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Tail-Dependence in Stock-Return Pairs

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Abstract

The empirical joint distribution of return-pairs on stock indices displays high tail-dependence in the lower tail and low tail-dependence in the upper tail. The presence of tail-dependence is not compatible with the assumption of (conditional) joint normality. The presence of asymmetric-tail dependence is not compatible with the assumption of a joint student-t distribution. A general test for one dependence structure versus another via the profile-likelihood is described and employed in a bivariate GARCH model, where the joint distribution of the disturbances is split into its marginals and its copula. The copula used is such that it allows for the presence of lower tail-dependence and for asymmetric tail-dependence, and that it encompasses the normal or t-copula. The model is estimated using bivariate data on a set of European stock indices. We find that the assumption of normal or student-t dependence is easily rejected in favour of an asymmetrically tail-dependent distribution.

Keywords

Value-at-Risk, copula, non-normal bivariate GARCH, asymmetric dependence, profile likelihood-ratio test

JEL Classifications

C12, C32, C52, C51, G15

Comments

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Contents

1 Introduction

The dependence structure of international financial markets has always attracted attention from various fields of finance including portfolio selection, pricing of complex financial products, and risk management. One of the crucial questions in risk management is how to aggregate individual risk into overall portfolio risk. At some point in the aggregation process, one has to make assumptions about the dependence structure between the factors which drive individual risk.

The standard practice in assessing the overall risk of a portfolio is to assume that asset prices are driven by jointly normal random variables. The assumption of joint normality (or ellipticity) is often implicitly made through the use of linear correlation as the measure of dependence. One example is JPMorgans CreditMetrics (1997), where credit ratings are driven by unobserved jointly normal distributions. However, different joint distributions with the same correlation matrix can well give rise to different Values-at-Risk (see for example Embrechts *et al.* 2002).

One approach in the Value-at-Risk literature to circumvent the dependency problem is to look at return series of an entire portfolio rather than at the set of univariate return series. Since it is then possible to investigate the distribution of the portfolio return and its Value-at-Risk directly, dependence or correlation are not an issue. Examples of this approach are Engle and Manganelli (1999), and McNeil and Frey (2000). Engle and Manganelli (1999) propose a modified GARCH model to model the evolution of the Value-at-Risk directly, while McNeil and Frey (2000) suggest using a GARCH model to estimate the conditional mean and variance of the portfolio return first, and then modelling the distribution of the residuals by employing extreme value theory and historical simulation, to provide estimates for the Value-at-Risk.

In considering problems such as the selection of optimal portfolio weights, however, it is necessary to understand the dependence structure between individual assets. One approach to address dependency is to model correlation itself as changing over time. Studies on international equity markets such as Longin and Solnik (1995) document that correlation is higher in periods of larger volatilities. Boyer, Gibson and Loretan (1999) suggest that the widely observed difference of correlations during periods of high and low market volatilities, the so-called correlation breakdown, may reflect time-changing (conditional) volatilities rather than a structural break in the underlying distribution. They show that the observed sample correlations, conditional on one variable falling below/above a certain threshold value, may differ substantially even if the true correlation is constant. Loretan and English (2000) find that this theoretical relationship can account for a large part of empirical correlation movements.

Acknowledging the correlation breakdown critique, Longin and Solnik (2001) still confirm time-changing correlations in international equity markets. In fact, their results suggest that the crucial condition for high correlation is not high volatility itself, but high volatility coupled with negative returns. Longin and Solnik show that the correlation between stock return series tends to be higher in market downturns than in market upturns, a fact for which standard symmetric models of multivariate stock returns cannot account. Indeed, the authors reject joint normality for the negative tail of the multivariate distribution, but not for the positive tail. In other words, there seems to be significant dependence in the lower tail of the joint distribution, which cannot be explained by assuming joint normality with its implied zero tail-dependence. One drawback of the approach of Longin and Solnik (2001) is that it uses extreme value theory and so concentrates on the tails of the distribution while neglecting the rest. Yet for many applications we want a complete model for the joint behaviour of the return series, which will describe both the tails and the central part of the distribution.

This paper introduces an alternative way of modelling (asymmetric) dependence in asset returns, which can also capture the return dynamics of the univariate time series. We propose a bivariate (multivariate) GARCH-model with a dependence structure which allows for the existence of lower tail-dependence by employing copulas. Our model is similar to a number of models recently put forward by Patton (2001), Rockinger and Jondeau (2001), Hu (2002), and Mashal and Zeevi (2002). They also suggest modelling financial return series through bivariate GARCH models with copulas, in slightly different versions. The main contribution of our paper is the provision of a general test procedure of testing various copulas against each other. This is the normal or t versus a few selected copulas which are able to display positive and asymmetric lower and upper tail-dependence. In addition, the paper presents empirical evidence on the dependence structure of European stock markets.

The starting point of our discussion is the observation that a multivariate distribution function F can be split into two parts. The first part is the set of univariate distribution functions of each of the random variables (marginals) involved, F_i, the second part is the dependence structure between the random variables, the copula C.

$$F(x_1, \dots, x_n) = C(F_1(x_1), \dots, F_n(x_n)) \tag{1}$$

It seems logical to make use of copulas in risk management in order to capture different natures of risk, the individual and the portfolio risk. Obviously, one particular copula has already been used extensively, that is the copula induced by the joint normal distribution. The normal or gaussian copula is entirely specified by its functional form and the correlation matrix of all random variables (assets) involved.

In this paper we are concerned with testing the hypothesis of normal or t-dependence against the alternative hypothesis of the presence of tail-dependence. Ideally, one might want to test joint ellipticity against non-ellipticity in financial data, since non-ellipticity and not non-normality causes concern when one relies on linear correlation in capturing dependence. One possible way of testing ellipticity could be a non-parametric approach to copula modelling, e.g. by means of Bernstein approximations to copulas as in Sancetta and Satchell (2001).

This paper, as a first step, only provides a test of a special case of ellipticity, the normal or the t-copula, against a special case of non-ellipticity which puts asymmetrically more probability on joint extreme outcomes, as discussed in Section 3. We provide a general test based on the profile likelihood to test one copula against another. There are two sources of complication in deriving the asymptotic distribution of the suggested test statistic. The first is the presence of nuisance parameters under the null hypothesis. The second is the fact that we are testing whether a certain parameter is on the boundary of the parameter space. To solve the first problem we make use of the profile likelihood, in which the nuisance parameters are taken as fixed at their estimated levels. Due to our large sample size the estimated parameters should be sufficiently accurate. The second problem is solved by appealing to a result in Chernoff (1954) which states that the distribution of the likelihood-ratio statistic can be found to be a mixture of the degenerate χ_0^2 and a χ_k^2 distribution, where k is the number of unestimated parameters under the null. A simulation exercise confirms that the presence of nuisance parameters does not bias the result in our large sample of close to 3,000 return pairs. An application of this test using data on European stock indices yields a significant rejection of the normal as well as the t-copula in favour of a more tail-dependent copula. The results are very robust both to the assumptions on the marginals and to the exact form of the alternative copula. The results are very similar for a set of different bivariate stock index series.

We then proceed to investigate whether the alternative model we propose is well-specified by employing the non-parametric hit-test of Patton (2001). We first test whether our model is correctly specified ignoring possible time-dependence or auto-dependence using a simplified hit-test. For data on the

DAX, FTSE, and CAC indices, we do not find the alternative model thus misspecified. However, as Patton (2001) pointed out, for the alternative model to be well specified we also need time-independence. Employing the hit-test with regressors including past hits, we test for misspecification of this kind and find one of our three return-pair series to be misspecified. We then propose to model the series of return-pairs by means of the same GARCH-process, but with a time-changing copula. The way the copula changes over time is slightly different from Patton's, which is due to the fact that in contrast to the exchange rate data Patton investigates, in our data there is hardly any upper tail-dependence.

Finally, our bivariate model is slightly modified to enable testing of volatility spillovers. The conditional variance of the return on one stock index is then modelled to depend additionally on the variance of the return on the second index. We employ likelihood-ratio tests to determine the direction of spillover effects.

The reminder of the paper is organized as follows. Section 2 presents informal evidence of asymmetric dependence in the DAX/FTSE return pairs. Section 3 provides an account of the concept of a copula and the properties it should have to be of interest for financial data and our testing exercise. Section 4 lays down a bivariate GARCH model with general copula dependence structure for the analysis of bivariate stock returns, and describes the various tests undertaken in this paper. The results of estimation and testing are presented in Section 5. Section 6 concludes.

2 Motivating Informal Evidence

In this section we present crude estimates of lower and upper tail-dependence for the series of DAX/FTSE return-pairs suggesting that joint normality (or even ellipticity) may not be an adequate model to explain bivariate stock-return data.

