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# The BDS Test as a Test for the Adequacy of a GARCH(1,1) Specification: A Monte Carlo Study

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## Impressum

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Founded in 1963 by two prominent Austrians living in exile – the sociologist Paul F. Lazarsfeld and the economist Oskar Morgenstern – with the financial support from the Ford Foundation, the Austrian Federal Ministry of Education and the City of Vienna, the Institute for Advanced Studies (IHS) is the first institution for postgraduate education and research in economics and the social sciences in Austria. The **Economics Series** presents research done at the Department of Economics and Finance and aims to share “work in progress” in a timely way before formal publication. As usual, authors bear full responsibility for the content of their contributions.

Das Institut für Höhere Studien (IHS) wurde im Jahr 1963 von zwei prominenten Exilösterreichern – dem Soziologen Paul F. Lazarsfeld und dem Ökonomen Oskar Morgenstern – mit Hilfe der Ford-Stiftung, des Österreichischen Bundesministeriums für Unterricht und der Stadt Wien gegründet und ist somit die erste nachuniversitäre Lehr- und Forschungsstätte für die Sozial- und Wirtschaftswissenschaften in Österreich. Die **Reihe Ökonomie** bietet Einblick in die Forschungsarbeit der Abteilung für Ökonomie und Finanzwirtschaft und verfolgt das Ziel, abteilungsinterne Diskussionsbeiträge einer breiteren fachinternen Öffentlichkeit zugänglich zu machen. Die inhaltliche Verantwortung für die veröffentlichten Beiträge liegt bei den Autoren und Autorinnen.

## **Abstract**

In this study, we examine the Brock, Dechert and Scheinkman (BDS) test when applied to the standardised residuals of an estimated GARCH(1,1) model as a test for the adequacy of this specification. We review the conditions derived by De Lima (1996, *Econometric Reviews*, 15, 237-259) for the nuisance-parameter free property to hold, and address the issue of their necessity, using the GARCH(1,1) model. By means of Monte Carlo simulations, we show that, provided that the unconditional mean exists, the BDS test statistic still approximates the standard null distribution even when the majority of the conditions are violated. Further, the test performs reasonably well, as its empirical size is rather close to the nominal one. As a by-product of this study, we also examine the related issue of consistency of the QML estimators of the conditional variance parameters under various parameter configurations and alternative distributional assumptions on the innovation process.

## **Keywords**

BDS Test, Nuisance-Parameter Free Property, Monte Carlo Analysis, GARCH(1,1) Model, QML estimator

## **JEL Classification**

C15, C22

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## 1 Introduction

The Brock, Dechert and Scheinkman (1987) test for nonlinearity (BDS test henceforth) is widely used as a misspecification test for parametric models capturing the dynamics of time series. For this purpose, it is applied to the estimated residuals from the model of interest. As these can still exhibit some form of dependence, even if the true innovations are *i.i.d.*, the asymptotic distribution of the test statistic may be affected by the estimation procedure. This possible distortion has led researchers to consider test statistics exhibiting the so-called nuisance-parameter free property, i.e. whose asymptotic distribution is not affected by the intermediate step of parameter estimation.

Brock, Dechert and Scheinkman (1987) and Brock, Hsieh and LeBaron (1991) were the first to derive conditions for the BDS test to be nuisance-parameter free, and to carry out Monte Carlo simulations to corroborate their theoretical results. However, they were not able to prove that the BDS statistic that uses estimated residuals is asymptotically normal. Following the work of Randles (1982), De Lima (1996) investigated further the invariance property of the BDS test, and showed that, under appropriate sufficient conditions for the series under scrutiny, the BDS test is nuisance parameter-free for linear additive models or models that can be cast into this format. This family includes the ARCH class of models, introduced by Engle (1982) and generalised by Bollerslev (1986), as long as the BDS test is applied to a modified residual series. Examining the ‘necessity’ of De Lima’s conditions is a difficult task analytically, and can be more conveniently achieved by means of suitably designed Monte Carlo simulations in the context of an appropriate model. This paper analyses the “necessity” of De Lima’s conditions for the BDS test statistic to have an asymptotic  $N(0,1)$  distribution when it is computed using estimated standardised residuals of a GARCH(1,1) model. More specifically, consider a martingale difference GARCH(1,1) process, defined as follows:

$$u_t = h_t z_t \tag{1}$$

$$z_t \sim IID(0,1) \tag{2}$$

where

$$h_t^2 = c + \mu u_{t-1}^2 + \delta h_{t-1}^2 \tag{3}$$

denotes the variance of  $u_t$  conditional on the  $\sigma$ -field,  $F_{t-1}$ , generated by all information available at time  $t - 1$ . We are interested in testing the null hypothesis

$$H_0 : \{z_t\} \text{ is an } i.i.d \text{ process}$$

by means of the BDS statistic constructed on the basis of  $\hat{z}_t = \hat{u}_t / \hat{h}_t$ , where  $\hat{h}_t$  is the Quasi Maximum Likelihood (QML) estimate of  $h_t$ . Our aim is to analyse the properties



of the BDS test when applied to  $\hat{z}_t$  for various distributional assumptions on  $z_t$ , and alternative values of the GARCH parameters  $\mu$  and  $\delta$ . Given that the GARCH(1,1) model is versatile enough to produce a range of stochastic processes, with very different moment and memory characteristics, depending on the parameter settings and the distribution of the innovations  $z_t$ , we shall be able to examine cases that violate some or all of De Lima's conditions. Indeed, the GARCH(1,1) model provides a flexible framework allowing the researcher to control for the amount of temporal dependence, the degree of time-heterogeneity and the number of unconditional moments that characterise the process by simply changing the values of the model parameters and/or the distributional assumption on the innovations  $z_t$ . The exact De Lima's conditions will be discussed in detail in the next section. However, it is already apparent from the previous discussion that the 'invariance property' of the BDS test is closely related to the question of consistency and the rate of convergence of the Quasi Maximum Likelihood (QML) estimator of the conditional variance parameters.

The paper is structured as follows. Section 2 briefly summarises the conditions that must be satisfied by the process of interest in order for the test to be nuisance-parameter free. Section 3 outlines the moment, memory and heterogeneity properties of the GARCH(1,1) model. In addition, it discusses the issue of consistently estimating its parameters, including the case where non-Gaussian innovations drive the GARCH process. Section 4 describes the Monte Carlo setup and reports the main findings. Finally, Section 5 offers a brief summary of the main results and some concluding remarks.

## 2 The Brock, Dechert and Scheinkman (BDS) Test

A detailed description of the BDS test can be found in Brock, Dechert and Scheinkman (1987) or De Lima (1996). Initially, the test was designed to be applied to raw series in order to test whether they are i.i.d. It was soon realised that it could also be applied to the estimated residuals from a model to test for omitted dynamics, i.e. to test model adequacy. Brock and Dechert (1988) and De Lima (1996) examined the conditions under which the BDS statistic is nuisance-parameter free, that is its asymptotic distribution does not change when it is applied to the estimated residuals from a model, rather than the raw series itself. The invariance property of the BDS statistic is ensured by a set of sufficient conditions that are more stringent than in the case of smooth U-statistics (a function of which the BDS test is). This is because the indicator kernel,  $I_\varepsilon(\cdot, \cdot)$ , used in the definition of the correlation integral, is not a differentiable smooth kernel. As a result, special sufficient conditions, ensuring the reversibility between the operations of differentiation and taking the limiting mean, are required in order to guarantee the invariance property of the BDS statistic. Specifically, following Randles (1982), De Lima (1996) derives five sufficient conditions (Assumptions A-D, pp. 240-241, and Assumption E, pp. 245) that ensure the 'nuisance-parameter free' property of

the BDS test for linear additive models, such as  $y_t = G(Y_{t-1}; \theta) + e_t$ , or for models that can be transformed into this format. In this paper, we focus on the effects of the failure of some of these conditions on the invariance property of the BDS test. Assumption A requires  $y_t$  to be a strong mixing process with summable mixing coefficients,  $a(k)$ , that is  $\sum_{k=1}^{\infty} a(k)^{1/2} < \infty$ . Assumption B and C impose moment conditions on the difference between the kernel  $I_\varepsilon$  evaluated at two different points of the residual function. Of these two assumptions, B is the most important one, and may be verified in a case by case framework by evaluating the supremum of the relevant random variable. Assumption C is automatically satisfied (by assumption B) for bounded kernels. Assumption D requires the parameters,  $\theta$ , of the model to be consistently estimated. The consistency of the estimator of  $\theta$  is usually based on some memory restrictions on the errors of the model, such as strong mixing with summable mixing coefficients. Of course, if the errors are *i.i.d.* these restrictions are automatically satisfied. Moreover, for the same reason (consistent estimation of  $\theta$ ) moment conditions on the error term (or sometimes on the raw series itself) should be imposed. Therefore, the nuisance-parameter free property of the BDS does require moment restrictions when the BDS test is applied to estimated residuals as opposed to raw data. Finally, Assumption E requires the distribution of the innovations to be absolutely continuous and differentiable, with a bounded density function.

### 3 Consistent Estimation and Moment and Memory Properties of the GARCH(1,1) Process

We shall investigate the necessity of conditions (A-E) in the context of the martingale-difference (MD) GARCH(1,1) model defined in the introduction. Note that no specific distributional characteristics of  $z_t$  have been assumed yet, other than the variance of the unspecified distribution is equal to one. The examination of the necessity of conditions (A-E) within the context of the MD-GARCH(1,1) process is conducted for two reasons. First, because of its empirical relevance: most studies that aim at capturing non-linear dynamics in economic or financial time series specify and estimate such a model. Second, the MD-GARCH(1,1) process is a versatile stochastic process, which, depending on the distribution of the innovations process,  $z_t$ , and the values of the coefficients  $\mu$  and  $\delta$ , is able to reproduce processes that range from  $b$ -mixing with finite unconditional moments of the fourth order to processes that are neither mixing nor possess any unconditional moments. Note that the error term in (1) is not additive. However, this does not generate any serious difficulties since, by raising both sides of (1) to the square power and taking logs, we can transform (1) into a model that contains an additive error,  $\nu_t = \ln z_t^2 = \ln u_t^2 - \ln h_t^2$ . The asymptotic distribution of the BDS test is the same whether it is applied to the estimated residuals  $\hat{\nu}_t$  or to  $\nu_t$  itself, provided that  $\hat{\nu}_t$  is a consistent estimator of  $\nu_t$ .

