IHS Economics Series
Working Paper 164
January 2005

A Diffusion Approximation for the Riskless Profit Under Selling of Discrete Time Call Options: Non-identically Distributed Jumps

Alexander V. Nagaev
Sergej Nagaev
Robert M. Kunst

Impressum

Author(s):
Alexander V. Nagaev, Sergej Nagaev, Robert M. Kunst

Title:
A Diffusion Approximation for the Riskless Profit Under Selling of Discrete Time Call Options: Non-identically Distributed Jumps

ISSN: Unspecified

2005 Institut für Höhere Studien - Institute for Advanced Studies (IHS)
Josefstädter Straße 39, A-1080 Wien
E-Mail: office@ihs.ac.at
Web: www.ihs.ac.at

164
Reihe Ökonomie
Economics Series

A Diffusion Approximation for the Riskless Profit Under Selling of Discrete Time Call Options

Non-identically Distributed Jumps

Alexander V. Nagaev, Sergei A. Nagaev, Robert M. Kunst

164

Reihe Ökonomie

Economics Series

A Diffusion Approximation for the Riskless Profit Under Selling of Discrete Time Call Options

Non-identically Distributed Jumps

Alexander V. Nagaev, Sergei A. Nagaev, Robert M. Kunst

January 2005

Institut für Höhere Studien (IHS), Wien
Institute for Advanced Studies, Vienna

Contact:

Alexander V. Nagaev
Faculty of Mathematics and Computer Sciences
Nicolaus Copernicus University
Chopin str. 12/18
87-100 Torun, POLAND
email: nagaev@mat.uni.torun.pl

Sergei A. Nagaev
Department of Economics and Finance
Institute for Advanced Studies
Stumpergasse 56
1060 Vienna, AUSTRIA
email: sergej@ihs.ac.at

Robert M. Kunst
Department of Economics and Finance
Institute for Advanced Studies
Stumpergasse 56
1060 Vienna, AUSTRIA
and
University of Vienna
Department of Economics
Brünner Straße 72
1210 Vienna, AUSTRIA
email: robert.kunst@univie.ac.at

Abstract

A discrete time model of financial markets is considered. It is assumed that the relative jumps of the risky security price are independent non-identically distributed random variables. In the focus of attention is the expected non-risky profit of the investor that arises when the jumps of the stock price are bounded while the investor follows the upper hedge. The considered discrete time model is approximated by a continuous time model that generalizes the classical geometrical Brownian motion.

Keywords

Asymptotic uniformity, local limit theorem, volatility

JEL Classification

G12, G11, G13

Comments
This article is prepared within the project "Risk-less profit of a writer under selling discrete time options" supported by the Jubiläumsfonds of the Austrian National Bank (No. 10712).

Contents

1 Introduction

Consider the simplest financial market in which securities of two types are circulating. The price evolution of the securities of the first type is given by the equations

$$b_k = b_0 \rho_k, \;\; k = 0, 1, 2, \ldots,$$

where $b_0 > 0, \;\; \rho_k \geq 1$. The prices are registered at the equidistant moments of time $t_k = a + kh$. With no loss of generality we put $a = 0, \;\; h = 1$, i.e. $t_k = k$.

The price of the security of the second type at the moment k is represented as

$$s_k = s_0 \xi_1 \cdots \xi_k, \;\; k = 0, 1, 2, \ldots,$$

where the relative jumps ξ_k are random.

The securities of the first type are *riskless* having the interest rates $(\rho_k - 1) \cdot 100\%$. Let us call them conventionally *bonds.* It is clear that possessing the securities of the second type is concerned with a risk of their devaluation. We call them conditionally *stocks.*

Taken together in certain amounts β and γ the securities of both types constitute a so-called *portfolio (writer's investment portfolio)* whose worth at the time moment k is $\beta b_k + \gamma s_k$. *Playing* in the considered financial market consists of successive changing of the portfolio content at the moments $k = 1, 2, \ldots, n-1$. The successive pairs $(\beta_0, \gamma_0), (\beta_1, \gamma_1), \ldots, (\beta_{n-1}, \gamma_{n-1})$ constitute a so-called *strategy* of the game or a *trading* strategy. Obviously, as a basis for choosing (β_k, γ_k) serves the evolution of the stock price up to this moment i. e. $s_0, s_1, \ldots, s_k$. In other words

$$\beta_k = \beta_k(s_0, s_1, \ldots, s_k), \;\; \gamma_k = \gamma_k(s_0, s_1, \ldots, s_k).$$

The player is called a *writer (seller, investor).*

A trading strategy is called *self-financing* if the changing of the portfolio content does not affect its value i.e.

$$\beta_k b_k + \gamma_k s_k = \beta_{k-1} b_k + \gamma_{k-1} s_k, \;\; k = 1, \ldots, n-1.$$

The final goal of the game is to meet the condition

$$x_n = \beta_{n-1} b_n + \gamma_{n-1} s_n \geq f(s_n) \tag{1.1}$$

where $f(s)$ is a so-called *pay-off function* of the simplest option of the *European* type having n as a *maturity date.* For more about the mathematical and substantial aspects of the option pricing theory see e.g. Shiryaev (1999).

Basic problems of the mathematical theory of options are the evaluation of the so-called *rational option price* and a corresponding to it strategy leading to (1.1).

Both the problems are easily solved within the framework of the so-called *binary* model that is in the case where $\rho = const \in (d, u)$ and ξ_k take only two values d and u. In this case (see e.g. Ch. VI in Shiryaev (1999)),

$$x_0 = \rho^{-n} \sum_{k=0}^{n} C_n^k p_*^k (1 - p_*)^{n-k} f(s_0 u^k d^{n-k}) \tag{1.2}$$

where

$$p_* = \frac{\rho - d}{u - d}.$$

It is worth emphasizing that (1.2) does not assume any restrictions on the measure which governs the evolution of the stock price $(\xi_1, \dots, \xi_n)$. Furthermore, there exists the unique self-financing trading strategy $(\beta_0, \gamma_0),\ (\beta_1, \gamma_1), \dots, (\beta_{n-1}, \gamma_{n-1})$ leading to the equality

$$x_n = \beta_{n-1} b_n + \gamma_{n-1} s_n = f(s_n). \tag{1.3}$$

The strategy is defined by the formulae

$$\beta_k = \frac{u f_{k+1}(s_k d) - d f_{k+1}(s_k u)}{\rho b_k (u - d)} \tag{1.4}$$

and

$$\gamma_k = \frac{f_{k+1}(s_k u) - f_{k+1}(s_k d)}{s_k (u - d)} \tag{1.5}$$

where

$$f_k(s) = \rho^{-(n-k)} \sum_{j=0}^{n-k} C_{n-k}^j p_*^j (1 - p_*)^{n-k-j} f(s u^j d^{n-k-j}). \tag{1.6}$$

The successive values of the portfolio are

$$x_k = f_k(s_k),\ k = 0, 1, \dots, n-1.$$

In particular, $x_0 = f_0(s_0)$ is the rational or *fair* price. The rational option price is the minimal initial capital x_0 which always allows the investor to meet contract terms under proper behavior. Note that any smaller initial capital never ensures the required pay-off.

Now, assume that the market model is binary but d and u are not constant. More precisely, assume that ξ_k takes the values d_k and u_k.

Proposition 1.1 *In order to guarantee the equality*

$$x_n = f(s_n) \tag{1.7}$$

the investor must have at the preceding moment the capital

$$x_{n-1} = \rho_n^{-1} (p_n f(s_{n-1} u_n) + (1 - p_n) f(s_{n-1} d_n))$$

where

$$p_n = \frac{\rho_n - d_n}{u_n - d_n}.$$

Furthermore, x_{n-1} must be distributed between bonds and stocks in the following way

$$\beta_{n-1} = \frac{u_n f(s_{n-1}d_n) - d_n f(s_{n-1}u_n)}{\rho_n b_{n-1}(u_n - d_n)}, \ \gamma_{n-1} = \frac{f(s_{n-1}u_n) - f(s_{n-1}d_n)}{s_{n-1}(u_n - d_n)}.$$

Proof. Let x and (β, γ) be respectively the investor capital and its distribution in the portfolio at moment $n-1$. For the sake of simplicity we omit the subscript $n-1$. So,

$$x = \beta b_{n-1} + \gamma s_{n-1}.$$

The value of the potfolio at the moment n equals

$$x_n = \beta b_n + \gamma s_n = \beta b_{n-1}\rho + \gamma s_{n-1}\xi_n.$$

Taking into account the condistion (1.7) we obtain

$$f(s_{n-1}\xi_n) = \beta b_{n-1}\rho_n + \gamma s_{n-1}\xi_n$$

or

$$\begin{cases} f(s_{n-1}u_n) = & \beta b_{n-1}\rho_n + \gamma s_{n-1}u_n \\ \\ f(s_{n-1}d_n) = & \beta b_{n-1}\rho_n + \gamma s_{n-1}d_n. \end{cases}$$

Solving the system we find out that

$$\beta_{n-1} = \frac{u_n f(s_{n-1}d_n) - d_n f(s_{n-1}u_n)}{\rho_n b_{n-1}(u_n - d_n)}, \ \gamma_{n-1} = \frac{f(s_{n-1}u_n) - f(s_{n-1}d_n)}{s_{n-1}(u_n - d_n)}. \quad (1.8)$$

It is easily verified that this portfolio contains the capital

$$x = \rho_n^{-1}\left(\frac{\rho_n - d_n}{u_n - d_n} f(s_{n-1}u_n) + \frac{u_n - \rho_n}{u_n - d_n} f(s_{n-1}d_n)\right) = \rho_n^{-1}(p_n f(s_{n-1}u_n) + (1-p_n) f(s_{n-1}d_n)).$$

The proposition is proved.