Figure 1 shows a scatter plot of DAX versus FTSE returns for the period August 3, 1990 through December 31, 2001. Considering the different pattern of joint negative and joint positive extreme values, the graph suggests a non-linear dependence structure between DAX and FTSE returns. For the moment let us call a DAX return extreme if it exceeds 4% in absolute value, while a FTSE return is denoted extreme if it exceeds 3% in absolute value. Different threshold values are used for the DAX and the FTSE as the two index series display different variances. The cut-off values of 4% and

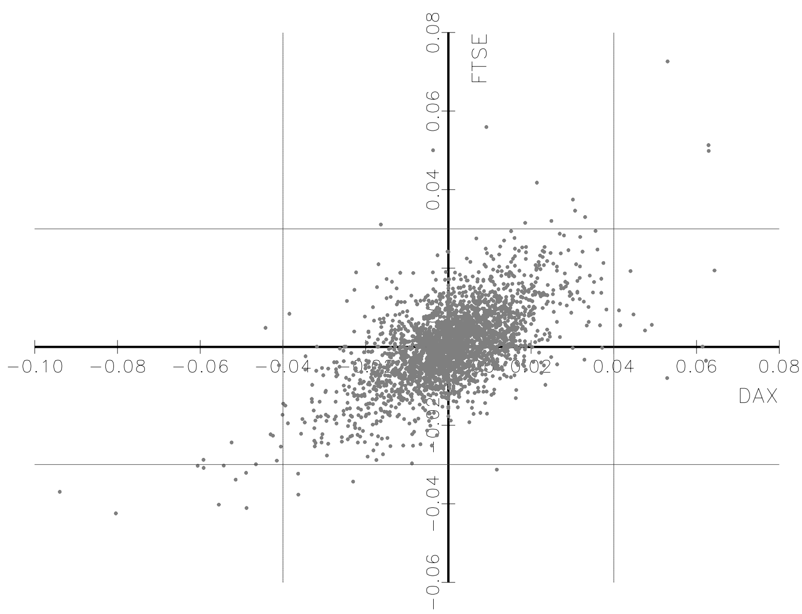

Figure 1: Scatter Plot of DAX/FTSE Returns.

3% equal approximately three times the standard deviations, respectively[1]. Given these definitions, we observe 39 extreme DAX and 28 extreme FTSE returns in the slightly more than 11 years under consideration.

Let us now consider the occurrence of an extreme return event of one index given the return of the other index is also extreme. This yields crude empirical estimates of lower and upper tail-dependence (for a rigorous definition see Section 3) in the bivariate equity returns. Consider first the event that the FTSE return is greater than +3%. This event is observed 13 times in our sample period. Of these 13 positive FTSE extremes there are 4 which are classified as (positively) extreme also for the DAX, the remaining 9 being ordinary returns. A rough estimate for the upper tail-dependence in DAX/FTSE returns is thus $\frac{4}{13} \approx 0.3077$. Conditioning on DAX returns, the estimate for upper tail-dependence would be $\frac{4}{16} = 0.25$, since there is a total of 16 positive extreme returns in the DAX index series.

On the negative side, consider all return-pairs, where the FTSE return is lower than −3%. There are 15 such instances, of which 11 are considered (negatively) extreme also for the DAX. This yields a crude estimate of lower tail-dependence of $\frac{11}{15} = 0.7333$. Conditioning on DAX returns, we have an

[1] The two average returns are below 0.0015%.

estimate of lower tail-dependence of $\frac{11}{23} \approx 0.4783$, since there is a total of 23 extreme negative FTSE returns. Values of 0.25 and 0.31 for the positive quadrant versus values of 0.48 and 0.73 for the negative quadrant suggest that there is substantially higher dependence in the lower tail of the distribution (negative extremes) than in the higher tail (positive extremes). This observation is not compatible with the presence of linear dependence structures like the one implied by the bivariate normal or the bivariate t-distribution. The bivariate normal distribution implies zero tail-dependence in both tails, while the bivariate t-distribution does display non-zero tail-dependence, but the same on both ends.

Table 1: **Empirical Tail Dependence for DAX/FTSE Returns**
This table lists empirical conditional upper and lower tail-probabilities $\lambda_L^\alpha(Y_D|Y_F) = P(Y_D \leq \alpha|Y_F \leq \alpha)$ and $\lambda_U^\alpha(Y_D|Y_F) = P(Y_D > \alpha|Y_F > \alpha)$, where Y_D and Y_F are the GARCH(1,1) standardized DAX and FTSE returns, respectively. The last column shows the corresponding values for the bivariate normal with a correlation of 0.56, the estimated value.

	Empirical Probabilities				Bivariate Normal
α	$\lambda_L^\alpha(Y_D\|Y_F)$	$\lambda_L^\alpha(Y_F\|Y_D)$	$\lambda_U^\alpha(Y_D\|Y_F)$	$\lambda_U^\alpha(Y_F\|Y_D)$	$\lambda_L^\alpha = \lambda_U^\alpha$
0	0.7050	0.7050	0.6841	0.6841	0.6908
0.5	0.6000	0.5791	0.5448	0.5358	0.5626
1	0.5061	0.4725	0.4248	0.4042	0.4337
1.5	0.4432	0.4271	0.2564	0.2339	0.3159
2	0.3333	0.3571	0.1633	0.1667	0.2172
2.5	0.3333	0.4118	0.1000	0.0909	0.1408
3	0.3684	0.4118	0.1000	0.1429	0.0859

It is well-known that stock returns are generally not identically distributed over all time periods. In fact, variances may change considerably over time. Let Y_{DAX} and Y_{FTSE} denote the standardized returns, where the time-changing variances have been separately estimated by a GARCH(1,1) process for each return series. The returns are now in units of their respective standard deviations. Table 1 shows empirical estimates of lower and upper tail-dependence and compares them to the tail-dependence implied by the joint normal distribution with a linear correlation of 0.56, the estimate for our data. These numbers indicate that the assumption of joint normality is seriously violated in a dangerous direction. The true joint distribution of the standardized returns of DAX and FTSE seems to display far heavier, especially lower, tail-dependence than the normal distribution implies. The asymmetry between lower and upper tail-dependence is also pronounced.

Assuming a normal distribution, the probability of a standardized return falling below the threshold of -3% is about 0.0014. In our sample such an event should only happen about 4 times. Yet it is observed 19 times for the DAX and 17 times for the FTSE return data, that is, 4 to 5 times the amount suggested by the normal distribution. This suggests that the marginal distributions should be modelled as student-t rather than normal. Let us assume that the probability of a return falling short of -3% is 0.006 as suggested by our numbers above. Under normal dependence (normal copula), given a correlation parameter of 0.56, the probability of both return series realizing below -3% is 8.2 in $10,000$. In a sample of $3,000$ we would thus expect roughly 2.5 jointly extremely negative returns. Our DAX/FTSE data offers 7 such instances. Conversely, the event of both returns exceeding $+3\%$ is observed only once in our sample. Using the empirical distributions for the marginals and the t_4-copula implies an expected number of roughly 5.7 jointly extreme negative return-pairs, as well as 5.7 jointly extreme positive return-pairs. Of course the presence of 7 jointly negative extreme return pairs and of 1 jointly positive extreme return pair in the DAX/FTSE data could be due to sampling error. Hence, we have to resort to more powerful tests than the simple one just undertaken. This is what this paper is about. Still we find Table 1 highly suggestive, and it motivates our testing exercise in this paper.

3 Copula Choice

This section gives the definition of a copula, and definitions of tail-dependence and Spearman's rank correlation in terms of copulas. A few useful transformations of copulas and their properties are stated. Then some well-known families of copulas and their properties are discussed. Finally, a flexible, parameterized copula is constructed from these copulas, which meets a set of requirements we believe a copula should have in order to be of interest for financial data in general and for our empirical study in Section 5 in particular. The definitions and results in this section are mostly taken from Nelsen (1999), Joe (1997), and Embrechts *et al.* (2002).

Definition 1 (Copula) *Let F be the joint distribution function of random variables X and Y with marginal distribution functions F_x and F_y, respectively. The copula $C : I^2 \to I$ is defined so as to satisfy*

$$F(x, y) = C(F_x(x), F_y(y))$$

I is the closed unit interval. If F_x and F_y are continuous, then C is unique.

Definition 2 (Symmetry) *A copula is said to be symmetric if* $C(u,v) = C(v,u)$.

Definition 3 (Independence Copula) *U and V are independent if and only if* $C(u,v) = uv = \Pi$.