As already mentioned, the moment and memory characteristics of  $u_t$  depend on a) the distribution of the innovations  $z_t$  and b) the values of the parameters  $\mu$  and  $\delta$ . The following subsection summarises the main theoretical results found in the literature.

### 3.1 Moment and Memory Properties of the GARCH(1,1) Process

#### Case I: Fourth-order stationary process.

He and Terasvirta (1999) provide a necessary and sufficient condition for  $E(u_t^4) < \infty$ .<sup>1</sup> This condition requires minimum restrictions on the innovation process  $z_t$ , which is assumed to be a zero mean *i.i.d.* process with second and fourth unconditional moments denoted by  $v_2$  and  $v_4$  respectively. The condition is

$$\mu^2 v_4 + 2\mu\delta v_2 + \delta^2 < 1 \quad (4)$$

which, under the assumption that  $z_t \sim N(0, 1)$ , reduces to Bollerslev's (1986) condition

$$3\mu^2 + 2\mu\delta + \delta^2 < 1 \quad (5)$$

As for the memory properties, Davidson (2002) retains the unit variance assumption ( $v_2 = 1$ ) and demonstrates that condition (4) is necessary and nearly sufficient for  $u_t$  to be near-epoch dependent on  $z_t$  in  $L_2$ -norm ( $L_2 - NED$ ).<sup>2</sup> This is sufficient for the series to obey the central limit theorem or the invariance principle.

#### Case II: Covariance stationary process.

In this case, condition (4) is violated, but we assume that the unconditional second moment exists, which amounts to

$$\mu v_2 + \delta < 1 \quad (6)$$

Under the unit variance assumption, the necessary and sufficient condition for covariance stationarity reduces to the well-known condition

$$\mu + \delta < 1 \quad (7)$$

Davidson (2002) proves that, under (7),  $u_t$  is  $L_1$ -NED on  $z_t$ , that is the sufficient condition for the series to obey the law of large numbers holds, but not the corresponding one for the Central Limit Theorem (CLT). Nevertheless, as shown by De Lima (1996), since covariance stationarity holds, the condition for assumption B is satisfied. Moreover, Carrasco and Chen (2002) prove that (7) is sufficient for  $u_t$  to be  $b$ -mixing. Since

<sup>1</sup>See also Ling and MacAleer (2002) for the necessary and sufficient condition for the existence of higher order moments of the GARCH(r,s) model.

<sup>2</sup>Davidson shows that the fourth moment condition is necessary and nearly sufficient for the  $L_2$ -NED property, irrespectively of the distribution of the innovations process. The  $L_2$ -NED property of a GARCH(1,1) process was first proved by Hansen (1991) under the additional assumption of normality of  $z_t$ .

$b$ -mixing is stronger than  $a$ -mixing (but weaker than  $\varphi$ -mixing), we may conclude that condition (7) is sufficient to guarantee the required memory property.

**Case III: IGARCH process, with  $E(u_t) = 0$ .**

Nelson (1990) proves that a necessary and sufficient condition for  $u_t$  to be strictly stationary and ergodic is given by

$$E \left[ \ln(\delta + \mu z^2) \right] < 0 \quad (8)$$

regardless of the distribution of the *i.i.d.* innovations  $z_t$ . This condition holds even if

$$\mu v_2 + \delta = 1 \quad (9)$$

Under (9),  $E(u_t) = 0$  and  $Var(u_t) = \infty$ . In such a case, the  $L_2$ -NED measure of memory is unavailable, since the unconditional second moment of  $u_t$  does not exist. Moreover, there is no guarantee that  $u_t$  is  $a$ -mixing.

**Case IV: Mildly explosive process, with  $E(u_t) = 0$ .**

Nelson (1990) shows that there might be an area in the  $(\mu - \delta)$  plane for which  $\mu v_2 + \delta > 1$  and condition (8) holds. In such a case,  $u_t$  is mildly explosive but still strictly stationary and ergodic. If, in addition,

$$E(\delta + \mu z^2)^{1/2} < 1 \quad (10)$$

the process  $u_t$  has an unconditional mean equal to zero.

**Case V: Mildly explosive process, with  $E(u_t) = \infty$ .**

This is a case where  $\mu$  and  $\delta$  are such that (8) is satisfied but (10) fails. By comparing this case with the previous one, one can discern the importance of the finite (zero) mean property of  $u_t$  for the properties of the BDS statistic.

Let us now turn to the problem of estimating the conditional variance parameters,  $\theta = (c, \mu, \delta)$ .

### 3.2 Consistent Estimation of the GARCH(1,1) Parameters.

When  $z_t \sim N(0, 1)$ , the employment of the Gaussian likelihood function results in Maximum Likelihood (ML) estimates,  $\hat{\theta}_{ML}$ , of  $\theta = (c, \mu, \delta)$ . If the distribution of  $z_t$  is not Normal, then the Gaussian likelihood function produces the so-called Quasi Maximum Likelihood (QML) estimates of  $\theta$ . The asymptotic properties of the QML estimates of  $\theta$  have been studied in Bollerslev and Wooldridge (1992), Lumsdaine (1996) and Lee and Hansen (1994). All these studies require the validity of (8), that is they assume that  $u_t$  is strictly stationary and ergodic so that the relevant laws of large numbers apply. They differ in the moment restrictions that they impose either on  $u_t$  or on  $z_t$ . The less restrictive moment condition is that of Lee and Hansen (1994) which requires the fourth moment of  $z_t$  to be finite. More recently, Jensen and Rahbek

(2003) relax the stationarity condition (8) and prove the  $\sqrt{T}$ -consistency of the QML estimates in cases where  $u_t$  is a non-stationary process with unbounded low order moments. The only condition they impose is the existence of the fourth moment of  $z_t$  (not  $u_t$ ). Interestingly, they prove that the rate of convergence is faster in the non-stationary than in the stationary case. These results imply that the QML estimates of  $\theta$  are  $\sqrt{T}$ -consistent for all the five cases defined above provided that  $\nu_4 < \infty$ . However, as will be shown below,  $\sqrt{T}$ -consistency of the QML estimates of  $\theta$  seems to be achieved even in cases where  $\nu_4$  is infinite, as for example when  $z_t$  follows a  $t(4)$  or  $t(3)$  distribution. Therefore, as a by-product of this study, we provide evidence that the existing conditions for  $\sqrt{T}$ -consistency of the QML estimates of  $\theta$  may be relaxed.

## 4 Monte Carlo Simulations

In this section we carry out Monte Carlo simulations that aim at examining the distribution of the BDS statistic and the resulting performance of the BDS test in cases where some of the sufficient conditions for the invariance property of the BDS test fail. Since the performance of the BDS test statistic is closely linked to the quality of estimation of the GARCH parameters, we examine both issues jointly. The simulations are designed as follows. First, we choose the distribution of the innovation  $z_t$ , and generate *i.i.d.* series with mean zero and variance one. Second, we select the GARCH parameters,  $\delta$  and  $\mu$ , in such a way as to generate a series  $u_t$  with specific moment and memory properties as outlined in cases I to V. Third, we estimate a GARCH(1,1) model, take the logarithm of the squared standardised residuals and compute the BDS statistic for various values of the embedding dimension  $m$  and for alternative sample sizes,  $T$ , namely  $T = 50$  to 950 by steps of 50. We repeat this procedure 2000 times and calculate the mean, variance, skewness and kurtosis coefficients of the BDS statistic. We also compute the 5% empirical size. Further, for each parameter configuration and for each sample size, we compute the average absolute bias of the QML estimates of  $c$ ,  $\mu$  and  $\delta$ . By doing this, we are able not only to examine whether the QML estimates are consistent, but also to investigate their rate of convergence. This issue is particularly important in cases where  $\nu_4$  is infinite, that is when the sufficient condition for  $\sqrt{T}$ -consistency is violated. In such a case the QML estimates may converge at a rate slower than  $\sqrt{T}$ , or they might not converge at all. In both cases the convergence of the BDS statistic to  $N(0, 1)$  will be affected.

### 4.1 The case of Gaussian innovations

First, we assume that  $z_t$  is an *i.i.d.* Gaussian process. Figure 1 shows the regions in the  $(\mu - \delta)$  plane that correspond to the five cases defined above, when  $z_t \sim N(0, 1)$ . Regions 1 and 2 correspond to cases I and II, respectively; the boundary between regions 2 and 4 to case III; and finally regions 4 and 5 to cases IV and V respectively. We selected

values of  $(\mu, \delta)$  ranging from the upper left area of each region to the bottom right one, in order to achieve full coverage of each region.<sup>3</sup> However, we report in the Tables only the results for two alternative values of  $(\mu, \delta)$ . The first pair of values is representative of those typically reported in the relevant literature, i.e. a very small  $\mu$  and a much larger  $\delta$ , while the second pair includes a much larger value of  $\mu$ . We mention in the text any cases where other points from the same region produce different results. The reported results are those for the following pairs of values for  $(\mu, \delta)$  corresponding to the five regions mentioned above.