Set

$$f_{n-1}(s) = \rho_n^{-1}(p_n f(su_n) + (1 - p_n) f(sd_n)).$$

From Proposition 1.1 it follows that

$$x_{n-1} = f_{n-1}(s_{n-1}). \quad (1.9)$$

So, we derived a pay-off function for a new European option with the maturity time $n-1$. By the proposition, in order to guarantee (1.9) the investor must have at the moment $n-2$ the capital

$$x_{n-2} = \rho_{n-1}^{-1}(p_n f_{n-1}(s_{n-2}u_n) + (1 - p_n) f_{n-1}(s_{n-2}d_n))$$

or substituting the formula for f_{n-1}

$$x_{n-2} = \rho_{n-1}^{-1}\rho_n^{-1}(p_{n-1}p_n f(s_{n-2}u_{n-1}u_n) + p_{n-1}(1-p_n)f(s_{n-2}u_{n-1}d_n)+$$

$$(1-p_{n-1})p_n f(s_{n-2}d_{n-1}u_n) + (1-p_{n-1})(1-p_n)f(s_{n-2}d_{n-1}d_n)) = f_{n-2}(s_{n-2})$$

where

$$f_{n-2}(s) = \rho_{n-1}^{-1}\rho_n^{-1}(p_{n-1}p_n f(su_{n-1}u_n) + p_{n-1}(1-p_n)f(su_{n-1}d_n)+$$

$$(1-p_{n-1})p_n f(sd_{n-1}u_n) + (1-p_{n-1})(1-p_n)f(sd_{n-1}d_n)).$$

Obviously, in order to meet the contract obligations at the moment k the investor must have the capital

$$x_k = f_k(s_k) \tag{1.10}$$

where

$$f_k(s) = \rho_{k+1}^{-1}\cdots\rho_n^{-1}\sum_{\mathbf{i}_k\in\{0,1\}^{n-k}} p(\mathbf{i}_k)f(sa(\mathbf{i}_k)), \tag{1.11}$$

while

$$\mathbf{i}_k = (i_{k+1},\ldots,i_n)$$

and

$$a(\mathbf{i}_k) = u_{k+1}^{i_{k+1}}d_{k+1}^{1-i_{k+1}}\cdots u_n^{i_n}d_n^{1-i_n}, \qquad p(\mathbf{i}_k) = p_{k+1}^{i_{k+1}}(1-p_{k+1})^{1-i_{k+1}}\cdots p_n^{i_n}(1-p_n)^{1-i_n}.$$

This capital must be distributed in accordance with

$$\beta_k = \frac{u_{k+1}f_{k+1}(s_kd_{k+1}) - d_{k+1}f_{k+1}(s_ku_{k+1})}{\rho_{k+1}b_k(u_{k+1}-d_{k+1})}, \qquad \gamma_k = \frac{f_{k+1}(s_ku_{k+1}) - f_{k+1}(s_kd_{k+1})}{s_k(u_{k+1}-d_{k+1})}. \tag{1.12}$$

In particular, the rational price is given by the formula

$$x_0 = \rho_1^{-1}\cdots\rho_n^{-1}\sum_{\mathbf{i}_0\in\{0,1\}^n} p(\mathbf{i}_0)f(s_0a(\mathbf{i}_0)) \tag{1.13}$$

and the initial portfolio must be of the form

$$\beta_0 = \frac{u_1f_1(s_0d_1) - d_1f_1(s_0u_1)}{\rho b_0(u_1-d_1)},\ \gamma_0 = \frac{f_1(s_0u_1) - f_1(s_0d_1)}{s_0(u_1-d_1)}.$$

If $\xi_k,\ k = 1,2,\ldots,n$, take more than two values then it is impossible to guarantee the desired relation (1.3) with probability 1. However, sometimes it is possible to guarantee (1.1). For example, this is possible when $f(s) = f_n(s)$ is convex. If $f(s)$ is convex so are all the functions $f_k(s),\ k = 0,1,\ldots,n-1$. If, furthermore, $\xi_k \in [d_k, u_k]$

then the hedging capital sequence is evaluated by the same formulae (1.10), (1.11) and (1.2).

This fact was, first, proven in Tessitore and Zabczyk (1996) for the case of constant d and u by the methods of control theory (see also Zabczyk (1996) and Motoczyński and Stettner (1998)). Later on in Shiryaev (1999) the rational price is derived as the solution of a extreme problem (see Theorem V.1c.1 ibidem).

Consider the sequence

$$\bar{x}_k = f_k(s_k),\ k = 0, \dots, n-1, \tag{1.14}$$

and let (β_k, γ_k) be defined as in (1.12).

Possessing after $(k-1)-$th step the capital $\bar{x}_{k-1}$ distributed in portfolio in accordance with (1.12) at the next step k the investor gains the capital

$$x_k = \beta_{k-1} b_k + \gamma_{k-1} s_k = \frac{u_k - \xi_k}{u_k - d_k} f_k(s_{k-1} d_k) + \frac{\xi_k - d_k}{u_k - d_k} f_k(s_{k-1} u_k).$$

If $\xi_k \in [d_k, u_k],\ k = 1, \dots, n$, then due to convexity of $f_k(s)$ we have

$$\delta_k = x_k - \bar{x}_k = f_k(s_{k-1} d_k) \frac{u_k - \xi_k}{u_k - d_k} + f_k(s_{k-1} u_k) \frac{\xi_k - d_k}{u_k - d_k} - f_k(s_{k-1} \xi_k) \geq 0. \tag{1.15}$$

If $f_k(s_{k-1}\xi)$ is strictly convex in $[d_k, u_k]$ then $\delta_k = 0$ if and only if $\xi_k = d_k$ or $\xi_k = u_k$. Otherwise $\delta_k > 0$. Thus, if ξ_k takes at least one value lying in (d_k, u_k) then a profit can arise. If the extreme values d_k and u_k belong to the support of the distribution of ξ_k then $\bar{x}_{k-1}$ is the minimal capital that allows such a profit. It implies that

$$\bar{x}_0 = \rho_1^{-1} \cdots \rho_n^{-1} \sum_{\mathbf{i}_0 \in \{0,1\}^n} p(\mathbf{i}_0) f(s_0 a(\mathbf{i}_0)) \tag{1.16}$$

is the minimal starting capital that allows the investor to meet his contract obligations with probability 1 provided he follows the strategy determined by (1.4) and (1.5). This strategy forms the so-called *upper hedge.* It determines the sequence $(\bar{x}_0, \bar{x}_1, \dots, \bar{x}_{n-1})$ of the *hedging capitals.* Here, $\bar{x}_0$ is called the *upper rational price.*

The investor may dispose of the so arisen profit in various ways. The simplest one is to withdraw from the game the superfluous quota δ_k which to the maturity date acquires the value $\delta_k \rho_{k+1} \cdots \rho_n$. So, the self-financing condition is fulfilled only in the part which bans any capital inflow.

Having withdrawn unnecessary quota one should follow the "binary" optimal strategy determined by (1.4) and (1.5). As a result to the maturity date the investor accumulates a riskless profit

$$\Delta_n = \delta_1 \rho_2 \dots \rho_n + \delta_2 \rho_3 \dots \rho_n + \cdots + \delta_n. \tag{1.17}$$

It should be emphasized that the upper hedge admits an *arbitrage* opportunity in the sense that the investor always meet his obligations, i.e.

$$\mathrm{P}(x_n \geq f(s_n)) = 1,$$

and may have a riskless profit in the sense that

$$\mathrm{P}(\Delta_n > 0) > 0.$$

It seems hopeless to find an acceptable formula for the expected value of riskless profit $\mathrm{E}\Delta_n$. So, the question arises how to approximate it. It is one of such approximations that is a basic goal of the paper.

It is worth emphasizing that a similar problem was studied in A. Nagaev and S. Nagaev (2003) (see also S. Nagaev (2003)). In these papers the authors considered the simplest case where the random variables $\xi_k,\ k = 1, 2, \ldots, n$, were i.i.d. Here, if the pay-off function is not smooth then chaotic phenomena arise. The typical example of such a function is provided by the call option. Unfortunately, the considered models do not take into account such intrinsic property of the stock price evolution as *volatility*. The basic goal of the present paper is to extend the main results of the latter work to the case where the stock price jumps are non-identically distributed.

The paper is organized as follows. In Section 2 the basic results concerning the expected value of the riskless profit under selling the call and put options are formulated. The "local" profit in the case where the model converges to a geometrical Gaussian process with independent increments is studied in Section 3. In Section 4 the limit value for the expected value of the total riskless profit is established. The limit value for the upper rational price is given in Section 5. Auxiliary facts concerning limit theorems for sums of independent variables are given in Section 6.

