U-shape of weights: first ($\hat{y} \leq \hat{y}^*$) the optimal weight of the riskier asset decreases with an increasing

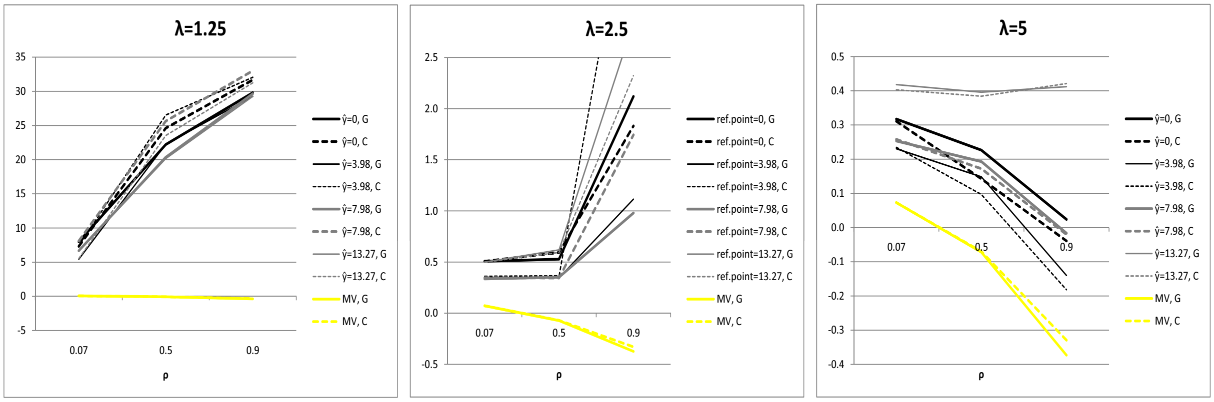

Figure 4: Sensitivity analysis of riskier asset's optimal weight with respect to ρ for different λ Returns are simulated using the Gaussian (solid line) and Clayton (dashed line) dependence models with $\mu_1 = 7.98\%$, $\sigma_1^2 = 5.66\%$ (safer asset) and $\mu_2 = 13.27\%$, $\sigma_2^2 = 16.92\%$ (riskier asset). These parameters are estimates for the bond and overall stock index returns in our empirical EU dataset (1982 – 2008). We use values of zero, the risk-free interest rate and the reported means of the safer and riskier assets for the reference point. The sample size of the simulation is 1,000. G (C) indicates that the returns were simulated with the Gaussian (Clayton) copula, MV indicates that the optimization procedure used was mean-variance.

reference point, beyond some threshold $\hat{y}^*$, i.e., $\hat{y} \geq \hat{y}^*$, the optimal weight increases with increasing $\hat{y}$. So, when the investor's reference return is below some threshold level, he/she wants to invest less into the riskier asset with an increasing $\hat{y}$, as the alternative (less risky) asset yields a high enough return anyway. On the other hand, when the investor's reference return is above the threshold level, he/she will want to invest more into the riskier (and more profitable) asset with an increasing $\hat{y}$ in order to meet the return target. For smaller penalty parameters this threshold $\hat{y}^*$ seems to coincide with the mean return of the safer asset (bond); i.e, $\hat{y}^* = 7.98\%$, while for larger penalty parameters the threshold seems to coincide with the risk-free interest rate ($\hat{y}^* = 3.98\%$). Note that according to Theorem 2.5 this threshold corresponds to the return of the (truly) safe asset. This property also explains the ranking of the size of the risky weights as presented in Figure 3, for both the Gaussian and the Clayton copulas provided the penalty parameter λ is large enough.

When focusing on performance measures such as the Sharpe ratio (SR) and the relative Sharpe ratio (RSR),[16] we can see that the property of RSR as stated in Remark 2.2(iii), namely RSR being an increasing function of the penalty parameter λ for $\hat{y} \leq \hat{y}^*$ and being a decreasing function of λ for

[16] Note that SR $= \frac{\mu' x^*}{\sqrt{x^*)'\Sigma x^*}}$ and RSR $= \frac{\mu' x^* - \hat{y}}{\sqrt{x^*)'\Sigma x^*}}$

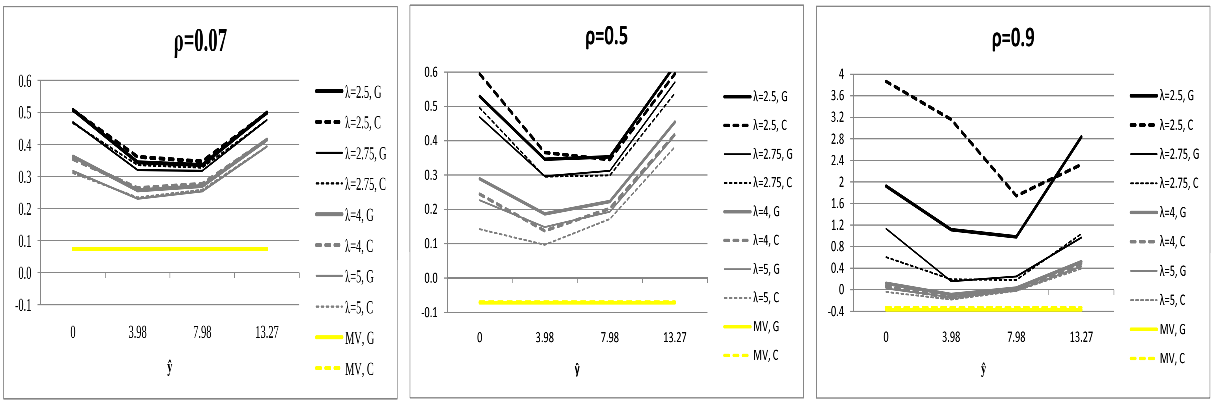

Figure 5: Sensitivity analysis of riskier asset's optimal weight with respect to $\hat{y}$ for different ρ Returns are simulated using the Gaussian (solid line) and Clayton (dashed line) dependence models with $\mu_1 = 7.98\%$, $\sigma_1^2 = 5.66\%$ (safer asset) and $\mu_2 = 13.27\%$, $\sigma_2^2 = 16.92\%$ (riskier asset). These parameters are estimates for the bond and overall stock index returns in our empirical EU dataset (1982 – 2008). We use values of zero, the risk-free interest rate and the reported means of the safer and riskier assets for the reference point. The sample size of the simulation is 1,000. G (C) indicates that the returns were simulated with the Gaussian (Clayton) copula, MV indicates that the optimization procedure used was mean-variance.

$\hat{y} > \hat{y}^*$, holds also in our simulation set-up where the underlying assumptions are slightly different than those valid in the remark. In our simulation set-up $\hat{y}^* \in [3.98\%, 7.98\%)$, which corresponds quite well to Remark 2.2(iii) where $\hat{y}^* = r^0$, as 3.98% is the mean of the risk-free interest rate and 7.98% is the mean of the bond return from our EU dataset. Thus, an increasing level of loss aversion (in terms of the penalty parameter λ) enhances the portfolio performance in terms of the RSR when the reference point is below a certain threshold value $\hat{y}^*$, and worsens it when the reference point exceeds $\hat{y}^*$.

With respect to the Sharpe ratio, LA portfolios outperform MV portfolios for a higher dependence parameter ($\rho = 0.5$ and 0.9) and for a higher penalty parameter ($\lambda \geq 4$ when $\rho = 0.5$ and $\lambda \geq 3.75$ when $\rho = 0.9$) when returns are generated by the Clayton dependence model. When returns are generated by the Gaussian dependence model, however, MV portfolios outperform LA portfolios. On the other hand, LA portfolios outperform MV portfolios with respect to the relative Sharpe ratio for $\hat{y} > 3.98\%$ for any $\lambda > 0$. If $\hat{y} = 3.98\%$ then LA portfolios outperform MV portfolios (with respect to RSR) when $\lambda \geq 2.75$ for $\rho = 0.07$, $\lambda \geq 2.25$ for $\rho = 0.5$ and $\lambda \geq 2.75$ for $\rho = 0.9$ for both types of dependence models.