Model Ia:  $(\mu, \delta) = (0.1, 0.85)$  and model Ib:  $(\mu', \delta') = (0.4, 0.4)$

Model IIa:  $(\mu, \delta) = (0.1, 0.895)$  and model IIb:  $(\mu', \delta') = (0.4, 0.5)$

Model IIIa:  $(\mu, \delta) = (0.1, 0.9)$  and model IIIb:  $(\mu', \delta') = (0.4, 0.6)$

Model IVa:  $(\mu, \delta) = (0.1, 0.904)$  and model IVb:  $(\mu', \delta') = (0.4, 0.63)$

Model Va:  $(\mu, \delta) = (0.1, 0.907)$  and model Vb:  $(\mu', \delta') = (0.4, 0.67)$

Before presenting the results on the performance of the BDS test statistic, let us first discuss the ML estimates of the GARCH parameters for each of the five models defined above. Note that, since  $z_t$  is Gaussian, there is no reason to believe that consistency is threatened in any of the cases. Figures 2.I refers to Models Ia and Ib and reports the mean absolute bias of the ML estimates of  $\mu$  and  $\delta$  for sample sizes ranging from 50 to 950 by steps of 50.<sup>4</sup> More specifically, the upper panel of figure 2.I corresponds to model Ia, that is  $(\mu, \delta) = (0.1, 0.85)$ , while the lower panel of figure 2.I corresponds to model Ib, that is  $(\mu', \delta') = (0.4, 0.4)$ . Similarly, Figures 2.II, 2.III, 2.IV and 2.V refer to Model II, III, IV and V respectively. In each case we also report simulated biases that would have been produced, had the rate of convergence of the ML estimates been exactly  $\sqrt{T}$ . We observe the following:

i) The ML estimators of the GARCH parameters seem consistent in all cases under examination, since the bias decreases with the sample size. Moreover, the mean absolute bias of the estimates decreases in general as we move from region 1 (i.e. model I) to region 5 (i.e. model V). The only exception is found when  $T = 50$ , in which case the bias of  $\delta$  increases.

ii) The rate of convergence of the ML estimators increases as we move from region 1 to region 5. However, the rate of convergence of the ML estimates of  $\mu$  and  $\delta$  differs substantially. To be more specific, the former is always very close to  $\sqrt{T}$  and only minor increases are observed as we move from region 1 to region 5. By contrast, the latter increases substantially. Moreover, when  $\mu = 0.1$ , the ML estimator of  $\delta$  becomes  $T$ -consistent when the fourth-order stationarity of  $u_t$  does not hold. However, the rate of convergence of the ML estimators depends on the pair of values  $(\mu, \delta)$ , and it generally decreases for increasing  $\mu$  and decreasing  $\delta$ . This is more obvious for the

<sup>3</sup>Throughout the simulations we set the constant term  $c$  equal to unity.

<sup>4</sup>The mean absolute bias of  $\mu$  is displayed on the left, while the mean absolute bias of  $\delta$  is displayed on the right. The results for the constant term are not reported to save space.

estimator of  $\delta$ , if we compare the rate of convergence for the two cases a and b.

We now examine the behaviour of the BDS test. The results, reported in Table 1, can be summarised as follows<sup>5</sup>:

i) For small sample sizes  $T < 250$ , the BDS test is slightly oversized.<sup>6</sup> For example, the empirical size of the test is about 13 percent when  $T = 50$  for all the values of the embedding dimension  $m$  under consideration. However, it approaches the nominal size of 5 percent as the sample size increases. For samples of 250 observations or more, it is generally less than 6 percent.

ii) The empirical size of BDS is affected neither by the moment characteristics of  $u_t$  nor by the relative values of  $\mu$  and  $\delta$ . Even for regions 3, 4 and 5, where the sufficient conditions for the invariance property of BDS are violated, the distribution of the BDS statistic approaches the standard normal  $N(0, 1)$ , providing empirical sizes that are almost equal to the nominal size.

## 4.2 Symmetric, heavy-tailed distributions: The case of t-innovations

In this set of experiments, we assume that the innovations  $z_t$  follow a standardised  $t$ -distribution with  $\kappa$  degrees of freedom, that is  $z_t = (\xi_t / \sqrt{k/k - 2})$ , where  $\xi_t$  is an *i.i.d.*  $t(k)$  variate. The new distributional assumption implies that some of the aforementioned cases (I) to (V) are no longer defined. Indeed, the number of valid cases, under  $t$ -innovations, depends on the degrees-of-freedom parameter,  $k$ . In particular, for  $k = 5$ , all the cases (I)-(V) exist. For  $k = 4$  and  $k = 3$ , fourth moments of the innovations do not exist, which in turn implies that case I is not defined. As already mentioned, the existence of the fourth moment of  $z_t$  was the condition imposed by Jensen and Rahbek (2003) to prove the  $\sqrt{T}$ -consistency of the QML estimates. It would therefore be very interesting to examine the consistency of QML estimates and the behaviour of the BDS statistic when the fourth moment condition of  $z_t$  is violated. Figure 3 defines the corresponding cases 2 to 5 for  $k = 4$ .

We first investigate the consistency property of the QML estimates of  $\theta$ , when  $z_t$  follows a standardised  $t(3)$ - or  $t(4)$ -distribution. In general, the results are similar for the two distributions. Therefore, we only report the results for the latter. We consider values of  $(\mu, \delta)$  corresponding to case II to V, and, once again, two pairs of values for each region, i.e. case  $a = (\mu, \delta)$  and  $b = (\mu', \delta')$ . More specifically, the models under investigation are the following:

Model VIa:  $(\mu, \delta) = (0.1, 0.8)$  and model VIb:  $(\mu', \delta') = (0.4, 0.5)$

Model VIIa:  $(\mu, \delta) = (0.1, 0.9)$  and model VIIb:  $(\mu', \delta') = (0.4, 0.6)$

Model IIXa:  $(\mu, \delta) = (0.1, 0.91)$  and model IIXb:  $(\mu', \delta') = (0.4, 0.65)$

<sup>5</sup>We only report the empirical size of the BDS test. Mean, standard deviation, skewness and kurtosis of the BDS test are available on request.

<sup>6</sup>This mainly reflects the effect of the positive skewness we observe in the empirical distribution of the BDS statistic.

Model IXa:  $(\mu, \delta) = (0.1, 0.915)$  and model IXb:  $(\mu', \delta') = (0.4, 0.72)$

The mean absolute bias of the GARCH parameters for these four models is shown in Figures 4.II, 4.III, 4.IV and 4.V respectively. The results can be summarised as follows:

i) The bias decreases as the sample size increases, which supports the consistency of the QML estimators. In general, it decreases as we move from region 2 to region 5. Few exceptions are observed for small samples ( $T < 200$ ).

ii) The rate of convergence of the QML estimators depends on both the moment properties of  $u_t$  and the value of  $(\mu, \delta)$ . More specifically, the estimators of all the GARCH parameters converge faster to  $\theta$  as we move from region 2 to region 5. Initially, when second-order stationarity of  $u_t$  holds, the rate of convergence is very slow (slower than  $\sqrt{T}$  in most cases). However, as we move to regions 3, 4 and 5, it increases, reaching  $T$  in some cases. For example, the estimator of  $\delta$  becomes  $T$ -consistent when  $\mu = 0.1$ , and the variances of  $u_t$  is infinite (for regions 3 to 5). On the other hand, the rate of convergence of the estimates of  $\mu$  is always slower than  $T$ . In general, the rate of convergence is slower for large values of  $\mu$  and small values of  $\delta$ . Similarly to the Gaussian case, the estimates of  $\delta$  converge faster than those of  $\mu$ .

We now examine the distribution of the BDS statistic. The results are presented in Table 2. On the whole, the results are similar to the Gaussian case. That is, the BDS test is slightly oversized for small sample sizes but it approaches the nominal size as the sample increases. For example, for  $T = 50$  and  $m = 2$ , the empirical size of BDS is 13.2 and 13.8 percent for the Gaussian and  $t(4)$ -distribution respectively. It can be noted that the empirical sizes in the case of the  $t(3)$ -distribution are about 1 percent larger than those of the  $t(4)$ -distribution. Finally, the distribution of the BDS statistic seems to be invariant of the moment characteristics of  $u_t$ .

### 4.3 Asymmetric Distributions: The case of $\chi^2$ innovations

So far, we have considered innovations that follow a symmetric distribution. We now analyse the behaviour of the BDS statistic when the innovations  $z_t$  have an asymmetric distribution. To shed some light on this issue, we carry out some additional simulations in which the innovations  $z_t$  follow a standardised  $\chi^2$ -distribution, that is  $z_t = \frac{\xi_t - 1}{\sqrt{2}}$ , where  $\xi_t$  is an *i.i.d.*  $\chi^2(1)$  variate. Under this assumption, all five cases (I to V) are meaningful. Figure 5 shows the corresponding regions in the  $(\mu, \delta)$  plane.

We consider the following models:

Model Xa:  $(\mu, \delta) = (0.1, 0.8)$  and model Xb:  $(\mu', \delta') = (0.2, 0.4)$

Model XIa:  $(\mu, \delta) = (0.1, 0.85)$  and model XIb:  $(\mu', \delta') = (0.4, 0.5)$

Model XIIa:  $(\mu, \delta) = (0.1, 0.9)$  and model XIIb:  $(\mu', \delta') = (0.4, 0.6)$

Model XIIVa:  $(\mu, \delta) = (0.1, 0.91)$  and model XIIVb:  $(\mu', \delta') = (0.4, 0.65)$

Model XIVa:  $(\mu, \delta) = (0.1, 0.92)$  and model XIVb:  $(\mu', \delta') = (0.4, 0.75)$

Figures 6.I, 6.II, 6.III, 6.IV and 6.V show the mean absolute bias of  $\mu$  and  $\delta$  for



models X, XI, XII, XIII and XIV respectively. As before, the mean absolute bias of the QML estimators decreases with the sample size. Few exceptions are found when  $\mu$  is large. In that case, the bias of the estimators initially increases with the sample size, but starts to decrease with the sample size when  $T > 100$ . Once again, the bias decreases as we move from region 1 to region 5. The rate of convergence appears to depend on the moment properties of  $u_t$  and the value of  $(\mu, \delta)$ . More specifically, the rate of convergence of the estimator of  $\mu$  is not affected by the moment characteristics of  $u_t$ , but it decreases with the value of  $\mu$ . For example, when  $\mu = 0.1$ , the rate of convergence of the estimates of  $\mu$  is slightly faster than  $\sqrt{T}$ . However, it becomes slower than  $\sqrt{T}$  as the value of  $\mu$  increases. On the other hand, the rate of convergence of the estimates of  $\delta$  increases as we move from region 1 to region 5, but it strongly depends on the value of  $(\mu, \delta)$ . The smaller is  $\mu$  and the larger is  $\delta$ , the faster the rate of convergence becomes, without exceeding a  $\sqrt{T}$  rate. However, when  $\mu = 0.1$  and the unconditional mean of  $u_t$  becomes infinite (region V), the estimator of  $\delta$  becomes  $T$ -consistent.