Fact 1 (Invariance) *If (X,Y) has copula C and h_1, h_2 are increasing, continuous functions, then $(h_1(X), h_2(Y))$ also has copula C.*

Fact 2 (Convex Combination) *A convex combination $C(u,v) = \sum_{i=1}^{n} \lambda_i C_i(u,v)$ of n copulas, with $\sum_{i=1}^{n} \lambda_i = 1$ and $\lambda_i \geq 0$ is again a copula.*

Fact 3 (Density) *Let U and V be standard uniform random variables with copula $C(u,v)$. Then the joint density of U and V is given by*

$$c(u,v) = \frac{\partial^2 C}{\partial u \partial v}(u,v). \tag{2}$$

Fact 4 (Rotation) *Let $\bar{U} = 1 - U$ and $\bar{V} = 1 - V$. Then $\bar{U}$ and $\bar{V}$ are also standard uniform random variables and the following statements are true:*

- *$\bar{U}$ and $\bar{V}$ have copula $C^{--}(u,v) = u + v - 1 + C(1-u, 1-v)$ and density $c^{--}(u,v) = c(1-u, 1-v)$*
- *$\bar{U}$ and V have copula $C^{-+}(u,v) = v - C(1-u, v)$ and density $c^{-+}(u,v) = c(1-u, v)$*
- *U and $\bar{V}$ have copula $C^{+-}(u,v) = u - C(u, 1-v)$ and density $c^{+-}(u,v) = c(u, 1-v)$*

If $C(u,v)$ is symmetric, then $C^{+-}(u,v) = C^{-+}(v,u)$.

Definition 4 (Tail-Dependence) *Let X and Y be random variables with continuous marginal distribution functions F_x and F_y and copula C. The coefficient of upper tail-dependence of X and Y is*

$$\begin{aligned} \tau_U &= \lim_{u \to 0^+} P\left(F_x(X) > 1 - u | F_y(Y) > 1 - u\right) \\ &= \lim_{u \to 0^+} \frac{2u - 1 + C(1-u, 1-u)}{u}. \end{aligned} \tag{3}$$

The coefficient of lower tail-dependence is given by

$$\tau_L = \lim_{u \to 0^+} P(F_x(X) < u | F_y(Y) < u) = \lim_{u \to 0^+} \frac{C(u,u)}{u} \tag{4}$$

If $\tau_U(\tau_L) \in (0,1]$ X and Y are said to be asymptotically dependent in the upper (lower) tail. If $\tau_U = 0$ $(\tau_L = 0)$ they are asymptotically independent in the upper (lower) tail.

Definition 5 (Spearman's Rho) *Let X and Y be random variables with distribution functions F_x and F_y and copula C. Spearman's rank correlation is given by*

$$\rho_S(X,Y) = \rho(F_x(X), F_y(Y)), \tag{5}$$

where ρ is the usual linear correlation operator. Spearman's rank correlation can be expressed in terms of the copula C:

$$\rho_S(X,Y) = 12 \int_0^1 \int_0^1 C(x,y)dxdy - 3. \tag{6}$$

For the independence copula, Π, Spearman's Rho is 0. If two random variables X and Y display a linear correlation of -1 or 1, then their Spearman's Rho is also given by -1 or 1, respectively.

Fact 5 (Properties of Convex Combinations and Rotations) *Let C_1 and C_2 be copulas and let $C = \lambda C_1 + (1-\lambda)C_2$, for $\lambda \in (0,1)$ be a convex combination of the two copulas. By fact 2 C is a copula. It is true that its lower (upper) tail-dependence is the λ-convex combination of the individual coefficients of lower (upper) tail-dependence. The same is true for Spearman's Rho and the density of C. Let C be a copula and let C^{--} be its $180°$-rotation as given in fact 4. Then the lower (upper) tail-dependence of C^{--} is the same as the upper (lower) tail-dependence of C. Also Spearman's Rho of C^{--} is the same as Spearman's Rho of C.*

Definition 6 (Well-Known Copulas) *Various well-known copulas are given below. The first three belong to the class of archimedean copulas, the remaining two are elliptical. Φ^{-1} denotes the inverse of the cumulative distribution function of a standard normal random variable, T_ν^{-1} denotes the inverse of the cumulative distribution function of a student-t random variable with ν degrees of freedom, and Γ is the gamma function. The α in C_α is the vector of all parameters of the copula C_α.*

Copula	$C_\alpha(u,v)$	α
Clayton	$(u^{-\theta}+v^{-\theta}-1)^{-1/\theta}$	$\theta>0$
Gumbel	$\exp(-[(-\ln u)^\theta+(-\ln v)^\theta]^{1/\theta})$	$\theta>1$
Joe	$1-[(1-u)^\theta+(1-v)^\theta-(1-u)^\theta(1-v)^\theta]^{1/\theta}$	$\theta>1$
Gaussian	$\int_{-\infty}^{\phi^{-1}(u)}\int_{-\infty}^{\phi^{-1}(v)}\frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left\{-\frac{s^2-2\rho st+t^2}{2(1-\rho^2)}\right\}dsdt$	$\rho\in(-1,1)$
Student-t	$\int_{-\infty}^{T_\nu^{-1}(u)}\int_{-\infty}^{T_\nu^{-1}(v)}\frac{\Gamma(\frac{\nu+2}{2})}{\frac{\nu}{2}\Gamma(\frac{\nu}{2})\pi(1-\rho^2)^{1/2}}\cdot$	$\rho\in(-1,1)$
	$\cdot\left\{1+\frac{s^2-2\rho st+t^2}{\nu(1-\rho^2)}\right\}^{-(\nu+2)/2}dsdt$	$\nu\geq 2$

Fact 6 (Properties of the above Copulas)

Copula	*range of* ρ_S	τ_L	τ_U
Clayton	$(0,1)$	$2^{-1/\theta}$	0
Gumbel	$(0,1)$	0	$2-2^{1/\theta}$
Joe	$(0,1)$	0	$2-2^{1/\theta}$
		$\tau_L=\tau_U$	
Gaussian	$(-1,1)$	0	
Student-t	$(-1,1)$	$2\left(1-t_{\nu+1}(\frac{\sqrt{\nu+1}\sqrt{1-\rho}}{\sqrt{1+\rho}})\right)$	

The definitions and facts stated above are the basic tools we need in order to construct flexible copulas to be applied to financial data. The brief investigation of the DAX/FTSE return series in Section 2 as well as the results of e.g. Longin and Solnik (2001) suggest that financial data might well show significant non-zero tail-dependence. This tail-dependence may be higher in the lower than in the upper tail. We should thus allow for the existence of asymmetric tail-dependence when specifying a copula model. Neither the normal copula nor the t-copula (see Fact 6) have this feature. Obviously, any degree of (positive) correlation should be possible. Finally, since we are interested in testing normality or t-dependence versus asymmetric tail-dependence, a copula of interest should nest the normal or student-t copula.

Hence, we believe a copula of interest for positively correlated financial data should be flexible enough to allow for (i) asymmetric tail-dependence, (ii) the whole theoretical range of lower tail-dependence ($\tau_L\in[0,1)$), (iii) any degree of (positive) Spearman's rank correlation ($\rho_S\in[0,1)$), and (iv) it should nest either the normal or the student-t copula as a special case. Unfortunately, to our knowledge there is no nice and simple copula which would combine all the above-mentioned properties. We will thus take advantage of the fact that any convex combination of copulas is again a copula (Fact 2) with nice properties (Fact 5) to construct our ideal copula by combining different copulas, of which each exhibits at least one of the desired properties.

One possible choice for a copula which satisfies all the above-mentioned criteria is the convex combination, as given in Definition 6, of e.g. the Clayton and gaussian copulas. Both the Gumbel or Joe copulas (as given in Definition 6) must be rotated by 180° as done in Fact 4, before use, as both copulas exhibit tail-dependence only in the upper tail. A convex combination of the rotated copula and the gaussian copula would then meet all the criteria. The copulas used to test joint normal (joint student-t) dependence in this paper are thus λ-convex combinations of the gaussian (student-t) copula and one the copulas mentioned above, the Clayton, the rotated Gumbel (Gumbel_r), or the rotated Joe (Joe_r) copula.

4 Model, Likelihood and Tests

In this section we first describe the econometric model for the joint distribution of asset returns, which serves as the framework for our testing exercise. We then provide the profile likelihood-ratio test employed for testing one dependence specification versus another and give a brief account of Patton's (2001) hit-test for goodness-of-fit, which we employ to test whether our alternative hypothesis is correctly specified. The model we introduce is a generalization and specialization of the multivariate GARCH model of Bollerslev (1990), and is similar to the bivariate GARCH models suggested by Patton (2001), Rockinger and Jondeau (2001), Hu (2002), and Mashal and Zeevi (2002). Yet our model differs from theirs in the specific copula assumed for the joint distribution of the disturbances, which is tailored (see Section 3) to allow likelihood-based testing of the null hypothesis of normal or student-t dependence versus the alternative hypothesis of asymmetric tail-dependence.