2 Basic results

In what follows we consider the simplest case of the standard call and put options determined, respectively, by the pay-off functions

$$f(s) = (s - K)_+, \qquad f(s) = (K - s)_+. \tag{2.18}$$

Since the random variables $\xi_k,\ k = 1, 2, \ldots, n$, are not identically distributed it is convenient to build an approximation based on a small parameter. This parameter should be linked with the maturity time. Let ε be such that

$$\varepsilon \to 0, \qquad n\varepsilon \to T,\ 0 < T < \infty.$$

Assume that

$$\xi_k = \xi_{k,\varepsilon} = \exp(h(k\varepsilon)\varepsilon + \eta_{k,\varepsilon}\varepsilon^{1/2}),\ k = 1, 2, \ldots, n, \tag{2.19}$$

where independent random variables $\eta_{k,\varepsilon} \in [-y, x]$ and $h(t)$ is a function continuous in $[0, T]$. Further, x and y are positive constants. Obviously, $\xi_{k,\varepsilon} \in [d_{k,\varepsilon}, u_{k,\varepsilon}]$, where

$$u_k = u_{k,\varepsilon} \exp(h(k\varepsilon)\varepsilon + x\varepsilon^{1/2}) \quad d_k = d_{k,\varepsilon} = \exp(h(k\varepsilon)\varepsilon - y\varepsilon^{1/2}),\ k = 1, 2, \ldots, n, \tag{2.20}$$

and

$$s_k = s_{k,n} = s_0 \xi_{1,\varepsilon} \cdots \xi_{k,\varepsilon}, \;\; k = 1, 2, \ldots, n. \tag{2.21}$$

Consider the random process

$$x_n(t) = \sum_{j=1}^{k-1} h(j\varepsilon)\varepsilon + \varepsilon^{1/2} \sum_{j=1}^{k-1} \eta_{j,\varepsilon}, \;\; \frac{k-1}{n} \le t < \frac{k}{n}, \;\; k = 1, 2, \ldots, n. \tag{2.22}$$

It is easily seen that the trajectories of the process belong to $D[0,1]$.

Definition 2.1 *We say that the sequence of independent variables* $\eta_{1,\varepsilon}, \eta_{2,\varepsilon}, \ldots, \eta_{n,\varepsilon}$, *satisfies Condition* $\mathcal{A}$ *if:*

(A1)

$$\mathrm{E}\eta_{k,\varepsilon} = 0, \;\; k = 1, 2, \ldots, n;$$

(A2) there exists a strictly positive continuous function $\sigma(t)$ *defined on* $[0, T]$ *such that*

$$\mathrm{Var}\ \eta_{k,\varepsilon} = \sigma^2(k\varepsilon) + \omega_{k,\varepsilon}, \;\; k = 1, 2, \ldots, n,$$

where

$$\lim_{\varepsilon \to 0} \sup_{1 \le k \le n} |\omega_{k,\varepsilon}| = 0;$$

(A3) $[-y, x]$ *is the minimal interval that contains the supports of all the distributions* $F_{k,\varepsilon}(u) = \mathrm{P}(\eta_{k,\varepsilon} < u), \;\; k = 1, 2, \ldots, n.$

In particular, condition (A2) implies that for all sufficiently small ε we have

$$\mathrm{Var}\ \eta_{k,\varepsilon} > \frac{1}{2} \min_{0 \le t \le T} \sigma^2(t) > 0.$$

If $\eta_{1,\varepsilon}, \eta_{2,\varepsilon}, \ldots, \eta_{n,\varepsilon}$, satisfy Condition $\mathcal{A}$ then the Lindeberg condition holds and, therefore, by the Central Limit Theorem the finite dimensional distributions of the process $x_n(t)$ converge to those of the process

$$x(t) = \int_0^{tT} h(u)\mathrm{d}u + y(tT), \;\; 0 \le t \le 1,$$

where $y(t)$ is the Gaussian process such that

$$y(0) = 0,\ \mathrm{E}y(t) \equiv 0,\ \mathrm{E}y(s)y(t) = B(t,s) = \int\limits_0^{\min(s,t)} \sigma^2(u)\mathrm{d}u.$$

It is easily seen that $y(t)$ has independent increments.

Actually, the process $x_n(t)$ weakly converges to $x(t)$ in $D[0,1]$. However, dealing with the expected value of the total profit Δ_n it suffices to have the weaker convergence.

Consider the sums

$$\zeta_{k,\varepsilon} = \sum_{j=1}^{k} \eta_{j,\varepsilon},\ \ k = 1, \dots, n.$$

It is evident that as $k \to \infty$

$$\mathrm{Var}\ \zeta_{k,\varepsilon} = b_{k,\varepsilon}^2(1 + o(1)).$$

where

$$b_{k,\varepsilon}^2 = \sum_{j=1}^{k} \sigma^2(j\varepsilon).$$

As in Nagaev and S. Nagaev (2003), the following form of the local limit theorem plays a crucial role. There exists ε_0 such that as $k \to \infty$

$$b_{k,\varepsilon}\mathrm{P}(z \le \zeta_{k,\varepsilon} < z + h) = h\varphi(z/b_{k,\varepsilon}) + o(1) \tag{2.23}$$

uniformly in $z \in I\!\!R^1,\ \varepsilon \in [0, \varepsilon_0]$ and $h, 0 < h' \le h \le h'' < \infty$. Here, $\varphi(z)$ is the density of the standard normal law.

The most general, though not very convenient, condition guaranteeing (2.23) is the following: for all sufficiently small ε

$$\sup_{1 \le k \le n} \sup_{0 < \delta \le |t| \le \Delta < \infty} |\mathrm{E}e^{\imath t \eta_{k,\varepsilon}}| = \rho(\delta, \Delta) < 1. \tag{2.24}$$

In Section 6 we discuss more convenient conditions stated in terms of the distribution functions $F_{k,\varepsilon}(u)$.

In addition to (2.20) and (2.19) assume that

$$\rho_k = \rho_{k,\varepsilon} = \exp(\alpha(k\varepsilon)\varepsilon) \tag{2.25}$$

where $\alpha(t) \ge 0$ is continuous in $[0,T]$. Let $\Delta_n = \Delta_{n,\varepsilon}$ be determined by (1.15) and (1.17) with u_k, d_k, ρ_k, ξ_k and s_k replaced, respectively, $u_{k,\varepsilon}, d_{k,\varepsilon}, \rho_{k,\varepsilon}, \xi_{k,\varepsilon}$ and $s_{k,\varepsilon}$.

Define for $t \in [0,1]$

$$\psi(t,z) = \frac{x+y}{\sqrt{xyT(1-t)}}\varphi\left(\frac{\ln K - z - \int\limits_{tT}^{T}\alpha(u)\mathrm{d}u + \frac{1}{2}\sqrt{xy(1-t)T}}{\sqrt{xy(1-t)T}}\right) \tag{2.26}$$

and for $t \in [0,T]$

$$I(t) = \frac{1}{x+y}\mathrm{E}\psi(t, x(t/T) + \ln s_0) =$$

$$\frac{1}{\sqrt{B(t,t)+xy(T-t)}}\varphi\left(\frac{\ln(K/s_0)-\int\limits_0^t h(u)\mathrm{d}u-\int\limits_t^T \alpha(u)\mathrm{d}u+\frac{1}{2}(T-t)xy}{\sqrt{B(t,t)+xy(T-t)}}\right). \qquad (2.27)$$

The following theorem contains the basic result of the present paper.

Theorem 2.2 *Assume that the sequence $\eta_{j,\varepsilon},\ j = 1, 2, \ldots, n$, satisfies Condition $\mathcal{A}$. If (2.23) is also fulfilled, then as $\varepsilon \to 0,\ n\varepsilon \to T$*

$$\mathrm{E}\Delta_{n,\varepsilon} = \frac{K}{2}\int\limits_0^T (xy - \sigma^2(t))I(t)\mathrm{d}t + o(1)$$

where K is the strike price from (2.18).

It is worth reminding that if the random variables $\eta_1, \eta_2, \ldots$ are independent then $\sigma_\pi = \sigma$.

Note that Var $\eta_{k,\varepsilon} \le xy,\ k = 1, 2, \ldots, n$. This implies that $\sup\limits_{0\le t\le T} \sigma^2(t) \le xy$. So, the limit value of the sequence $\mathrm{E}\Delta_n$ is non-negative. It should be emphasized that this limit value depends on x and y through xy. Furthermore, in the case of the call option, the upper rational price corresponding to x and y as $n \to \infty$ converges to (see (5.47))

$$\bar{x}_0 \to c(xy) =$$

$$s_0\Phi\left(\frac{\ln(s_0/K)+\int\limits_0^T \alpha(t)\mathrm{d}t+Txy/2}{\sqrt{Txy}}\right) - K\exp\left(-\int\limits_0^T \alpha(t)\mathrm{d}t\right)\Phi\left(\frac{\ln(s_0/K)+\int\limits_0^T \alpha(t)\mathrm{d}t-Txy/2}{\sqrt{Txy}}\right).$$

As to the lower rational price given by the formula

$$\underline{x}_0 = \rho_1^{-1}\cdots\rho_n^{-1}(s_0\rho_1\cdots\rho_n - K)_+$$

it converges to

$$c(0) = s_0\left(1 - \frac{K}{s_0}\cdot\exp\left(-\int\limits_0^T \alpha(t)\mathrm{d}t\right)\right)_+.$$

So, the interval of the rational prices converges as $n \to \infty$ to $(c(0), c(xy))$.