Surprisingly, although the bias of the QML estimators is larger in the case of the  $\chi^2$  as opposed to that of the Normal distribution, the empirical sizes of the BDS test are similar in the two cases. That is, the BDS test is slightly oversized for small sample sizes, but it reaches the nominal size very fast as the sample size increases. These results are reported in Table 3.

## 5 Conclusions

In this study we carry out Monte Carlo simulations to examine the behaviour of the widely used Brock, Dechert and Scheinkman (BDS) test when applied to the standardised residuals of an estimated GARCH(1,1) model as a test for the adequacy of this specification. More in detail, we consider a variety of distributions for the innovations (implying different moment and memory characteristics of the error term) to examine the consistency of the QML estimators of the GARCH parameters, and the "necessity" of De Lima's (1996) conditions for the invariance property of the BDS test statistic. The results suggest that the distribution of the BDS statistic is not generally affected by the moment properties of the innovations. The test is slightly oversized for small sample sizes ( $T < 200$ ), but it gradually reaches the nominal size as the sample size increases. As far as the consistency of the QML estimators is concerned, the mean absolute bias of the estimates decreases with the sample size. The rate of convergence is faster in the non-stationary case than in the stationary case, and the estimators sometimes become  $T$ -consistent. However, the rate of convergence strongly depends on the pair of values  $(\mu, \delta)$ : it is faster for small values of  $\mu$  and large values of  $\delta$ .

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Table 1: Empirical Size of the BDS Test -  $N(0,1)$  Distribution

Case a						Case b					
<b>T=50</b>	Region 1	Region 2	Region 3	Region 4	Region 5	<b>T=50</b>	Region 1	Region 2	Region 3	Region 4	Region 5
m=2	0.132	0.139	0.123	0.127	0.136	m=2	0.136	0.125	0.138	0.123	0.124
m=3	0.132	0.149	0.133	0.123	0.134	m=3	0.127	0.122	0.143	0.139	0.135
m=4	0.132	0.121	0.141	0.133	0.144	m=4	0.127	0.140	0.121	0.134	0.132
m=5	0.128	0.133	0.139	0.128	0.136	m=5	0.123	0.144	0.127	0.127	0.126
<b>T=150</b>						<b>T=150</b>					
m=2	0.081	0.071	0.065	0.064	0.067	m=2	0.068	0.072	0.067	0.069	0.073
m=3	0.070	0.076	0.072	0.073	0.069	m=3	0.076	0.069	0.059	0.064	0.076
m=4	0.070	0.078	0.073	0.067	0.067	m=4	0.073	0.072	0.071	0.069	0.087
m=5	0.072	0.076	0.072	0.057	0.072	m=5	0.066	0.073	0.074	0.068	0.084
<b>T=250</b>						<b>T=250</b>					
m=2	0.068	0.059	0.052	0.070	0.059	m=2	0.060	0.060	0.059	0.060	0.059
m=3	0.069	0.051	0.054	0.065	0.065	m=3	0.059	0.065	0.066	0.058	0.055
m=4	0.065	0.048	0.058	0.071	0.060	m=4	0.059	0.065	0.059	0.058	0.051
m=5	0.061	0.052	0.058	0.067	0.059	m=5	0.058	0.062	0.051	0.051	0.054
<b>T=950</b>						<b>T=950</b>					
m=2	0.054	0.051	0.054	0.050	0.054	m=2	0.062	0.052	0.049	0.053	0.053
m=3	0.060	0.052	0.058	0.053	0.057	m=3	0.061	0.059	0.051	0.052	0.046
m=4	0.058	0.052	0.061	0.056	0.058	m=4	0.062	0.056	0.051	0.049	0.043
m=5	0.055	0.049	0.062	0.051	0.059	m=5	0.059	0.052	0.050	0.051	0.045

Table 2: Empirical Size of the BDS Test – t-Student(4) Distribution

Case a					Case b				
<b>T=50</b>	Region 2	Region 3	Region 4	Region 5	<b>T=50</b>	Region 2	Region 3	Region 4	Region 5
m=2	0.138	0.139	0.129	0.129	m=2	0.140	0.132	0.141	0.160
m=3	0.155	0.139	0.126	0.152	m=3	0.148	0.138	0.148	0.147
m=4	0.154	0.148	0.142	0.145	m=4	0.159	0.139	0.152	0.148
m=5	0.145	0.148	0.149	0.155	m=5	0.162	0.145	0.168	0.153
<b>T=150</b>					<b>T=150</b>				
m=2	0.071	0.078	0.062	0.078	m=2	0.066	0.080	0.073	0.080
m=3	0.075	0.070	0.069	0.071	m=3	0.069	0.081	0.076	0.079
m=4	0.066	0.069	0.075	0.076	m=4	0.074	0.075	0.086	0.069
m=5	0.066	0.069	0.069	0.072	m=5	0.074	0.076	0.087	0.071
<b>T=250</b>					<b>T=250</b>				
m=2	0.060	0.059	0.064	0.057	m=2	0.060	0.059	0.066	0.067
m=3	0.054	0.062	0.060	0.058	m=3	0.060	0.059	0.059	0.069
m=4	0.059	0.064	0.075	0.060	m=4	0.058	0.066	0.060	0.064
m=5	0.053	0.061	0.077	0.058	m=5	0.059	0.067	0.056	0.063
<b>T=950</b>					<b>T=950</b>				
m=2	0.050	0.045	0.048	0.049	m=2	0.061	0.051	0.052	0.060
m=3	0.054	0.046	0.057	0.045	m=3	0.066	0.047	0.052	0.059
m=4	0.058	0.054	0.057	0.057	m=4	0.056	0.046	0.045	0.057
m=5	0.063	0.057	0.057	0.061	m=5	0.058	0.048	0.048	0.059

Table 3: Empirical Size of the BDS Test –  $\chi^2(1)$  Distribution

Case a						Case b					
T=50	Region 1	Region 2	Region 3	Region 4	Region 5	T=50	Region 1	Region 2	Region 3	Region 4	Region 5
m=2	0.121	0.114	0.121	0.108	0.108	m=2	0.121	0.114	0.105	0.105	0.124
m=3	0.120	0.105	0.121	0.116	0.116	m=3	0.117	0.116	0.103	0.120	0.122
m=4	0.115	0.105	0.121	0.117	0.115	m=4	0.114	0.107	0.099	0.109	0.119
m=5	0.118	0.104	0.113	0.117	0.113	m=5	0.112	0.108	0.089	0.102	0.117
T=150						T=150					
m=2	0.083	0.073	0.077	0.071	0.084	m=2	0.079	0.066	0.071	0.074	0.083
m=3	0.086	0.070	0.082	0.071	0.092	m=3	0.086	0.065	0.065	0.070	0.085
m=4	0.080	0.061	0.081	0.077	0.086	m=4	0.077	0.059	0.065	0.062	0.083
m=5	0.079	0.066	0.088	0.074	0.085	m=5	0.079	0.058	0.056	0.068	0.080
T=250						T=250					
m=2	0.061	0.061	0.081	0.067	0.074	m=2	0.063	0.058	0.063	0.067	0.055
m=3	0.068	0.061	0.076	0.061	0.070	m=3	0.070	0.069	0.058	0.064	0.057
m=4	0.060	0.061	0.076	0.062	0.073	m=4	0.074	0.062	0.065	0.063	0.058
m=5	0.064	0.058	0.074	0.062	0.076	m=5	0.069	0.067	0.064	0.060	0.064
T=950						T=950					
m=2	0.046	0.062	0.059	0.046	0.054	m=2	0.051	0.047	0.047	0.056	0.070
m=3	0.053	0.054	0.053	0.044	0.052	m=3	0.053	0.040	0.051	0.050	0.070
m=4	0.055	0.053	0.057	0.049	0.054	m=4	0.052	0.043	0.056	0.050	0.068
m=5	0.050	0.054	0.053	0.048	0.059	m=5	0.049	0.046	0.055	0.045	0.059