4 Empirical application

In this section we investigate the performance of an optimal asset portfolio constructed by a linear loss-averse investor. We study the *benchmark scenario*, where the penalty parameter is constant and the reference point is equal to zero percent, as well as two modified versions of the benchmark scenario. The first modification uses the risk-free interest rate as the reference point (*risk-free scenario*), the second modification employs time-changing versions of the penalty parameter and the reference point which both depend on previous gains and losses (*dynamic scenario*). If the investor has experienced recent gains, his/her penalty parameter is equal to the prespecified λ while his/her reference point is lower than the risk-free interest rate due to the investor's decreasing loss aversion. On the other hand, if the investor has experienced recent losses, his/her loss aversion and thus his/her penalty parameter increases. At the same time his/her reference point is equal to the risk-free interest rate. In setting up the specific form of the dynamic model we closely follow Barberis and Huang (2001).

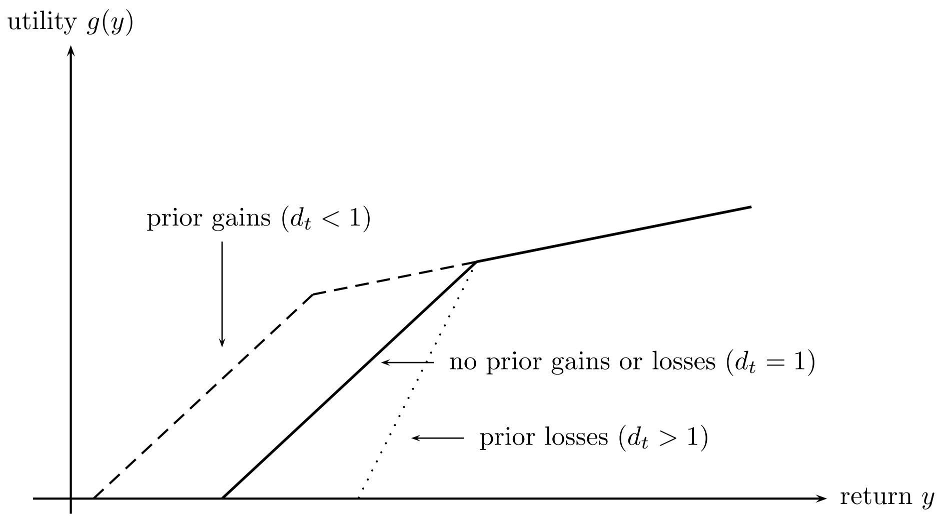

Figure 6: Utility of gains and losses

Let $d_t = r_B/r_t$ be a state variable describing the investor's sentiment with respect to prior gains or losses, which depends on the prior benchmark return $r_B = 1/T \sum_{i=1}^{T} r_{t-i}$ and the current portfolio return r_t. The benchmark return, which is the average value of the latest T realized portfolio returns, is compared with the current portfolio return. If $d_t \leq 1$, then the current portfolio return is greater than or equal to the benchmark return, making the investor feel that his/her portfolio has performed well and that he has accumulated gains; if $d_t > 1$, then the current portfolio return is lower than the benchmark return, making the investor feel he has experienced losses. We take $T = 1$ because in general investors seem to be the most sensitive to recent losses, and thus the current portfolio return is compared to the previous period's portfolio return. The linear loss-averse utility function adjusted for a time-changing penalty parameter and reference point is

$$g(r_t) = \begin{cases} r_t, & r_t \geq \hat{y}_t \\ (1+\lambda_t) r_t - \lambda_t \hat{y}_t, & r_t < \hat{y}_t \end{cases} \tag{4.26}$$

where

$$\lambda_t = \begin{cases} \lambda, & r_t \geq r_{t-1} \ \text{(prior gains)} \\ \lambda + \left(\frac{r_{t-1}}{r_t} - 1\right), & r_t < r_{t-1} \ \text{(prior losses)} \end{cases} \qquad \hat{y}_t = \begin{cases} \frac{r_{t-1}}{r_t} r_t^0, & r_t \geq r_{t-1} \ \text{(prior gains)} \\ r_t^0, & r_t < r_{t-1} \ \text{(prior losses)} \end{cases} \tag{4.27}$$

and r_t^0 is the risk-free interest rate. Note that $\lambda_t \geq \lambda$ and $\hat{y}_t \leq r_t^0$, where higher values of the loss aversion parameters reflect a higher degree of loss aversion.

We use different values of λ in the benchmark, the risk-free and the dynamic scenarios to allow for different degrees of loss aversion. Specifically, we let the penalty parameter be equal to $0.3, 0.5, 1, 2$ and 5. For the European and U.S. linear loss-averse investors we report optimization results for all three scenarios. In particular, we present descriptive statistics including mean, standard deviation, CVaR, and the Sharpe ratio of the optimal linear loss-averse portfolio return as well as the average optimal portfolio weights. To be able to compare the new linear loss-averse portfolio optimization to other, more standard, approaches, we also report optimization results for the MV and the CVaR methods.

The investor is assumed to re-optimize his/her portfolio each month using monthly closing prices and an optimization sample of 36 months, i.e., three years. This yields an out-of-sample evaluation period from February 1985 until December 2008. We have experimented with other, longer optimization samples, e.g., five years, but the performance of the resulting optimal LA portfolio is generally better for shorter periods indicating that changing market conditions should be taken into immediate account.

We consider two geographical markets, the European and the U.S. markets, each including different types of financial assets among which the investor may select. These assets include sectoral stock indices, government bonds and the two commodities gold and crude oil, yielding a total of 13 assets. Tables 1 and 5 report the summary statistics of the considered European and U.S. financial assets. In general, the stock indices exhibit comparatively high risk and return, the government bonds show a low risk and return, and gold exhibits moderate risk and a low return while crude oil shows high risk and a moderate return. Returns are computed as $r_t = p_t/p_{t-1} - 1$, where p_t is the monthly closing price at time t. All prices are extracted from Thomson Reuters Datastream from January 1982 to December 2008. The sectoral stock indices follow the Datastream classification for EMU and U.S. stock markets and cover the following 10 sectors: oil and gas, basic materials, industrials, consumer goods, health care, consumer services, telecom, utilities, financials, and technology. We use Brent and WTI crude oil quotations for the European and U.S. markets, respectively. Prices in the European markets are quoted in, or transformed to, Euro; prices in the U.S. markets are quoted in U.S. dollar, hence we consider European and U.S. investors who completely hedge their respective currency risk.[17]

Considering the empirical results of the benchmark scenario (see Table 2), where the reference point is equal to zero percent, the optimal LA portfolios generally display a higher expected return and higher risk (in terms of standard deviation and conditional value-at-risk) than the optimal MV and CVaR portfolios. In particular, the Sharpe ratio of all LA portfolios is significantly larger than that of the MV or CVaR portfolios, suggesting a clear outperformance of LA portfolios over the MV and CVaR portfolios. Taking a closer look at the LA results with respect to the penalty parameter, we note that with an increasing loss aversion, i.e., with increasing values of λ, the expected return, the risk and the Sharpe ratio of the optimal LA portfolio decrease as soon as some benchmark level of λ has been exceeded, where the maximum mean return among the different LA utility functions – as well as the maximum Sharpe ratio – is achieved for a penalty parameter of 0.5. So with an increasing value of the loss aversion parameter, the LA portfolio gets more similar to the MV portfolio. Comparing the optimal CVaR and MV portfolios we find only very small differences, and thus the results we report on the comparison of LA portfolios and the MV portfolio also hold for the LA portfolios and the CVaR portfolio. Setting the reference point equal to the risk-free interest rate (risk-free scenario) alters the LA results only marginally (see Table 3), so we do not

[17] The gold price, which is quoted in U.S. dollar, is transformed to Euro for the European investor. Differences between the descriptive statistics of the U.S. and the European gold price are thus entirely due to fluctuations in the USD/EUR exchange rate.

separately discuss the results for the risk-free scenario.

If we consider the dynamic scenario, however, the LA results change significantly, as long as the constant penalty parameter which applies after gains is not too large (see Table 4). This observation is in line with the observation that shorter optimization samples tend to yield "better" LA portfolios (e.g., in terms of the Sharpe ratio), i.e., it seems to be important that recent market developments are quickly taken into account. If λ is equal to 0.3, 0.5 or 1, then the mean of the optimal LA portfolio in the dynamic scenario is larger by more than 2 percentage points than the mean of the LA portfolio of the benchmark scenario. If λ is equal to 2 or 5, then the mean of the dynamic LA portfolio is only larger by about 0.6 percentage points. For all five values of the loss aversion parameter the optimal LA portfolios of the dynamic scenario outperform the optimal LA portfolios of the benchmark scenario in terms of the Sharpe ratio. A similar result is true if we compare the dynamic and the risk-free scenarios. In summary LA portfolios seem to generally outperform the MV and CVaR portfolios in terms of the Sharpe ratio, where among the different LA specifications the benchmark and the risk-free scenarios are clearly dominated by the dynamic scenario, and the "best" penalty parameter seems to be $\lambda = 0.5$.