3 "Local" profit of investor

Let us denote by c any positive constant whose concrete value is of no importance. Under such a convention we have e.g. $c+c=c$, $c^2=c$ etc. By θ we denote any variable taking values in $[-1,1]$.

Denote

$$p_{k,\varepsilon} = \frac{\rho_{k,\varepsilon} - d_{k,\varepsilon}}{u_{k,\varepsilon} - d_{k,\varepsilon}}, \ \lambda_{k,\varepsilon} = \frac{\xi_{k,\varepsilon} - d_{k,\varepsilon}}{u_{k,\varepsilon} - d_{k,\varepsilon}}.$$

From (1.6) it follows that the discounted "local" profit of the investor takes the form

$$\Delta_{k,n} = \delta_{k,n}\rho_{k+1,\varepsilon}\cdots\rho_{n,\varepsilon} = \sum_{\mathbf{i}_k\in\{0,1\}^{n-k}} p(\mathbf{i}_k)(\lambda_{k,\varepsilon} f(s_{k-1,\varepsilon}u_{k,\varepsilon}a(\mathbf{i}_k)) + (1-\lambda_{k,\varepsilon})f(s_{k-1,\varepsilon}d_{k,\varepsilon}a(\mathbf{i}_k)) -$$

$$-f(s_{k-1,\varepsilon}\xi_{k,\varepsilon}a(\mathbf{i}_k)), \tag{3.28}$$

where $p(\mathbf{i}_k)$ and $a(\mathbf{i}_k)$ are as in (1.11). For the time being we suppress the dependence of λ_k, d, u, ξ_k and s_k on ε.

Let, first, $f(s) = (s-K)_+$. Consider $\mathbf{i}_k$ such that $s_{k-1}d_k a(\mathbf{i}_k) > K$. Then

$$\lambda_k f(s_{k-1}u_k a(\mathbf{i}_k)) + (1-\lambda_k) f(s_{k-1}d_k a(\mathbf{i}_k)) - f(s_{k-1}\xi_k a(\mathbf{i}_k)) = s_{k-1}(\lambda_k u_k + (1-\lambda_k)d_k - \xi_k)a(\mathbf{i}_k) = 0.$$

If $s_{k-1}d_k a(\mathbf{i}_k) \le K$ then

$$0 = f(s_{k-1}u_k a(\mathbf{i}_k)) \ge f(s_{k-1}\xi_k a(\mathbf{i}_k)) \ge f(s_{k-1}d_k a(\mathbf{i}_k)).$$

It is worth reminding that $d_{k-1} \le \xi_{k-1} \le u_{k-1}$. Thus,

$$\Delta_{k,n} = \delta_{k,n}\rho_{k+1}\cdots\rho_n = \sum_{(\mathbf{i}_k:\ s_{k-1}d_k a(\mathbf{i}_k) \le K < s_{k-1}d_k a(\mathbf{i}_k)} p(\mathbf{i}_k)(\lambda_k(s_{k-1}u_k a(\mathbf{i}_k) - K)_+ +$$

$$(1-\lambda_k)(s_{k-1}d_k a(\mathbf{i}_k) - K)_+ - (s_{k-1}\xi_k a(\mathbf{i}_k) - K)_+).$$

Denote

$$|\mathbf{i}_k| = i_{k+1} + \cdots + i_n.$$

Define for $d_k \le z \le u_k$

$$\bar{r}_{n-k}(z, Z) = (n-k)p + \varepsilon^{-1/2}R^{-1}\left(Z - \sum_{j=k}^{n} h(j\varepsilon)\varepsilon\right) - R^{-1}w$$

where

$$R = x + y, \ p = \frac{y}{x+y}, \ \ln z = h(k\varepsilon)\varepsilon + w\varepsilon^{1/2}.$$

Let

$$r_{n-k}(z) = \bar{r}_{n-k}(z, \ln(K/s_{k-1})).$$

The following lemma plays an important role.

Lemma 3.1 *Let* $\mathbf{i}_k$ *and* z *satisfy the equation*

$$s_{k-1} z a(\mathbf{i}_k) = K$$

where $\ln z = w\varepsilon^{1/2} + h(k\varepsilon)\varepsilon,\ -y \le w \le x$. *If* $0 < x' \le \min(x, y) \le \max(x, y) \le x'' < \infty$ *Then*

$$|\mathbf{i}_k| = r_{n-k}(z) = [r_{n-k}(d_k)].$$

Proof. From the equation

$$s_{k-1} z a(\mathbf{i}_k) = K$$

it follows that

$$\ln(K/s_{k-1}) = i_{k+1} \ln \frac{u_{k+1}}{d_{k+1}} + \cdots + i_n \ln \frac{u_n}{d_n} + \ln d_{k+1} + \cdots + \ln d_n + \ln z.$$

According to (2.20)

$$\ln \frac{u_k}{d_k} = R\varepsilon^{1/2}$$

and, therefore,

$$\ln(K/s_{k-1}) = \varepsilon^{1/2} R(i_{k+1} + \cdots + i_n) - (n-k)y\varepsilon^{1/2} + w\varepsilon^{1/2} + \sum_{j=k}^{n} h(j\varepsilon)\varepsilon.$$

or

$$\ln(K/s_{k-1}) = \varepsilon^{1/2} R|\mathbf{i}_k| - (n-k)y\varepsilon^{1/2} + w\varepsilon^{1/2} + \sum_{j=k}^{n} h(j\varepsilon)\varepsilon.$$

So, $|\mathbf{i}_k| = r_{n-k}(z)$. If z varies within $[d_k, u_k]$, then w stays in $[-y, x]$. It is easily seen that $r_m(d_k) - r_m(u_k) = 1$. It implies that

$$\#\{j:\ r_{n-k}(u_k) < j \le r_{n-k}(d_k)\} = 1. \tag{3.29}$$

So, $r_{n-k}(z) = [r_{n-k}(d_k)]$. The lemma is proved.

From the lemma it follows that

$$\Delta_{k,n} = \delta_{k,n}\rho_{k+1}\cdots\rho_n = \sum_{(\mathbf{i}_k:\ |\mathbf{i}_k| = [r_{n-k}(d_k)])} p(\mathbf{i}_k)(\lambda_k(s_{k-1}u_k a(\mathbf{i}_k) - K)_+ +$$

$$(1-\lambda_k)(s_{k-1}d_k a(\mathbf{i}_k) - K)_+ - (s_{k-1}\xi_k a(\mathbf{i}_k) - K)_+).$$

Let $z_k = \exp(w_k\varepsilon^{1/2} + h(k\varepsilon)\varepsilon) \in [d_k, u_k]$ be determined by the equality $r_{n-k}(z_k) = [r_{n-k}(d_k)]$. It is easily seen that

$$p - \{r_{n-k}(d_k)\} = -\frac{w_k}{R}. \tag{3.30}$$

Furthermore,

$$s_{k-1}u_k a(\mathbf{i}_k) = K\exp((x-w_k)\varepsilon^{1/2})$$

and

$$s_{k-1}d_k a(\mathbf{i}_k) = K\exp(-(y+w_k)\varepsilon^{1/2}).$$

Further,

$$\Delta_k = \sum_{(\mathbf{i}_k:\ |\mathbf{i}_k|=[r_{n-k}(d_k)],\ s_{k-1}\xi_k a(\mathbf{i}_k)>K)} p(\mathbf{i}_k)(\lambda_k(s_{k-1}u_k a(\mathbf{i}_k)-K)-(s_{k-1}\xi_k a(\mathbf{i}_k)-K))+$$

$$\lambda_k \sum_{(\mathbf{i}_k:\ |\mathbf{i}_k|=[r_{n-k}(d_k)]),\ s_{k-1}\xi_k a(\mathbf{i}_k)\le K)} p(\mathbf{i}_k)s_{k-1}u_k a(\mathbf{i}_k-K) = \Delta'_k+\Delta''_k. \tag{3.31}$$

Since $\lambda_k u - \xi_k = -d(1-\lambda_k)$ we conclude that

$$\Delta'_k = K(1-\lambda_k)(1-\exp(-(y+w_k)\varepsilon^{1/2})) \sum_{(\mathbf{i}_k:\ |\mathbf{i}_k|=[r_{n-k}(d_k)],\ s_{k-1}\xi_k a(\mathbf{i}_k)>K)} p(\mathbf{i}_k) \tag{3.32}$$

while

$$\Delta''_k = K\lambda_k(\exp((x-w_k)\varepsilon^{1/2})-1) \sum_{(\mathbf{i}_k:\ |\mathbf{i}_k|=[r_{n-k}(d_k)]),\ s_{k-1}\xi_k a(\mathbf{i}_k)\le K)} p(\mathbf{i}_k). \tag{3.33}$$

The inequality $s_{k-1}\xi_k a(\mathbf{i}_k) > K$ is equivalent to $\eta_k > w_k$. Taking into account (2.20)S and (2.19) we conclude that

$$\lambda_k = R^{-1}(\eta_k + y) + O(\varepsilon^{1/2}),\ 1-\lambda_k = R^{-1}(x-\eta_k) + O(\varepsilon^{1/2}).$$

Then we obtain

$$\Delta_k = K\varepsilon^{1/2}(\sigma^*_k + O(\varepsilon^{1/2}))\pi_k \tag{3.34}$$

where

$$\pi_k = \sum_{|\mathbf{i}_k|=[r_{n-k}(d_k)]} p(\mathbf{i}_k)$$

and

$$\sigma^*_k = \begin{cases} (x-\eta_k)\{r_{n-k}(d_k)\} & \text{if } \eta_k > R(\{r_{n-k}(d_k)\}-p) \\ (\eta_k+y)(1-\{r_{n-k}(d_k)\}) & \text{otherwise.} \end{cases} \tag{3.35}$$

Curiously, if $f(s) = (K-s)_+$ then the asymptotic formula (3.34) remains valid. In order to verify this one should slightly modify the calculations leading to (3.34).