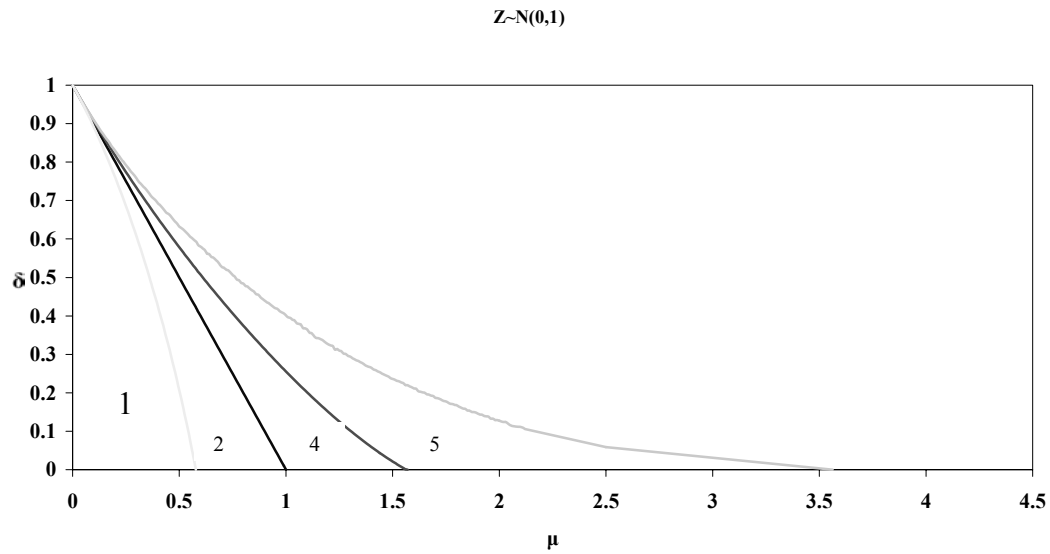
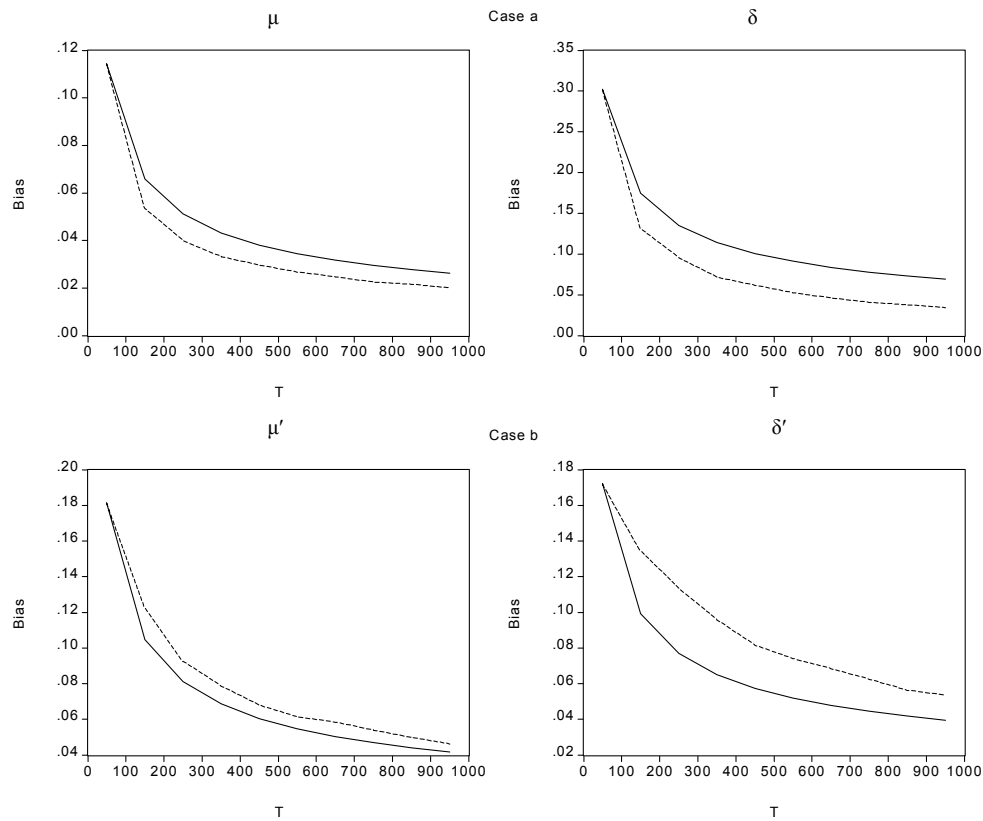
Figure 1: Regions for the  $N(0,1)$  distribution

Figure 2.I: Mean Absolute Bias (dashed line) for Models Ia, Ib.\*



\* The solid line refers to the simulated biases that would have been produced, had the rate of convergence been exactly  $\sqrt{T}$ .

Figure 2.II: Mean Absolute Bias (dashed line) for Models IIa, IIb.\*

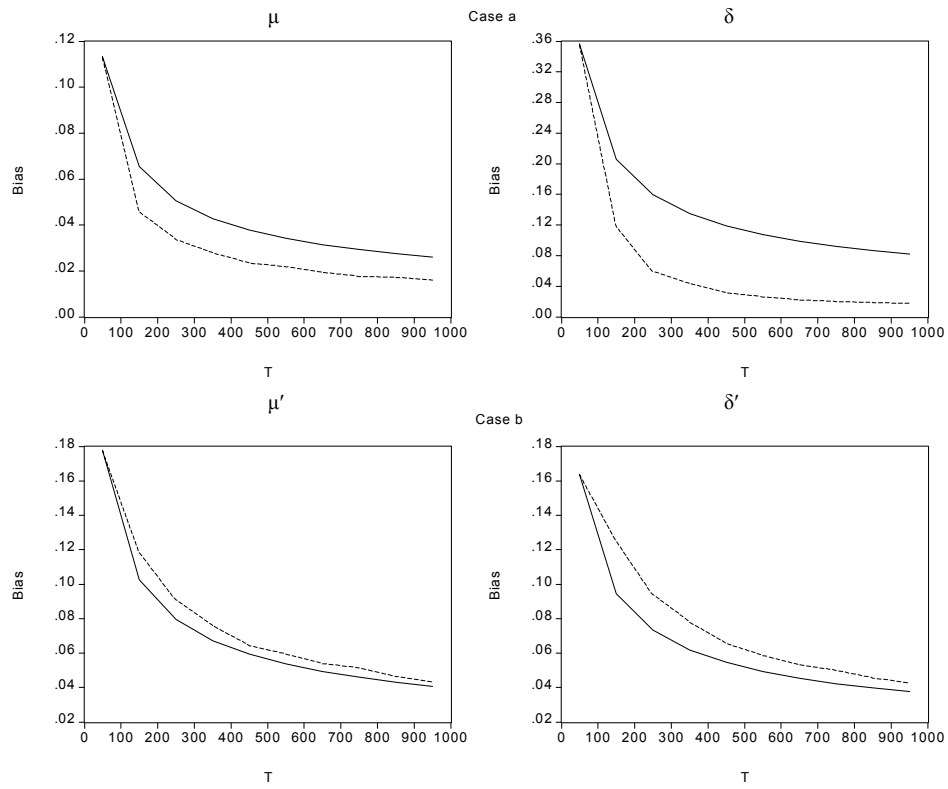
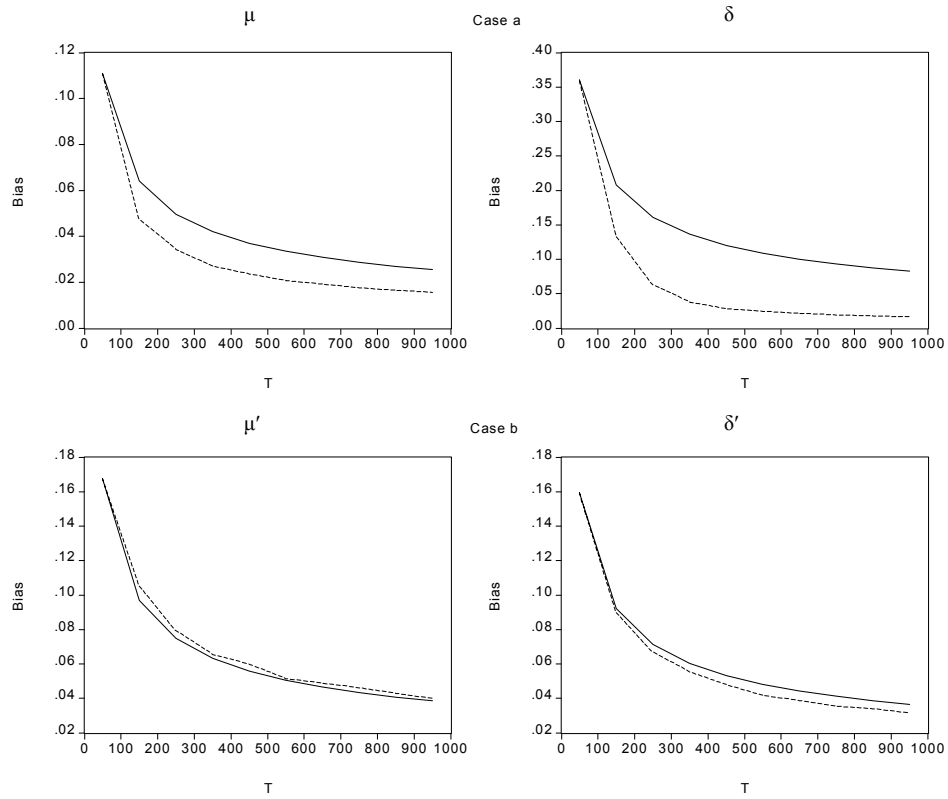


Figure 2.III: Mean Absolute Bias (dashed line) for Models IIIa, IIIb.\*



\* See Figure 2.I.