The results for the U.S. markets are similar to those for the European markets, however, the clear outperformance of LA portfolios, which is observed in European markets, can only be found at a lower degree in U.S. markets. For example, dynamic LA portfolios only partially outperform the benchmark LA portfolios, and also the degree of outperformance is lower. When comparing LA portfolios and the MV portfolio, the LA portfolios outperform MV portfolios only for selected values of the loss aversion parameter. See Tables 6, 7 and 8 in the appendix.

We also check on robustness of the reported empirical results by running a set of simulations that reflect the investigated European and U.S. markets in the way that the first and second moments of the simulated and empirical data are equal. We thus simulate 13 returns, which are distributed $N(\mu, \Sigma)$, where μ and Σ are the sample mean and sample covariance matrix of the European and U.S. datasets, respectively. The simulation results (using a zero percent reference point or a reference point equal to the risk-free interest rate) support the empirical observation that the Sharpe ratio first increases with the penalty parameter and then decreases sightly; and that the LA portfolio outperforms the MV portfolio in terms of the Sharpe ratio if the loss aversion parameter is not too small ($\lambda \geq 1$).

	OIL	BASICMAT	INDUS	CONSGDS	HEALTH	CONSSVS	TELE	UTIL	FIN	TECH	BOND	GOLD	CRUDEOIL
Performance of 1-Month Returns (in percent p.a.)													
Mean	17.31	14.48	13.01	12.76	15.36	13.22	16.43	15.36	11.81	20.31	7.98	3.79	6.97
StDev	18.45	18.74	19.89	21.36	14.64	18.44	24	13.86	19.18	30.36	5.66	16.27	38.27
VaR	-61.33	-61.16	-65.32	-65.02	-55.64	-61.82	-67.78	-47.08	-63.39	-83.97	-22.31	-56.35	-88.02
CVaR	-74.61	-82.13	-84.58	-83.93	-67.73	-79.57	-83.65	-64.45	-83.99	-91.36	-30.96	-69.19	-95.87
Percentiles (in percent p.a.)													
5	-61.33	-61.16	-65.32	-65.02	-55.64	-61.82	-67.78	-47.08	-63.39	-83.97	-22.31	-56.35	-88.02
10	-45.47	-47.73	-46.96	-53.64	-39.61	-44.66	-59.33	-38.42	-46.67	-67.02	-16.34	-47.71	-79.17
25	-21.34	-18.05	-22.09	-23.56	-10.03	-19.67	-28.76	-12.74	-17.82	-27.43	-4.9	-24.83	-48.1
50	21.48	20.6	17.74	13.52	19.87	19	20.04	17.89	16.72	16.98	9.98	0.56	5.97
75	70.64	67.03	70.24	72.58	54.02	61.42	84.46	56.48	56.89	107.92	22.65	39.82	122.99
90	137.22	124.25	130.55	160.43	102.99	116.99	184.03	99.36	124.95	268.68	37.59	99.63	285.98
95	204.67	182.92	190.81	247.96	141.13	170.57	250.6	133.24	197.23	527.78	48.57	157.09	485.79

Table 1: Summary statistics for European data (January 1982 - December 2008)
Statistics are calculated on the basis of monthly returns and then annualized using discrete compounding. The annualized standard deviation is calculated by multiplying the monthly standard deviation with $\sqrt{12}$.

	Riskfree	Benchmark	MV	CVaR	Linear loss-averse (LA)					
					λ=0	λ=0.3	λ=0.5	λ=1	λ=2	λ=5
Performance of 1-Month Returns (in percent p.a.)										
Mean	4.45	11.23	6.45	5.22	18.11	17.47	18.41	15.77	13.05	8.99
Std.Dev.	0.64	17.27	5.11	6.18	30.26	23.70	22.55	19.82	15.79	9.73
CVaR	0.43	-79.85	-27.87	-36.32	-91.15	-84.27	-81.10	-77.33	-68.39	-52.59
Minimum	0.00	-93.99	-43.15	-58.83	-97.67	-97.34	-96.20	-94.97	-94.53	-94.19
Sharpe's ratio (in percent)		37.59	37.25	11.85	43.28	52.59	59.26	54.69	52.07	44.55
Total Realized Return (in percent p.a.)										
Last 10 Years	0.27	0.52	4.72	3.86	5.56	6.85	7.54	7.79	11.60	8.38
Last 5 Years	0.26	1.87	5.10	4.54	6.35	4.95	6.54	8.70	12.85	9.16
Last 3 Years	0.31	-9.27	1.33	0.52	-3.38	4.55	4.54	5.73	13.90	7.88
Last Year	0.34	-43.79	-0.68	-2.01	-28.30	-23.99	-25.98	-23.61	-15.71	-13.43
Percentiles (in percent p.a.)										
5	2.06	-64.03	-21.60	-25.16	-79.36	-68.82	-66.90	-58.92	-54.35	-32.65
10	2.11	-43.48	-16.65	-18.94	-64.34	-52.47	-50.29	-43.78	-38.12	-22.09
25	3.13	-17.48	-3.36	-7.64	-28.64	-21.07	-19.87	-14.77	-8.11	-5.28
50	3.93	18.30	7.31	6.43	14.93	16.98	16.37	16.32	12.70	9.49
75	5.04	55.39	18.06	21.20	99.35	68.60	63.38	48.99	40.64	24.94
90	8.38	105.22	30.09	33.32	253.15	177.43	148.19	124.68	86.47	52.10
95	9.15	150.92	40.61	46.62	435.12	333.29	292.63	243.40	128.48	78.71
Mean Allocation (in percent)										
OIL			0.15	2.76	6.62	9.54	9.12	8.65	4.68	3.71
BASICMAT			1.71	1.08	5.92	5.91	6.19	4.66	2.48	2.59
INDUS			0.67	0.16	2.09	1.39	0.90	0.13	0.47	0.22
CONSGDS			0.68	1.47	0.35	1.07	1.06	1.07	1.21	0.48
HEALTH			3.62	3.78	3.83	4.56	4.19	6.29	5.64	3.89
CONSSVS			0.99	0.53	0.00	0.23	0.39	0.59	1.69	1.93
TELE			0.52	1.92	6.27	5.21	5.29	5.95	6.97	3.62
UTIL			3.39	7.03	5.23	11.48	14.20	19.60	19.83	13.52
FIN			0.17	2.06	0.70	1.73	2.44	3.50	3.74	0.70
TECH			0.49	0.72	39.72	36.45	33.74	23.33	10.33	3.61
BOND			74.76	62.48	5.23	9.38	12.58	18.94	35.53	56.43
GOLD			10.43	12.16	1.39	2.10	2.23	2.70	3.54	5.93
CRUDEOIL			2.41	3.85	22.65	10.93	7.66	4.59	3.89	3.37

Table 2: Out-of-sample evaluation of EU portfolios: Benchmark scenario
The table reports statistics of a monthly reallocated optimal linear loss-averse portfolio based on an optimization period of 36 months as well as the average of the optimal asset weights. The benchmark scenario assumes a constant loss aversion parameter λ and a zero reference point. The evaluation period covers February 1985 to December 2008. Statistics are calculated on the basis of monthly returns and then annualized assuming discrete compounding. The annual standard deviation is computed as $\sigma_{pa} = \sqrt{12}\sigma_{pm}$.