Lemma 3.2 *Let $\psi(t,z)$ be defined as in (2.26). Under the conditions of Theorem 2.2*

$$\pi_k = \varepsilon^{1/2}\psi(kn^{-1}, \ln s_{k-1}) + o(\varepsilon^{1/2})$$

uniformly in $k,\ k \le (1-\delta)n$.

Proof. Consider independent variables $\zeta_k,\ k = 1, \dots, n$ such that

$$P(\zeta_k = 1) = p_k,\ P(\zeta_k = 0) = 1 - p_k.$$

It is evident that

$$\pi_k = P(\zeta_{k+1} + \cdots + \zeta_n = j)$$

where $j = [r_{n-k}(d_k)]$.

Taking into account (2.20) we obtain

$$u_k - d_k = (x + y)\varepsilon^{1/2} + \frac{x^2 - y^2}{2}\varepsilon + O(\varepsilon^{3/2})$$

while

$$\rho_k - d_k = y\varepsilon^{1/2} + (\alpha(k\varepsilon) - h(k\varepsilon) - y^2/2)\varepsilon + O(\varepsilon^{3/2}).$$

Therefore,

$$p_k = p + \frac{\alpha(k\varepsilon) - h(k\varepsilon) - xy/2}{R}\varepsilon^{1/2} + O(\varepsilon).$$

Denote

$$a_k = p_{k+1} + \cdot + p_n,\ b_k^2 = p_{k+1}(1 - p_{k+1}) + \cdot + p_n(1 - p_n).$$

Obviously,

$$a_k = (n - k)p + \varepsilon^{-1/2}R^{-1}\left(\sum_{j=k+1}^{n} \alpha(j\varepsilon)\varepsilon - \sum_{j=k+1}^{n} h(j\varepsilon)\varepsilon - \frac{(n - k)xy\varepsilon}{2}\right) + O(1)$$

and

$$b_k^2(n - k)p(1 - p) + O(\varepsilon{-}1/2).$$

If $n - k \to \infty$ then by (5.44) we obtain

$$\pi_k = \frac{1}{b_k}\varphi\left(\frac{r_{n-k}(d_k) - a_k}{b_k}\right) + o(b_k^{-1}).$$

By Lemma 3.1

$$r_{n-k}(d_k) - a_k = \varepsilon^{-1/2}R^{-1}\left(\ln(K/s_{k-1}) - \sum_{j=k+1}^{n} \alpha(j\varepsilon)\varepsilon\right) + \frac{(n - k)\varepsilon^{1/2}xy}{2R} + O(1)$$

and, therefore,

$$\frac{r_{n-k}(d_k) - a_k}{b_k} = \frac{1}{\sqrt{(n - k)\varepsilon xy}}\left(\ln(K/s_{k-1}) - \sum_{j=k+1}^{n} \alpha(j\varepsilon)\varepsilon\right) + \frac{\sqrt{(n - k)\varepsilon xy}}{2} + O(\varepsilon^{1/2}).$$

Since $n\varepsilon \to T$

$$\frac{r_{n-k}(d_k)-a_k}{b_k} = \frac{1}{\sqrt{T(1-kn^{-1})xy}}\left(\ln(K/s_{k-1}) - \int\limits_{kn^{-1}T}^{T} \alpha(u)\mathrm{d}u\right) + \frac{\sqrt{T(1-kn^{-1})xy}}{2} + o(1). \tag{3.36}$$

Since

$$b_k = \varepsilon^{-1/2}R^{-1}\sqrt{(n-k)\varepsilon xy}$$

it remains to recall (2.26). The lemma is proved.

From Lemma 3.2 and (3.34) it follows that

$$\Delta_k = K\varepsilon\psi(kn^{-1}, \ln s_{k-1})\sigma_k^* + o(\varepsilon) \tag{3.37}$$

uniformly in $k,\ k \le (1-\delta)n$. This is the starting point for the proof of Th. 2.2.

4 Proof of Theorem 2.2

Represent the total profit Δ_n as

$$\Delta_n = \sum_{1\le k<\delta n} \Delta_{k,n} + \sum_{\delta n\le k\le(1-\delta)n} \Delta_{k,n} + \sum_{(1-\delta)n\le k\le n} \Delta_{k,n} = \Delta_n' + \Delta_n'' + \Delta_n''' \tag{4.38}$$

and estimate the expectations $\mathrm{E}\Delta_n',\ \mathrm{E}\Delta_n''$ and $\mathrm{E}\Delta_n'''$ one after another.

According to (3.37) we have

$$\mathrm{E}\Delta_n'' = K\varepsilon \sum_{\delta n\le k\le(1-\delta)n} \mathrm{E}\psi(kn^{-1}, \ln s_{k-1,n})\sigma_k^* + o(1)$$

or in view of (2.19)

$$\mathrm{E}\Delta_n'' = K\varepsilon\mathrm{E}\psi(kn^{-1}, \varepsilon^{1/2}(\eta_1+\cdots+\eta_{k-1}) + \sum_{j=0}^{k-1} h(j\varepsilon)\varepsilon + \ln s_0)\sigma_k^*.$$

Consider

$$A(u,v) = (x-v)u\chi(u,v) + (v+y)(1-u)(1-\chi(u,v)),\ (u,v)\in[0,1]\times[-y,x], \tag{4.39}$$

where

$$\chi(u,v) = \begin{cases} 1 & \text{if} \quad R(u-p) < v \le x,\ 0\le u\le 1 \\ 0 & \text{if} \quad -y < v \le R(u-p),\ 0\le u\le 1 \end{cases}$$

In view of (3.35) we have

$$\sigma_k^* = A(\{r_{n-k}(d_k)\}, \eta_k).$$

It is evident that $\chi(u,v)$ admits a monotone $\varepsilon-$approximation by means of $\chi_+(u,v)$ and $\chi_-(u,v)$ where

$$\chi_+(u,v) = \begin{cases} \frac{v-R(u-p)}{\varepsilon'}+1 & \text{if } R(u-p)-\varepsilon' \le v \le R(u-p),\ 0 \le u \le 1 \\ 0 & \text{if } -y \le v \le R(u-p)-\varepsilon',\ 0 \le u \le 1 \\ 1 & \text{if } R(u-p) \le v \le x,\ 0 \le u \le 1 \end{cases}$$

and

$$\chi_-(u,v) = \begin{cases} \frac{v-R(u-p)}{\varepsilon'} & \text{if } R(u-p) \le v \le R(u-p)+\varepsilon',\ 0 \le u \le 1 \\ 0 & \text{if } -y \le v \le R(u-p),\ 0 \le u \le 1 \\ 1 & \text{if } R(u-p)+\varepsilon' \le v \le x,\ 0 \le u \le 1. \end{cases}$$

Obviously, $\chi_\pm(u,v)$ are continuous in $[0,1]\times[-y,x]$ and

$$\chi_-(u,v) \le \chi(u,v) \le \chi_+(u,v).$$

Furthermore,

$$0 \le \int\limits_{[0,1]\times[-y,x]} (\chi_+(u,v)-\chi_-(u,v))\mathrm{d}u\mathrm{d}F_k(v) \le \int\limits_{U_{\varepsilon'}} \mathrm{d}u\mathrm{d}F_k(v) \le (2\varepsilon'/R) \tag{4.40}$$

where

$$U_{\varepsilon'} = ((u,v):\ u \in (0,1),\ -y < v < x,\ |v-R(u-p)| \le \varepsilon').$$

Therefore,

$$\mathrm{E}\psi(kn^{-1},\varepsilon^{1/2}(\eta_1+\cdots+\eta_{k-1}) + \sum_{j=0}^{k-1} h(j\varepsilon)\varepsilon + \ln s_0)A_-(\{r_{n-k}(d_k)\},\eta_k) \le$$

$$\mathrm{E}\psi(kn^{-1},\varepsilon^{1/2}(\eta_1+\cdots+\eta_{k-1}) + \sum_{j=0}^{k-1} h(j\varepsilon)\varepsilon + \ln s_0)\sigma_k^* =$$

$$\mathrm{E}\psi(kn^{-1},\varepsilon^{1/2}(\eta_1+\cdots+\eta_{k-1}) + \sum_{j=0}^{k-1} h(j\varepsilon)\varepsilon + \ln s_0)A(\{r_{n-k}(d_k)\},\eta_k) \le$$

$$\mathrm{E}\psi(kn^{-1},\varepsilon^{1/2}(\eta_1+\cdots+\eta_{k-1}) + \sum_{j=0}^{k-1} h(j\varepsilon)\varepsilon + \ln s_0)A_+(\{r_{n-k}(d_k)\},\eta_k)$$

where

$$A_\pm(u,v) = (x-y)u\chi_\pm(u,v) + (y+x)(1-u)(1-\chi_\mp(u,v)).$$

Obviously, the family $\psi(t,z)$, $\delta \le t \le 1-\delta$, is contained in the class $\mathcal{G}$ defined in Section 6. So, we may apply Corollary 6.4.