Let $y_{i,t}$, $i = 1, 2$ denote the return series of financial assets. Each return series is assumed to marginally follow a GARCH(1,1)-process (see Bollerslev, 1986). The joint distribution of any two time t disturbances is given by assumed marginal distributions, F_i, and the λ-parameterized convex combination, C, of two copulas.

$$
\begin{aligned}
y_{i,t} &= \mu_i + \sigma_{i,t}\epsilon_{i,t}, && (7)\\
{\sigma_{i,t}}^2 &= \gamma_i + \alpha_i\,(y_{i,t-1} - \mu_i)^2 + \beta_i\,(\sigma_{i,t-1})^2, && (8)\\
F(\epsilon_{1,t}, \epsilon_{2,t}) &= C(F_1(\epsilon_{1,t}), F_2(\epsilon_{2,t})), && (9)\\
C &= (1-\lambda)C^{\text{trad}} + \lambda C^{\text{tail}}. && (10)
\end{aligned}
$$

In our testing exercise we assume the marginal distributions, F_1 and F_2, to be either student-t, as suggested e.g. by Bollerslev (1987), or to be given by the empirical distribution of the fitted disturbances. The normal distribution does not describe the univariate tail-probabilities very well, due to the existence of fat tails in our financial data, and so fails to qualify for a marginal distribution. The actual copula used in our testing exercise is a convex combination of a traditional copula, C^{trad}, and an archimedean copula, C^{tail}. The traditional copula is assumed to be either the gaussian copula, C_ρ with parameter ρ, with zero tail-dependence in both tails, or the t-copula, $C_{\rho,\nu}$ with parameters ρ and ν, with positive but symmetric lower and upper tail-dependence. The archimedean copula is either the Clayton, the rotated Gumbel, or the rotated Joe copula, each with parameter θ, and is such that it generally displays asymmetric tail-dependence. In fact, the archimedean copulas proposed exhibit non-zero tail-dependence only in the lower tail (see Section 3).

Let I_t denote all information available at time t. In the model above this information includes asset returns and variances up to time t. Then conditional on I_{t-1}, the joint density for the observed returns $(y_{1,t}, y_{2,t})$ is given by[2]

$$\begin{aligned} f(y_{1,t}, y_{2,t}|I_{t-1}) &= c\left(F_1\left(\frac{y_{1,t}-\mu_1}{\sigma_{1,t}}\right), F_2\left(\frac{y_{2,t}-\mu_2}{\sigma_{2,t}}\right)\right) \\ &\cdot f_1\left(\frac{y_{1,t}-\mu_1}{\sigma_{1,t}}\right)\frac{1}{\sigma_{1,t}} \cdot f_2\left(\frac{y_{2,t}-\mu_2}{\sigma_{2,t}}\right)\frac{1}{\sigma_{2,t}}, \end{aligned} \tag{11}$$

where $c(u,v) = \frac{\partial^2 C}{\partial u \partial v}(u,v)$ is the density of copula C.

The log-likelihood is now easily obtained as the sum of the logarithm of the above joint densities over all t. Let $\mathbf{\Theta}$ denote the vector of all parameters used in the model given by equations (7-10). The log-likelihood function can then be written as

$$\ell(\mathbf{\Theta}; \{y_{1,t}\}_{t=1}^T, \{y_{2,t}\}_{t=1}^T) = \sum_{t=2}^T \ln\left(f(y_{1,t}, y_{2,t}|I_{t-1})\right), \tag{12}$$

where the density $f(y_{1,t}, y_{1,t}|I_{t-1})$ is given by expression (11).

In the model given by equations (7-10) we take the dependence structure C and thus tail-dependence to be time-invariant. One might want to test whether this assumption is too restrictive. We will do so by allowing λ to change over time and test for the nested case of a constant λ via a likelihood-ratio test. First, we have to specify a functional form for the evolution of λ. In the constant λ situation, the larger λ the larger the potential lower tail-dependence in the resulting convex copula. Consider now the Euclidean distance between a realization in the unit square and the origin. The smaller this distance the closer the return realization is to the negative extreme situation, suggesting higher probability in the negative tail and thus a convex copula which puts more weight on the copula with asymmetrically higher lower tail-dependence. We thus propose the following equation[3] for the evolution of λ, where u and v are the 'uniformed' student t or empirical disturbances from the univariate GARCH(1,1)-models.

$$\lambda_t = \Lambda\left(\delta_1 + \delta_2 \Lambda^{-1}(\lambda_{t-1}) + \delta_3 \frac{1}{M}\sum_{i=1}^M \sqrt{u_{t-i}^2 + v_{t-i}^2}\right), \tag{13}$$

[2]Note that the following is true: If X is a random variable with density f_X and $Y = \mu + \sigma X$, then $f_Y(y) = f_X(\frac{y-\mu}{\sigma})\frac{1}{\sigma}$

[3]This is slightly different from Patton's (2001) time changing copulas. First, Patton makes the tail-dependence itself change over time. Second, the changes are driven by the distance of the uniformed disturbances to the diagonal in the unit square. This makes sense for Patton's data on exchange rates where both tails show dependence, but is not appropriate for our data on stock returns.

where $\Lambda(x) = \frac{1}{1+e^{-x}}$ is the logistic transformation, which maps the real line $\mathbb{R}$ into the unit interval I. At any time t, λ_t is then explained by a constant δ_1, by an autoregressive term $\delta_2\Lambda^{-1}(\lambda_{t-1})$, and by the average distance of the disturbance pairs and the origin over the last M days.

We are interested in the structure of the dependence between extreme events, and in particular whether either the traditional gaussian copula with its implied zero tail-dependence or the t-copula with positive but still symmetric lower and upper tail-dependence are sufficient to capture the dependence structure in bivariate stock returns. Preliminary results from the exploration of the DAX/FTSE data in Section 2 suggest that the observed tail-dependence may indeed be higher than the one implied by the normal copula, and the dependence structure may well be asymmetric. In order to test rigorously for the presence of positive tail-dependence and/or asymmetric dependence in bivariate stock returns, we suggest performing a profile likelihood-ratio (pLR) test. We want to test the null hypothesis that copula C in our econometric model given by equations (7-10) is the gaussian or the t-copula, respectively, i.e. we want to test the null $H_0 : \lambda = 0$ versus the alternative $H_1 : \lambda > 0$.

In this case, the derivation of the asymptotic distribution of the LR-statistic is complicated by two things. First, there are nuisance parameters present under the null hypothesis. These are the parameters in the univariate GARCH models. Second, the subset of the parameter space where $\lambda = 0$ is on the boundary of the parameter space, which means that the distribution of the LR-statistic is not simply asymptotically Chi-square.

To deal with the first problem, we use the profile likelihood (see e.g. Barndorff-Nielsen and Cox, 1994), i.e. the likelihood as a function of λ only, where the random parameter estimates are assumed to be fixed at their estimated levels given λ. Under the null, these parameter estimates will be very accurate given our sample size of close to $3,000$ return pairs. For the profile likelihood the result of Chernoff (1954) holds, that the asymptotic distribution of the pLR-statistic is an even mixture of the degenerate χ^2_0-distribution and a χ^2_2-distribution. The degrees of freedom in the second distribution are 2 since, under the null, there are two parameters which are not estimated, λ and the parameter of the archimedean copula, θ. To give an indication of how well the asymptotic distribution of the pLR-statistic is approximated by this χ^2-mixture, given the true nuisance parameters are replaced by their estimates, we perform a small simulation exercise for the DAX/FTSE model with the rotated Gumbel copula. A pair of return series is generated 100 times from the model with parameters as estimated under the null. For each of these pairs of return series the unrestricted as well as the restricted model is estimated and their pLR-statistic computed. Out of the 100 realizations of the pLR-statistic, 47 were (virtually) zero, the remaining 53 are given in ascending order in table 2. The average of these 53 numbers is 2.2, the estimated variance 4.2. These numbers are close to the theoretical values for the mean and variance of the χ^2_2-distribution of 2 and 4, respectively. Also the higher quantiles are very much in line with the theoretical ones.