	Riskfree	Benchmark	MV	CVaR	Linear loss-averse (LA)					
					λ=0	λ=0.3	λ=0.5	λ=1	λ=2	λ=5
Performance of 1-Month Returns (in percent p.a.)										
Mean	4.45	11.23	6.45	5.22	18.11	17.54	18.74	15.63	13.15	9.14
Std.Dev.	0.64	17.27	5.11	6.18	30.26	23.68	22.55	19.72	15.79	9.62
CVaR	0.43	-79.85	-27.87	-36.32	-91.15	-84.12	-80.88	-77.05	-67.56	-51.19
Minimum	0.00	-93.99	-43.15	-58.83	-97.67	-97.21	-95.96	-95.12	-94.88	-93.21
Sharpe's ratio (in percent)		37.59	37.25	11.85	43.28	52.96	60.69	54.26	52.70	46.55
Total Realized Return (in percent p.a.)										
Last 10 Years	0.27	0.52	4.72	3.86	5.56	6.88	7.38	7.58	11.86	8.29
Last 5 Years	0.26	1.87	5.10	4.54	6.35	4.86	6.78	7.99	12.80	9.15
Last 3 Years	0.31	-9.27	1.33	0.52	-3.38	4.55	4.69	4.24	12.86	7.91
Last Year	0.34	-43.79	-0.68	-2.01	-28.30	-23.93	-25.39	-25.34	-18.27	-12.64
Percentiles (in percent p.a.)										
5	2.06	-64.03	-21.60	-25.16	-79.36	-68.82	-64.25	-58.97	-54.51	-33.90
10	2.11	-43.48	-16.65	-18.94	-64.34	-53.01	-51.45	-44.08	-36.34	-20.69
25	3.13	-17.48	-3.36	-7.64	-28.64	-21.07	-18.17	-14.50	-9.74	-7.05
50	3.93	18.30	7.31	6.43	14.93	16.51	16.34	15.32	12.13	9.49
75	5.04	55.39	18.06	21.20	99.35	65.23	63.38	51.29	41.59	25.70
90	8.38	105.22	30.09	33.32	253.15	173.97	154.39	122.68	81.03	54.75
95	9.15	150.92	40.61	46.62	435.12	322.40	291.44	202.76	145.21	72.19
Mean Allocation (in percent)										
OIL			0.15	2.76	6.62	9.56	8.60	8.16	5.06	3.72
BASICMAT			1.71	1.08	5.92	5.99	6.27	4.74	2.59	2.39
INDUS			0.67	0.16	2.09	1.54	0.89	0.14	0.46	0.22
CONSGDS			0.68	1.47	0.35	1.09	1.12	1.24	1.44	0.66
HEALTH			3.62	3.78	3.83	4.40	3.79	7.32	5.95	4.20
CONSSVS			0.99	0.53	0.00	0.23	0.40	0.57	2.07	1.78
TELE			0.52	1.92	6.27	5.40	5.58	6.42	7.06	3.65
UTIL			3.39	7.03	5.23	11.70	14.81	19.07	19.50	14.48
FIN			0.17	2.06	0.70	1.70	2.38	3.40	3.42	0.64
TECH			0.49	0.72	39.72	36.17	33.33	22.40	9.50	3.14
BOND			74.76	62.48	5.23	9.44	12.91	19.56	35.97	56.76
GOLD			10.43	12.16	1.39	2.05	2.39	2.59	3.24	5.13
CRUDEOIL			2.41	3.85	22.65	10.74	7.54	4.40	3.75	3.21

Table 3: Out-of-Sample evaluation of EU portfolios: Risk-free scenario
The table reports statistics of a monthly reallocated optimal linear loss-averse portfolio based on an optimization period of 36 months as well as the average of the optimal asset weights. The risk-free scenario assumes a constant loss aversion parameter λ and a reference point which is equal to the risk-free interest rate. The evaluation period covers February 1985 to December 2008. Statistics are calculated on the basis of monthly returns and then annualized assuming discrete compounding. The annual standard deviation is computed as $\sigma_{pa} = \sqrt{12}\sigma_{pm}$.

	Riskfree	Benchmark	MV	CVaR	Linear loss-averse (LA)					
					λ=0	λ=0.3	λ=0.5	λ=1	λ=2	λ=5
Performance of 1-Month Returns (in percent p.a.)										
Mean	4.45	11.23	6.45	5.22	18.11	20.02	21.13	17.70	13.73	9.59
Std.Dev.	0.64	17.27	5.11	6.18	30.26	20.86	19.92	17.58	15.96	10.52
CVaR	0.43	-79.85	-27.87	-36.32	-91.15	-78.48	-74.80	-70.15	-69.90	-54.25
Minimum	0.00	-93.99	-43.15	-58.83	-97.67	-95.19	-95.19	-95.00	-94.71	-94.18
Sharpe's ratio (in percent)		37.59	37.25	11.85	43.28	71.47	80.10	72.07	55.63	46.77
Total Realized Return (in percent p.a.)										
Last 10 Years	0.27	0.52	4.72	3.86	5.56	13.16	15.75	13.67	12.22	8.04
Last 5 Years	0.26	1.87	5.10	4.54	6.35	7.81	9.88	10.14	14.43	10.16
Last 3 Years	0.31	-9.27	1.33	0.52	-3.38	8.53	6.69	8.25	15.80	9.68
Last Year	0.34	-43.79	-0.68	-2.01	-28.30	-21.98	-25.14	-23.33	-13.28	-9.56
Percentiles (in percent p.a.)										
5	2.06	-64.03	-21.60	-25.16	-79.36	-64.71	-53.73	-50.90	-47.60	-37.10
10	2.11	-43.48	-16.65	-18.94	-64.34	-42.31	-39.21	-37.53	-29.11	-23.63
25	3.13	-17.48	-3.36	-7.64	-28.64	-14.61	-12.19	-10.86	-6.80	-6.27
50	3.93	18.30	7.31	6.43	14.93	16.98	14.93	14.59	10.93	8.52
75	5.04	55.39	18.06	21.20	99.35	63.96	59.76	48.84	38.23	25.56
90	8.38	105.22	30.09	33.32	253.15	148.61	132.37	102.18	94.63	59.14
95	9.15	150.92	40.61	46.62	435.12	254.72	254.75	174.11	148.87	79.91
Mean Allocation (in percent)										
OIL			0.15	2.76	6.62	8.29	7.50	6.70	4.50	3.39
BASICMAT			1.71	1.08	5.92	5.40	5.15	3.96	2.76	2.31
INDUS			0.67	0.16	2.09	1.04	0.88	0.29	0.56	0.27
CONSGDS			0.68	1.47	0.35	0.99	0.95	0.95	1.10	0.70
HEALTH			3.62	3.78	3.83	5.37	5.93	6.25	5.31	3.96
CONSSVS			0.99	0.53	0.00	0.39	0.55	0.80	1.59	2.11
TELE			0.52	1.92	6.27	5.35	5.61	5.82	6.10	3.88
UTIL			3.39	7.03	5.23	13.23	15.40	19.24	19.31	13.94
FIN			0.17	2.06	0.70	1.87	2.35	3.31	3.02	0.64
TECH			0.49	0.72	39.72	30.42	27.06	19.60	9.69	3.96
BOND			74.76	62.48	5.23	16.15	19.77	25.33	38.10	54.73
GOLD			10.43	12.16	1.39	2.50	2.65	3.41	3.92	6.15
CRUDEOIL			2.41	3.85	22.65	9.01	6.19	4.34	4.03	3.96

Table 4: Out-of-sample evaluation of EU portfolios: Dynamic scenario

The table reports statistics of a monthly reallocated optimal linear loss-averse portfolio based on an optimization period of 36 months as well as the average of the optimal asset weights. The dynamic scenario assumes a time-changing loss aversion parameter λ and a time-changing reference point, where both depend on previous gains and losses. The evaluation period covers February 1985 to December 2008. Statistics are calculated on the basis of monthly returns and then annualized assuming discrete compounding. The annual standard deviation is computed as $\sigma_{pa} = \sqrt{12}\sigma_{pm}$.