By the corollary

$$\mathrm{E}\psi(kn^{-1}, \varepsilon^{1/2}(\eta_1 + \cdots + \eta_{k-1}) + \sum_{j=0}^{k-1} h(j\varepsilon)\varepsilon + \ln s_0) A_{\pm}(\{r_{n-k}(d_k)\}, \eta_k) =$$

$$\mathrm{E}\psi(kn^{-1}, \nu\sqrt{B(k\varepsilon, k\varepsilon)} + \int_0^{k\varepsilon} h(u)\mathrm{d}u + \ln s_0) \int_{[0,1]\times[-y,x]} A_{\pm}(u,v)\mathrm{d}u\mathrm{d}F_k(v) + o(1)$$

uniformly in k, $\delta \le kn^{-1} \le 1-\delta$. Here ν has the standard $(0,1)-$normal distribution and F_k is the distribution function of η_k.

In view of (4.40)

$$\int_{[0,1]\times[-y,x]} A_{\pm}(u,v)\mathrm{d}u\mathrm{d}F_k(v) = \int_{[0,1]\times[-y,x]} A(u,v)\mathrm{d}u\mathrm{d}F_k(v) + 2\theta\varepsilon'.$$

It is easily verified that

$$\int_{[0,1]\times[-y,x]} A(u,v)\mathrm{d}u\mathrm{d}F_k(v) = \frac{1}{2(x+y)}\left(xy - \mathrm{Var}\ \eta_k\right).$$

Since ε' is arbitrary we obtain

$$\mathrm{E}\psi\left(kn^{-1}, \nu\sqrt{B(k\varepsilon, k\varepsilon)} + \int_0^{k\varepsilon} h(u)\mathrm{d}u + \ln s_0\right)\sigma_k^* =$$

$$\tfrac{1}{2(x+y)}(xy - \mathrm{Var}\ \eta_k)\mathrm{E}\psi\left(kn^{-1}, \nu\sqrt{B(k\varepsilon, k\varepsilon)} + \int_0^{k\varepsilon} h(u)\mathrm{d}u + \ln s_0\right) + o(1)$$

uniformly in k, $\delta \le kn^{-1} \le 1-\delta$.
Thus,

$$\mathrm{E}\Delta_n'' = \frac{K\varepsilon}{2R} \sum_{\delta n \le k \le (1-\delta)n} (xy - \sigma^2(k\varepsilon))\mathrm{E}\psi\left(kn^{-1}, \nu\sqrt{B(k\varepsilon, k\varepsilon)} + \int_0^{k\varepsilon} h(u)\mathrm{d}u + \ln s_0\right) + o(1)$$

or

$$\mathrm{E}\Delta_n'' = \frac{KT}{2R} \int_\delta^{1-\delta} (xy - \sigma^2(tT))\mathrm{E}\psi\left(t, \nu\sqrt{B(tT, tT)} + \int_0^{tT} h(u)\mathrm{d}u + \ln s_0\right).$$

After simple calculations

$$\mathrm{E}\psi\left(t,\nu\sqrt{B(tT,tT)}+\int\limits_0^{tT}h(u)\mathrm{d}u+\ln s_0\right)=$$

$$\frac{R}{\sqrt{B(tT,Tt)+xyT(1-t)}}\varphi\left(\frac{\ln(K/s_0)-\int\limits_0^{tT}h(u)\mathrm{d}u-\int\limits_{tT}^{T}\alpha(u)\mathrm{d}u+\frac{1}{2}T(1-t)xy}{\sqrt{B(tT,tT)+xyT(1-t)}}\right)=I(tT)$$

where $I(t)$ is as in (2.27). Therefore,

$$\mathrm{E}\Delta_n''=\frac{KT}{2}\int\limits_{\delta}^{1-\delta}(xy-\sigma^2(tT))I(tT)\mathrm{d}t+o(1)=\frac{K}{2}\int\limits_{\delta T}^{(1-\delta)T}(xy-\sigma^2(t))I(t)\mathrm{d}t+o(1). \quad (4.41)$$

Now, we are going to estimate $\mathrm{E}\Delta_n'''$.

For the extreme "local" profit $\Delta_{n,\varepsilon}$ we obtain

$$\Delta_{n,\varepsilon}=\delta_{n,\varepsilon}=(s_{n-1,\varepsilon}d_{n,\varepsilon}-K)_+\frac{u_{n,\varepsilon}-\xi_{n,\varepsilon}}{u_{n,\varepsilon}-d_{n,\varepsilon}}+(s_{n-1,\varepsilon}u_{n,\varepsilon}-K)_+\frac{\xi_{n,\varepsilon}-d_{n,\varepsilon}}{u_{n,\varepsilon}-d_{n,\varepsilon}}-(s_{n-1,\varepsilon}\xi_{n,\varepsilon}-K)_+$$

whence

$$\Delta_{n,\varepsilon}=\begin{cases}0 & \text{if} \quad s_{n-1,\varepsilon}u_{n,\varepsilon}\le K \text{ or } s_{n-1,\varepsilon}d_{n,\varepsilon}>K\\ \theta(s_{n-1,\varepsilon}u_{n,\varepsilon}-K) & \text{if} \quad K/u_{n,\varepsilon}<s_{n-1,\varepsilon}\le K/d_{n,\varepsilon}.\end{cases}$$

Therefore,

$$\Delta_{n,\varepsilon}\le K(u_{n,\varepsilon}/d_{n,\varepsilon}-1)\le c\varepsilon^{1/2}.$$

For $m=n-k\ge 1$ in view of (3.31) – (3.33)

$$\Delta_{n-m,\varepsilon}\le c\sum_{(\mathbf{i}_{n-m}:\ |\mathbf{i}_{n-m}|=[r_m(d_{n-m})])}p(\mathbf{i}_{n-m})((s_{n-m-1,\varepsilon}u_{n-m,\varepsilon}a(\mathbf{i}_{n-m})-K)$$

or

$$\Delta_{n-m,\varepsilon}\le c\sum_{(\mathbf{i}_{n-m}:\ |\mathbf{i}_{n-m}|=[r_m(d_{n-m})])}p(\mathbf{i}_{n-m})(\exp((x-w_{n-m})\varepsilon^{1/2})-1).$$

Taking into account (2.19) and (2.23) we obtain

$$\Delta_{n-m,\varepsilon}\le cm^{-1/2}\varepsilon^{1/2}.$$

Thus, for all sufficiently small ε

$$\Delta_n'''\le c\varepsilon^{1/2}\sum_{0\le m\le\delta n}m^{-1/2}\le c\delta^{1/2}. \quad (4.42)$$

For $1 \le k \le \delta n$ we have

$$\Delta_{k,\varepsilon} \le \pi_k(s_{k-1,\varepsilon}u_{k,\varepsilon}a(\mathbf{i}_k) - K) = K\pi_k(\exp((x - w_{n-m})\varepsilon^{1/2}) - 1) \le cn^{-1/2}\varepsilon^{1/2}.$$

Thus,

$$\Delta'_n \le c\delta. \tag{4.43}$$

Since δ is arbitrary in view of (4.38), (4.41), (4.42) and (4.43) the theorem follows.

5 The limit value of the upper rational price

Let $\xi_1, \xi_2, \dots, \xi_n$ be independent random variables such that

$$\mathrm{P}(\xi_k = 1) = p_k,\ \mathrm{P}(\xi_k = 0) = 1 - p_k,\ k = 1, 2, \dots, n.$$

Let $P^{(n)}$ denote the measure on $\{0,1\}^n$, generated by $\xi_1, \dots, \xi_n$. Consider the sum

$$\zeta_n = \xi_1 + \cdots + \xi_n.$$

It is easily verified that (see e.g. Ch. 7 in Petrov (1975))

$$\sup_{\delta \le p_{(1)},\ p_{(n)} \le 1-\delta} \sup_k |P^{(n)}(\zeta_n = k) - \frac{1}{b_n\sqrt{n}}\varphi\left(\frac{k - a_n}{b_n\sqrt{n}}\right)| = O(1/n), \tag{5.44}$$

where $\varphi(x)$ is the density function of the standard normal law,

$$p_{(1)} = \min_{1\le k\le n} p_j,\ p_{(n)} = \max_{1\le j\le n} p_j$$

and

$$a_n = \mathrm{E}^{(n)}\zeta_n = \sum_{k=1}^n p_k,\ b_n^2 = \mathrm{Var}^{(n)}\zeta_n = \sum_{k=1}^n p_k(1 - p_k).$$

Here $\mathrm{E}^{(n)}$ and $\mathrm{Var}^{(n)}$ correspond to $P^{(n)}$.