Figure 2.IV: Mean Absolute Bias (dashed line) for Models IVa, IVb.\*

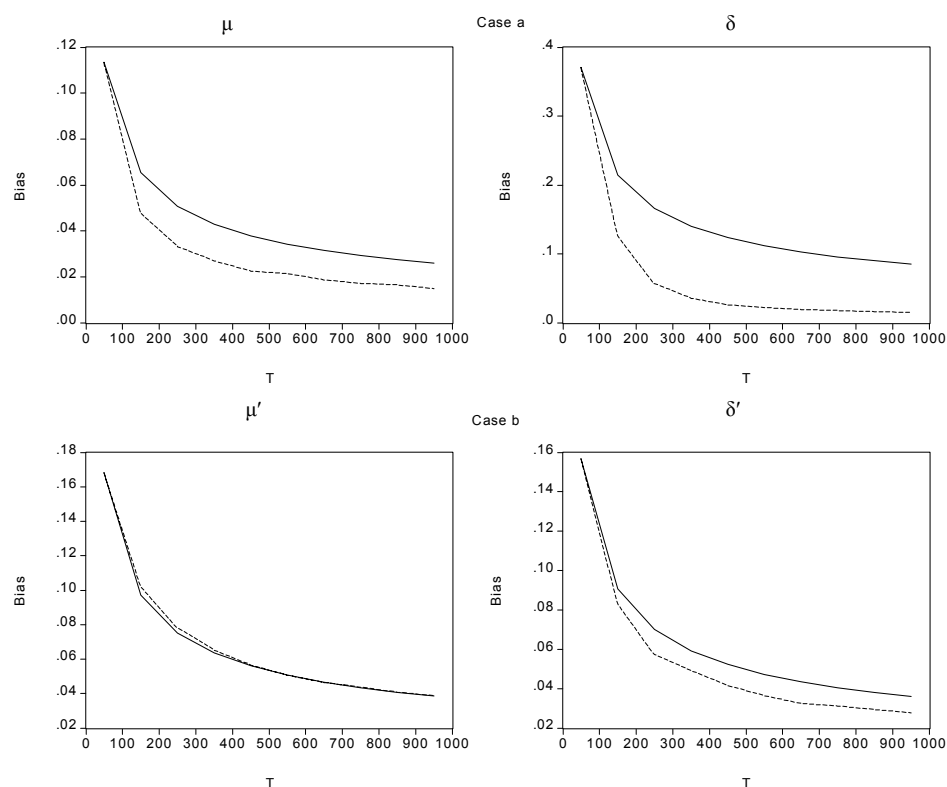
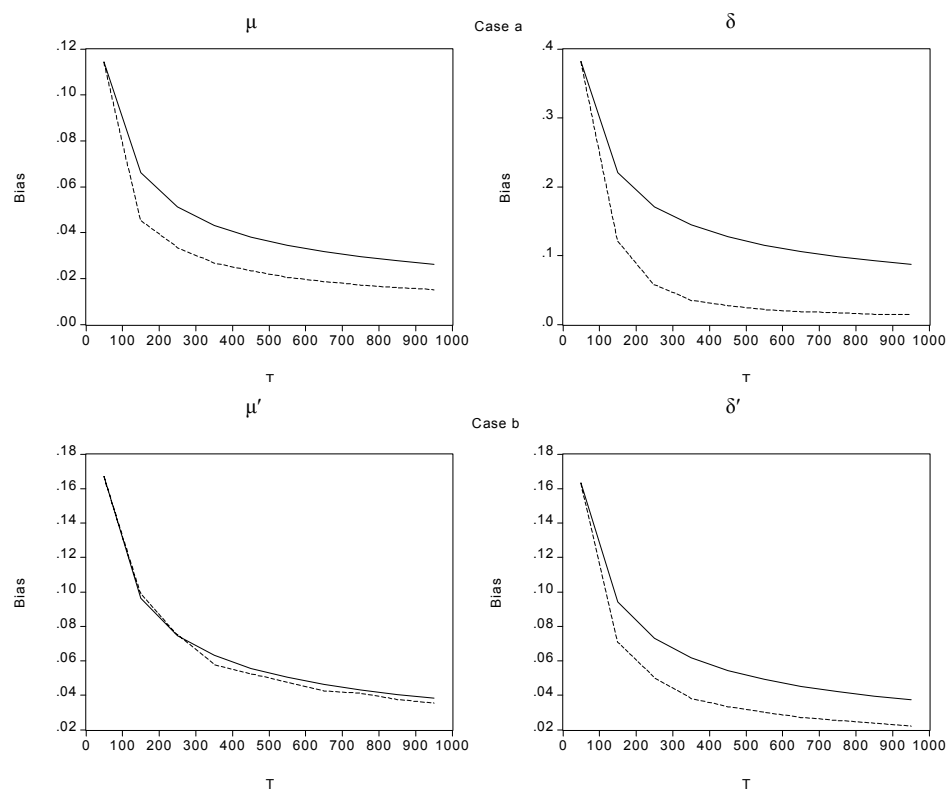


Figure 2.V: Mean Absolute Bias (dashed line) for Models Va, Vb.\*



\* See Figure 2.I.

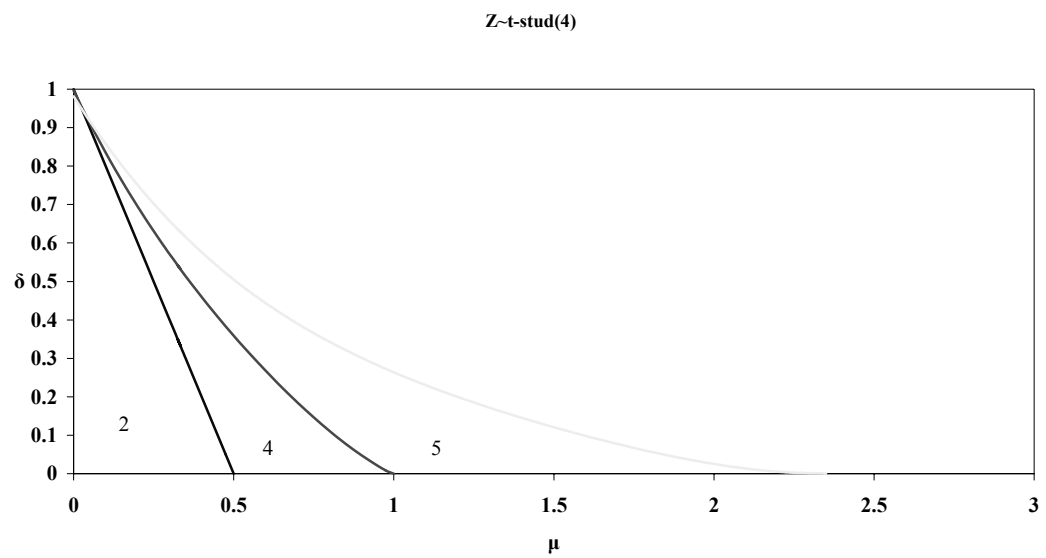
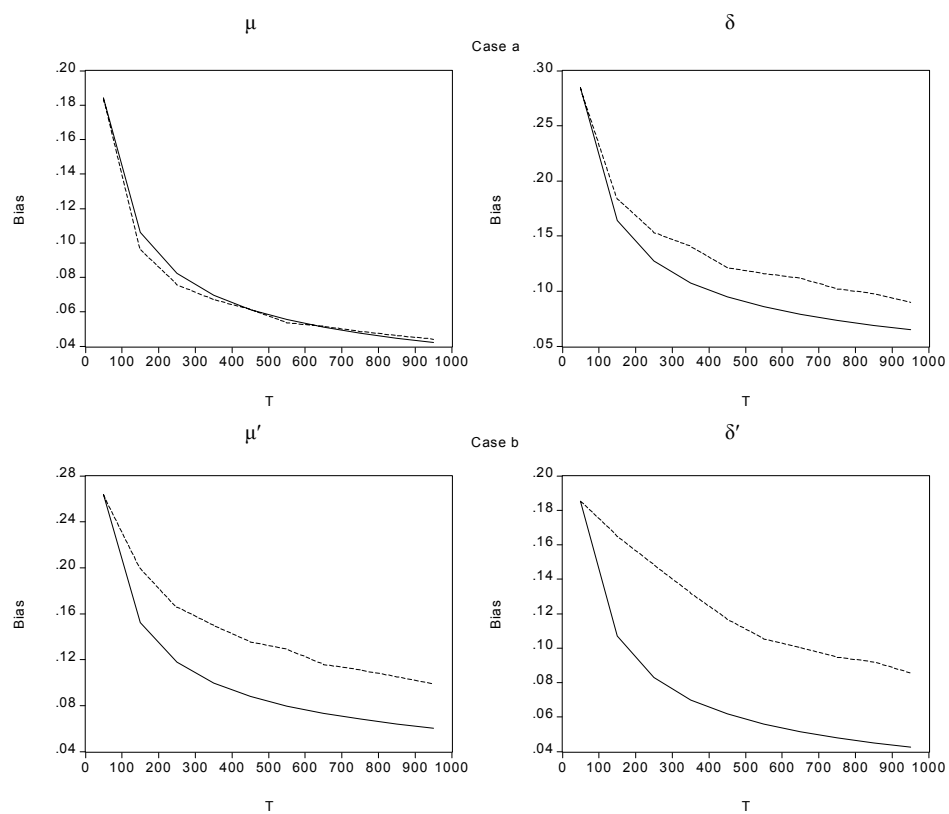
Figure 3: Regions for the  $t(4)$ -distribution

Figure 4.II: Mean Absolute Bias (dashed line) for Models VIa, VIb.\*



\* See Figure 2.I.