To evaluate whether our proposed copula models are correctly specified, we employ the non-parametric hit-test introduced by Patton (2001). In its simplest form this test involves comparing the theoretical and empirical number of realizations of uniformed disturbance-pairs in a set of specifically designed regions of the unit square. These regions are illustrated in Figure 2. We use the regions suggested by Patton (2001), which are chosen to capture potential misspecification in the lower and upper tails. In addition, the more elaborate version of this test allows testing the null of (residual) independence over time. This is of particular interest for our model with a time-invariant copula. In this case any hit in a particular region is regressed on past hits (one day, one week, one month past) using maximum likelihood. The null of no (residual) time-dependence can then be tested for by means of a likelihood-ratio test, testing whether all the coefficients of past hits are zero.

5 Empirical Results

In this section we use data on stock return-pairs to estimate our model given by equations (7-10), and to test our hypotheses. The data consists of daily returns as of 4pm UK time of the DAX 30, the FTSE 100 and the CAC 40 indices for the period August 3, 1990 through December 31, 2001, as reported by Thomson Financial Datastream. We explicitly chose to use only stock indices for which simultaneous price quotations were available, so as to avoid problems resulting from non-synchronicity of price observations.[4] Our sample period covers a total of 2976 observations for each index and includes the stock market crashes following the Asian and Russian crises in 1998 and the terrorist attack in September 2001, as well as the period of internationally declining stock markets starting as of March 2000.

We use Ox version 2.20 (Doornik, 1999) and the return series on the three asset pairs DAX/FTSE, DAX/CAC, and FTSE/CAC to estimate the bivariate return model given by equations (7 - 10). Tables 3 to 6 present

[4]Patton (2001) shows that for the copula representation theorem (Sklar's theorem) to hold it is a sufficient and (often) necessary condition that the information sets in the marginal distributions and the copula be the same.

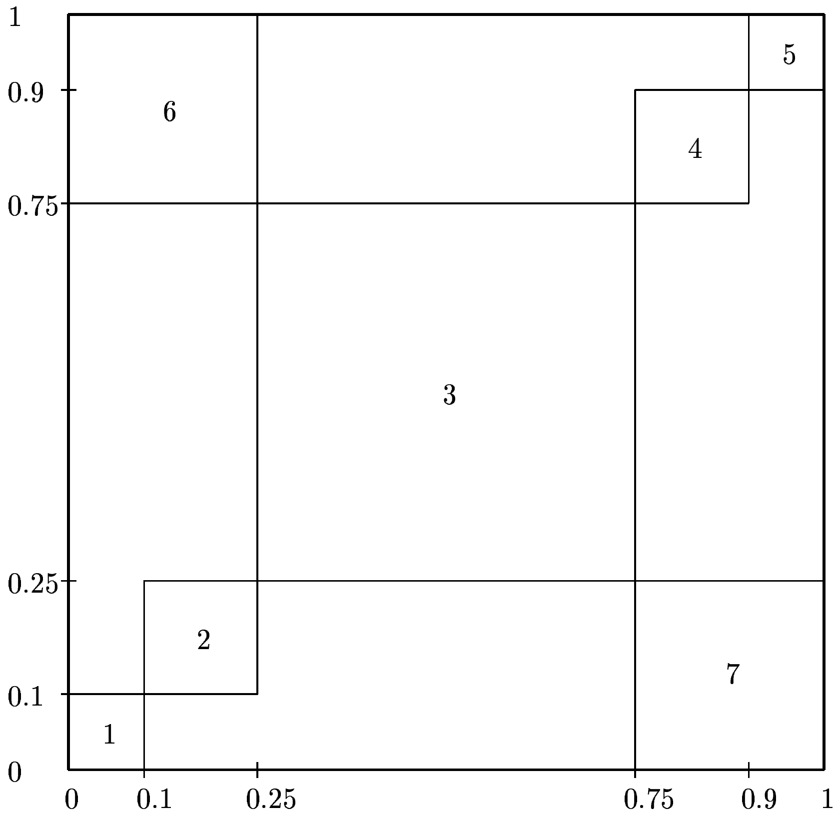

Figure 2: Test regions on the unit square for Patton's (2001) hit-test. There are 8 regions, regions 1 to 7, and one region consisting of the remaining (unnumbered) patches.

estimation results for the various models with a time-invariant copula. In addition to the parameter estimates we also report the coefficient of lower tail-dependence τ_L, the coefficient of upper tail-dependence τ_U in the models involving the t-copula[5], the profile log-likelihood[6], and the profile likelihood-ratio statistic.

Tables 3 and 4 give the results for the models with student-t marginals for the disturbances; Tables 5 and 6 state results for the models where we use

[5] A convex combination of the gaussian copula and any of the three archimedean copulas always displays zero upper tail-dependence, see Section 3.

[6] Not for the models with empirical disturbance distributions. In these cases, for computational convenience we added a constant to the profile log-likelihood, which does not affect the profile log-likelihood-ratio statistic.

the empirical distribution for the marginal distribution of the disturbances. For the latter we still apply the GARCH-filtering to account for the time-changing volatility of returns, but the two univariate GARCH(1,1)-processes and the copula-models are estimated separately now.

For each of the model specifications, we perform the profile likelihood-ratio (pLR) test as described in Section 4. We test the hypothesis that the copula C in the econometric model (7-10) is the gaussian or the t-copula, i.e. we test the null hypothesis $H_0 : \lambda = 0$ against the alternative hypothesis $H_1 : \lambda > 0$.

For any of the three data pairs, for any of the four different specifications of the model, and for any of the three different alternative copulas used, both the hypothesis of normal and that of student-t dependence are strongly rejected. This rejection together with an estimated coefficient for λ ranging from 0.26 to 0.65 for the gaussian, and 0.16 to 0.60 for the t-copula, in the unrestricted models implies that the symmetric copulas exhibit too little difference (in fact none) between lower and upper tail-dependence to describe adequately the dependence structure between DAX, FTSE, and CAC returns. Furthermore, the degree of lower tail-dependence is in general underestimated by both the gaussian and student-t copulas. Our results suggest that the "true" dependence is rather a mixture of the normal or t-copula and a second copula, which exhibits asymmetric tail-dependence, here the Clayton, the rotated Gumbel, or the rotated Joe copula.

Having established the need for a copula allowing for asymmetric dependence, we now turn to test the goodness-of-fit of the estimated asymmetric bivariate return models, using Patton's (2001) hit-test. We do this only for the apparently superior models where the marginal distribution of the disturbances is taken as their empirical distribution. Table 7 reports p-values for model (7-10), where C^{trad} is the gaussian or the t-copula, respectively. We consider any p-value of less than 0.05 as evidence of a model misspecification. Given the number of tests we undertake this is not very conservative. We first perform the hit-test with no lagged information. Of all the models estimated, only the model with the Clayton-gaussian copula for the DAX/CAC return data is misspecified.

The more interesting hit-test is the one where we test whether lagged hits can explain current hits. Even according to this test the majority of models is correctly specified. Three of the gaussian and one of the t-copula models are misspecified given our rejection criterion. Three of these four rejections are observed for the DAX/CAC returns, one for DAX/FTSE returns. Three of these are well-specified, however, if the t-copula instead of the normal one is used. The only misspecified model according to the time-varying version of the hit-test, when the t-copula is taken to be the symmetric copula, is the Clayton-t model for the DAX/CAC data.

At least in the DAX/CAC data, the bivariate return model should probably allow for a time-varying copula. We thus estimate a modified version of our return model as given by equations (7-10), where we let λ_t evolve over time according to equation (13). Parameter estimates for various specifications of this model are shown in Table 8. In addition, the profile likelihood-ratio statistic is given. Interestingly, this test easily rejects all the time-invariant models. The fact that the hit-test in most cases does not reject the time-invariant models, while the profile likelihood-ratio test does, is due to the fact that the pLR-test has much higher power than any non-parametric test, such as the hit-test, if we have a particular parametric alternative model at hand. Of course pLR-tests cannot be used if no such parametric alternative is assumed. For the time-varying models parameterized through equation (13) the hit-test cannot not find any additional time-dependence. Table 7 shows that all the time-changing models pass the goodness-of-fit test.

The bivariate return model as given by equations (7-10) allows to test for volatility spillovers among the three markets if we introduce appropriate exogenous variables in the variance equations (8). The modified variance equation of stock return i is then given by

$$\sigma_{i,t}{}^2 = \gamma_i + \alpha_i \left(y_{i,t-1} - \mu_i\right)^2 + \beta_i \left(\sigma_{i,t-1}\right)^2 + \delta_j \left(y_{j,t-1} - \mu_j\right)^2, \; j \neq i. \quad (14)$$

The general model now allows both conditional return variances to depend on the other market's volatility, respectively. We employ likelihood-ratio tests to determine the direction of spillover effects. Consider for example the DAX/FTSE return pair. We first test the hypothesis that spillovers occur only in one direction, i.e. either from London to Frankfurt, $H_0 : \delta_{DAX} = 0$ (in the FTSE-variance equation), or from Frankfurt to London, $H_0 : \delta_{FTSE} = 0$ (in the variance equation of the DAX returns) against the alternative that both spillover terms are present. We then proceed to test the new null of no spillovers, i.e. $\tilde{H}_0 : \delta_{DAX} = \delta_{FTSE} = 0$, against the alternative that spillovers work in one direction only, where the direction is determined by the outcome of the above tests.