5 Conclusion

A large body of experimental evidence suggests that loss aversion plays an important role in the asset allocation decision. In this paper we have investigated the linear loss-averse utility maximization along different dimensions. First we examined the theoretical relationship between the optimal asset allocation under linear loss aversion and more traditional asset allocation methods, i.e., the MV and CVaR methods. We have formulated assumptions under which the LA, MV and CVaR problems are equivalent, provided that portfolio returns are normally or t-distributed. We have thus created a link between two fundamentally different ways of portfolio optimization, namely between maximizing loss-averse utility whose specific form is motivated by experiments (LA problem) and between optimizing purely descriptive downside risk measures (CVaR problem). Then we investigated the two-asset case, involving one risky and one risk-free asset, and analytically derived the optimal risky asset's weight as well as the relative Sharpe ratio, under the assumption of binomially and normally distributed returns of the risky asset. Next, using a number of simulation experiments, we studied the properties of the optimal LA portfolios and their differences with respect to the optimal MV portfolios under slightly different assumptions then in the theoretical analysis. Specifically, both assets were supposed to be risky in the simulation experiment, with one being considerably safer (and less profitable) than the other. In particular, we investigated the impact of different structures of dependence on the results, where one dependence structure was symmetric (Gaussian copula) and the other asymmetric (Clayton copula). The asymmetric model assigns a higher probability to joint negative than to joint positive extreme co-movements. Most results related to the sensitivity analysis of the optimal riskier asset's weight and the corresponding relative Sharpe ratio with respect to the loss aversion parameter and the reference point were in line with our theoretical findings that were derived under stronger assumptions. With respect to the dependence structure, LA portfolios outperformed MV portfolios in terms of the Sharpe ratio for a higher degree of dependence and a higher degree of loss aversion when returns were generated by the asymmetric Clayton copula, but not when they were generated by the symmetric Gaussian copula. This suggests that investors should make an effort to select the "right" dependence model. In addition, a higher degree of (positive) dependence seems to induce a riskier behavior of less loss-averse investors in the sense that they invest more into the risky asset with an increasing degree of dependence. On the other hand, investors with a higher degree of loss aversion seem to become more conservative with an increasing degree of dependence.

Finally we implemented the trading strategy of a linear loss-averse investor who reallocates his/her portfolio on a monthly basis. In addition to the benchmark LA scenario, which uses a constant loss aversion parameter and a constant reference point, we have introduced a dynamic LA scenario, where both the loss aversion parameter and the reference point are updated conditional on previous gains and losses. The assets available for portfolio selection include sectoral stock indices, government bonds as well as the two commodities gold and crude oil, yielding a total of 13 assets, and we considered a European and a U.S. investor. Our empirical results suggest that – independent of the loss aversion parameter's value – the optimal LA portfolio clearly outperforms the optimal MV and CVaR portfolios, when the Sharpe ratio is used as a performance measure. Among the different LA scenarios, the dynamic method achieves by far the highest Sharpe ratios, which indicates that investors reacting to changing market conditions perform better than investors behaving the same all the time.

An interesting topic for further research would be to consider different (more general) forms of loss-averse utility, which are no longer bilinear, and investigate the properties of the corresponding optimal portfolios and some of their performance measures with respect to the loss aversion parameters and the (asymmetric) dependence structure.

Appendix A

Proof of Theorem 2.3: We are going to show that $\mathrm{LA}_{\lambda,\hat{y}}(R(x))$ is (a) increasing in $I_1 \equiv \left(-\infty, \max\left\{\frac{\hat{y}-r^0}{r_g-r^0}, \frac{r^0-\hat{y}}{r^0-r_b}\right\}\right]$, (b) increasing in $I_2 \equiv \left[\max\left\{\frac{\hat{y}-r^0}{r_g-r^0}, \frac{r^0-\hat{y}}{r^0-r_b}\right\}, +\infty\right)$ if $\lambda < \hat{\lambda}$, (c) constant in I_2 if $\lambda = \hat{\lambda}$ and (d) decreasing in I_2 if $\lambda > \hat{\lambda}$. Then statements of the theorem follow directly as function $\mathrm{LA}_{\lambda,\hat{y}}(x)$ is continuous.

It follows from (2.13), (2.14) and (2.15) that for $x \in \left(-\infty, \min\left\{\frac{\hat{y}-r^0}{r_g-r^0}, \frac{r^0-\hat{y}}{r^0-r_b}\right\}\right)$

$$\begin{aligned}\mathrm{LA}_{\lambda,\hat{y}}(R(x)) &= r^0 + (p(r_g - r_b) - r^0 + r_b)x - \lambda p(\hat{y} - r^0 - (r_g - r^0)x) \\ &= r^0 - \lambda p(\hat{y} - r^0) + [p(r_g - r_b) - r^0 + r_b + \lambda p(r_g - r^0)]x\end{aligned}$$

which is increasing if $p(r_g - r_b) - r^0 + r_b + \lambda p(r_g - r^0) > 0$. The latter condition is satisfied as $\lambda \geq 0 > \frac{r^0-r_b-p(r_g-r_b)}{p(r_g-r^0)}$ which follows from $p > \frac{r^0-r_b}{r_g-r_b}$.[18]

Let $x \in \left[\min\left\{\frac{\hat{y}-r^0}{r_g-r^0}, \frac{r^0-\hat{y}}{r^0-r_b}\right\}, \max\left\{\frac{\hat{y}-r^0}{r_g-r^0}, \frac{r^0-\hat{y}}{r^0-r_b}\right\}\right] \equiv I$. If $\hat{y} < r^0$ then $I = \left[\frac{\hat{y}-r^0}{r_g-r^0}, \frac{r^0-\hat{y}}{r^0-r_b}\right]$ and (2.11)-(2.15) imply that $\mathrm{LA}_{\lambda,\hat{y}}(R(x)) = r^0 + (p(r_g - r_b) - r^0 + r_b)x = \mathbb{E}(R(x))$. Thus, $\mathrm{LA}_{\lambda,\hat{y}}(R(x))$ is increasing in I if $p(r_g - r_b) - r^0 + r_b > 0$ and thus $p > \frac{r^0-r_b}{r_g-r_b}$ which holds by the assumption of the theorem. On the other hand, if $\hat{y} > r^0$ then $I = \left[\frac{r^0-\hat{y}}{r^0-r_b}, \frac{\hat{y}-r^0}{r_g-r^0}\right]$ and (2.11)-(2.15) imply that $\mathrm{LA}_{\lambda,\hat{y}}(R(x)) = -\lambda\hat{y} + (1+\lambda)[r^0 + (p(r_g - r_b) - r^0 + r_b)x] = -\lambda\hat{y} + (1+\lambda)\mathbb{E}(R)$. Thus, $\mathrm{LA}_{\lambda,\hat{y}}(R(x))$ is increasing in I under the same conditions as when $\hat{y} < r^0$. The case $\hat{y} = r^0$ is trivial as then $I = \{0\}$.

Finally, (2.11)-(2.15) imply that for $x \in I_2$

$$\begin{aligned}\mathrm{LA}_{\lambda,\hat{y}}(R(x)) &= r^0 + (p(r_g - r_b) - r^0 + r_b)x - \lambda(1-p)(\hat{y} - r^0 - (r_b - r^0)x) \\ &= r^0 - \lambda(1-p)(\hat{y} - r^0) + [p(r_g - r_b) - r^0 + r_b + \lambda(1-p)(r_b - r^0)]x\end{aligned}$$

Thus, $\mathrm{LA}_{\lambda,\hat{y}}(R(x))$ is increasing in I_2 if $p(r_g - r_b) - r^0 + r_b + \lambda(1-p)(r_b - r^0) > 0$ which is equivalent to $\lambda < \hat{\lambda}$. Similarly, $\mathrm{LA}_{\lambda,\hat{y}}(R(x))$ is constant (decreasing) in I_2 if $\lambda = \hat{\lambda}$ ($\lambda > \hat{\lambda}$). This concludes the proof of the theorem. $\square$

[18] Note that condition $p > \frac{r^0-r_b}{r_g-r_b}$ is equivalent to $\mathbb{E}(r) = pr_g + (1-p)r_b > r^0$.