Consider the moment generating function

$$f_k(t) = \mathrm{E}e^{t(\xi_k - p_k)} = p_k e^{t(1-p_k)} + (1 - p_k)e^{-tp_k}.$$

Let x_0 be the root of the equation $xe^x = 3$. Then for $|x| \le x_0$ we obtain

$$e^x = 1 + x + \theta x^2$$

where $\theta \in [0, 1]$. Then for

$$0 \le t \le T_0 = x_0 \min\left(\frac{1}{p_{(n)}}, \frac{1}{1 - p_{(1)}}\right)$$

we obtain

$$f_k(t) \le 1 + p_k(1-p_k)\theta_k t^2,\ \theta_k \in [0,1],\ k = 1,\dots,n.$$

Thus, for $0 \le t \le T$ we have

$$f_k(t) \le 1 + \frac{t^2}{4} \le \exp(t^2/4).$$

Applying Th. III.16 in Petrov (1975) yields

$$\mathrm{P}^{(n)}(\zeta_n - a_n \ge u) \le \begin{cases} \exp(-n^{-1}u^2) & \text{if } 0 \le u \le (T_0/2)n \\ \exp(-T_0 u/2) & \text{if } u \ge (T_0/2)n. \end{cases} \tag{5.45}$$

Similarly,

$$\mathrm{P}^{(n)}(\zeta_n - a_n \le -u) \le \begin{cases} \exp(-n^{-1}u^2) & \text{if } 0 \le u \le (T_0/2)n \\ \exp(-T_0 u/2) & \text{if } u \ge (T_0/2)n. \end{cases} \tag{5.46}$$

Lemma 5.1 *Let ζ_n is defined as above. Let a non-negative function $g(z)$ satisfy the inequality $g(z) \le e^{c|z|}$ for some $c > 0$. If $p_k \in [\delta, 1-\delta],\ k = 1,\dots,n,$ then as $n \to \infty$*

$$\mathrm{E}^{(n)} g\left(\frac{\zeta_n - a_n}{b_n}\right) \to \int g(z)\varphi(z)\mathrm{d}z.$$

Proof. Denote

$$F_n(z) = \mathrm{P}(\zeta_n - a_n < z b_n).$$

From the Central Limit Theorem it follows that for any fixed $Z > 0$

$$\int\limits_{|z|\le Z} g(z)\mathrm{d}F_n(z) \to \int\limits_{|z|\le Z} g(z)\varphi(z)\mathrm{d}z.$$

Integrating by parts yields

$$I_+ = \int\limits_{z>Z} g(z)\mathrm{d}F_n(z) = (1 - F_n(Z))e^{cZ} + \int\limits_{z>Z} g(z)(1 - F_n(z))\mathrm{d}z.$$

Note that $\zeta_n \le n$. Utilizing (5.45) we obtain

$$I_+ \le \exp(cZ - n^{-1}b_n^2 Z^2) + \int\limits_{z>Z} \exp\left(cz - z b_n \min\left(n^{-1}b_n z, \frac{T_0}{2}\right)\right)\mathrm{d}z.$$

It is easily seen that for all sufficiently large n we have $I_+ \le \omega(Z)$ where $\lim\limits_{Z\to\infty} \omega(Z) = 0$. Similarly, making use of (5.46) we obtain

$$I_- = \int\limits_{z<-Z} g(z)\mathrm{d}F_n(z) \le \omega(Z).$$

These estimates implies the required convergence. The lemma is proven.

Let $p_k = p_{k,\varepsilon},\ k = 1, 2, \dots, n$, where $p_{k,\varepsilon}$ are defined as in Section 3. Then (1.16) is rewritten as

$$\bar{x}_0 = \rho_1^{-1}\rho_2^{-1}\cdots\rho_n^{-1}\mathrm{E}^{(n)} f(s_0)a(\zeta_n).$$

As in the proof of Lemma 3.2 we obtain

$$a_n = np + \frac{\varepsilon^{-1/2}}{x+y}\sum_{j=1}^{n}(\alpha(j\varepsilon) - h(j\varepsilon) - xy/2)\varepsilon + O(1)$$

and

$$b_n^2 = np(1-p)(1+o(1)) = nR^{-2}xy(1+o(1)).$$

Taking into account (2.19) we obtain

$$a(\zeta_n) = \exp\left(\frac{\zeta_n - a_n}{b_n}\sqrt{n\varepsilon xy} + \sum_{j=1}^{n}\alpha(j\varepsilon)\varepsilon - n\varepsilon\cdot\frac{xy}{2} + O(\varepsilon^{1/2})\right).$$

Since

$$f(s_0 a(\zeta_n)) \le c\exp\left(c\left|\frac{\zeta_n' - a_n}{b_n}\right|\right)$$

we may apply Lemma 5.1. Applying the lemma yields

$$\bar{x}_0 \to \exp\left(-\int\limits_0^T \alpha(t)\mathrm{d}t\right)\int \varphi(v) f(s_0\exp(v\sqrt{Txy} + \int\limits_0^T \alpha(t)\mathrm{d}t - Txy/2))\mathrm{d}v.$$

If $f(s) = (s-K)_+$ then

$\bar{x}_0 \to c(xy) =$

$$s_0\Phi\left(\frac{\ln(s_0/K) + \int\limits_0^T \alpha(t)\mathrm{d}t + Txy/2}{\sqrt{Txy}}\right) - K\exp\left(-\int\limits_0^T \alpha(t)\mathrm{d}t\right)\Phi\left(\frac{\ln(s_0/K) + \int\limits_0^T \alpha(t)\mathrm{d}t - Txy/2}{\sqrt{Txy}}\right). \tag{5.47}$$

But if $f(s) = (K - s)_+$ then

$$\bar{x}_0 \to c_1(xy) =$$

$$K \exp\left(-\int\limits_0^T \alpha(t)\mathrm{d}t\right) \Phi\left(\frac{\ln(K/s_0) - \int\limits_0^T \alpha(t)\mathrm{d}t + Txy/2}{\sqrt{Txy}}\right) - s_0 \Phi\left(\frac{\ln(s_0/K) - \int\limits_0^T \alpha(t)\mathrm{d}t - Txy/2}{\sqrt{Txy}}\right). \tag{5.48}$$

6 The local limit theorem and its applications

First, consider the case where (2.24) is ensured by atoms.

Proposition 6.1 *Let $\eta_{k,\varepsilon}$, $k = 1, 2, \ldots, n$, take at least three values $-y, 0, x$ so that*

$$\mathrm{P}(\eta_{k,\varepsilon} = 0) = p_{k0},\ \mathrm{P}(\eta_{k,\varepsilon} = -y) = p_{k1},\ \mathrm{P}(\eta_{k,\varepsilon} = x) = p_{k2}$$

where

$$\pi_k = p_{k0} + p_{k1} + p_{k2} \le 1,\ \min(p_{k0}, p_{k1}, p_{k2}) \ge p_0 > 0.$$

If the ratio $\frac{y}{x}$ is an irrational number then condition (2.24) is fulfilled.

Proof. Denote by $\psi_k(t)$ the characteristic function of $\eta_{k,\varepsilon}$. Represent it as follows

$$\psi_k(t) = p_{k0} + p_{k1} e^{-\imath ty} + p_{k2} e^{\imath tx} + \psi_{k1}(t)$$

where, obviously, $|\psi_{k1}(t)| \le 1 - \pi_k$. Denote

$$\psi_{k0}(t) = \pi_{k0} + \pi_{k1} e^{-\imath ty} + \pi_{k2} e^{\imath tx}.$$

where

$$\pi_{k0} = \frac{p_{k0}}{\pi_k},\ \pi_{k1} = \frac{p_{k1}}{\pi_k},\ \pi_{k2} = \frac{p_{k2}}{\pi_k}.$$

Obviously,

$$\min(\pi_{k0}, \pi_{k1}, \pi_{k2}) > p_0$$

and

$$|\psi_k(t)| \le \pi_k |\psi_{k0}(t)| + 1 - \pi_k.$$

It is easily seen that

$$1 - |\psi_{k0}(t)|^2 = 2\pi_{k0}\pi_{k1}(1 - \cos(yt)) + 2\pi_{k0}\pi_{k2}(1 - \cos(xt)) +$$

$$2\pi_{k1}\pi_{k2}(1 - \cos((x+y)t)) \ge 4p_0^2 \left(1 - \tfrac{\cos(yt) + \cos(xt)}{2}\right).$$

Since $\frac{y}{x}$ is irrational it follows that for all sufficiently small $\delta > 0$ and for all $\delta < \Delta < \infty$

$$1 < \sup_{\delta \le |t| \le \Delta} (\cos(yt) + \cos(xt)) = c(\delta, \Delta) < 2. \tag{6.49}$$

Assume that there exist $t \in I\!R^1$ and $m, l \in \mathcal{Z}^1_+$ such that $t = \frac{2\pi}{y} m = \frac{2\pi}{x} l$. But this would imply that $\frac{y}{x}$ is rational. So, (6.49) holds. Further,

$$\sup_{\delta \le |t| \le \Delta} |\psi_k(t)| \le \pi_k \left(1 - 4p_0^2 \left(1 - \tfrac{c(\delta,\Delta)}{2}\right)\right) + 1 - \pi_k =$$

$$1 - 4\pi_k p_0^2 \left(1 - \tfrac{c(\delta,\Delta)}{2}\right) \le 1 - 12p_0^3 \left(1 - \tfrac{c(\delta,\Delta)}{2}\right) = \rho(\delta, \Delta) < 1.$$

It is worth noting that

$$0 < 12p_0^3 \left(1 - \frac{c(\delta,\Delta)}{2}\right) < \frac{6}{27}.$$

Thus, the proposition is proven.

In the following proposition the condition (2.24) is secured by the absolutely continuous components.