In the DAX/FTSE and the DAX/CAC return data, the results of the test procedure described above are as follows. In both bivariate models the null that spillover effects exist only in one direction, from Frankfurt to London or from Frankfurt to Paris, i.e. lagged volatility in the DAX helps to explain the current variability in both the FTSE and CAC stock indices, cannot be rejected at a significance level of 0.01. When testing the null of zero spillover effects in either direction against these semi-restricted models, the hypothesis of no crossover effects whatsoever is rejected at a significance level of 0.01. Our results, which do not depend on the specific heavy tail-dependent copula used, therefore indicate that volatility transmissions occur from Frankfurt to London, and from Frankfurt to Paris, and not vice versa. These results, at least those for the Frankfurt/London case, contrast with the evidence reported by Kanas (1998), who examines volatility transmissions across London, Frankfurt and Paris, and finds unidirectional spillovers from London to Frankfurt.

To present a thorough picture of the directions of volatility spillovers among European stock exchanges, acknowledging asymmetric dependence at the same time, a more detailed analysis should be undertaken. Exogenous volatility shocks, for example, might better be modelled as averages over a certain period than simply as lagged volatilities. It could also be interesting to analyze the changes in different volatility coefficients, depending on whether and which exogenous shocks are found to be present. Also when analyzing spillovers between three markets one should probably model them simultaneously in a three-variate model.

6 Conclusion

We study the nature of dependence between return pairs on European stock indices. The model of stock-return pairs we use is a bivariate GARCH(1,1)-model with a fairly general dependence structure similar to recent models of Patton (2001), Rockinger and Jondeau (2001), Hu (2002), and Mashal and Zeevi (2002). The dependence between the disturbances is characterized by their marginal distribution, assumed to be either student-t or taken as their empirical distribution, and their copula. The copula in this paper is such that it allows for (i) asymmetric tail-dependence, (ii) any degree of lower tail-dependence, (iii) any positive value for Spearman's Rho, and (iv) such that it nests either the gaussian or the student-t copula, copulas usually used when analyzing financial data.

This model allows us to use write down a profile likelihood test of the hypothesis of either normal or student-t dependence against the alternative of asymmetric tail-dependence. Since we are testing for a parameter being on the boundary of the parameter space, the pLR-statistic has an asymptotic distribution given by an even mixture of the degenerate χ_0^2 and a χ_2^2-distribution. In using the profile likelihood we are assuming away the stochastic nature of the estimates of the nuisance parameters in the model, which are the parameters describing the GARCH-processes for the marginals. A simulation exercise is undertaken to illustrate the validity of this assumption.

The profile likelihood-ratio test easily rejects the assumption of both normal and student-t dependence in any specification of our basic model. The data displays significantly asymmetric tail-dependence as well as higher lower tail-dependence than it would under the null hypothesis of either normal or student-t dependence. Our findings, which are in line with the results of Longin and Solnik (2001), are important for a number of financial applications. The 1-day Value-at-Risk may well be seriously underestimated if normal or student-t dependence is assumed. Also optimal portfolio weights may well differ substantially from the normal or student-t case.

Finally we use Patton's (2001) hit-test to test for misspecification of our alternative model with a time-invariant copula. Most models cannot be rejected on evidence from the hit-test. For the DAX/CAC return pair series, however, the hit-test indicates that at least one model is seriously misspecified. In order to account for the apparent dependence over time we adopt another model which differs from the ones used previously in one respect. The copula is allowed to change over time, in a slightly different manner than in Patton (2001), due to the different nature of our data on stock-returns as opposed to Patton's data on exchange rates. Interestingly, profile likelihood-ratio tests of the null of time-invariance reject the hypothesis even in the models the hit-test did not identify as misspecified. This is due to the fact that tests based on the likelihood function are generally more powerful than any non-parametric tests such as the hit-test. Patton's hit-test, however, is applicable against any alternative. We therefore employ it again for our time-varying model to see whether any additional dependence over time is present in the data and find that according to the hit-test all models are finally correctly specified.

References

[1] Barndorff-Nielsen, O. E. and Cox, D. R. (1994), *Inference and asymptotics* (Chapman & Hall, London).

[2] Bollerslev, T. (1986), Generalized autoregressive conditional heteroskedasticity, *Journal of Econometrics* **31**, 307-327.

[3] Bollerslev, T. (1987), A conditional heteroskedastic time series model for speculative prices and rates of return, *Review of Economics and Statistics* **69**, 542-547.

[4] Bollerslev, T. (1990), Modelling the coherence in short-run nominal exchange rates: A multivariate generalized ARCH model, *Review of Economics and Statistics* **72**, 498-505.

[5] Boyer, B.H., Gibson, M.S. and Loretan, M. (1999): Pitfalls in tests for changes in correlations, International Finance Discussion Papers 597, Board of Governors of the Federal Reserve System.

[6] Chernoff, H. (1954), On the distribution of the likelihood ratio, *Annals of Mathematical Statistics* **25**, 573-578.

[7] Doornik, J.A. (1999), *Object-Oriented Matrix Programming Using Ox*, 3rd ed. London: Timberlake Consultants Press and Oxford: www.nuff.ox.ac.uk/Users/Doornik.

[8] Embrechts, P., McNeil, A. and Straumann, D. (2002), Correlation and dependence in risk management: properties and pitfalls, in Dempster, M.A.H. (ed.), *Risk Management: Value at Risk and Beyond* (Cambridge University Press, Cambridge) 176-223.

[9] Engle, R.F. and Manganelli, S. (1999), CAViaR: Conditional autoregressive value at risk by regression quantiles, NBER working paper, 7341.

[10] Hu, Ling (2002): Dependence patterns across financial markets: Methods and evidence, mimeo, Department of Economics, Ohio State University.

[11] Joe, H. (1997). *Multivariate models and dependence concepts* (Chapman & Hall, London).

[12] Kanas, A. (1998): Volatility spillovers across equity markets: European evidence, *Applied Financial Economics* **8**, 245-256.

[13] Longin, F. and Solnik, B. (2001), Extreme correlation of international equity markets, *Journal of Finance* **56**, 649-76.

[14] Longin, F. and Solnik, B. (1995), Is the Correlation in international equity returns constant: 1960-1990?, *Journal of International Money and Finance* **14**, 3-26.

[15] Loretan, M. and English, W.B. (2000): Evaluating "correlation breakdowns" during periods of market volatility, International Finance Discussion Papers 658, Board of Governors of the Federal Reserve System.

[16] Mashal, R. and Zeevi, Assaf (2002): Beyond Correlation: Extreme Co-movements Between Financial Assets, Working Paper, Columbia Business School, New York.

[17] McNeil, A.J. and Frey, R. (2000), Estimation of tail-related risk measures for heteroscedastic financial time series: An extreme value approach, *Journal of Empirical Finance* **7**, 271-300.

[18] J.P.Morgan (1997), Introduction to CreditMetrics, Technical Document, New York: J.P.Morgan.

[19] Nelsen, R.B. (1999), *An Introduction to copulas*, Springer, New York.

[20] Patton, A. J. (2001), Modelling time-varying exchange rate dependence using the conditional copula, Discussion Paper 01-09, University of California, San Diego.

[21] Rockinger, M. and Jondeau, E. (2001), Conditional dependency of financial series: An application of copulas, Cahier de Recherche du Groupe HEC 723/2001.

[22] Sancetta, A. and Satchell, S. (2001), Bernstein approximations to the copula function and portfolio optimisation, Working Paper, Faculty of Economics and Politics, University of Cambridge, UK.

7 Tables

Table 2: **Simulation Results** for the pLR-statistic, where parameters are taken from the restricted (purely gaussian) bivariate GARCH model for the DAX/FTSE series. Only the 53 positive values (out of the total sample of 100) are shown. The copula of the unrestricted model is the convex combination of the normal and the rotated Gumbel copula.