Lemma 5.1 *Let $LA_{\lambda,\hat{y}}(R(x))$ be defined by (2.20) and $R(x) = r^0 + (r - r^0)x$. Then*

$$\frac{d\mathrm{LA}_{\lambda,\hat{y}}(R(x))}{dx} = \mu - r^0 + \lambda\sigma\left[\frac{\mu - r^0}{\sigma}F\left(\frac{\hat{y} - r^0}{\sigma x} - \frac{\mu - r^0}{\sigma}\right) - f\left(\frac{\hat{y} - r^0}{\sigma x} - \frac{\mu - r^0}{\sigma}\right)\right] \quad \textit{for } x \neq 0, \tag{5.28}$$

$$\frac{d\mathrm{LA}_{\lambda,\hat{y}}(R(0))}{dx^+} = \begin{cases} \mu - r^0, & \textit{if} \quad \hat{y} < r^0 \\ \mu - r^0 + \lambda\sigma\left[\frac{\mu - r^0}{\sigma}F\left(-\frac{\mu - r^0}{\sigma}\right) - f\left(-\frac{\mu - r^0}{\sigma}\right)\right], & \textit{if} \quad \hat{y} = r^0 \\ (1+\lambda)(\mu - r^0), & \textit{if} \quad \hat{y} > r^0 \end{cases} \tag{5.29}$$

$$\frac{d\mathrm{LA}_{\lambda,\hat{y}}(R(0))}{dx^-} = \begin{cases} (1+\lambda)(\mu - r^0), & \textit{if} \quad \hat{y} < r^0 \\ \mu - r^0 + \lambda\sigma\left[\frac{\mu - r^0}{\sigma}F\left(-\frac{\mu - r^0}{\sigma}\right) - f\left(-\frac{\mu - r^0}{\sigma}\right)\right], & \textit{if} \quad \hat{y} = r^0 \\ \mu - r^0, & \textit{if} \quad \hat{y} > r^0 \end{cases}$$

Proof: The results are obtained by straightforward differentiation of (2.20) using

$$\frac{dF(x)}{dx} = f(x) \quad \text{and} \quad \frac{df(x)}{dx} = -xf(x)$$

□

Proof of Theorem 2.4: Note that under the normality assumption of the risky asset's return the loss-averse utility is defined by (2.20). Part (i) follows from Lemma 5.1 by obtaining $\frac{d\mathrm{LA}_{\lambda,\hat{y}}(R(x))}{dx} < 0$ for $\mu \leq r^0$, $x \in \mathbb{R}\backslash\{0\}$, and $\lim_{x\to-\infty}\mathrm{LA}_{\lambda,\hat{y}}(R(x)) = +\infty > \lim_{x\to 0^+}\mathrm{LA}_{\lambda,\hat{y}}(R(x))$.

Note that $\frac{d^2\mathrm{LA}_{\lambda,\hat{y}}(R(x))}{dx^2} = -\lambda\frac{(\hat{y}-r^0)^2}{\sigma x^3}f\left(\frac{\hat{y}-r^0}{\sigma x} - \frac{\mu - r^0}{\sigma}\right)$ for $x \neq 0$. Thus, for $\lambda > 0$ and $\hat{y} \neq r^0$ the utility function $\mathrm{LA}_{\lambda,\hat{y}}(R(x))$ is strictly concave for $x > 0$ and strictly convex for $x < 0$.

Let $\hat{y} > r^0$, $H(x) \equiv \frac{\mu - r^0}{\sigma}F\left(\frac{\hat{y}-r^0}{\sigma x} - \frac{\mu - r^0}{\sigma}\right) - f\left(\frac{\hat{y}-r^0}{\sigma x} - \frac{\mu - r^0}{\sigma}\right)$, $x \neq 0$. Note that with respect to this notation $h = -H(\pm\infty)$. It can be shown that function $H(x)$ has the following properties: there exists $\tilde{x} > 0$ such that $H(\tilde{x}) = 0$, $H(x) < 0$ and is strictly increasing on $(-\infty, 0)$, $H(x) > 0$ on $(0, \tilde{x})$, and $H(x) < 0$ and is strictly decreasing on $(\tilde{x}, +\infty)$. Thus, $\inf\{H(x)\,|\,x \in (-\infty, 0) \cup (0, +\infty)\} = H(\pm\infty) = -h$. These properties of $H(x)$, $\mu > r^0$, assumptions of the theorem and Lemma 5.1 imply that $\frac{d\mathrm{LA}_{\lambda,\hat{y}}(R(x))}{dx} = \mu - r^0 + \lambda\sigma H(x) > 0$ for $x \in (0, \tilde{x}]$ and also $\frac{d\mathrm{LA}_{\lambda,\hat{y}}(R(0))}{dx^-} > 0$ as well as $\frac{d\mathrm{LA}_{\lambda,\hat{y}}(R(0))}{dx^+} > 0$. Thus, $\mathrm{LA}_{\lambda,\hat{y}}(R(x))$ is strictly increasing on $[0, \tilde{x}]$. In addition, $\mathrm{LA}_{\lambda,\hat{y}}(R(x))$ is strictly increasing also on $(-\infty, 0) \cup (\tilde{x}, +\infty)$ if $\lambda < \inf\left\{\frac{\mu - r^0}{\sigma(-H(x))}\,|\,x \in (-\infty, 0) \cup (\tilde{x}, +\infty)\right\} = \frac{\mu - r^0}{\sigma h} = \hat{\lambda}_N$. Thus, $\mathrm{LA}_{\lambda,\hat{y}}(R(x))$ is increasing over all its domain when $\lambda \leq \hat{\lambda}_N$. Finally, as $\lim_{x\to 0^-}\mathrm{LA}_{\lambda,\hat{y}}(R(x)) = r^0 < +\infty = \lim_{x\to+\infty}\mathrm{LA}_{\lambda,\hat{y}}(R(x))$ then this implies that $x^* = +\infty$. The same can be shown for $\hat{y} < r^0$ and $\mu > r^0$, and a similar statement, namely $x^* = +\infty$ when $\lambda < \hat{\lambda}_N$ holds also for $\hat{y} = r^0$. This concludes proves of parts (ii) and (iii).

Part (iv) follows from the fact that $\mathrm{LA}_{\hat{\lambda}_N, r^0}(R(x)) = r^0$ for any $x \in \mathbb{R}$. In addition, Lemma 5.1 implies that $\frac{d\mathrm{LA}_{\lambda,r^0}(R(x))}{dx} = \mu - r^0 + \lambda\sigma(-h)$ is negative and thus $\mathrm{LA}_{\lambda,r^0}(R(x))$ is decreasing for $x \in \mathbb{R}$; i.e., $x^* = -\infty$, if $\lambda > \hat{\lambda}_N$ which is part of (v) for the case when $\hat{y} = r^0$.

Part (v): Let $\hat{y} > r^0$ and let function $\lambda(x) \equiv \frac{\mu - r^0}{\sigma(-H(x))}$ be defined on $(-\infty, 0) \cup (\tilde{x}, +\infty)$. Note that $\lambda(x)$ is defined by setting the first derivative with respect to x of $\mathrm{LA}_{\lambda,\hat{y}}(R(x))$ equal to zero, and it has to hold at the local optima of LA. Note that properties of $H(x)$ imply that $\lambda(x)$ is continuous and increasing on $(-\infty, 0)$ and is continuous and decreasing on $(\tilde{x}, +\infty)$ and that both intervals $(-\infty, 0)$ and $(\tilde{x}, +\infty)$ map to $(\hat{\lambda}_N, +\infty)$. From this it follows then that for any $\lambda > \hat{\lambda}_N$ there exist $\tilde{x}_1 \in (-\infty, 0)$ and $\tilde{x}_2 \in (\tilde{x}, +\infty)$ such that $\lambda = \lambda(\tilde{x}_i) = \frac{\mu - r^0}{\sigma(-H(\tilde{x}_i))}$ for $i = 1, 2$. This, definition of $H(x)$, properties of $\mathrm{LA}_{\lambda,\hat{y}}(R(x))$, namely convexity for $x < 0$ and concavity on $x > 0$, and Lemma 5.1 imply that $\tilde{x}_1$ is the point of the local minimum of $\mathrm{LA}_{\lambda,\hat{y}}(R(x))$ and $\tilde{x}_2$ is the point of local maximum of $\mathrm{LA}_{\lambda,\hat{y}}(R(x))$. Thus, $\mathrm{LA}_{\lambda,\hat{y}}(R(x))$ has an S-shape such that $\lim_{x\to-\infty}\mathrm{LA}_{\lambda,\hat{y}}(R(x)) = +\infty$ and $\lim_{x\to+\infty}\mathrm{LA}_{\lambda,\hat{y}}(R(x)) = -\infty$ which implies that for any $\lambda > \hat{\lambda}_N$ is $x^* = -\infty$. The same can be shown for the case when $\hat{y} < r^0$. This concludes the proof of part (v). □