Figure 4.III: Mean Absolute Bias (dashed line) for Models VIIa, VIIb.\*

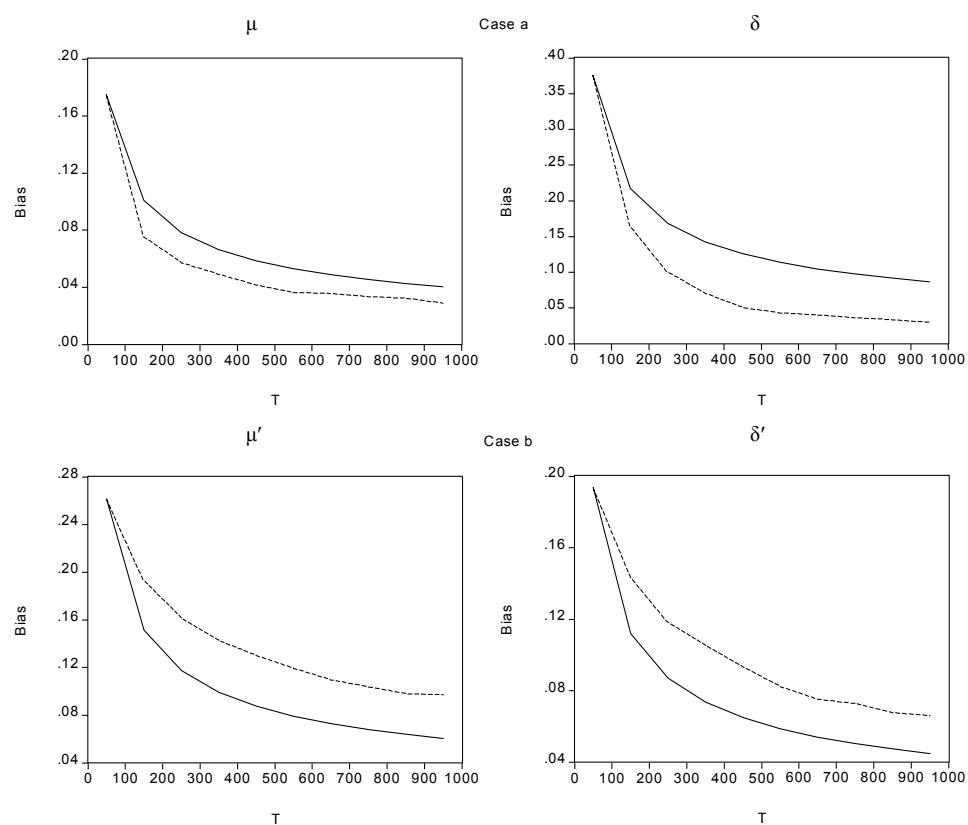
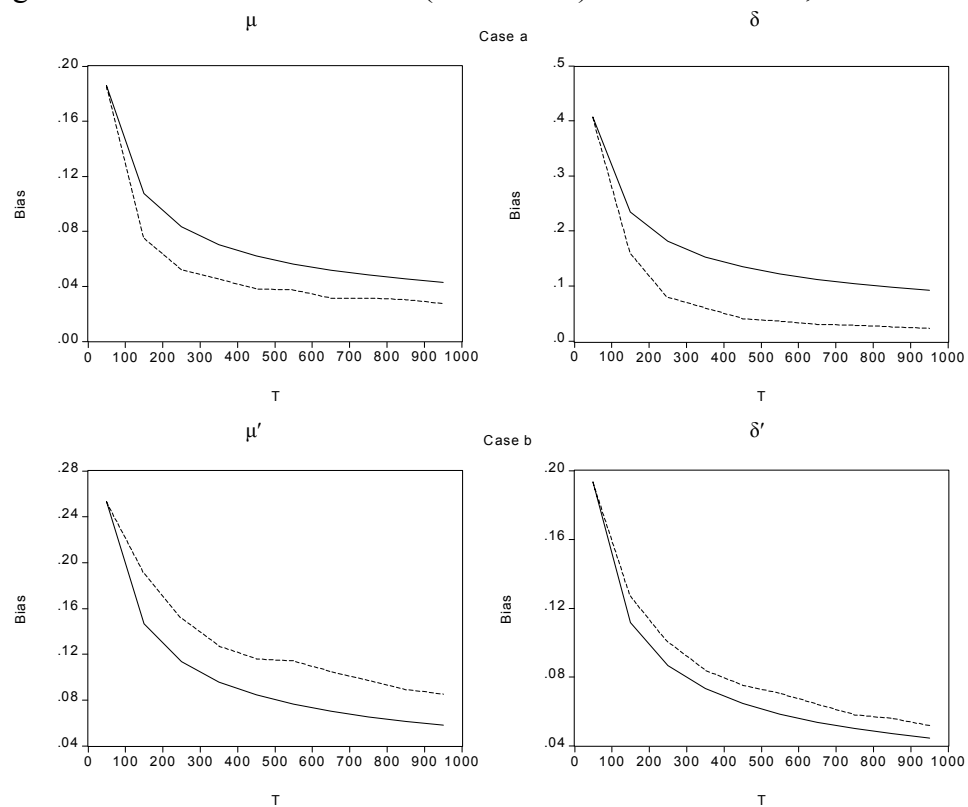
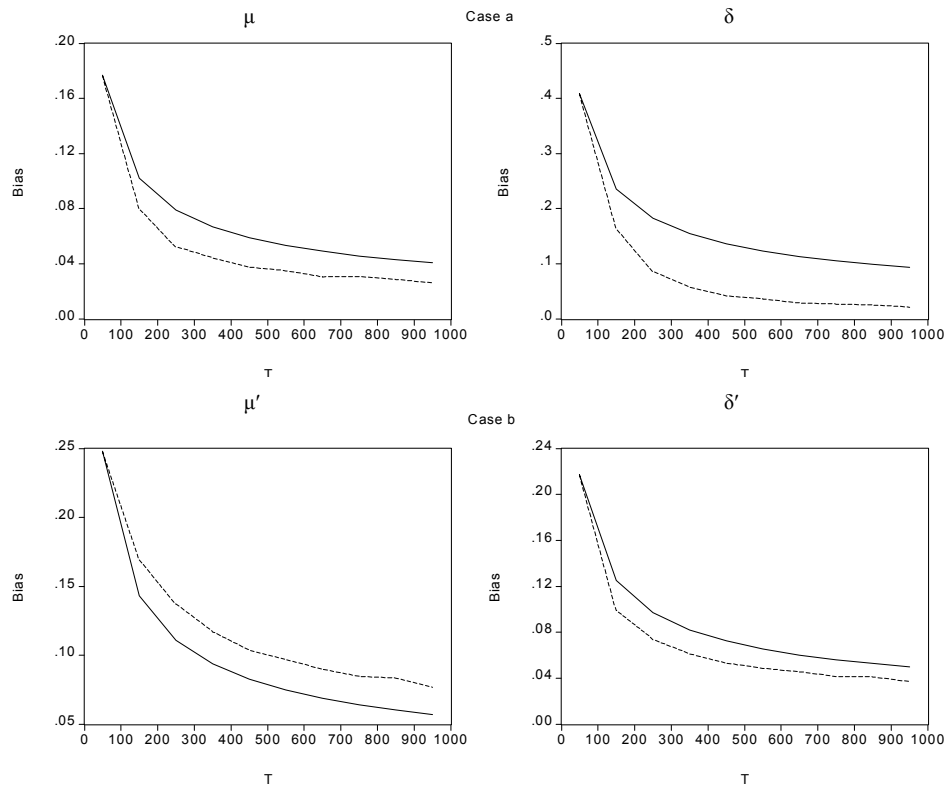


Figure 4.IV: Mean Absolute Bias (dashed line) for Models IIXa, IIXb.\*

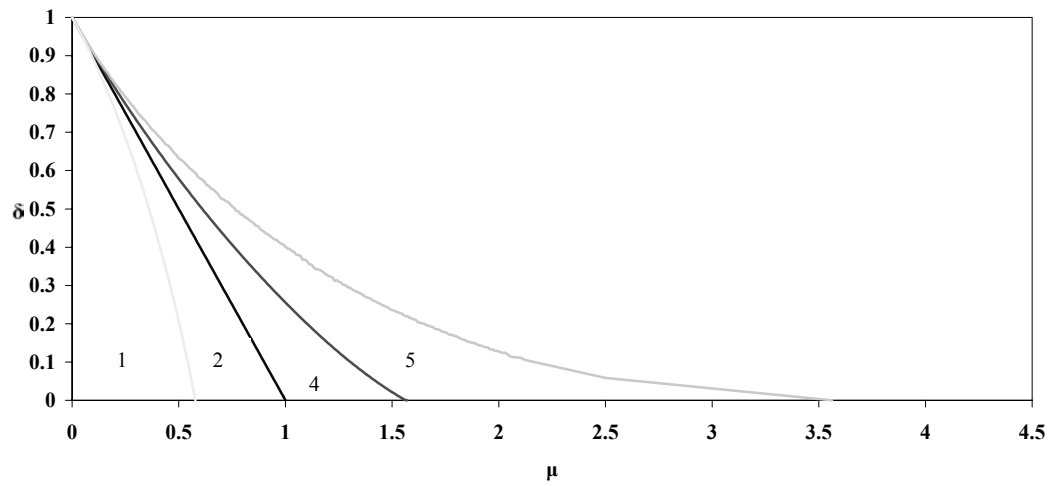


\* See Figure 2.I.

Figure 4.V: Mean Absolute Bias (dashed line) for Models IXa, IXb.\*

Figure 5: Regions for the  $\chi^2(1)$ -distribution

$$Z \sim \chi^2(1)$$



\* See Figure 2.I.

Figure 6.I: Mean Absolute Bias (dashed line) for Models Xa, Xb.\*

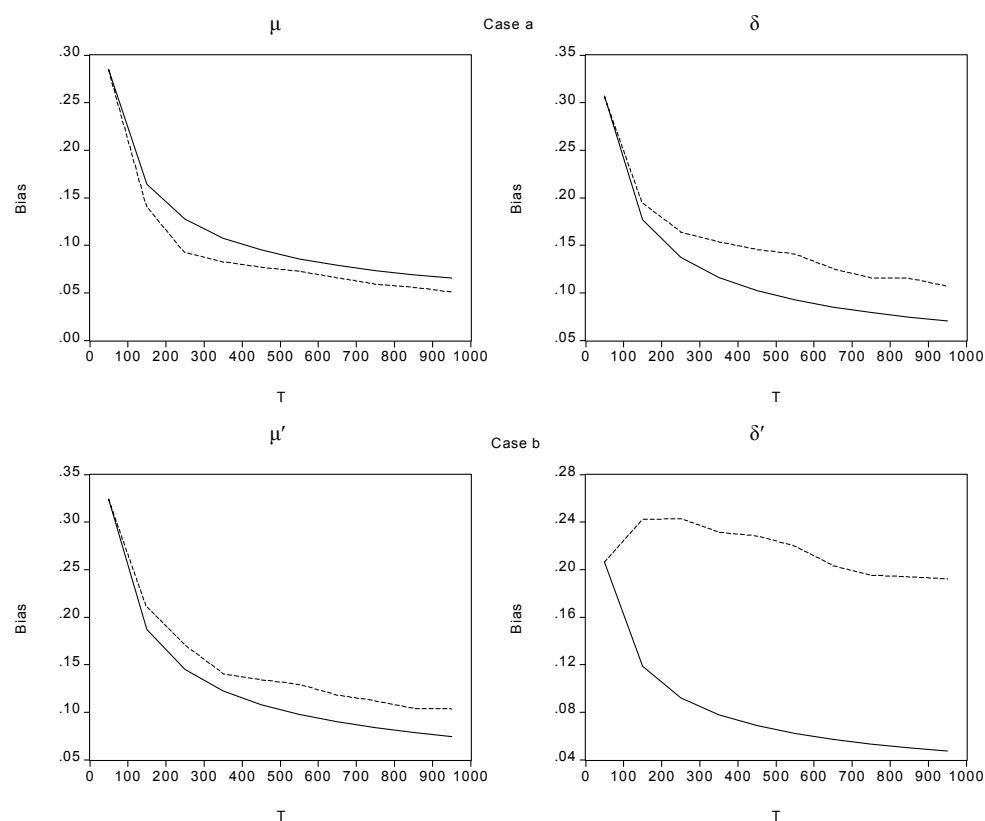
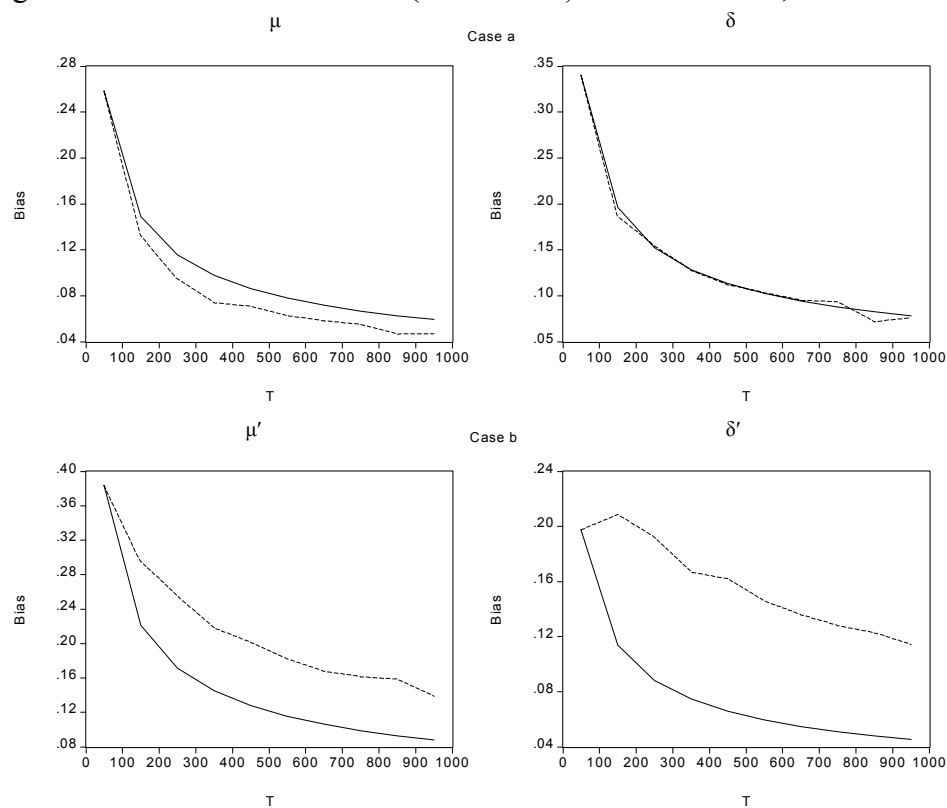