0.1062	0.11412	0.15676	0.27262	0.3207
0.42113	0.57949	0.60932	0.64102	0.68202
0.7199	0.76298	0.77278	0.82365	0.90562
0.90947	0.94683	1.0357	1.0454	1.0842
1.1885	1.2247	1.2904	1.3902	1.4419
1.4606	1.4795	1.5146	1.5205	1.5875
1.684	1.6899	1.7682	1.7877	2.3405
2.3495	2.6168	2.899	3.0661	3.1596
3.2356	3.4551	3.5562	3.726	3.741
3.8981	3.9807	4.3574	4.4467	5.913
7.085	8.7867	9.7493		

Table 3: **Estimation Results for the Gaussian Copula (1)** for the bivariate return model given by equations (7-10), where margins are student-t, and the copula is a convex combination of the gaussian and one of the Clayton, rotated Gumbel, or rotated Joe copulas.

		Unrestricted			Restricted
		Clayton	Gumbel_r	Joe_r	$(\lambda = 0)$
DAX, FTSE	$\hat{\mu_1}$	0.0715	0.0689	0.0686	0.0667
	$\hat{\mu_2}$	0.0477	0.0448	0.0442	0.0417
	$\hat{\gamma_1}$	0.0167	0.0147	0.0150	0.0182
	$\hat{\gamma_2}$	0.0143	0.0116	0.0118	0.0172
	$\hat{\alpha_1}$	0.0413	0.0365	0.0364	0.0439
	$\hat{\alpha_2}$	0.0404	0.0328	0.0327	0.0413
	$\hat{\beta_1}$	0.9231	0.9308	0.9308	0.9190
	$\hat{\beta_2}$	0.9250	0.9384	0.9380	0.9193
	$\hat{t_1}$	5.8744	5.9291	6.0459	6.4846
	$\hat{t_2}$	9.1996	8.4636	8.3423	8.5529
	$\hat{\lambda}$	0.2835	0.5877	0.3848	0
	$\hat{\theta}$	2.2866	1.3560	1.3599	-
	$\hat{\rho}$	0.5032	0.7894	0.7399	0.5708
	$\hat{\tau_L}$	0.2094	0.1956	0.1290	0
	pLL	-7,816.57	-7,793.57	-7,796.00	-7847.72
	pLR	62.30	108.30	103.44	
DAX, CAC	$\hat{\mu_1}$	0.0718	0.0726	0.0729	0.0655
	$\hat{\mu_2}$	0.0498	0.0506	0.0505	0.0464
	$\hat{\gamma_1}$	0.0181	0.0179	0.0182	0.0201
	$\hat{\gamma_2}$	0.0410	0.0405	0.0411	0.0510
	$\hat{\alpha_1}$	0.0344	0.0338	0.0336	0.0360
	$\hat{\alpha_2}$	0.0303	0.0300	0.0296	0.0327
	$\hat{\beta_1}$	0.9302	0.9306	0.9309	0.9278
	$\hat{\beta_2}$	0.9222	0.9229	0.9232	0.9110
	$\hat{t_1}$	6.2204	5.9783	6.1235	6.7224
	$\hat{t_2}$	8.0029	7.9933	8.0481	8.0105
	$\hat{\lambda}$	0.3918	0.5187	0.3442	0
	$\hat{\theta}$	0.5314	1.4042	1.4063	-
	$\hat{\rho}$	0.8335	0.8560	0.8132	0.6532
	$\hat{\tau_L}$	0.1063	0.1876	0.1249	0
	pLL	-8428.24	-8425.62	-8428.87	-8509.71
	pLR	162.94	168.18	161.68	

Table 3: **Estimation Results for the Gaussian Copula (1)**, ctd.

			Unrestricted		Restricted
		Clayton	Gumbel_r	Joe_r	($\lambda = 0$)
FTSE, CAC	$\hat{\mu_1}$	0.0467	0.0461	0.0466	0.0452
	$\hat{\mu_2}$	0.0624	0.0626	0.0616	0.0529
	$\hat{\gamma_1}$	0.0126	0.0125	0.0124	0.0136
	$\hat{\gamma_2}$	0.0318	0.0306	0.0317	0.0309
	$\hat{\alpha_1}$	0.0367	0.0361	0.0362	0.0374
	$\hat{\alpha_2}$	0.0316	0.0305	0.0313	0.0326
	$\hat{\beta_1}$	0.9331	0.9341	0.9339	0.9299
	$\hat{\beta_2}$	0.9303	0.9321	0.9306	0.9304
	$\hat{t_1}$	9.0482	9.1453	8.9544	8.4245
	$\hat{t_2}$	8.0179	7.7296	7.8796	8.3574
	$\hat{\lambda}$	0.2878	0.4091	0.2588	0
	$\hat{\theta}$	0.5555	1.4410	1.4277	-
	$\hat{\rho}$	0.8048	0.8189	0.7947	0.6816
	$\hat{\tau_L}$	0.0826	0.1564	0.0970	0
	pLL	-7637.22	-7632.78	-7637.87	-7686.21
	pLR	97.98	106.86	96.68	

Table 4: **Estimation Results for the t-Copula (1)** for the bivariate return model given by equations (7-10), where margins are student-t, and the copula is a convex combination of the t-copula and one of the Clayton, rotated Gumbel, or rotated Joe copulas.

		Unrestricted			Restricted
		Clayton	Gumbel_r	Joe_r	$(\lambda = 0)$
DAX, FTSE	$\hat{\mu_1}$	0.0697	0.0696	0.0695	0.0717
	$\hat{\mu_2}$	0.0458	0.0461	0.0459	0.0481
	$\hat{\gamma_1}$	0.0146	0.0144	0.0146	0.0151
	$\hat{\gamma_2}$	0.0111	0.0111	0.0111	0.0118
	$\hat{\alpha_1}$	0.0351	0.0349	0.0350	0.0354
	$\hat{\alpha_2}$	0.0320	0.0320	0.0320	0.0328
	$\hat{\beta_1}$	0.9321	0.9324	0.9322	0.9316
	$\hat{\beta_2}$	0.9400	0.9399	0.9399	0.9380
	$\hat{\lambda}$	0.2146	0.4722	0.2268	0
	$\hat{\theta}$	1.3934	1.4400	1.9074	-
	$\hat{\nu}$	5.7740	5.2996	5.9673	5.6976
	$\hat{\rho}$	0.5681	0.6689	0.5949	0.5765
	$\hat{t_1}$	5.7134	5.6333	5.6817	5.8203
	$\hat{t_2}$	8.3946	8.3381	8.2813	8.1468
	$\hat{\tau_L}$	0.2173	0.2611	0.2170	0.1142
	$\hat{\tau_U}$	0.0867	0.0808	0.0896	0.1142
	pLL	-7793.43	-7791.96	-7792.72	-7802.18
	pLR	17.50	20.44	18.92	
DAX, CAC	$\hat{\mu_1}$	0.0738	0.0741	0.0739	0.0769
	$\hat{\mu_2}$	0.0508	0.0509	0.0509	0.0542
	$\hat{\gamma_1}$	0.0182	0.0181	0.0181	0.0179
	$\hat{\gamma_2}$	0.0402	0.0401	0.0401	0.0392
	$\hat{\alpha_1}$	0.0318	0.0318	0.0316	0.0311
	$\hat{\alpha_2}$	0.0291	0.0293	0.0288	0.0280
	$\hat{\beta_1}$	0.9320	0.9321	0.9324	0.9338
	$\hat{\beta_2}$	0.9239	0.9241	0.9243	0.9264
	$\hat{\lambda}$	0.2435	0.3663	0.2134	0
	$\hat{\theta}$	0.6959	1.4306	1.5815	-
	$\hat{\nu}$	4.3621	4.4879	4.7612	4.2666
	$\hat{\rho}$	0.7347	0.7650	0.7263	0.6605
	$\hat{t_1}$	5.6689	5.6710	5.6477	5.7087
	$\hat{t_2}$	7.5760	7.7140	7.5554	7.5069
	$\hat{\tau_L}$	0.2438	0.2747	0.2467	0.1735
	$\hat{\tau_U}$	0.1538	0.1368	0.1507	0.1735
	pLL	-8420.94	-8421.28	-8421.72	-8431.67
	pLR	21.46	20.78	19.90	

Table 4: **Estimation Results for the t-Copula (1)**, ctd.