Proposition 6.2 *Let $F_{k,\varepsilon}(u)$ contain an absolutely continuous component $R_k(u)$ such that*

$$\inf_{u \in [a_k, b_k]} R_k'(u) \ge p_0 > 0, \;\; b_k - a_k \ge h > 0.$$

Then (2.24) holds.

Proof. Consider the density

$$p_k(u) = \frac{R_k'(u)}{R_k(b_k) - R_k(a_k)} \mathbf{1}_{[a_k, b_k]}(u).$$

It is evident that that

$$p_k(u) \ge p_0.$$

Consider the symmetrized density

$$\bar{p}_k(v) = \int_{a_k}^{b_k} p_k(v+u) p_k(u) \mathrm{d}u, \;\; |v| \le b_k - a_k = h_k.$$

Recall that $\bar{p}_k(v) \ge p_0$ for $|v| \le h_k$. Denote

$$\psi_{k0}(t) = \int e^{\imath tu} p_k(u) \mathrm{d}u.$$

It is evident that

$$1-|\psi_{k0}(t)|^2 = 2\int_0^{h_k}(1-\cos(tu))\bar{p}_k(u)\mathrm{d}u \geq 2p_0\int_0^{h}(1-\cos(tu))\mathrm{d}u$$

and, therefore,

$$1-|\psi_{k0}(t)|^2 \geq 2p_0 h\left(1-\frac{\sin(\delta h)}{\delta h}\right) = c(\delta) > 0.$$

It implies that that

$$\sup_{1\leq k\leq n}\sup_{|t|\geq\delta}|\psi_{k0}(t)| \leq c(\delta) < 1.$$

As before, let $\psi_k(t)$ be the characteristic function of $\eta_{k,\varepsilon}$. It is evident that

$$|\psi_k(t)| \leq \pi_k|\psi_{k0}(t)| + 1 - \pi_k$$

where

$$\pi_k \geq \int_{a_k}^{b_k} p_k(u)\mathrm{d}u \geq p_0(b_k-a_k) \geq p_0 h.$$

So, (2.24) holds and, therefore, the proposition follows.

Under the conditions of the just proven propositions the relation (2.24) takes place, i.e. the local limit theorem in the form (2.23) holds. In order to verify it one should slightly modify the argument used, say, in Nagaev (1973).

Consider the sequence of the measures

$$Q_{k,\varepsilon}(A) = b_{k,\varepsilon}\sqrt{2\pi}\mathrm{P}(\zeta_{k,\varepsilon}\in A).$$

The statement (2.23) implies that $Q_{k,\varepsilon}$ weakly converge, as $k\to\infty$ uniformly in $\varepsilon\in[0,\varepsilon_0]$ to the Lebesgue measure that is for any continuous compactly supported function $g(u)$

$$\sup_{\varepsilon\in[0,\varepsilon_0]}|\int g(u)Q_{k,\varepsilon}(\mathrm{d}u) - \int g(u)\mathrm{d}u| \to 0. \tag{6.50}$$

Let $\mathcal{G}$ be the class of equicontinuous functions defined on $(-\infty,\infty)$ such that

$$\lim_{t\to\infty}\sup_{g\in\mathcal{G}}\int_{|u|>t}|g(u)|\mathrm{d}u = 0.$$

It is easily seen that (6.50) holds uniformly in $g\in\mathcal{G}$. More precisely,

$$\lim_{k\to\infty}\sup_{\varepsilon\in[0,\varepsilon_0]}\sup_{g\in\mathcal{G}}|\int g(u)Q_{k,\varepsilon}(\mathrm{d}u) - \int g(u)\mathrm{d}u| = 0. \tag{6.51}$$

Consider the family of the random variables $\tau_{k,\varepsilon}(a) = \{\lambda\zeta_{k,\varepsilon}+a\}$ where $a\in I\!R^1$ and $\lambda\neq 1$ is constant. It is worth comparing the following statement with the basic result in S. V. Nagaev and Mukhin (1966).

Lemma 6.3 *If (2.23) takes place then for any fixed* u', u'', $0 < u' < u'' < 1$ *and* z', z'', $-\infty < z' < z'' < \infty$ *as* $k \to \infty$

$$\sup_a |\mathrm{P}(u' \le \tau_{k,\varepsilon}(a) < u'',\ z' \le b_{k,\varepsilon}^{-1}\zeta_{k,\varepsilon} < z'') - (u'' - u')\,(\Phi(z'') - \Phi(z'))\,| = o(1).$$

Proof. Let $k = k(a) = [a],\ \theta = \theta(a) = \{a\}$. Suppose that $\lambda > 0$. It is easily seen that

$$P_{k,\varepsilon} = \mathrm{P}(u' \le \tau_{k,\varepsilon}(a) < u'',\ z' \le b_{k,\varepsilon}^{-1}\zeta_{k,\varepsilon} < z'') =$$

$$\sum_j \mathrm{P}(j + u' \le \lambda\zeta_{k,\varepsilon} + a < j + u'',\ z' b_{k,\varepsilon} \le \zeta_{k,\varepsilon} < z'' b_{k,\varepsilon}) =$$

$$\sum_{j' \le j \le j''} \mathrm{P}(\tfrac{j+u'-\theta}{\lambda} \le \zeta_{k,\varepsilon} < \tfrac{j+u''-\theta}{\lambda}) + \mathrm{P}(\tfrac{j''+u''-\theta}{\lambda} \le \zeta_{k,\varepsilon} < z'' b_{k,\varepsilon}) +$$

$$\mathrm{P}(z' b_{k,\varepsilon} \le \zeta_{k,\varepsilon} < \tfrac{j'+u'-\theta}{\lambda})$$

where

$$j' = \min(j :\ \frac{j + u' - \theta}{\lambda} \ge z' b_{k,\varepsilon}),\ j'' = \max(j :\ \frac{j + u'' - \theta}{\lambda} \le z'' b_{k,\varepsilon}).$$

According to (2.23)

$$P_{k,\varepsilon} = \frac{u'' - u'}{\lambda b_{k,\varepsilon}} \sum_{j' \le j \le j''} \varphi\left(\frac{j}{\lambda b_{k,\varepsilon}}\right) + o(1).$$

It remains to recall that

$$j' = z'\lambda b_{k,\varepsilon}(1 + o(1)),\ j'' = z''\lambda b_{k,\varepsilon}(1 + o(1)).$$

Lemma 6.3 has the following evident corollary (cf. Corollary 7.1 in A. Nagaev and S. Nagaev (2003)).

Corollary 6.4 *Let* $\chi(u, v)$ *be a bounded continuous function defined on* $[0, 1] \times I\!R^1$. *Under the conditions of Theorem 2.2*

$$\lim_{n\to\infty} \sup_{g\in\mathcal{G}} \sup_a |\mathrm{E}g(b_{k,\varepsilon}^{-1}\zeta_{k,\varepsilon})\chi(\{\lambda\zeta_{k,\varepsilon} + a\}, \eta_k) - \int g(z)\varphi(z)\mathrm{d}z \int_{[0,1]\times I\!R^1} \chi(u, v)\mathrm{d}u\mathrm{d}F_k(v)| = 0$$

uniformly in k, $k \ge \delta n$, $\delta > 0$. *Here,* F_k *is the distribution function of* $\eta_{k,\varepsilon}$.

References

Nagaev, A. V. (1973) Remarks on multidimensional local theorems. *Math. Notes* **14**, 4, 559–563 (in Russian).

Nagaev, A. V. and Nagaev, S. A. (2003) Asymptotics of riskless profit under selling of discrete time call options. *Applicationnes Mathematicae* **30**, 2, 173 – 191.

Nagaev, S. A. (2003) A diffusion approximation for the riskless profit under selling of discrete time call options. *Preprint.* Institute of Advanced Studies, Vienna.

Nagaev S. V. and Mukhin A. B (1966) A case of convergence to the uniform distribution on a segment. In: Limit Theorems and Statistical Inference. Fan, Tashkent, 113–117 (in Russian).

Petrov, V. V. (1975) *Sums of Independent Variables,* Springer, Berlin etc.

Shiryaev, A. N. (1999) *Essentials of Stochastic Finance.* World Scientific Publ., Singapore.

Tessitore, G. and Zabczyk, J. (1996) Pricing options for multinomial models. Bull. Polish Acad. Sci. **44**, 3, 363–380.

Zabczyk, J. (1996) Pricing options by dynamic programming. *In: Stochastic Processes and Related Topics, Ed. H.J. Engelbert, H. Föllmer, J. Zabczyk, Stochastic Monographs Vol. 10, Gordon and Breach Publ.*, 55–72.

Motoczyński, M. and Stettner, L. (1998) On option pricing in the multinomial Cox-Ross-Rubinstein model. *Applications Math.* **25**, 1, 55–72.

Authors: Alexander V. Nagaev, Sergei A. Nagaev, Robert M. Kunst

Title: A Diffusion Approximation for the Riskless Profit Under Selling of Discrete Time Call Options. – Non-identically Distributed Jumps

Reihe Ökonomie / Economics Series 164

Editor: Robert M. Kunst (Econometrics)
Associate Editors: Walter Fisher (Macroeconomics), Klaus Ritzberger (Microeconomics)

ISSN: 1605-7996

Stumpergasse 56, A-1060 Vienna • ☎ +43 1 59991-0 • Fax +43 1 59991-555 • http://www.ihs.ac.at

ISSN: 1605-7996