Figure 6.II: Mean Absolute Bias (dashed line) for Models XIa, XIb.\*



\* See Figure 2.I.

Figure 6.III: Mean Absolute Bias (dashed line) for Models XIIa, XIIb.\*

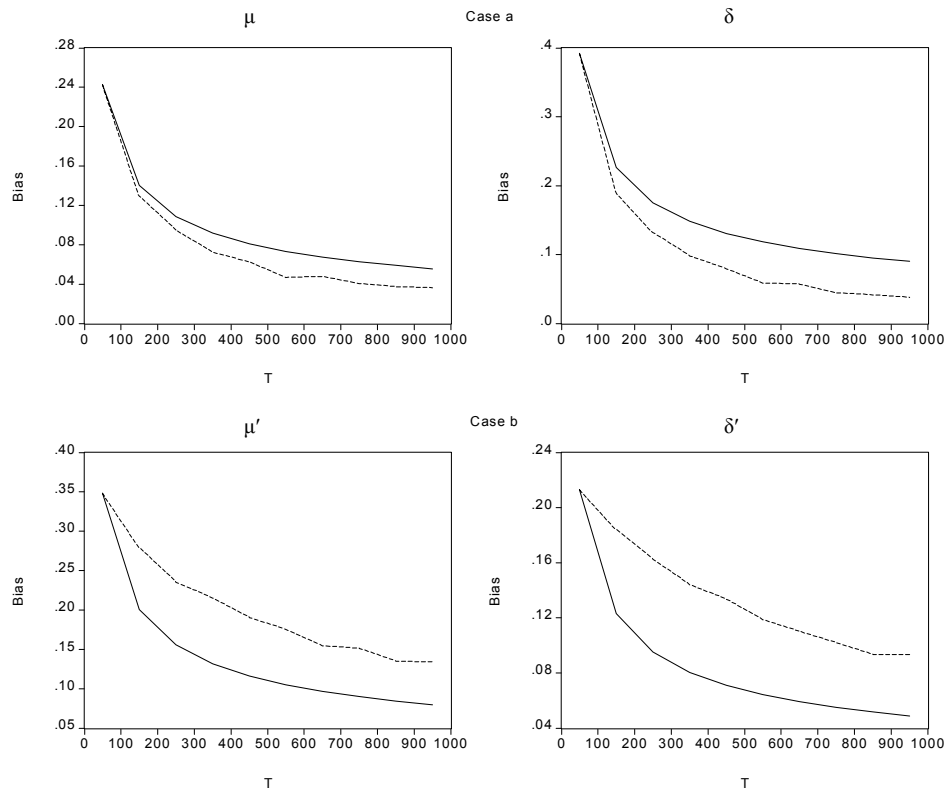
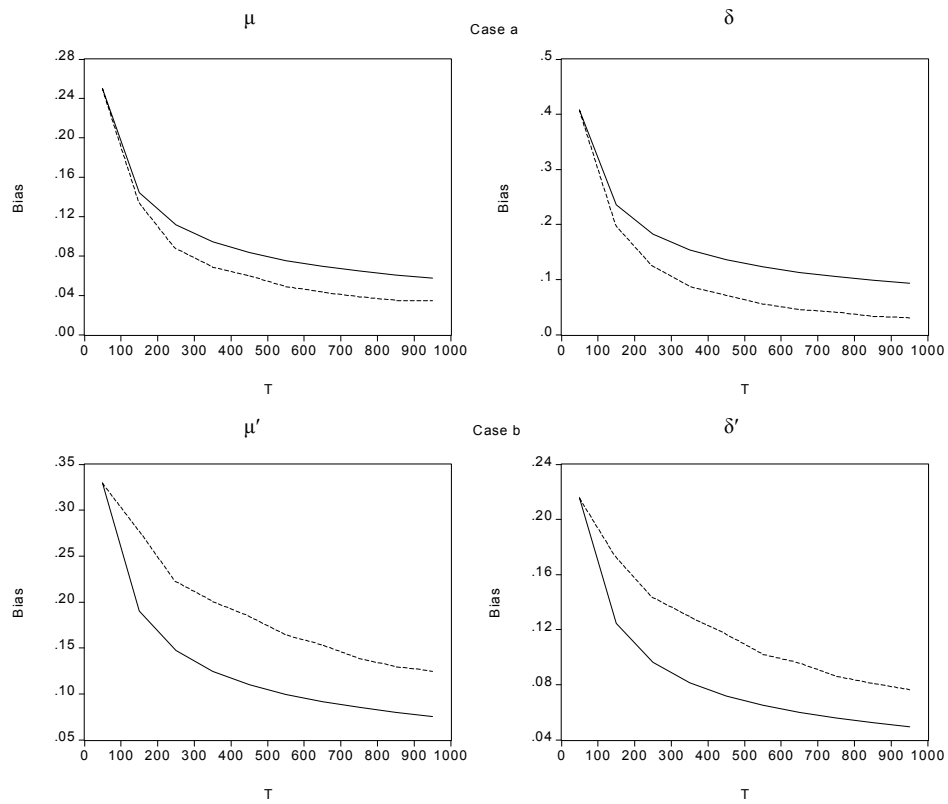
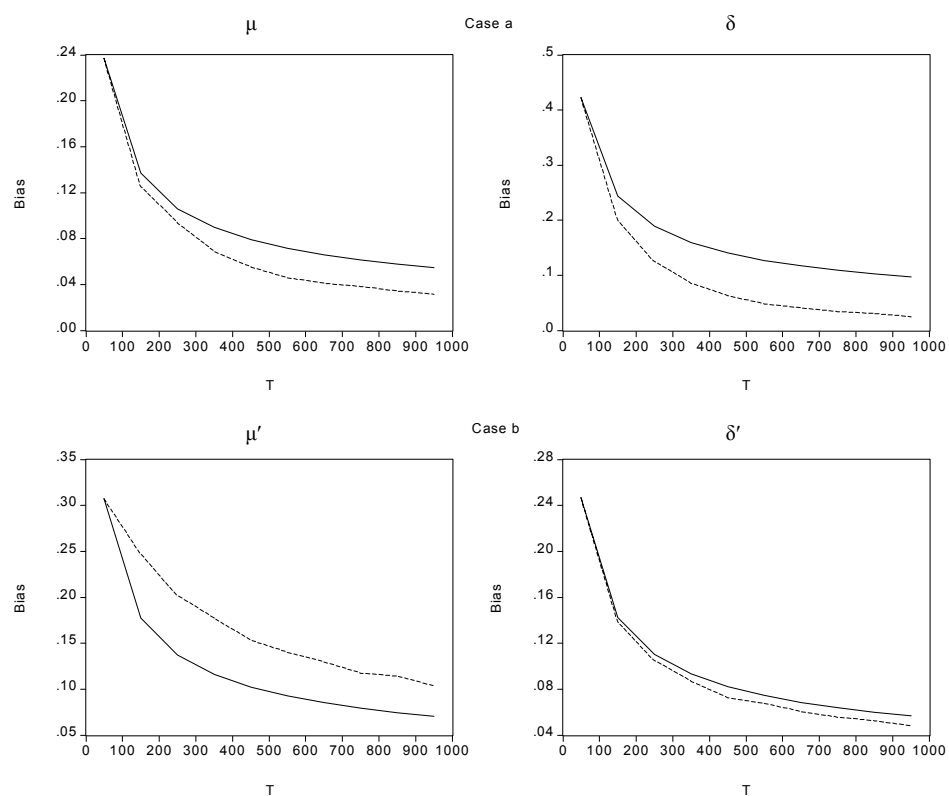


Figure 6.VI: Mean Absolute Bias (dashed line) for Models XIIIa, XIIIb.\*



\* See Figure 2.I.

Figure 6.V: Mean Absolute Bias (dashed line) for Models XIVa, XIVb.\*



\* See Figure 2.I.

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