		Unrestricted			Restricted
		Clayton	Gumbel_r	Joe_r	$(\lambda = 0)$
FTSE, CAC	$\hat{\mu}_1$	0.0459	0.0453	0.0459	0.0466
	$\hat{\mu}_2$	0.0629	0.0631	0.0627	0.0627
	$\hat{\gamma}_1$	0.0119	0.0122	0.0120	0.0120
	$\hat{\gamma}_2$	0.0292	0.0291	0.0293	0.0292
	$\hat{\alpha}_1$	0.0348	0.0351	0.0347	0.0349
	$\hat{\alpha}_2$	0.0292	0.0291	0.0292	0.0295
	$\hat{\beta}_1$	0.9358	0.9351	0.9358	0.9353
	$\hat{\beta}_2$	0.9336	0.9338	0.9335	0.9332
	$\hat{\lambda}$	0.1847	0.3039	0.1616	0
	$\hat{\theta}$	1.0057	1.5226	1.8433	-
	$\hat{\nu}$	5.9629	6.4342	6.3187	5.5644
	$\hat{\rho}$	0.7249	0.7542	0.7222	0.6863
	$\hat{t}_1$	8.6629	8.7853	8.6234	8.4303
	$\hat{t}_2$	7.0187	7.0221	7.0139	7.0238
	$\hat{\tau}_L$	0.2283	0.2475	0.2191	0.1557
	$\hat{\tau}_U$	0.1355	0.1188	0.1313	0.1557
	pLL	-7629.03	-7628.75	-7629.78	-7637.30
	pLR	16.54	17.10	15.04	

Table 5: **Estimation Results for the Gaussian Copula (2)** for the bivariate return model given by equations (7-10), where margins are given by the empirical distributions, and the copula is a convex combination of the gaussian and one of the Clayton, rotated Gumbel, or rotated Joe copulas. GARCH(1,1) parameter estimates are not given.

		Unrestricted			Restricted
		Clayton	Gumbel$_r$	Joe$_r$	($\lambda = 0$)
DAX, FTSE	$\hat{\lambda}$	0.3265	0.6475	0.3316	0
	$\hat{\theta}$	2.3320	1.9522	2.9007	-
	$\hat{\rho}$	0.4756	0.3163	0.4927	0.5549
	$\hat{\tau_L}$	0.2426	0.3715	0.2421	0
	pLR	97.84	124.70	98.34	
DAX, CAC	$\hat{\lambda}$	0.3264	0.5572	0.3968	0
	$\hat{\theta}$	2.7394	2.5027	1.4833	-
	$\hat{\rho}$	0.5732	0.4254	0.8178	0.6355
	$\hat{\tau_L}$	0.2534	0.3794	0.1604	0
	pLR	97.71	156.96	165.09	
FTSE, CAC	$\hat{\lambda}$	0.3273	0.6075	0.2916	0
	$\hat{\theta}$	0.6804	2.1651	1.5498	-
	$\hat{\rho}$	0.8018	0.5671	0.7888	0.6728
	$\hat{\tau_L}$	0.1182	0.3783	0.1272	0
	pLR	101.35	89.12	101.23	

Table 6: **Estimation Results for the t-Copula (2)** for the bivariate return model given by equations (7-10), where the margins are given by the empirical distributions, and the copula is a convex combination of the t-copula and one of the Clayton, rotated Gumbel, or rotated Joe copulas. GARCH(1,1) parameter estimates are not given.

		Unrestricted			Restricted
		Clayton	Gumbel$_r$	Joe$_r$	($\lambda = 0$)
DAX, FTSE	$\hat{\lambda}$	0.2527	0.5434	0.2525	0
	$\hat{\theta}$	2.2377	1.9595	2.8908	-
	$\hat{\rho}$	0.5116	0.4080	0.5212	0.5665
	$\hat{\nu}$	7.6303	10.9758	7.5469	6.4276
	$\hat{\tau_L}$	0.2852	0.3340	0.2884	0.1948
	$\hat{\tau_U}$	0.0998	0.0212	0.1044	0.1948
	pLR	42.07	46.02	43.32	
DAX, CAC	$\hat{\lambda}$	0.3369	0.6011	0.2993	0
	$\hat{\theta}$	0.7343	1.4745	1.6209	-
	$\hat{\rho}$	0.7573	0.8625	0.7443	0.6479
	$\hat{\nu}$	4.7486	10.1154	5.4849	4.7058
	$\hat{\tau_L}$	0.4054	0.3937	0.3969	0.3198
	$\hat{\tau_U}$	0.2743	0.1533	0.2573	0.3198
	pLR	38.47	51.19	36.89	
FTSE, CAC	$\hat{\lambda}$	0.2308	0.4330	0.2075	0
	$\hat{\theta}$	1.1635	1.6010	1.9882	-
	$\hat{\rho}$	0.7199	0.7689	0.7189	0.6796
	$\hat{\nu}$	6.4997	6.9617	7.0901	6.0591
	$\hat{\tau_L}$	0.3623	0.3921	0.3454	0.2839
	$\hat{\tau_U}$	0.2350	0.1937	0.2245	0.2839
	pLR	27.04	33.07	25.59	

Table 7: **Specification Tests**
This table reports p-values for the test that the models are correctly specified (joint test in 8 regions, see Figure 2). We consider any p-value smaller than 0.05 (asterisked) a rejection of the hypothesis that the model is correctly specified.

	Constant Regressor			Time Series Regressors		
	Clayton	Gumbel_r	Joe_r	Clayton	Gumbel_r	Joe_r
DAX, FTSE						
Gaussian copula	0.0915	0.8109	0.0564	0.0561	0.2655	0.0413*
T-copula	0.7727	0.8609	0.7691	0.2523	0.2848	0.2511
T-copula, λ_t	0.8342	0.8843	0.8411	0.4152	0.4822	0.4145
DAX, CAC						
Gaussian copula	0.0000*	0.1803	0.3716	0.0001*	0.0477*	0.0812
T-copula	0.1240	0.6541	0.3401	0.0367*	0.1313	0.0758
T-copula, λ_t	0.4800	0.4211	0.3683	0.0867	0.1432	0.0635
FTSE, CAC						
Gaussian copula	0.9476	0.5094	0.8938	0.6853	0.4650	0.6481
T-copula	0.9653	0.9333	0.9776	0.7014	0.6741	0.7152
T-copula, λ_t	0.9494	0.8533	0.9116	0.6921	0.6498	0.6577

Table 8: **Estimation Results for the t-Copula (3)** for the bivariate return model given by equations (7-10), where the margins are given by the empirical distributions, the copula is a convex combination of the t-copula and one of the Clayton, rotated Gumbel, or rotated Joe copulas, and λ is modelled to be time-changing as given by equation (13). Profile likelihood ratios for the hypothesis that λ is not time-varying, i.e. $\delta_2 = \delta_3 = 0$, are reported. GARCH(1,1) parameter estimates are not given.

	C^{tail} = Clayton		C^{tail} = Gumbel$_r$		C^{tail} = Joe$_r$	
	unrest.	restricted $\delta_2 = \delta_3 = 0$	unrest.	restricted $\delta_2 = \delta_3 = 0$	unrest.	restricted $\delta_2 = \delta_3 = 0$
DAX, FTSE						
$\hat{\delta}_1$	0.2455	0.2319	0.3303	0.5310	0.2396	0.2381
$\hat{\delta}_2$	0.9631	0	0.9733	0	0.9618	0
$\hat{\delta}_3$	-0.4063	0	-0.4604	0	-0.4023	0
$\hat{\theta}$	3.0490	2.5081	2.1992	1.9722	3.8283	3.0758
$\hat{\rho}$	0.4991	0.5072	0.4109	0.4088	0.5028	0.5156
$\hat{\nu}$	8.6698	7.9684	14.0922	11.1789	8.5905	7.7957
pLR	15.99		29.49		14.86	
DAX, CAC						
$\hat{\delta}_1$	-0.2393	0.3393	0.3673	0.3816	-0.2582	0.3043
$\hat{\delta}_2$	0.9700	0	0.9823	0	0.9670	0
$\hat{\delta}_3$	0.3161	0	-0.5458	0	0.3279	0
$\hat{\theta}$	0.6257	0.7025	3.2765	2.6338	1.4863	1.5747
$\hat{\rho}$	0.8258	0.7625	0.5760	0.5280	0.7990	0.7526
$\hat{\nu}$	8.0473	5.0114	6.7362	7.5470	8.4876	5.9714
pLR	26.24		51.77		21.24	
FTSE, CAC						
$\hat{\delta}_1$	-0.2111	0.2306	-0.2248	0.4333	-0.2046	0.2074
$\hat{\delta}_2$	0.9818	0	0.9821	0	0.9819	0
$\hat{\delta}_3$	0.2647	0	0.3077	0	0.2515	0
$\hat{\theta}$	0.7996	1.1612	1.5444	1.5949	1.6342	1.9833
$\hat{\rho}$	0.7689	0.7200	0.8161	0.7710	0.7626	0.7192
$\hat{\nu}$	8.6176	6.5342	11.9961	7.1570	9.9144	7.1339
pLR	15.31		21.83		14.14	

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