Proof of Theorem 2.5: The fact that $x^* > 0$ follows from the proof of Theorem 2.4. The remaining proof is based on implicit function differentiation. Let $G(\lambda, \hat{y}, x) \equiv \frac{dLA_{\lambda,\hat{y}}(R(x))}{dx} = 0$. Then

$$\frac{dx}{d\lambda} = -\frac{\partial G/\partial\lambda}{\partial G/\partial x} \quad \text{and} \quad \frac{dx}{d\hat{y}} = -\frac{\partial G/\partial\hat{y}}{\partial G/\partial x} \tag{5.30}$$

when $\hat{y}$ is fixed in the first case and λ is fixed in the second case. Thus, we need to obtain $\partial G/\partial\lambda$, $\partial G/\partial x$ and $\partial G/\partial\hat{y}$. Lemma 5.1 implies that

$$\partial G/\partial\lambda = (\mu - r^0)F\left(\frac{\hat{y} - r^0}{\sigma x} - \frac{\mu - r^0}{\sigma}\right) - \sigma f\left(\frac{\hat{y} - r^0}{\sigma x} - \frac{\mu - r^0}{\sigma}\right) \tag{5.31}$$

$$\begin{aligned}\partial G/\partial\hat{y} &= \lambda(\mu - r^0)f\left(\frac{\hat{y} - r^0}{\sigma x} - \frac{\mu - r^0}{\sigma}\right)\frac{1}{\sigma x} + \lambda\sigma\left(\frac{\hat{y} - r^0}{\sigma x} - \frac{\mu - r^0}{\sigma}\right)f\left(\frac{\hat{y} - r^0}{\sigma x} - \frac{\mu - r^0}{\sigma}\right)\frac{1}{\sigma x} \\ &= \frac{\lambda(\hat{y} - r^0)}{\sigma x^2}f\left(\frac{\hat{y} - r^0}{\sigma x} - \frac{\mu - r^0}{\sigma}\right) \end{aligned} \tag{5.32}$$

$$
\begin{aligned}
\partial G/\partial x &= \lambda(\mu - r^0) f\left(\frac{\hat{y} - r^0}{\sigma x} - \frac{\mu - r^0}{\sigma}\right)\left(-\frac{\hat{y} - r^0}{\sigma x^2}\right) \\
&\quad +\lambda\sigma\left(\frac{\hat{y} - r^0}{\sigma x} - \frac{\mu - r^0}{\sigma}\right) f\left(\frac{\hat{y} - r^0}{\sigma x} - \frac{\mu - r^0}{\sigma}\right)\left(-\frac{\hat{y} - r^0}{\sigma x^2}\right) \\
&= -\frac{\lambda(\hat{y} - r^0)^2}{\sigma x^3} f\left(\frac{\hat{y} - r^0}{\sigma x} - \frac{\mu - r^0}{\sigma}\right) \qquad (5.33)
\end{aligned}
$$

Finally, (5.30)–(5.33) conclude the proof. □

Appendix B

	OIL	BASICMAT	INDUS	CONSGDS	HEALTH	CONSSVS	TELE	UTIL	FIN	TECH	BOND	GOLD	CRUDEOIL
Performance of 1-Month Returns (in percent p.a.)													
Mean	14.54	12.98	14.07	12.5	15.58	13	12.01	12.37	14.04	14.55	10.19	4.62	6.75
StDev	18.5	20.82	18.12	19.52	15.26	18.62	19.71	14.8	19.2	26.07	6.96	16.11	34
VaR	-58.09	-62.46	-57.32	-64.62	-53.1	-63.37	-68.64	-52.13	-57.24	-78.42	-25.23	-53.25	-82.53
CVaR	-75.29	-82.65	-78.1	-80.21	-68.73	-77.05	-78.87	-67.01	-79.63	-88.69	-33.39	-68.69	-94.24
Percentiles (in percent p.a.)													
5	-58.09	-62.46	-57.32	-64.62	-53.1	-63.37	-68.64	-52.13	-57.24	-78.42	-25.23	-53.25	-82.53
10	-41.91	-47.84	-45.29	-49.86	-37.48	-46.02	-56.37	-44.36	-45.54	-60.57	-18.78	-43.7	-73.6
25	-21.99	-27.13	-19.16	-21.66	-14.1	-21.14	-24.75	-16.88	-22.53	-32.12	-5.56	-25.23	-47.18
50	12.27	16.17	18.65	11.7	17.68	13.47	18.98	15.7	17.48	19.01	9.66	-1.87	5.69
75	69.21	70.34	63.51	68.67	57.59	64.33	63.96	53.13	65.45	100.46	27.85	36.78	109.72
90	130	141	127.37	141.08	108.17	142.99	129.09	103.74	128.12	230.54	46.93	103.68	248.6
95	207.58	210.31	179.25	221.13	157.5	206.39	184.94	127.88	207.54	362.39	68	157.73	412.66

Table 5: Summary statistics for U.S. data (January 1982 - December 2008)
Statistics are calculated on the basis of monthly returns and then annualized using discrete compounding. The annualized standard deviation is calculated by multiplying the monthly standard deviation with $\sqrt{12}$.

	Riskfree	Benchmark	MV	CVaR	Linear loss-averse (LA)					
					λ=0	λ=0.3	λ=0.5	λ=1	λ=2	λ=5
Performance of 1-Month Returns (in percent p.a.)										
Mean	4.83	11.44	7.74	6.51	14.87	15.78	13.79	11.09	10.04	10.05
Std.Dev.	0.64	15.46	5.83	7.12	28.13	22.86	20.53	17.15	13.00	8.96
CVaR	0.27	-74.25	-30.63	-39.36	-90.41	-85.28	-81.86	-77.92	-65.40	-49.52
Minimum	0.00	-93.86	-55.82	-78.07	-97.93	-95.46	-94.05	-94.05	-91.61	-61.15
Sharpe's ratio (in percent)		41.04	47.18	22.43	34.17	45.85	41.84	34.99	38.41	55.56
Total Realized Return (in percent p.a.)										
Last 10 Years	0.30	-1.14	6.52	5.38	4.27	7.80	5.39	3.88	5.74	7.45
Last 5 Years	0.29	-1.58	6.59	6.87	12.64	12.07	6.92	4.43	6.63	8.68
Last 3 Years	0.35	-8.00	6.91	5.62	7.44	11.03	6.75	3.68	7.41	10.16
Last Year	0.21	-37.21	1.42	-2.07	-6.55	-12.76	-18.88	-16.88	-10.42	1.22
Percentiles (in percent p.a.)										
5	1.10	-59.96	-21.53	-25.20	-77.30	-73.60	-66.17	-55.29	-43.18	-30.70
10	1.37	-41.72	-13.78	-17.59	-64.22	-54.75	-51.75	-43.33	-34.38	-20.43
25	3.19	-17.68	-4.18	-6.69	-34.74	-25.97	-23.91	-19.53	-12.37	-5.30
50	5.32	17.28	6.71	6.94	20.13	20.15	18.42	12.43	10.36	10.04
75	6.13	59.22	20.22	21.78	107.22	85.87	72.42	56.40	40.98	29.82
90	8.06	100.10	33.61	35.76	219.19	175.84	157.02	126.39	84.40	61.21
95	8.52	131.64	42.79	46.52	371.47	267.56	228.74	161.68	119.68	87.83
Mean Allocation (in percent)										
OIL			0.65	2.27	8.04	11.17	11.17	5.63	3.73	4.22
BASICMAT			0.33	2.02	1.40	1.44	2.03	2.43	1.77	1.25
INDUS			2.41	0.81	0.00	0.01	0.02	1.20	2.18	2.38
CONSGDS			1.84	1.43	5.24	3.06	2.47	2.18	1.82	1.40
HEALTH			4.18	6.17	15.38	18.74	18.83	17.04	13.94	11.23
CONSSVS			3.93	4.85	0.00	0.00	0.00	0.25	0.75	1.18
TELE			0.87	3.04	4.55	4.90	5.61	8.37	7.42	3.70
UTIL			1.23	4.05	0.35	2.09	5.17	9.28	9.62	6.42
FIN			0.84	3.69	10.49	11.87	11.55	9.36	5.79	2.82
TECH			1.65	2.41	26.22	23.55	21.68	16.14	10.67	6.50
BOND			60.78	43.59	1.40	5.28	6.62	14.73	27.92	43.34
GOLD			16.04	19.23	5.59	6.41	6.60	6.92	8.37	9.89
CRUDEOIL			5.25	6.43	21.33	11.47	8.24	6.46	6.00	5.67

Table 6: Out-of-sample evaluation of U.S. portfolios: Benchmark scenario
The table reports statistics of a monthly reallocated optimal linear loss-averse portfolio based on an optimization period of 36 months as well as the average of the optimal asset weights. The benchmark scenario assumes a constant loss aversion parameter λ and a zero reference point. The evaluation period covers February 1985 to December 2008. Statistics are calculated on the basis of monthly returns and then annualized assuming discrete compounding. The annual standard deviation is computed as $\sigma_{pa} = \sqrt{12}\sigma_{pm}$.

	Riskfree	Benchmark	MV	CVaR	Linear loss-averse (LA)					
					λ=0	λ=0.3	λ=0.5	λ=1	λ=2	λ=5
Performance of 1-Month Returns (in percent p.a.)										
Mean	4.83	11.44	7.74	6.51	14.87	15.78	13.79	11.09	10.04	10.05
Std.Dev.	0.64	15.46	5.83	7.12	28.13	22.86	20.53	17.15	13.00	8.96
CVaR	0.27	-74.25	-30.63	-39.36	-90.41	-85.28	-81.86	-77.92	-65.40	-49.52
Minimum	0.00	-93.86	-55.82	-78.07	-97.93	-95.46	-94.05	-94.05	-91.61	-61.15
Sharpe's ratio (in percent)		41.04	47.18	22.43	34.17	45.85	41.84	34.99	38.41	55.56
Total Realized Return (in percent p.a.)										
Last 10 Years	0.30	-1.14	6.52	5.38	4.27	7.80	5.39	3.88	5.74	7.45
Last 5 Years	0.29	-1.58	6.59	6.87	12.64	12.07	6.92	4.43	6.63	8.68
Last 3 Years	0.35	-8.00	6.91	5.62	7.44	11.03	6.75	3.68	7.41	10.16
Last Year	0.21	-37.21	1.42	-2.07	-6.55	-12.76	-18.88	-16.88	-10.42	1.22
Percentiles (in percent p.a.)										
5	1.10	-59.96	-21.53	-25.20	-77.30	-73.60	-66.17	-55.29	-43.18	-30.70
10	1.37	-41.72	-13.78	-17.59	-64.22	-54.75	-51.75	-43.33	-34.38	-20.43
25	3.19	-17.68	-4.18	-6.69	-34.74	-25.97	-23.91	-19.53	-12.37	-5.30
50	5.32	17.28	6.71	6.94	20.13	20.15	18.42	12.43	10.36	10.04
75	6.13	59.22	20.22	21.78	107.22	85.87	72.42	56.40	40.98	29.82
90	8.06	100.10	33.61	35.76	219.19	175.84	157.02	126.39	84.40	61.21
95	8.52	131.64	42.79	46.52	371.47	267.56	228.74	161.68	119.68	87.83
Mean Allocation (in percent)										
OIL			0.65	2.27	8.04	11.17	11.17	5.63	3.73	4.22
BASICMAT			0.33	2.02	1.40	1.44	2.03	2.43	1.77	1.25
INDUS			2.41	0.81	0.00	0.01	0.02	1.20	2.18	2.38
CONSGDS			1.84	1.43	5.24	3.06	2.47	2.18	1.82	1.40
HEALTH			4.18	6.17	15.38	18.74	18.83	17.04	13.94	11.23
CONSSVS			3.93	4.85	0.00	0.00	0.00	0.25	0.75	1.18
TELE			0.87	3.04	4.55	4.90	5.61	8.37	7.42	3.70
UTIL			1.23	4.05	0.35	2.09	5.17	9.28	9.62	6.42
FIN			0.84	3.69	10.49	11.87	11.55	9.36	5.79	2.82
TECH			1.65	2.41	26.22	23.55	21.68	16.14	10.67	6.50
BOND			60.78	43.59	1.40	5.28	6.62	14.73	27.92	43.34
GOLD			16.04	19.23	5.59	6.41	6.60	6.92	8.37	9.89
CRUDEOIL			5.25	6.43	21.33	11.47	8.24	6.46	6.00	5.67

Table 7: Out-of-sample evaluation of U.S. portfolios: Risk-free scenario
The table reports statistics of a monthly reallocated optimal linear loss-averse portfolio based on an optimization period of 36 months as well as the average of the optimal asset weights. The risk-free scenario assumes a constant loss aversion parameter λ and a reference point which is equal to the risk-free interest rate. The evaluation period covers February 1985 to December 2008. Statistics are calculated on the basis of monthly returns and then annualized assuming discrete compounding. The annual standard deviation is computed as $\sigma_{pa} = \sqrt{12}\sigma_{pm}$.

	Riskfree	Benchmark	MV	CVaR	Linear loss-averse (LA)					
					λ=0	λ=0.3	λ=0.5	λ=1	λ=2	λ=5
Performance of 1-Month Returns (in percent p.a.)										
Mean	4.83	11.44	7.74	6.51	14.87	14.75	15.94	11.18	8.88	12.69
Std.Dev.	0.64	15.46	5.83	7.12	28.13	21.53	19.20	17.07	15.01	12.90
CVaR	0.27	-74.25	-30.63	-39.36	-90.41	-81.62	-77.63	-78.87	-75.39	-50.07
Minimum	0.00	-93.86	-55.82	-78.07	-97.93	-94.42	-94.42	-94.42	-94.27	-71.19
Sharpe's ratio (in percent)		41.04	47.18	22.43	34.17	44.12	55.42	35.62	25.82	58.42
Total Realized Return (in percent p.a.)										
Last 10 Years	0.30	-1.14	6.52	5.38	4.27	9.23	11.36	7.30	6.21	7.76
Last 5 Years	0.29	-1.58	6.59	6.87	12.64	9.95	15.20	7.51	7.26	10.97
Last 3 Years	0.35	-8.00	6.91	5.62	7.44	8.89	11.42	4.97	12.44	13.75
Last Year	0.21	-37.21	1.42	-2.07	-6.55	-16.59	-7.53	-15.05	-0.84	1.28
Percentiles (in percent p.a.)										
5	1.10	-59.96	-21.53	-25.20	-77.30	-67.32	-59.66	-60.43	-50.18	-30.24
10	1.37	-41.72	-13.78	-17.59	-64.22	-52.32	-46.90	-42.27	-35.34	-21.39
25	3.19	-17.68	-4.18	-6.69	-34.74	-23.91	-20.37	-17.72	-13.46	-5.92
50	5.32	17.28	6.71	6.94	20.13	16.70	17.47	12.27	9.20	8.50
75	6.13	59.22	20.22	21.78	107.22	74.09	71.60	59.20	42.40	29.77
90	8.06	100.10	33.61	35.76	219.19	160.56	150.53	110.43	84.93	61.85
95	8.52	131.64	42.79	46.52	371.47	267.56	236.20	156.59	123.15	83.08
Mean Allocation (in percent)										
OIL			0.65	2.27	8.04	9.68	9.55	5.62	4.15	4.42
BASICMAT			0.33	2.02	1.40	1.89	2.19	2.40	1.57	1.53
INDUS			2.41	0.81	0.00	0.29	0.46	1.33	2.25	2.53
CONSGDS			1.84	1.43	5.24	3.53	2.60	2.66	1.84	1.17
HEALTH			4.18	6.17	15.38	17.13	17.62	16.27	13.71	11.31
CONSSVS			3.93	4.85	0.00	0.15	0.10	0.28	0.67	1.26
TELE			0.87	3.04	4.55	4.82	5.69	7.37	8.22	4.03
UTIL			1.23	4.05	0.35	3.75	5.92	8.58	9.03	6.48
FIN			0.84	3.69	10.49	10.56	9.86	8.32	5.78	3.36
TECH			1.65	2.41	26.22	21.85	20.55	15.66	11.35	6.53
BOND			60.78	43.59	1.40	9.07	11.11	16.66	26.92	40.96
GOLD			16.04	19.23	5.59	6.61	7.17	7.60	7.79	10.27
CRUDEOIL			5.25	6.43	21.33	10.66	7.20	7.24	6.71	6.15

Table 8: Out-of-sample evaluation of U.S. portfolios: Dynamic scenario
The table reports statistics of a monthly reallocated optimal linear loss-averse portfolio based on an optimization period of 36 months as well as the average of the optimal asset weights. The dynamic scenario assumes a time-changing loss aversion parameter λ and a time-changing reference point, where both depend on previous gains and losses. The evaluation period covers February 1985 to December 2008. Statistics are calculated on the basis of monthly returns and then annualized assuming discrete compounding. The annual standard deviation is computed as $\sigma_{pa} = \sqrt{12}\sigma_{pm}$.

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